c01.fm Page 3 Friday, December 8, 2006 10:08 AM

1

CONTINUOUS IMAGE MATHEMATICAL CHARACTERIZATION

In the design and analysis of image processing systems, it is convenient and often necessary mathematically to characterize the image to be processed. There are two basic mathematical characterizations of interest: deterministic and statistical. In *deterministic image representation*, a mathematical image function is defined and point properties of the image are considered. For a *statistical image representation*, the image is specified by average properties. The following sections develop the deterministic and statistical characterization of continuous images. Although the analysis is presented in the context of visual images, many of the results can be extended to general two-dimensional time-varying signals and fields.

1.1. IMAGE REPRESENTATION

Let $C(x, y, t, \lambda)$ represent the spatial energy distribution of an image source of radiant energy at spatial coordinates (x, y), at time t and wavelength λ. Because light intensity is a real positive quantity, that is, because intensity is proportional to the modulus squared of the electric field, the image light function is real and nonnegative. Furthermore, in all practical imaging systems, a small amount of background light is always present. The physical imaging system also imposes some restriction on the maximum intensity of an image, for example, film saturation and *cathode ray tube* (*CRT*) phosphor heating. Hence it is assumed that

$$0 < C(x, y, t, \lambda) \le A \tag{1.1-1}$$

Digital Image Processing: PIKS Scientific Inside, Fourth Edition, by William K. Pratt

where A is the maximum image intensity. A physical image is necessarily limited in extent by the imaging system and image recording media. For mathematical simplicity, all images are assumed to be nonzero only over a rectangular region for which

$$-L_x \leq x \leq L_x \tag{1.1-2a}$$

$$-L_y \leq y \leq L_y \tag{1.1-2b}$$

The physical image is, of course, observable only over some finite time interval. Thus, let

$$-T \leq t \leq T \tag{1.1-2c}$$

The image light function $C(x, y, t, \lambda)$ is, therefore, a bounded four-dimensional function with bounded independent variables. As a final restriction, it is assumed that the image function is continuous over its domain of definition.

The intensity response of a standard human observer to an image light function is commonly measured in terms of the instantaneous luminance of the light field as defined by

$$Y(x, y, t) = \int_0^\infty C(x, y, t, \lambda) V(\lambda) \, d\lambda \tag{1.1-3}$$

where $V(\lambda)$ represents the *relative luminous efficiency function*, that is, the spectral response of human vision. Similarly, the color response of a standard observer is commonly measured in terms of a set of tristimulus values that are linearly proportional to the amounts of red, green and blue light needed to match a colored light. For an arbitrary red–green–blue coordinate system, the instantaneous tristimulus values are

$$R(x, y, t) = \int_0^\infty C(x, y, t, \lambda) R_S(\lambda) \, d\lambda \tag{1.1-4a}$$

$$G(x, y, t) = \int_0^\infty C(x, y, t, \lambda) G_S(\lambda) \, d\lambda \tag{1.1-4b}$$

$$B(x, y, t) = \int_0^\infty C(x, y, t, \lambda) B_S(\lambda) \, d\lambda \tag{1.1-4c}$$

where $R_S(\lambda)$, $G_S(\lambda)$, $B_S(\lambda)$ are spectral tristimulus values for the set of red, green and blue primaries. The spectral tristimulus values are, in effect, the tristimulus values required to match a unit amount of narrowband light at wavelength λ. In a multispectral imaging system, the image field observed is modeled as a spectrally weighted integral of the image light function. The ith spectral image field is then given as

$$F_i(x, y, t) = \int_0^{\infty} C(x, y, t, \lambda) S_i(\lambda)\, d\lambda \tag{1.1-5}$$

where $S_i(\lambda)$ is the spectral response of the ith sensor.

For notational simplicity, a single image function $F(x, y, t)$ is selected to represent an image field in a physical imaging system. For a monochrome imaging system, the image function $F(x, y, t)$ nominally denotes the image luminance, or some converted or corrupted physical representation of the luminance, whereas in a color imaging system, $F(x, y, t)$ signifies one of the tristimulus values, or some function of the tristimulus value. The image function $F(x, y, t)$ is also used to denote general three-dimensional fields, such as the time-varying noise of an image scanner.

In correspondence with the standard definition for one-dimensional time signals, the time average of an image function at a given point (x, y) is defined as

$$\langle F(x, y, t) \rangle_T = \lim_{T \to \infty} \left[\frac{1}{2T} \int_{-T}^{T} F(x, y, t) L(t)\, dt \right] \tag{1.1-6}$$

where $L(t)$ is a time-weighting function. Similarly, the average image brightness at a given time is given by the spatial average,

$$\langle F(x, y, t) \rangle_S = \lim_{\substack{L_x \to \infty \\ L_y \to \infty}} \left[\frac{1}{4 L_x L_y} \int_{-L_x}^{L_x} \int_{-L_y}^{L_y} F(x, y, t)\, dx\, dy \right] \tag{1.1-7}$$

In many imaging systems, such as image projection devices, the image does not change with time, and the time variable may be dropped from the image function. For other types of systems, such as movie pictures, the image function is time sampled. It is also possible to convert the spatial variation into time variation, as in television, by an image scanning process. In the subsequent discussion, the time variable is dropped from the image field notation unless specifically required.

