1

Introduction: Basic Concepts and Terminology

1.1 CONCEPT OF VIBRATION

Any repetitive motion is called *vibration* or *oscillation*. The motion of a guitar string, motion felt by passengers in an automobile traveling over a bumpy road, swaying of tall buildings due to wind or earthquake, and motion of an airplane in turbulence are typical examples of vibration. The theory of vibration deals with the study of oscillatory motion of bodies and the associated forces. The oscillatory motion shown in Fig. 1.1(a) is called *harmonic motion* and is denoted as

$$x(t) = X \cos \omega t \tag{1.1}$$

where X is called the *amplitude of motion*, ω is the *frequency of motion*, and t is the time. The motion shown in Fig. 1.1(b) is called *periodic motion*, and that shown in Fig. 1.1(c) is called *nonperiodic* or *transient motion*. The motion indicated in Fig. 1.1(d) is *random* or *long-duration nonperiodic vibration*.

The phenomenon of vibration involves an alternating interchange of potential energy to kinetic energy and kinetic energy to potential energy. Hence, any vibrating system must have a component that stores potential energy and a component that stores kinetic energy. The components storing potential and kinetic energies are called a *spring* or *elastic element* and a *mass* or *inertia element*, respectively. The elastic element stores potential energy and gives it up to the inertia element as kinetic energy, and vice versa, in each cycle of motion. The repetitive motion associated with vibration can be explained through the motion of a mass on a smooth surface, as shown in Fig. 1.2. The mass is connected to a linear spring and is assumed to be in equilibrium or rest at position 1. Let the mass m be given an initial displacement to position 2 and released with zero velocity. At position 2, the spring is in a maximum elongated condition, and hence the potential or strain energy of the spring is a maximum and the kinetic energy of the mass will be zero since the initial velocity is assumed to be zero. Because of the tendency of the spring to return to its unstretched condition, there will be a force that causes the mass m to move to the left. The velocity of the mass will gradually increase as it moves from position 2 to position 1. At position 1, the potential energy of the spring is zero because the deformation of the spring is zero. However, the kinetic energy and hence the velocity of the mass will be maximum at position 1 because of conservation of energy (assuming no dissipation of energy due to damping or friction). Since the velocity is maximum at position 1, the mass will

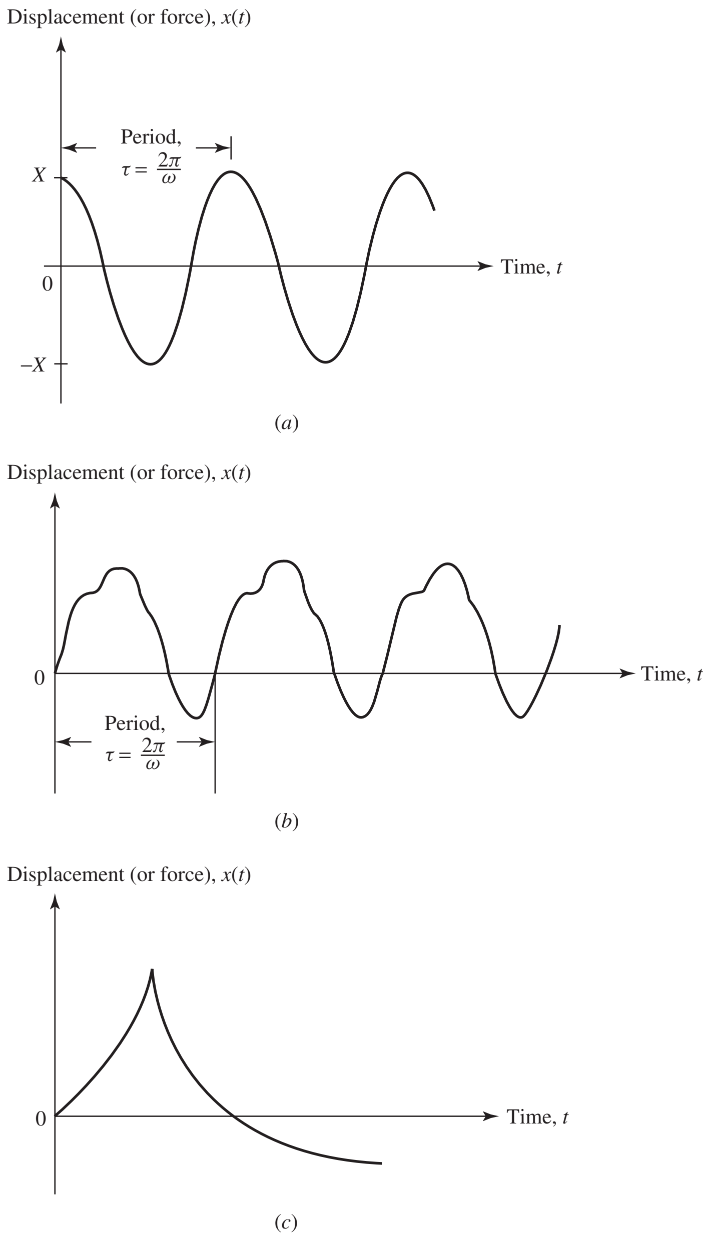
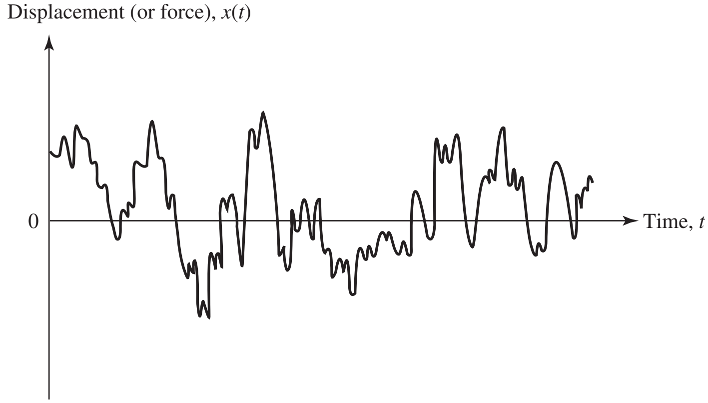


Figure 1.1 Types of displacements (or forces): (*a*) periodic simple harmonic; (*b*) periodic, nonharmonic; (*c*) nonperiodic, transient; (*d*) nonperiodic, random.



(d)

Figure 1.1 (*continued*)

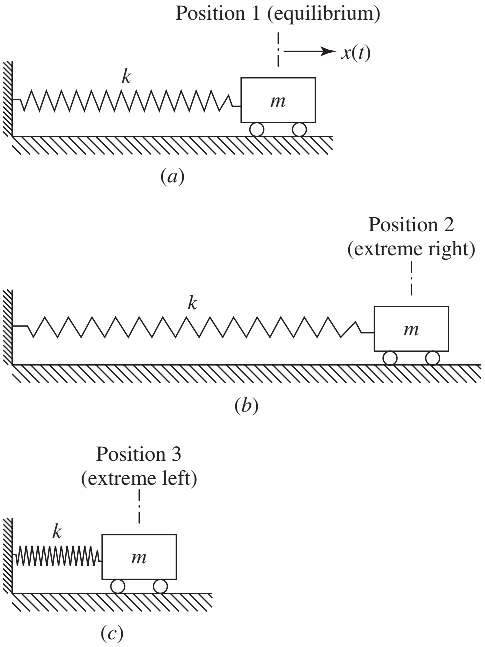


Figure 1.2 Vibratory motion of a spring–mass system: (a) system in equilibrium (spring undeformed); (b) system in extreme right position (spring stretched); (c) system in extreme left position (spring compressed).

continue to move to the left, but against the resisting force due to compression of the spring. As the mass moves from position 1 to the left, its velocity will gradually decrease until it reaches a value of zero at position 3. At position 3 the velocity and hence the kinetic energy of the mass will be zero and the deflection (compression) and hence the potential energy of the spring will be maximum. Again, because of the tendency of the spring to return to its uncompressed condition, there will be a force that causes the mass *m* to move to the right from position 3. The velocity of the mass will increase gradually as it moves from position 3 to position 1. At position 1, all of the potential energy of the spring has been converted to the kinetic energy of the mass, and hence the velocity of the mass will be maximum. Thus, the mass continues to move to the right against increasing spring resistance until it reaches position 2 with zero velocity. This completes one cycle of motion of the mass, and the process repeats; thus, the mass will have oscillatory motion.

The initial excitation to a vibrating system can be in the form of initial displacement and/or initial velocity of the mass element(s). This amounts to imparting potential and/or kinetic energy to the system. The initial excitation sets the system into oscillatory motion, which can be called *free vibration*. During free vibration, there will be exchange between potential and kinetic energies. If the system is conservative, the sum of potential energy and kinetic energy will be a constant at any instant. Thus, the system continues to vibrate forever, at least in theory. In practice, there will be some damping or friction due to the surrounding medium (e.g., air), which will cause loss of some energy during motion. This causes the total energy of the system to diminish continuously until it reaches a value of zero, at which point the motion stops. If the system is given only an initial excitation, the resulting oscillatory motion eventually will come to rest for all practical systems, and hence the initial excitation is called *transient excitation* and the resulting motion is called *transient motion*. If the vibration of the system is to be maintained in a steady state, an external source must replace continuously the energy dissipated due to damping.

1.2 IMPORTANCE OF VIBRATION

Any body having mass and elasticity is capable of oscillatory motion. In fact, most human activities, including hearing, seeing, talking, walking, and breathing, also involve oscillatory motion. Hearing involves vibration of the eardrum, seeing is associated with the vibratory motion of light waves, talking requires oscillations of the laryng (tongue), walking involves oscillatory motion of legs and hands, and breathing is based on the periodic motion of lungs. In engineering, an understanding of the vibratory behavior of mechanical and structural systems is important for the safe design, construction, and operation of a variety of machines and structures.

The failure of most mechanical and structural elements and systems can be associated with vibration. For example, the blade and disk failures in steam and gas turbines and structural failures in aircraft are usually associated with vibration and the resulting fatigue. Vibration in machines leads to rapid wear of parts such as gears and bearings, loosening of fasteners such as nuts and bolts, poor surface finish during metal cutting, and excessive noise. Excessive vibration in machines causes not only the failure of components and systems but also annoyance to humans. For example, imbalance in diesel engines can cause ground waves powerful enough to create a nuisance in urban areas. Supersonic aircraft create sonic booms that shatter doors and windows. Several spectacular failures of bridges, buildings, and dams are associated with wind-induced vibration, as well as oscillatory ground motion during earthquakes.

In some engineering applications, vibrations serve a useful purpose. For example, in vibratory conveyors, sieves, hoppers, compactors, dentist drills, electric toothbrushes, washing machines, clocks, electric massaging units, pile drivers, vibratory testing of materials, vibratory finishing processes, and materials processing operations such as casting and forging, vibration is used to improve the efficiency and quality of the process.

1.3 ORIGINS AND DEVELOPMENTS IN MECHANICS AND VIBRATION

The earliest human interest in the study of vibration can be traced to the time when the first musical instruments, probably whistles or drums, were discovered. Since that time, people have applied ingenuity and critical investigation to study the phenomenon of vibration and its relation to sound. Although certain very definite rules were observed in the art of music, even in ancient times, they can hardly be called science. The ancient Egyptians used advanced engineering concepts such as the use of dovetailed cramps and dowels in the stone joints of major structures such as the pyramids during the third and second millennia B.C.

As far back as 4000 B.C., music was highly developed and well appreciated in China, India, Japan, and perhaps Egypt [1, 6]. Drawings of stringed instruments such as harps appeared on the walls of Egyptian tombs as early as 3000 B.C. The British Museum also has a nanga, a primitive stringed instrument from 155 B.C. The present system of music is considered to have arisen in ancient Greece.

The scientific method of dealing with nature and the use of logical proofs for abstract propositions began in the time of Thales of Miletos (640–546 B.C.), who introduced the term *electricity* after discovering the electrical properties of yellow amber. The first person to investigate the scientific basis of musical sounds is considered to be the Greek mathematician and philosopher Pythagoras (582–507 B.C.). Pythagoras established the Pythagorean school, the first institute of higher education and scientific research. Pythagoras conducted experiments on vibrating strings using an apparatus called the monochord. Pythagoras found that if two strings of identical properties but different lengths are subject to the same tension, the shorter string produces a higher note, and in particular, if the length of the shorter string is one-half that of the longer string, the shorter string produces a note an octave above the other. The concept of pitch was known by the time of Pythagoras; however, the relation between the pitch and the frequency of a sounding string was not known at that time. Only in the sixteenth century, around the time of Galileo, did the relation between pitch and frequency become understood [2].

Daedalus is considered to have invented the pendulum in the middle of the second millennium B.C. One initial application of the pendulum as a timing device was made by Aristophanes (450–388 B.C.). Aristotle wrote a book on sound and music around 350 B.C. and documents his observations in statements such as “the voice is sweeter than the sound of instruments” and “the sound of the flute is sweeter than that of the lyre.” Aristotle recognized the vectorial character of forces and introduced the concept of vectorial addition of forces. In addition, he studied the laws of motion, similar to those of Newton. Aristoxenus, who was a musician and a student of Aristotle, wrote a three-volume book called *Elements of Harmony*. These books are considered the oldest books available on the subject of music. Alexander of Afrodisias introduced the ideas of potential and kinetic energies and the concept of conservation of energy. In about 300 B.C., in addition to his contributions to geometry, Euclid gave a brief description of music in a treatise called *Introduction to Harmonics*. However, he did not discuss the physical nature of sound in the book. Euclid was distinguished for his teaching ability, and his greatest work, the *Elements*, has seen numerous editions and remains one of the most influential books of mathematics of all time. Archimedes (287–212 B.C.) is called by some scholars the father of mathematical physics. He developed the rules of statics. In his *On Floating Bodies*, Archimedes developed major rules of fluid pressure on a variety of shapes and on buoyancy.