1.2. TWO-DIMENSIONAL SYSTEMS

A *two-dimensional system*, in its most general form, is simply a mapping of some input set of two-dimensional functions $F_1(x, y), F_2(x, y), \ldots, F_N(x, y)$ to a set of output two-dimensional functions $G_1(x, y), G_2(x, y), \ldots, G_M(x, y)$, where $(-\infty < x, y < \infty)$ denotes the independent, continuous spatial variables of the functions. This mapping may be represented by the operators $O_m\{\cdot\}$ for $m = 1, 2, \ldots, M$, which relate the input to output set of functions by the set of equations

$$\begin{bmatrix} G_1(x, y) = O_1\{F_1(x, y), F_2(x, y), \ldots, F_N(x, y)\} \\ \vdots \\ G_m(x, y) = O_m\{F_1(x, y), F_2(x, y), \ldots, F_N(x, y)\} \\ \vdots \\ G_M(x, y) = O_M\{F_1(x, y), F_2(x, y), \ldots, F_N(x, y)\} \end{bmatrix} \tag{1.2-1}$$

In specific cases, the mapping may be many-to-few, few-to-many, or one-to-one. The *one-to-one mapping* is defined as

$$G(x, y) = O\{F(x, y)\} \tag{1.2-2}$$

To proceed further with a discussion of the properties of two-dimensional systems, it is necessary to direct the discourse toward specific types of operators.

1.2.1. Singularity Operators

Singularity operators are widely employed in the analysis of two-dimensional systems, especially systems that involve sampling of continuous functions. The two-dimensional *Dirac delta function* is a singularity operator that possesses the following properties:

$$\int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \delta(x, y)\, dx\, dy = 1 \qquad \text{for } \varepsilon > 0 \tag{1.2-3a}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi, \eta)\delta(x - \xi, y - \eta)\, d\xi\, d\eta = F(x, y) \tag{1.2-3b}$$

In Eq. 1.2-3*a*, ε is an infinitesimally small limit of integration; Eq. 1.2-3*b* is called the *sifting property* of the Dirac delta function.

The two-dimensional delta function can be decomposed into the product of two one-dimensional delta functions defined along orthonormal coordinates. Thus

$$\delta(x, y) = \delta(x)\delta(y) \tag{1.2-4}$$

where the one-dimensional delta function satisfies one-dimensional versions of Eq. 1.2-3. The delta function also can be defined as a limit on a family of functions. General examples are given in References 1 and 2.

1.2.2. Additive Linear Operators

A two-dimensional system is said to be an *additive linear system* if the system obeys the law of additive superposition. In the special case of one-to-one mappings, the additive superposition property requires that

$$O\{a_1 F_1(x, y) + a_2 F_2(x, y)\} = a_1 O\{F_1(x, y)\} + a_2 O\{F_2(x, y)\} \tag{1.2-5}$$

where a_1 and a_2 are constants that are possibly complex numbers. This additive superposition property can easily be extended to the general mapping of Eq. 1.2-1.

A system input function $F(x, y)$ can be represented as a sum of amplitude-weighted Dirac delta functions by the sifting integral,

$$F(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta)\delta(x - \xi, y - \eta)\, d\xi\, d\eta \tag{1.2-6}$$

where $F(\xi, \eta)$ is the weighting factor of the impulse located at coordinates (ξ, η) in the x–y plane, as shown in Figure 1.2-1. If the output of a general linear one-to-one system is defined to be

$$G(x, y) = O\{F(x, y)\} \tag{1.2-7}$$

then

$$G(x, y) = O\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta)\delta(x - \xi, y - \eta)\, d\xi\, d\eta\right\} \tag{1.2-8a}$$

or

$$G(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi, \eta) O\{\delta(x - \xi, y - \eta)\}\, d\xi\, d\eta \tag{1.2-8b}$$

In moving from Eq. 1.2-8*a* to Eq. 1.2-8*b*, the application order of the general linear operator $O\{\cdot\}$ and the integral operator have been reversed. Also, the linear operator has been applied only to the term in the integrand that is dependent on the