China experienced many deadly earthquakes in ancient times. Zhang Heng, a historian and astronomer of the second century A.D., invented the world's first seismograph to measure earthquakes in A.D. 132 [3]. This seismograph was a bronze vessel in the form of a wine jar, with an arrangement consisting of pendulums surrounded by a group of eight lever mechanisms pointing in eight directions. Eight dragon figures, with a bronze ball in the mouth of each, were arranged outside the jar. An earthquake in any direction would tilt the pendulum in that direction, which would cause the release of the bronze ball in that direction. This instrument enabled monitoring personnel to know the direction, time of occurrence, and perhaps, the magnitude of the earthquake.

The foundations of modern philosophy and science were laid during the sixteenth century; in fact, the seventeenth century is called the *century of genius* by many. Galileo (1564–1642) laid the foundations for modern experimental science through his measurements on a simple pendulum and vibrating strings. During one of his trips to the church in Pisa, the swinging movements of a lamp caught Galileo's attention. He measured the period of the pendulum movements of the lamp with his pulse and was amazed to find that the time period was not influenced by the amplitude of swings. Subsequently, Galileo conducted more experiments on the simple pendulum and published his findings in *Discourses Concerning Two New Sciences* in 1638. In this work, he discussed the relationship between the length and the frequency of vibration of a simple pendulum, as well as the idea of sympathetic vibrations or resonance [4].

Although the writings of Galileo indicate that he understood the interdependence of the parameters—length, tension, density and frequency of transverse vibration—of a string, they did not offer an analytical treatment of the problem. Marinus Mersenne (1588–1648), a mathematician and theologian from France, described the correct behavior of the vibration of strings in 1636 in his book *Harmonicorum Liber*. For the first time, by knowing (measuring) the frequency of vibration of a long string, Mersenne was able to predict the frequency of vibration of a shorter string having the same density and tension. He is considered to be the first person to discover the laws of vibrating strings. The truth was that Galileo was the first person to conduct experimental studies on vibrating strings; however, publication of his work was prohibited until 1638, by order of the Inquisitor of Rome. Although Galileo studied the pendulum extensively and discussed the isochronism of the pendulum, Christian Huygens (1629–1695) was the person who developed the pendulum clock, the first accurate device developed for measuring time. He observed deviation from isochronism due to the nonlinearity of the pendulum, and investigated various designs to improve the accuracy of the pendulum clock.

The works of Galileo contributed to a substantially increased level of experimental work among many scientists and paved the way to the establishment of several professional organizations, such as the Academia Naturae in Naples in 1560, Academia dei Lincei in Rome in 1606, Royal Society in London in 1662, the French Academy of Sciences in 1766, and the Berlin Academy of Science in 1770.

The relation between the pitch and frequency of vibration of a taut string was investigated further by Robert Hooke (1635–1703) and Joseph Sauveur (1653–1716). The phenomenon of mode shapes during the vibration of stretched strings, involving no motion at certain points and violent motion at intermediate points, was observed independently by Sauveur in France (1653–1716) and John Wallis in England (1616–1703). Sauveur called points with no motion *nodes* and points with violent motion, *loops*. Also, he observed that vibrations involving nodes and loops had higher frequencies than those involving no nodes. After observing that the values of the higher frequencies were integral multiples of the frequency of simple vibration with no nodes, Sauveur termed the frequency of simple vibration the *fundamental frequency* and the higher frequencies, the *harmonics*. In addition, he found that the vibration of a stretched string can contain several harmonics simultaneously. The phenomenon of beats was also observed by Sauveur when two organ pipes, having slightly different pitches, were sounded together. He also tried to compute the frequency of vibration of a taut string from the measured sag of its middle point. Sauveur introduced the word *acoustics* for the first time for the science of sound [7].

Isaac Newton (1642–1727) studied at Trinity College, Cambridge and later became professor of mathematics at Cambridge and president of the Royal Society of London. In 1687 he published the most admired scientific treatise of all time, *Philosophia Naturalis Principia Mathematica.* Although the laws of motion were already known in one form or other, the development of differential calculus by Newton and Leibnitz made the laws applicable to a variety of problems in mechanics and physics. Leonhard Euler (1707–1783) laid the groundwork for the calculus of variations. He popularized the use of free-body diagrams in mechanics and introduced several notations, including $e = 2.71828\ldots$, $f(x)$, $\sum$, and $i = \sqrt{-1}$. In fact, many people believe that the current techniques of formulating and solving mechanics problems are due more to Euler than to any other person in the history of mechanics. Using the concept of inertia force, Jean D'Alembert (1717–1783) reduced the problem of dynamics to a problem in statics. Joseph Lagrange (1736–1813) developed the variational principles for deriving the equations of motion and introduced the concept of generalized coordinates. He introduced *Lagrange equations* as a powerful tool for formulating the equations of motion for lumped-parameter systems. Charles Coulomb (1736–1806) studied the torsional oscillations both theoretically and experimentally. In addition, he derived the relation between electric force and charge.

Claude Louis Marie Henri Navier (1785–1836) presented a rigorous theory for the bending of plates. In addition, he considered the vibration of solids and presented the continuum theory of elasticity. In 1882, Augustin Louis Cauchy (1789–1857) presented a formulation for the mathematical theory of continuum mechanics. William Hamilton (1805–1865) extended the formulation of Lagrange for dynamics problems and presented a powerful method (Hamilton's principle) for the derivation of equations of motion of continuous systems. Heinrich Hertz (1857–1894) introduced the terms *holonomic* and *nonholonomic* into dynamics around 1894. Jules Henri Poincaré

(1854–1912) made many contributions to pure and applied mathematics, particularly to celestial mechanics and electrodynamics. His work on nonlinear vibrations in terms of the classification of singular points of nonlinear autonomous systems is notable.

1.4 HISTORY OF VIBRATION OF CONTINUOUS SYSTEMS

The precise treatment of the vibration of continuous systems can be associated with the discovery of the basic law of elasticity by Hooke, the second law of motion by Newton, and the principles of differential calculus by Leibnitz. Newton's second law of motion is used routinely in modern books on vibrations to derive the equations of motion of a vibrating body.

Strings A theoretical (dynamical) solution of the problem of the vibrating string was found in 1713 by the English mathematician Brook Taylor (1685–1731), who also presented the famous Taylor theorem on infinite series. He applied the fluxion approach, similar to the differential calculus approach developed by Newton and Newton's second law of motion, to an element of a continuous string and found the true value of the first natural frequency of the string. This value was found to agree with the experimental values observed by Galileo and Mersenne. The procedure adopted by Taylor was perfected through the introduction of partial derivatives in the equations of motion by Daniel Bernoulli, Jean D'Alembert, and Leonhard Euler. The fluxion method proved too clumsy for use with more complex vibration analysis problems. With the controversy between Newton and Leibnitz as to the origin of differential calculus, patriotic Englishmen stuck to the cumbersome fluxions while other investigators in Europe followed the simpler notation afforded by the approach of Leibnitz.

In 1747, D'Alembert derived the partial differential equation, later referred to as the *wave equation*, and found the wave travel solution. Although D'Alembert was assisted by Daniel Bernoulli and Leonhard Euler in this work, he did not give them credit. With all three claiming credit for the work, the specific contribution of each has remained controversial.

The possibility of a string vibrating with several of its harmonics present at the same time (with displacement of any point at any instant being equal to the algebraic sum of displacements for each harmonic) was observed by Bernoulli in 1747 and proved by Euler in 1753. This was established through the dynamic equations of Daniel Bernoulli in his memoir, published by the Berlin Academy in 1755. This characteristic was referred to as the *principle of the coexistence of small oscillations*, which is the same as the *principle of superposition* in today's terminology. This principle proved to be very valuable in the development of the theory of vibrations and led to the possibility of expressing any arbitrary function (i.e., any initial shape of the string) using an infinite series of sine and cosine terms. Because of this implication, D'Alembert and Euler doubted the validity of this principle. However, the validity of this type of expansion was proved by Fourier (1768–1830) in his *Analytical Theory of Heat* in 1822.

It is clear that Bernoulli and Euler are to be credited as the originators of the modal analysis procedure. They should also be considered the originators of the Fourier expansion method. However, as with many discoveries in the history of science, the persons credited with the achievement may not deserve it completely. It is often the person who publishes at the right time who gets the credit.

The analytical solution of the vibrating string was presented by Joseph Lagrange in his memoir published by the Turin Academy in 1759. In his study, Lagrange assumed that the string was made up of a finite number of equally spaced identical mass particles, and he established the existence of a number of independent frequencies equal to the number of mass particles. When the number of particles was allowed to be infinite, the resulting frequencies were found to be the same as the harmonic frequencies of the stretched string. The method of setting up the differential equation of motion of a string (called the *wave equation*), presented in most modern books on vibration theory, was developed by D'Alembert and described in his memoir published by the Berlin Academy in 1750.

Bars Chladni in 1787, and Biot in 1816, conducted experiments on the longitudinal vibration of rods. In 1824, Navier, presented an analytical equation and its solution for the longitudinal vibration of rods.

Shafts Charles Coulomb did both theoretical and experimental studies in 1784 on the torsional oscillations of a metal cylinder suspended by a wire [5]. By assuming that the resulting torque of the twisted wire is proportional to the angle of twist, he derived an equation of motion for the torsional vibration of a suspended cylinder. By integrating the equation of motion, he found that the period of oscillation is independent of the angle of twist. The derivation of the equation of motion for the torsional vibration of a continuous shaft was attempted by Caughy in an approximate manner in 1827 and given correctly by Poisson in 1829. In fact, Saint-Venant deserves the credit for deriving the torsional wave equation and finding its solution in 1849.

Beams The equation of motion for the transverse vibration of thin beams was derived by Daniel Bernoulli in 1735, and the first solutions of the equation for various support conditions were given by Euler in 1744. Their approach has become known as the *Euler–Bernoulli* or *thin beam theory*. Rayleigh presented a beam theory by including the effect of rotary inertia. In 1921, Stephen Timoshenko presented an improved theory of beam vibration, which has become known as the *Timoshenko* or *thick beam theory*, by considering the effects of rotary inertia and shear deformation.

Membranes In 1766, Euler, derived equations for the vibration of rectangular membranes which were correct only for the uniform tension case. He considered the rectangular membrane instead of the more obvious circular membrane in a drumhead, because he pictured a rectangular membrane as a superposition of two sets of strings laid in perpendicular directions. The correct equations for the vibration of rectangular and circular membranes were derived by Poisson in 1828. Although a solution corresponding to axisymmetric vibration of a circular membrane was given by Poisson, a nonaxisymmetric solution was presented by Pagani in 1829.

Plates The vibration of plates was also being studied by several investigators at this time. Based on the success achieved by Euler in studying the vibration of a rectangular membrane as a superposition of strings, Euler's student James Bernoulli, the grandnephew of the famous mathematician Daniel Bernoulli, attempted in 1788 to derive an equation for the vibration of a rectangular plate as a gridwork of beams. However, the resulting equation was not correct. As the torsional resistance of the plate was not considered in his equation of motion, only a resemblance, not the real agreement, was noted between the theoretical and experimental results.

The method of placing sand on a vibrating plate to find its mode shapes and to observe the various intricate modal patterns was developed by the German scientist Chladni in 1802. In his experiments, Chladni distributed sand evenly on horizontal plates. During vibration, he observed regular patterns of modes because of the accumulation of sand along the nodal lines that had no vertical displacement. Napoléon Bonaparte, who was a trained military engineer, was present when Chladni gave a demonstration of his experiments on plates at the French Academy in 1809. Napoléon was so impressed by Chladni's demonstration that he gave a sum of 3000 francs to the French Academy to be presented to the first person to give a satisfactory mathematical theory of the vibration of plates. When the competition was announced, only one person, Sophie Germain, entered the contest by the closing date of October 1811 [8]. However, an error in the derivation of Germain's differential equation was noted by one of the judges, Lagrange. In fact, Lagrange derived the correct form of the differential equation of plates in 1811. When the academy opened the competition again, with a new closing date of October 1813, Germain entered the competition again with a correct form of the differential equation of plates. Since the judges were not satisfied, due to the lack of physical justification of the assumptions she made in deriving the equation, she was not awarded the prize. The academy opened the competition again with a new closing date of October 1815. Again, Germain entered the contest. This time she was awarded the prize, although the judges were not completely satisfied with her theory. It was found later that her differential equation for the vibration of plates was correct but the boundary conditions she presented were wrong. In fact, Kirchhoff, in 1850, presented the correct boundary conditions for the vibration of plates as well as the correct solution for a vibrating circular plate.

The great engineer and bridge designer Navier (1785–1836) can be considered the originator of the modern theory of elasticity. He derived the correct differential equation for rectangular plates with flexural resistance. He presented an exact method that transforms the differential equation into an algebraic equation for the solution of plate and other boundary value problems using trigonometric series. In 1829, Poisson extended Navier's method for the lateral vibration of circular plates.

Kirchhoff (1824–1887) who included the effects of both bending and stretching in his theory of plates published in his book *Lectures on Mathematical Physics*, is considered the founder of the extended plate theory. Kirchhoff's book was translated into French by Clebsch with numerous valuable comments by Saint-Venant. Love extended Kirchhoff's approach to thick plates. In 1915, Timoshenko presented a solution for circular plates with large deflections. Foppl considered the nonlinear theory of plates in 1907; however, the final form of the differential equation for the large deflection of plates was developed by von Kármán in 1910. A more rigorous plate theory that considers the effects of transverse shear forces was presented by Reissner. A plate theory that includes the effects of both rotatory inertia and transverse shear deformation, similar to the Timoshenko beam theory, was presented by Mindlin in 1951.

Shells The derivation of an equation for the vibration of shells was attempted by Sophie Germain, who in 1821 published a simplified equation, with errors, for the vibration of a cylindrical shell. She assumed that the in-plane displacement of the neutral surface of a cylindrical shell was negligible. Her equation can be reduced to the correct form for a rectangular plate but not for a ring. The correct equation for the vibration of a ring had been given by Euler in 1766.