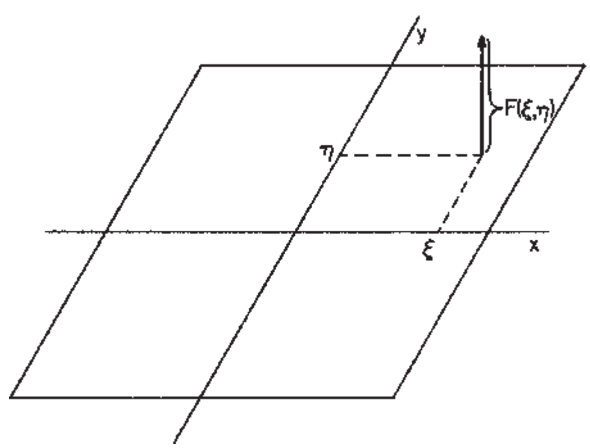

FIGURE 1.2-1. Decomposition of image function.

spatial variables (x, y). The second term in the integrand of Eq. 1.2-8*b*, which is redefined as

$$H(x, y; \xi, \eta) \equiv O\{\delta(x - \xi, y - \eta)\} \tag{1.2-9}$$

is called the *impulse response* of the two-dimensional system. In optical systems, the impulse response is often called the *point spread function* of the system. Substitution of the impulse response function into Eq. 1.2-8*b* yields the additive *superposition integral*

$$G(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi, \eta) H(x, y; \xi, \eta) \, d\xi \, d\eta \tag{1.2-10}$$

An additive linear two-dimensional system is called *space invariant* (isoplanatic) if its impulse response depends only on the factors $x - \xi$ and $y - \eta$. In an optical system, as shown in Figure 1.2-2, this implies that the image of a point source in the focal plane will change only in location, not in functional form, as the placement of the point source moves in the object plane. For a space-invariant system

$$H(x, y; \xi, \eta) = H(x - \xi, y - \eta) \tag{1.2-11}$$

and the superposition integral reduces to the special case called the *convolution integral*, given by

$$G(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi, \eta) H(x - \xi, y - \eta) \, d\xi \, d\eta \tag{1.2-12a}$$

Symbolically,

$$G(x, y) = F(x, y) \circledast H(x, y) \tag{1.2-12b}$$

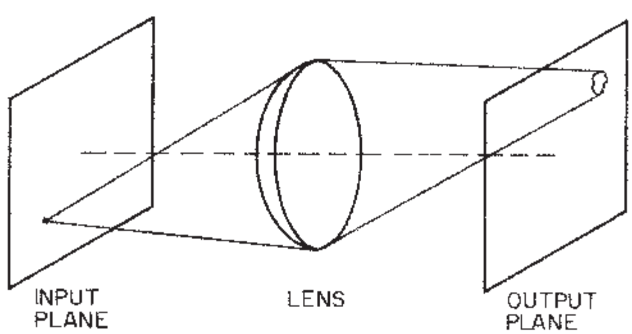

FIGURE 1.2-2. Point-source imaging system.

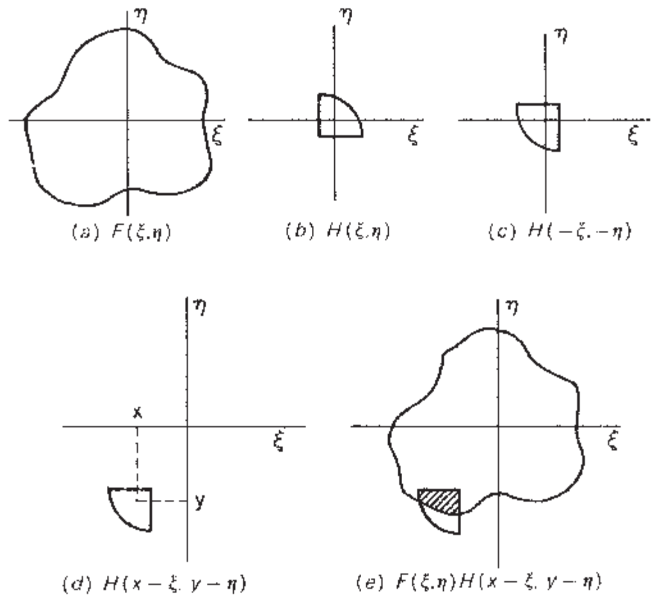

FIGURE 1.2-3. Graphical example of two-dimensional convolution.

denotes the *convolution operation*. The convolution integral is symmetric in the sense that

$$G(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x-\xi, y-\eta) H(\xi, \eta)\, d\xi\, d\eta \tag{1.2-13}$$

Figure 1.2-3 provides a visualization of the convolution process. In Figure 1.2-3*a* and *b*, the input function $F(x, y)$ and impulse response are plotted in the dummy coordinate system (ξ, η). Next, in Figures 1.2-3*c* and *d,* the coordinates of the impulse response are reversed, and the impulse response is offset by the spatial values (x, y). In Figure 1.2-3*e*, the integrand product of the convolution integral of Eq. 1.2-12 is shown as a crosshatched region. The integral over this region is the value of $G(x, y)$ at the offset coordinate (x, y). The complete function $F(x, y)$ could, in effect, be computed by sequentially scanning the reversed, offset impulse response across the input function and simultaneously integrating the overlapped region.