Aron, in 1874, derived the general shell equations in curvilinear coordinates, which were shown to reduce to the plate equation when curvatures were set to zero. The equations were complicated because no simplifying assumptions were made. Lord Rayleigh proposed different simplifications for the vibration of shells in 1882 and considered the neutral surface of the shell either extensional or inextensional. Love, in 1888, derived the equations for the vibration of shells by using simplifying assumptions similar to those of beams and plates for both in-plane and transverse motions. Love's equations can be considered to be most general in unifying the theory of vibration of continuous structures whose thickness is small compared to other dimensions. The vibration of shells, with a consideration of rotatory inertia and shear deformation, was presented by Soedel in 1982.

Approximate Methods Lord Rayleigh published his book on the theory of sound in 1877; it is still considered a classic on the subject of sound and vibration. Notable among the many contributions of Rayleigh is the method of finding the fundamental frequency of vibration of a conservative system by making use of the principle of conservation of energy—now known as *Rayleigh's method*. Ritz (1878–1909) extended Rayleigh's method for finding approximate solutions of boundary value problems. The method, which became known as the *Rayleigh–Ritz method*, can be considered to be a variational approach. Galerkin (1871–1945) developed a procedure that can be considered a weighted residual method for the approximate solution of boundary value problems.

Until about 40 years ago, vibration analyses of even the most complex engineering systems were conducted using simple approximate analytical methods. Continuous systems were modeled using only a few degrees of freedom. The advent of high-speed digital computers in the 1950s permitted the use of more degrees of freedom in modeling engineering systems for the purpose of vibration analysis. Simultaneous development of the finite element method in the 1960s made it possible to consider thousands of degrees of freedom to approximate practical problems in a wide spectrum of areas, including machine design, structural design, vehicle dynamics, and engineering mechanics. Notable contributions to the theory of the vibration of continuous systems are summarized in Table 1.1.

1.5 DISCRETE AND CONTINUOUS SYSTEMS

The *degrees of freedom* of a system are defined by the minimum number of independent coordinates necessary to describe the positions of all parts of the system at any instant of time. For example, the spring–mass system shown in Fig. 1.2 is a single-degree-of-freedom system since a single coordinate, $x(t)$, is sufficient to describe the position of the mass from its equilibrium position at any instant of time. Similarly, the simple pendulum shown in Fig. 1.3 also denotes a single-degree-of-freedom system. The reason is that the position of a simple pendulum during motion can be described by using a single angular coordinate, θ. Although the position of a simple pendulum can be stated in terms of the Cartesian coordinates x and y, the two coordinates x and y are not independent; they are related to one another by the constraint $x^2 + y^2 = l^2$, where l is the

Table 1.1 Notable Contributions to the Theory of Vibration of Continuous Systems

Period	Scientist	Contribution
582–507 B.C.	Pythagoras	Established the first school of higher education and scientific research. Conducted experiments on vibrating strings. Invented the monochord.
384–322 B.C.	Aristotle	Wrote a book on acoustics. Studied laws of motion (similar to those of Newton). Introduced vectorial addition of forces.
Third century B.C.	Alexander of Afrodisias	Kinetic and potential energies. Idea of conservation of energy.
325–265 B.C.	Euclid	Prominent mathematician. Published a treatise called *Introduction to Harmonics*.
A.D.		
1564–1642	Galileo Galilei	Experiments on pendulum and vibration of strings. Wrote the first treatise on modern dynamics.
1642–1727	Isaac Newton	Laws of motion. Differential calculus. Published the famous *Principia Mathematica.*
1653–1716	Joseph Sauveur	Introduced the term *acoustics*. Investigated harmonics in vibration.
1685–1731	Brook Taylor	Theoretical solution of vibrating strings. Taylor's theorem.
1700–1782	Daniel Bernoulli	Principle of angular momentum. Principle of superposition.
1707–1783	Leonhard Euler	Principle of superposition. Beam theory. Vibration of membranes. Introduced several mathematical symbols.
1717–1783	Jean D'Alembert	Dynamic equilibrium of bodies in motion. Inertia force. Wave equation.
1736–1813	Joseph Louis Lagrange	Analytical solution of vibrating strings. Lagrange's equations. Variational calculus. Introduced the term *generalized coordinates*.
1736–1806	Charles Coulomb	Torsional vibration studies.
1756–1827	E. F. F. Chladni	Experimental observation of mode shapes of plates.
1776–1831	Sophie Germain	Vibration of plates.
1785–1836	Claude Louis Marie Henri Navier	Bending vibration of plates. Vibration of solids. Originator of modern theory of elasticity.
1797–1872	Jean Marie Duhamel	Studied partial differential equations applied to vibrating strings and vibration of air in pipes. Duhamel's integral.
1805–1865	William Hamilton	Principle of least action. Hamilton's principle.

Table 1.1 (*continued*)

Period	Scientist	Contribution
1824–1887	Gustav Robert Kirchhoff	Presented extended theory of plates. Kirchhoff's laws of electrical circuits.
1842–1919	John William Strutt (Lord Rayleigh)	Energy method. Effect of rotatory inertia. Shell equations.
1874	H. Aron	Shell equations in curvilinear coordinates.
1888	A. E. H. Love	Classical theory of thin shells.
1871–1945	Boris Grigorevich Galerkin	Approximate solution of boundary value problems with application to elasticity and vibration.
1878–1909	Walter Ritz	Extended Rayleigh's energy method for approximate solution of boundary value problems.
1956	Turner, Clough, Martin, and Topp	Finite element method.

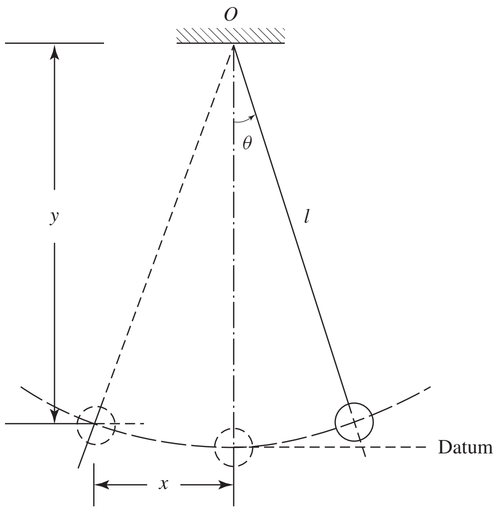


Figure 1.3 Simple pendulum.

constant length of the pendulum. Thus, the pendulum is a single-degree-of-freedom system. The mass–spring–damper systems shown in Fig. 1.4(a) and (b) denote two- and three-degree-of-freedom systems, respectively, since they have, two and three masses that change their positions with time during vibration. Thus, a multidegree-of-freedom system can be considered to be a system consisting of point masses separated by springs and dampers. The parameters of the system are discrete sets of finite numbers. These systems are also called *lumped-parameter*, *discrete*, or *finite-dimensional systems*.

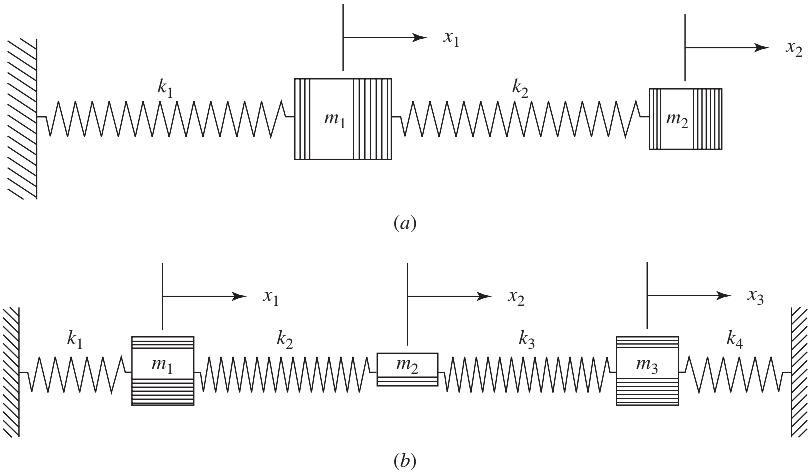


Figure 1.4 (*a*) Two- and (b) three-degree-of-freedom systems.

On the other hand, in a continuous system, the mass, elasticity (or flexibility), and damping are distributed throughout the system. During vibration, each of the infinite number of point masses moves relative to each other point mass in a continuous fashion. These systems are also known as *distributed, continuous*, or *infinite-dimensional systems*. A simple example of a continuous system is the cantilever beam shown in Fig. 1.5. The beam has an infinite number of mass points, and hence an infinite number of coordinates are required to specify its deflected shape. The infinite number of coordinates, in fact, define the elastic deflection curve of the beam. Thus, the cantilever beam is considered to be a system with an infinite number of degrees of freedom. Most mechanical and structural systems have members with continuous elasticity and mass distribution and hence have infinite degrees of freedom.

The choice of modeling a given system as discrete or continuous depends on the purpose of the analysis and the expected accuracy of the results. The motion of an n-degree-of-freedom system is governed by a system of n coupled second-order ordinary differential equations. For a continuous system, the governing equation of motion is in the form of a partial differential equation. Since the solution of a set of ordinary differential equations is simple, it is relatively easy to find the response of a discrete system that is experiencing a specified excitation. On the other hand, solution of a partial differential equation is more involved, and closed-form solutions are available for only a few continuous systems that have a simple geometry and simple, boundary conditions and excitations. However, the closed-form solutions that are available will often provide insight into the behavior of more complex systems for which closed-form solutions cannot be found.

For an n-degree-of-freedom system, there will be, at most, n distinct natural frequencies of vibration with a mode shape corresponding to each natural frequency. A continuous system, on the other hand, will have an infinite number of natural frequencies, with one mode shape corresponding to each natural frequency. A continuous system can be approximated as a discrete system, and its solution can be obtained in a simpler manner. For example, the cantilever beam shown in Fig. 1.5(a) can be

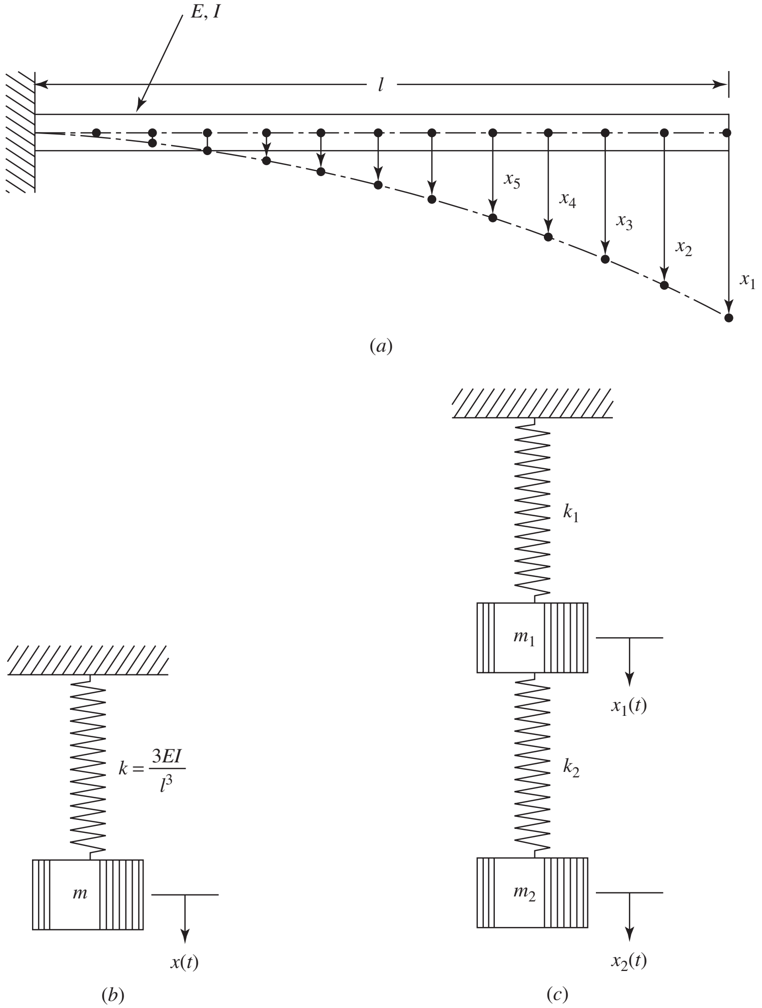


Figure 1.5 Modeling of a cantilever beam as (*a*) a continuous system, (*b*) a single-degree-of-freedom system, and (*c*) a two-degree-of-freedom system.

approximated as a single degree of freedom by assuming the mass of the beam to be a concentrated point mass located at the free end of the beam and the continuous flexibility to be approximated as a simple linear spring as shown in Fig. 1.5(*b*). The accuracy of approximation can be improved by using a two-degree-of-freedom model as shown in Fig. 1.5(*c*), where the mass and flexibility of the beam are approximated by two point masses and two linear springs.

1.6 VIBRATION PROBLEMS

Vibration problems may be classified into the following types [9]:

1. *Undamped and damped vibration*. If there is no loss or dissipation of energy due to friction or other resistance during vibration of a system, the system is

said to be *undamped*. If there is energy loss due to the presence of damping, the system is called *damped*. Although system analysis is simpler when neglecting damping, a consideration of damping becomes extremely important if the system operates near resonance.

2. *Free and forced vibration*. If a system vibrates due to an initial disturbance (with no external force applied after time zero), the system is said to undergo *free vibration*. On the other hand, if the system vibrates due to the application of an external force, the system is said to be under *forced vibration*.
3. *Linear and nonlinear vibration*. If all the basic components of a vibrating system (i.e., the mass, the spring, and the damper) behave linearly, the resulting vibration is called *linear vibration*. However, if any of the basic components of a vibrating system behave nonlinearly, the resulting vibration is called *nonlinear vibration*. The equation of motion governing linear vibration will be a linear differential equation, whereas the equation governing nonlinear vibration will be a nonlinear differential equation. Most vibratory systems behave nonlinearly as the amplitudes of vibration increase to large values.