1.2.3. Differential Operators

Edge detection in images is commonly accomplished by performing a spatial differentiation of the image field followed by a thresholding operation to determine points of steep amplitude change. Horizontal and vertical spatial derivatives are defined as

$$d_x = \frac{\partial F(x, y)}{\partial x} \tag{1.2-14a}$$

$$d_y = \frac{\partial F(x, y)}{\partial y} \tag{1.2-14b}$$

The directional derivative of the image field along a vector direction z subtending an angle ϕ with respect to the horizontal axis is given by (3, p. 106)

$$\nabla\{F(x, y)\} = \frac{\partial F(x, y)}{\partial z} = d_x \cos\phi + d_y \sin\phi \tag{1.2-15}$$

The gradient magnitude is then

$$|\nabla\{F(x, y)\}| = \sqrt{d_x^2 + d_y^2} \tag{1.2-16}$$

Spatial second derivatives in the horizontal and vertical directions are defined as

$$d_{xx} = \frac{\partial^2 F(x, y)}{\partial x^2} \tag{1.2-17a}$$

$$d_{yy} = \frac{\partial^2 F(x, y)}{\partial y^2} \tag{1.2-17b}$$

The sum of these two spatial derivatives is called the *Laplacian operator*:

$$\nabla^2\{F(x, y)\} = \frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2} \tag{1.2-18}$$

1.3. TWO-DIMENSIONAL FOURIER TRANSFORM

The two-dimensional *Fourier transform* of the image function $F(x, y)$ is defined as (1,2)

$$\mathcal{F}(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x, y) \exp\{-i(\omega_x x + \omega_y y)\}\, dx\, dy \tag{1.3-1}$$

where ω_x and ω_y are *spatial frequencies* and $i = \sqrt{-1}$. Notationally, the Fourier transform is written as

$$\mathcal{F}(\omega_x, \omega_y) = O_{\mathcal{F}}\{F(x, y)\} \tag{1.3-2}$$

In general, the Fourier coefficient $\mathcal{F}(\omega_x, \omega_y)$ is a complex number that may be represented in real and imaginary form,

$$\mathcal{F}(\omega_x, \omega_y) = \mathcal{R}(\omega_x, \omega_y) + iI(\omega_x, \omega_y) \tag{1.3-3a}$$

or in magnitude and phase-angle form,

$$\mathcal{F}(\omega_x, \omega_y) = \mathcal{M}(\omega_x, \omega_y)\exp\{i\phi(\omega_x, \omega_y)\} \tag{1.3-3b}$$

where

$$\mathcal{M}(\omega_x, \omega_y) = [\mathcal{R}^2(\omega_x, \omega_y) + I^2(\omega_x, \omega_y)]^{1/2} \tag{1.3-4a}$$

$$\phi(\omega_x, \omega_y) = \arctan\left\{\frac{I(\omega_x, \omega_y)}{\mathcal{R}(\omega_x, \omega_y)}\right\} \tag{1.3-4b}$$

A sufficient condition for the existence of the Fourier transform of $F(x, y)$ is that the function be absolutely integrable. That is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x, y)|\, dx\, dy < \infty \tag{1.3-5}$$

The input function $F(x, y)$ can be recovered from its Fourier transform by the inversion formula

$$F(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(\omega_x, \omega_y)\exp\{i(\omega_x x + \omega_y y)\}\, d\omega_x\, d\omega_y \tag{1.3-6a}$$

or in operator form

$$F(x, y) = O_{\mathcal{F}}^{-1}\{\mathcal{F}(\omega_x, \omega_y)\} \tag{1.3-6b}$$

The functions $F(x, y)$ and $\mathcal{F}(\omega_x, \omega_y)$ are called *Fourier transform pairs*.

The two-dimensional Fourier transform can be computed in two steps as a result of the separability of the kernel. Thus, let

$$\mathcal{F}_y(\omega_x, y) = \int_{-\infty}^{\infty} F(x, y)\exp\{-i(\omega_x x)\}\, dx \tag{1.3-7}$$

then

$$\mathcal{F}(\omega_x, \omega_y) = \int_{-\infty}^{\infty} \mathcal{F}_y(\omega_x, y) \exp\{-i(\omega_y y)\}\, dy \tag{1.3-8}$$

Several useful properties of the two-dimensional Fourier transform are stated below. Proofs are given in References 1 and 2.