1.7 VIBRATION ANALYSIS

A vibratory system is a dynamic system for which the response (output) depends on the excitations (inputs) and the characteristics of the system (e.g., mass, stiffness, and damping) as indicated in Fig. 1.6. The excitation and response of the system are both time dependent. Vibration analysis of a given system involves determination of the response for the excitation specified. The analysis usually involves mathematical modeling, derivation of the governing equations of motion, solution of the equations of motion, and interpretation of the response results.

The purpose of mathematical modeling is to represent all the important characteristics of a system for the purpose of deriving mathematical equations that govern the behavior of the system. The mathematical model is usually selected to include enough details to describe the system in terms of equations that are not too complex. The mathematical model may be linear or nonlinear, depending on the nature of the system characteristics. Although linear models permit quick solutions and are simple to deal with, nonlinear models sometimes reveal certain important behavior of the system which cannot be predicted using linear models. Thus, a great deal of engineering judgment is required to develop a suitable mathematical model of a vibrating system. If the mathematical model of the system is linear, the principle of superposition can be used. This means that if the responses of the system under individual excitations $f_1(t)$ and $f_2(t)$ are denoted as $x_1(t)$ and $x_2(t)$, respectively, the response of the system would be

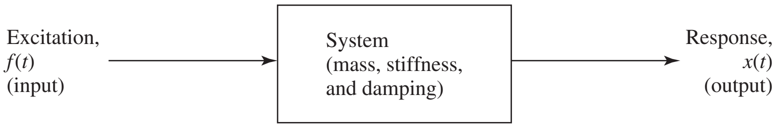


Figure 1.6 Input–output relationship of a vibratory system.

$x(t) = c_1 x_1(t) + c_2 x_2(t)$ when subjected to the excitation $f(t) = c_1 f_1(t) + c_2 f_2(t)$, where c_1 and c_2 are constants.

Once the mathematical model is selected, the principles of dynamics are used to derive the equations of motion of the vibrating system. For this, the free-body diagrams of the masses, indicating all externally applied forces (excitations), reaction forces, and inertia forces, can be used. Several approaches, such as D'Alembert's principle, Newton's second law of motion, and Hamilton's principle, can be used to derive the equations of motion of the system. The equations of motion can be solved using a variety of techniques to obtain analytical (closed-form) or numerical solutions, depending on the complexity of the equations involved. The solution of the equations of motion provides the displacement, velocity, and acceleration responses of the system. The responses and the results of analysis need to be interpreted with a clear view of the purpose of the analysis and the possible design implications.

1.8 EXCITATIONS

Several types of excitations or loads can act on a vibrating system. As stated earlier, the excitation may be in the form of initial displacements and initial velocities that are produced by imparting potential energy and kinetic energy to the system, respectively. The response of the system due to initial excitations is called *free vibration*. For real-life systems, the vibration caused by initial excitations diminishes to zero eventually and the initial excitations are known as *transient excitations*.

In addition to the initial excitations, a vibrating system may be subjected to a large variety of external forces. The origin of these forces may be environmental, machine induced, vehicle induced, or blast induced. Typical examples of environmentally induced dynamic forces include wind loads, wave loads, and earthquake loads. *Machine-induced loads* are due primarily to imbalance in reciprocating and rotating machines, engines, and turbines, and are usually periodic in nature. *Vehicle-induced loads* are those induced on highway and railway bridges from speeding trucks and trains crossing them. In some cases, dynamic forces are induced on bodies and equipment located inside vehicles due to the motion of the vehicles. For example, sensitive navigational equipment mounted inside the cockpit of an aircraft may be subjected to dynamic loads induced by takeoff, landing, or in-flight turbulence. *Blast-induced loads* include those generated by explosive devices during blast operations, accidental chemical explosions, or terrorist bombings.

The nature of some of the dynamic loads originating from different sources is shown in Fig. 1.1. In the case of rotating machines with imbalance, the induced loads will be harmonic, as shown in Fig. 1.1(a). In other types of machines, the loads induced due to the unbalance will be *periodic*, as shown in Fig. 1.1(b). A blast load acting on a vibrating structure is usually in the form of an overpressure, as shown in Fig. 1.1(c). The blast overpressure will cause severe damage to structures located close to the explosion. On the other hand, a large explosion due to underground detonation may even affect structures located far away from the explosion. Earthquake-, wave-, and wind-, gust-, or turbulence-, induced loads will be random in nature, as indicated in Fig. 1.1(d).

It can be seen that harmonic force is the simplest type of force to which a vibrating system can be subjected. The harmonic force also plays a very important role in the study of vibrations. For example, any periodic force can be represented as an infinite sum of harmonic forces using Fourier series. In addition, any nonperiodic force can be represented (by considering its period to be approaching infinity) in terms of harmonic forces using the Fourier integral. Because of their importance in vibration analysis, a detailed discussion of harmonic functions is given in the following section.

1.9 HARMONIC FUNCTIONS

In most practical applications, harmonic time dependence is considered to be same as sinusoidal vibration. For example, the harmonic variations of alternating current and electromagnetic waves are represented by sinusoidal functions. As an application in the area of mechanical systems, the motion of point S in the action of the Scotch yoke mechanism shown in Fig. 1.7 is simple harmonic. In this system, a crank of radius A rotates about point O. It can be seen that the amplitude is the maximum value of $x(t)$ from the zero value, either positively or negatively, so that $A = \max|x(t)|$. The frequency is related to the period τ, which is the time interval over which $x(t)$ repeats such that $x(t+\tau) = x(t)$.

The other end of the crank (P) slides in the slot of the rod that reciprocates in the guide G. When the crank rotates at the angular velocity ω, endpoint S of the slotted link is displaced from its original position. The displacement of endpoint S in time t is given by

$$x = A\sin\theta = A\sin\omega t \tag{1.2}$$

and is shown graphically in Fig. 1.7. The velocity and acceleration of point S at time t are given by

$$\frac{dx}{dt} = \omega A\,\cos\,\omega t \tag{1.3}$$

$$\frac{d^2x}{dt^2} = -\omega^2 A\,\sin\,\omega t = -\omega^2\,x \tag{1.4}$$

Equation (1.4) indicates that the acceleration of point S is directly proportional to the displacement. Such motion, in which the acceleration is proportional to the displacement and is directed toward the mean position, is called *simple harmonic motion*. According to this definition, motion given by $x = A\cos\omega t$ will also be simple harmonic.