Separability. If the image function is spatially separable such that

$$F(x, y) = f_x(x) f_y(y) \tag{1.3-9}$$

then

$$\mathcal{F}_y(\omega_x, \omega_y) = f_x(\omega_x) f_y(\omega_y) \tag{1.3-10}$$

where $f_x(\omega_x)$ and $f_y(\omega_y)$ are one-dimensional Fourier transforms of $f_x(x)$ and $f_y(y)$, respectively. Also, if $F(x, y)$ and $\mathcal{F}(\omega_x, \omega_y)$ are two-dimensional Fourier transform pairs, the Fourier transform of $F^*(x, y)$ is $\mathcal{F}^*(-\omega_x, -\omega_y)$. An asterisk* used as a superscript denotes complex conjugation of a variable (i.e. if $F = A + iB$, then $F^* = A - iB$). Finally, if $F(x, y)$ is symmetric such that $F(x, y) = F(-x, -y)$, then $\mathcal{F}(\omega_x, \omega_y) = \mathcal{F}(-\omega_x, -\omega_y)$.

Linearity. The Fourier transform is a linear operator. Thus

$$O_{\mathcal{F}}\{aF_1(x, y) + bF_2(x, y)\} = a\mathcal{F}_1(\omega_x, \omega_y) + b\mathcal{F}_2(\omega_x, \omega_y) \tag{1.3-11}$$

where *a* and *b* are constants.

Scaling. A linear scaling of the spatial variables results in an inverse scaling of the spatial frequencies as given by

$$O_{\mathcal{F}}\{F(ax, by)\} = \frac{1}{|ab|}\mathcal{F}\left(\frac{\omega_x}{a}, \frac{\omega_y}{b}\right) \tag{1.3-12}$$

Hence, stretching of an axis in one domain results in a contraction of the corresponding axis in the other domain plus an amplitude change.

Shift. A positional shift in the input plane results in a phase shift in the output plane:

$$O_{\mathcal{F}}\{F(x - a, y - b)\} = \mathcal{F}(\omega_x, \omega_y) \exp\{-i(\omega_x a + \omega_y b)\} \tag{1.3-13a}$$

Alternatively, a frequency shift in the Fourier plane results in the equivalence

$$O_{\mathcal{F}}^{-1}\{\mathcal{F}(\omega_x - a, \omega_y - b)\} = F(x, y)\exp\{i(ax + by)\} \tag{1.3-13b}$$

Convolution. The two-dimensional Fourier transform of two convolved functions is equal to the products of the transforms of the functions. Thus

$$O_{\mathcal{F}}\{F(x, y) \circledast H(x, y)\} = \mathcal{F}(\omega_x, \omega_y)\mathcal{H}(\omega_x, \omega_y) \tag{1.3-14}$$

The inverse theorem states that

$$O_{\mathcal{F}}\{F(x, y)H(x, y)\} = \frac{1}{4\pi^2}\mathcal{F}(\omega_x, \omega_y) \circledast \mathcal{H}(\omega_x, \omega_y) \tag{1.3-15}$$

Parseval's Theorem. The energy in the spatial and Fourier transform domains is related by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(x, y)|^2 dx\, dy = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathcal{F}(\omega_x, \omega_y)|^2 d\omega_x\, d\omega_y \tag{1.3-16}$$

Autocorrelation Theorem. The Fourier transform of the spatial autocorrelation of a function is equal to the magnitude squared of its Fourier transform. Hence

$$O_{\mathcal{F}}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\alpha, \beta)F^*(\alpha - x, \beta - y)\, d\alpha\, d\beta\right\} = |\mathcal{F}(\omega_x, \omega_y)|^2 \tag{1.3-17}$$

Spatial Differentials. The Fourier transform of the directional derivative of an image function is related to the Fourier transform by

$$O_{\mathcal{F}}\left\{\frac{\partial F(x, y)}{\partial x}\right\} = -i\omega_x \mathcal{F}(\omega_x, \omega_y) \tag{1.3-18a}$$

$$O_{\mathcal{F}}\left\{\frac{\partial F(x, y)}{\partial y}\right\} = -i\omega_y \mathcal{F}(\omega_x, \omega_y) \tag{1.3-18b}$$

Consequently, the Fourier transform of the Laplacian of an image function is equal to

$$O_{\mathcal{F}}\left\{\frac{\partial^2 F(x, y)}{\partial x^2} + \frac{\partial^2 F(x, y)}{\partial y^2}\right\} = -(\omega_x^2 + \omega_y^2)\,\mathcal{F}(\omega_x, \omega_y) \tag{1.3-19}$$

The Fourier transform convolution theorem stated by Eq. 1.3-14 is an extremely useful tool for the analysis of additive linear systems. Consider an image function $F(x, y)$ that is the input to an additive linear system with an impulse response $H(x, y)$. The output image function is given by the convolution integral

$$G(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \beta)H(x-\alpha, y-\beta)\, d\alpha\, d\beta \tag{1.3-20}$$

Taking the Fourier transform of both sides of Eq. 1.3-20 and reversing the order of integration on the right-hand side results in

$$\mathcal{G}(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \beta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-\alpha, y-\beta)\exp\{-i(\omega_x x + \omega_y y)\}\,dx\,dy\right]d\alpha\, d\beta \tag{1.3-21}$$

By the Fourier transform shift theorem of Eq. 1.3-13, the inner integral is equal to the Fourier transform of $H(x, y)$ multiplied by an exponential phase-shift factor. Thus

$$\mathcal{G}(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\alpha, \beta)\mathcal{H}(\omega_x, \omega_y)\exp\{-i(\omega_x\alpha + \omega_y\beta)\}\, d\alpha\, d\beta \tag{1.3-22}$$

Performing the indicated Fourier transformation gives

$$\mathcal{G}(\omega_x, \omega_y) = \mathcal{H}(\omega_x, \omega_y)\,\mathcal{F}(\omega_x, \omega_y) \tag{1.3-23}$$

Then an inverse transformation of Eq. 1.3-23 provides the output image function

$$G(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{H}(\omega_x, \omega_y)\,\mathcal{F}(\omega_x, \omega_y)\exp\{i(\omega_x x + \omega_y y)\}\, d\omega_x\, d\omega_y \tag{1.3-24}$$

Equations 1.3-20 and 1.3-24 represent two alternative means of determining the output image response of an additive, linear, space-invariant system. The analytic or operational choice between the two approaches, convolution or Fourier processing, is usually problem dependent.

1.4. IMAGE STOCHASTIC CHARACTERIZATION

The following presentation on the statistical characterization of images assumes general familiarity with probability theory, random variables and stochastic processes. References 2 and 4 to 7 can provide suitable background. The primary purpose of the discussion here is to introduce notation and develop stochastic image models.

It is often convenient to regard an image as a sample of a stochastic process. For continuous images, the image function $F(x, y, t)$ is assumed to be a member of a continuous three-dimensional stochastic process with space variables (x, y) and time variable (t).

The stochastic process $F(x, y, t)$ can be described completely by knowledge of its *joint probability density*

$$p\{F_1, F_2 \ldots, F_J; x_1, y_1, t_1, x_2, y_2, t_2, \ldots, x_J, y_J, t_J\}$$

for all sample points J, where (x_j, y_j, t_j) represent space and time samples of image function $F_j(x_j, y_j, t_j)$. In general, high-order joint probability densities of images are usually not known, nor are they easily modeled. The first-order probability density $p(F; x, y, t)$ can sometimes be modeled successfully on the basis of the physics of the process or histogram measurements. For example, the first-order probability density of random noise from an electronic sensor is usually well modeled by a *Gaussian density* of the form

$$p\{F; x, y, t\} = [2\pi\sigma_F^2(x, y, t)]^{-1/2} \exp\left\{-\frac{[F(x, y, t) - \eta_F(x, y, t)]^2}{2\sigma_F^2(x, y, t)}\right\} \qquad (1.4\text{-}1)$$

where the parameters $\eta_F(x, y, t)$ and $\sigma_F^2(x, y, t)$ denote the mean and variance of the process. The Gaussian density is also a reasonably accurate model for the probability density of the amplitude of unitary transform coefficients of an image. The probability density of the luminance function must be a one-sided density because the luminance measure is positive. Models that have found application include the *Rayleigh density*,

$$p\{F; x, y, t\} = \frac{F(x, y, t)}{\alpha^2} \exp\left\{-\frac{[F(x, y, t)]^2}{2\alpha^2}\right\} \qquad (1.4\text{-}2a)$$

the *log-normal density*,

$$p\{F; x, y, t\} = [2\pi F^2(x, y, t)\sigma_F^2(x, y, t)]^{-1/2} \exp\left\{-\frac{[\log\{F(x, y, t)\} - \eta_F(x, y, t)]^2}{2\sigma_F^2(x, y, t)}\right\} \qquad (1.4\text{-}2b)$$

and the *exponential density*,

$$p\{F; x, y, t\} = \alpha \exp\{-\alpha|F(x, y, t)|\} \qquad (1.4\text{-}2c)$$

all defined for $F \geq 0,$ where α is a constant. The *two-sided*, or *Laplacian density*,

$$p\{F;\ x, y, t\} = \frac{\alpha}{2} \exp\{-\alpha|F(x, y, t)|\} \tag{1.4-3}$$

where α is a constant, is often selected as a model for the probability density of the difference of image samples. Finally, the *uniform density*

$$p\{F; x, y, t\} = \frac{1}{2\pi} \tag{1.4-4}$$

for $-\pi \leq F \leq \pi$ is a common model for phase fluctuations of a random process. Conditional probability densities are also useful in characterizing a stochastic process. The *conditional density* of an image function evaluated at (x_1, y_1, t_1) given knowledge of the image function at (x_2, y_2, t_2) is defined as

$$p\{F_1; x_1, y_1, t_1 | F_2; x_2, y_2, t_2\} = \frac{p\{F_1, F_2; x_1, y_1, t_1, x_2, y_2, t_2\}}{p\{F_2; x_2, y_2, t_2\}} \tag{1.4-5}$$

Higher-order conditional densities are defined in a similar manner.

Another means of describing a stochastic process is through computation of its ensemble averages. The *first moment* or *mean* of the image function is defined as

$$\eta_F(x, y, t) = E\{F(x, y, t)\} = \int_{-\infty}^{\infty} F(x, y, t) p\{F; x, y, t\}\, dF \tag{1.4-6}$$

where $E\{\cdot\}$ is the *expectation operator*, as defined by the right-hand side of Eq. 1.4-6.

The *second moment* or *autocorrelation function* is given by

$$R(x_1, y_1, t_1; x_2, y_2, t_2) = E\{F(x_1, y_1, t_1) F^*(x_2, y_2, t_2)\} \tag{1.4-7a}$$

or in explicit form

$$R(x_1, y_1, t_1; x_2, y_2, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x_1, x_1, y_1) F^*(x_2, y_2, t_2)$$

$$\times p\{F_1, F_2; x_1, y_1, t_1, x_2, y_2, t_2\} dF_1 dF_2 \tag{1.4-7b}$$

The autocovariance of the image process is the autocorrelation about the mean, defined as

$$K(x_1, y_1, t_1; x_2, y_2, t_2) = E\{[F(x_1, y_1, t_1) - \eta_F(x_1, y_1, t_1)][F^*(x_2, y_2, t_2) - \eta_F^*(x_2, y_2, t_2)]\} \tag{1.4-8a}$$

or

$$K(x_1, y_1, t_1; x_2, y_2, t_2) = R(x_1, y_1, t_1; x_2, y_2, t_2) - \eta_F(x_1, y_1, t_1)\,\eta_F^*(x_2, y_2, t_2) \tag{1.4-8b}$$

Finally, the *variance* of an image process is

$$\sigma_F^2(x, y, t) = K(x, y, t; x, y, t) \tag{1.4-9}$$

An image process is called *stationary in the strict sense* if its moments are unaffected by shifts in the space and time origins. The image process is said to be *stationary in the wide sense* if its mean is constant and its autocorrelation is dependent on the differences in the image coordinates, $x_1 - x_2$, $y_1 - y_2$, $t_1 - t_2$, and not on their individual values. In other words, the image autocorrelation is not a function of position or time. For stationary image processes,

$$E\{F(x, y, t)\} = \eta_F \tag{1.4-10a}$$

$$R(x_1, y_1, t_1; x_2, y_2, t_2) = R(x_1 - x_2, y_1 - y_2, t_1 - t_2) \tag{1.4-10b}$$

The autocorrelation expression may then be written as

$$R(\tau_x, \tau_y, \tau_t) = E\{F(x + \tau_x, y + \tau_y, t + \tau_t)F^*(x, y, t)\} \tag{1.4-11}$$

Because

$$R(-\tau_x, -\tau_y, -\tau_t) = R^*(\tau_x, \tau_y, \tau_t) \tag{1.4-12}$$

then for an image function with F real, the autocorrelation is real and an even function of τ_x, τ_y, τ_t. The *power spectral density*, also called the *power spectrum*, of a stationary image process is defined as the three-dimensional Fourier transform of its autocorrelation function as given by

$$\mathcal{W}(\omega_x, \omega_y, \omega_t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\tau_x, \tau_y, \tau_t) \exp\{-i(\omega_x\tau_x + \omega_y\tau_y + \omega_t\tau_t)\}\, d\tau_x\, d\tau_y\, d\tau_t \tag{1.4-13}$$

In many imaging systems, the spatial and time image processes are separable so that the stationary correlation function may be written as

$$R(\tau_x, \tau_y, \tau_t) = R_{xy}(\tau_x, \tau_y) R_t(\tau_t) \tag{1.4-14}$$

Furthermore, the spatial autocorrelation function is often considered as the product of *x* and *y* axis autocorrelation functions,

$$R_{xy}(\tau_x, \tau_y) = R_x(\tau_x) R_y(\tau_y) \tag{1.4-15}$$

for computational simplicity. For scenes of manufactured objects, there is often a large amount of horizontal and vertical image structure, and the spatial separation approximation may be quite good. In natural scenes, there usually is no preferential direction of correlation; the spatial autocorrelation function tends to be rotationally symmetric and not separable.

An image field is often modeled as a sample of a first-order Markov process for which the correlation between points on the image field is proportional to their geometric separation. The *autocovariance* function for the two-dimensional Markov process is

$$R_{xy}(\tau_x, \tau_y) = C \exp\left\{-\sqrt{\alpha_x^2 \tau_x^2 + \alpha_y^2 \tau_y^2}\right\} \tag{1.4-16}$$

where C is an energy scaling constant and α_x and α_y are spatial scaling constants. The corresponding power spectrum is

$$\mathcal{W}(\omega_x, \omega_y) = \frac{1}{\sqrt{\alpha_x \alpha_y}} \frac{2C}{1 + [\omega_x^2/\alpha_x^2 + \omega_y^2/\alpha_y^2]} \tag{1.4-17}$$

As a simplifying assumption, the Markov process is often assumed to be of separable form with an autocovariance function

$$K_{xy}(\tau_x, \tau_y) = C \exp\{-\alpha_x|\tau_x| - \alpha_y|\tau_y|\} \tag{1.4-18}$$

The power spectrum of this process is

$$\mathcal{W}(\omega_x, \omega_y) = \frac{4\alpha_x \alpha_y C}{(\alpha_x^2 + \omega_x^2)(\alpha_y^2 + \omega_y^2)} \tag{1.4-19}$$

In the discussion of the deterministic characteristics of an image, both time and space averages of the image function have been defined. An ensemble average has also been defined for the statistical image characterization. A question of interest is: What is the relationship between the spatial-time averages and the ensemble averages? The answer is that for certain stochastic processes, which are called *ergodic processes*, the spatial-time averages and the ensemble averages are equal. Proof of the ergodicity of a process in the general case is often difficult; it usually suffices to determine second-order ergodicity in which the first- and second-order space-time averages are equal to the first- and second-order ensemble averages.

Often, the probability density or moments of a stochastic image field are known at the input to a system, and it is desired to determine the corresponding information at the system output. If the system transfer function is algebraic in nature, the output probability density can be determined in terms of the input probability density by a probability density transformation. For example, let the system output be related to the system input by

$$G(x, y, t) = O_F\{F(x, y, t)\} \tag{1.4-20}$$

where $O_F\{\cdot\}$ is a monotonic operator on $F(x, y)$. The probability density of the output field is then

$$p\{G; x, y, t\} = \frac{p\{F; x, y, t\}}{\left|dO_F\{F(x, y, t)\}/dF\right|} \tag{1.4-21}$$

The extension to higher-order probability densities is straightforward, but often cumbersome.

The moments of the output of a system can be obtained directly from knowledge of the output probability density, or in certain cases, indirectly in terms of the system operator. For example, if the system operator is additive linear, the mean of the system output is

$$E\{G(x, y, t)\} = E\{O_F\{F(x, y, t)\}\} = O_F\{E\{F(x, y, t)\}\} \tag{1.4-22}$$

It can be shown that if a system operator is additive linear, and if the system input image field is stationary in the strict sense, the system output is also stationary in the strict sense. Furthermore, if the input is stationary in the wide sense, the output is also wide-sense stationary.

Consider an additive linear space-invariant system whose output is described by the three-dimensional convolution integral

$$G(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x-\alpha, y-\beta, t-\gamma) H(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma \qquad (1.4\text{-}23)$$

where $H(x, y, t)$ is the system impulse response. The mean of the output is then

$$E\{G(x, y, t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{F(x-\alpha, y-\beta, t-\gamma)\} H(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma \qquad (1.4\text{-}24)$$

If the input image field is stationary, its mean η_F is a constant that may be brought outside the integral. As a result,

$$E\{G(x, y, t)\} = \eta_F \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma = \eta_F\, \mathcal{H}(0, 0, 0) \qquad (1.4\text{-}25)$$

where $\mathcal{H}(0, 0, 0)$ is the transfer function of the linear system evaluated at the origin in the spatial-time frequency domain. Following the same techniques, it can easily be shown that the autocorrelation functions of the system input and output are related by

$$R_G(\tau_x, \tau_y, \tau_t) = R_F(\tau_x, \tau_y, \tau_t) \circledast H(\tau_x, \tau_y, \tau_t) \circledast H^*(-\tau_x, -\tau_y, -\tau_t) \qquad (1.4\text{-}26)$$

Taking Fourier transforms on both sides of Eq. 1.4-26 and invoking the Fourier transform convolution theorem, one obtains the relationship between the power spectra of the input and output image,

$$\mathcal{W}_G(\omega_x, \omega_y, \omega_t) = \mathcal{W}_F(\omega_x, \omega_y, \omega_t)\mathcal{H}(\omega_x, \omega_y, \omega_t)\mathcal{H}^*(\omega_x, \omega_y, \omega_t) \qquad (1.4\text{-}27a)$$

or

$$\mathcal{W}_G(\omega_x, \omega_y, \omega_t) = \mathcal{W}_F(\omega_x, \omega_y, \omega_t)\left|\mathcal{H}(\omega_x, \omega_y, \omega_t)\right|^2 \qquad (1.4\text{-}27b)$$

This result is found useful in analyzing the effect of noise in imaging systems.

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