1.9.1 Representation of Harmonic Motion

Harmonic motion can be represented by means of a vector $\vec{OP}$ of magnitude A rotating at a constant angular velocity ω, as shown in Fig. 1.8. It can be observed that the projection of the tip of the vector $\vec{X} = \vec{OP}$ on the vertical axis is given by

$$y = A\,\sin\,\omega t \tag{1.5}$$

and its projection on the horizontal axis by

$$x = A\,\cos\,\omega t \tag{1.6}$$

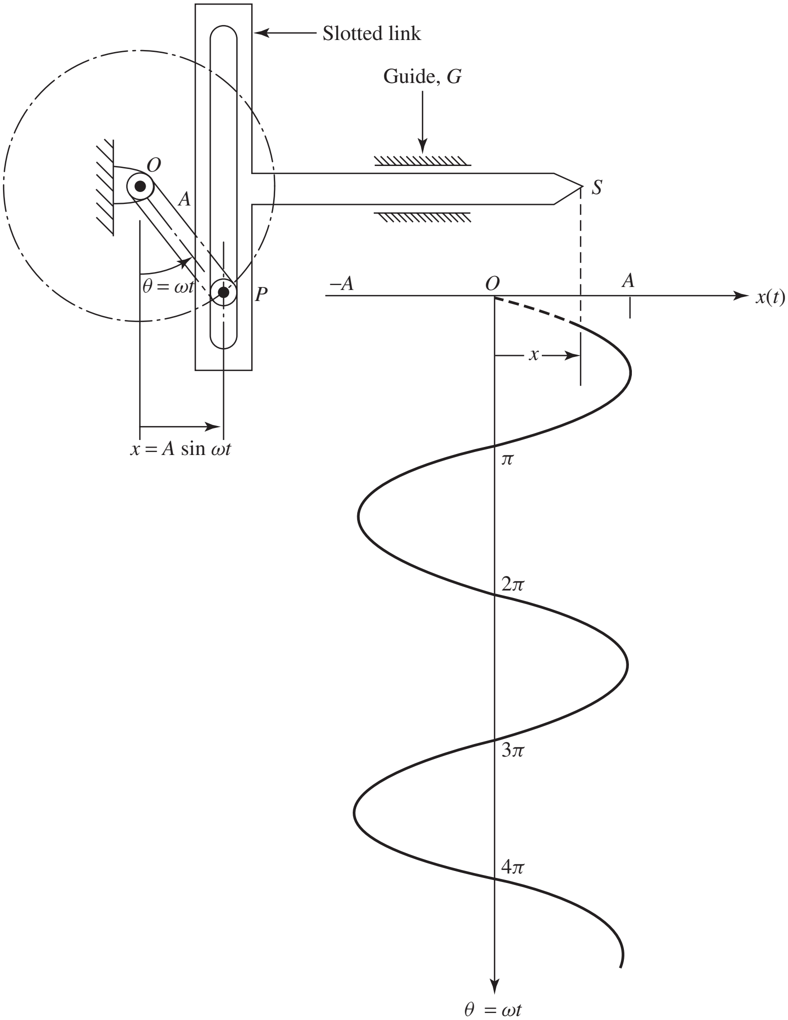


Figure 1.7 Simple harmonic motion produced by a Scotch yoke mechanism.

Equations (1.5) and (1.6) both represent simple harmonic motion. In the vectorial method of representing harmonic motion, two equations, Eqs. (1.5) and (1.6), are required to describe the vertical and horizontal components. Harmonic motion can be represented more conveniently using complex numbers. Any vector $\vec{X}$ can be represented as a complex number in the *xy* plane as

$$\vec{X} = a + ib \tag{1.7}$$

where $i = \sqrt{-1}$ and a and b denote the x and y components of $\vec{X}$, respectively, and can be considered as the *real* and *imaginary parts* of the vector $\vec{X}$. The vector $\vec{X}$ can also be expressed as

$$\vec{X} = A\,(\cos\theta + i\ \sin\theta) \tag{1.8}$$

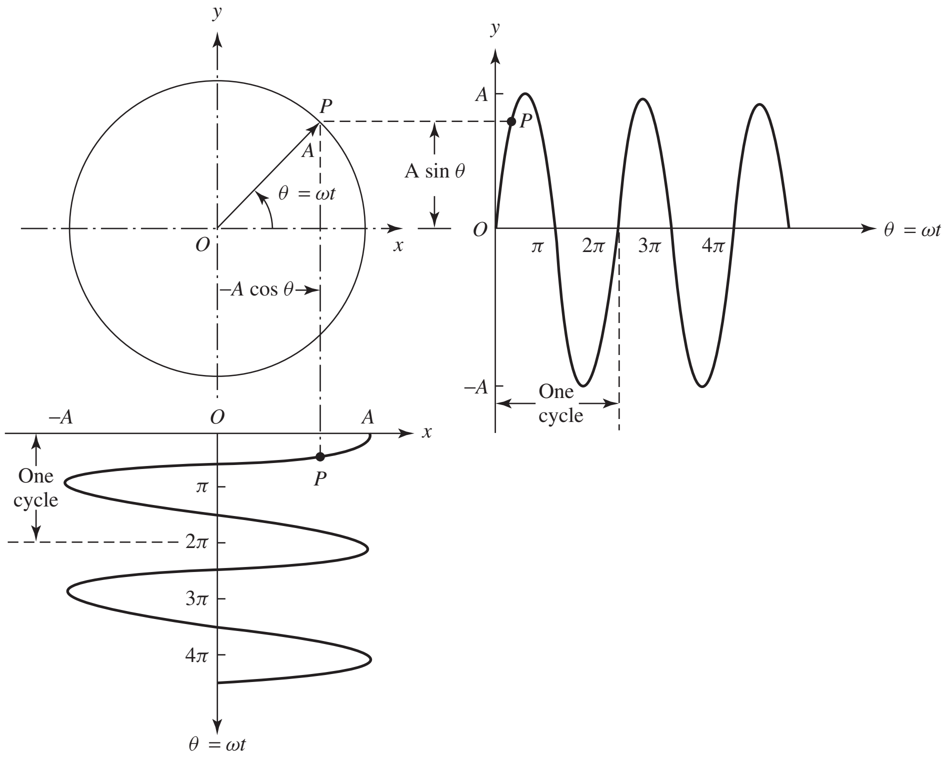


Figure 1.8 Harmonic motion: projection of a rotating vector.

where

$$A = (a^2 + b^2)^{1/2} \tag{1.9}$$

denotes the modulus or magnitude of the vector $\vec{X}$ and

$$\theta = \tan^{-1} \frac{b}{a} \tag{1.10}$$

indicates the argument or the angle between the vector and the x axis. Noting that

$$\cos\theta + i \sin\theta = e^{i\theta} \tag{1.11}$$

Eq. (1.8) can be expressed as

$$\vec{X} = A(\cos\theta + i \sin\theta) = Ae^{i\theta} \tag{1.12}$$

Thus, the rotating vector $\vec{X}$ of Fig. 1.8 can be written, using complex number representation, as

$$\vec{X} = Ae^{i\omega t} \tag{1.13}$$

where ω denotes the circular frequency (rad/sec) of rotation of the vector $\vec{X}$ in the counterclockwise direction. The harmonic motion given by Eq. (1.13) can be differentiated with respect to time as

$$\frac{d\vec{X}}{dt} = \frac{d}{dt}(Ae^{i\omega t}) = i\omega Ae^{i\omega t} = i\omega\vec{X} \tag{1.14}$$

$$\frac{d^2\vec{X}}{dt^2} = \frac{d}{dt}(i\omega Ae^{i\omega t}) = -\omega^2 Ae^{i\omega t} = -\omega^2\vec{X} \tag{1.15}$$

Thus, if $\vec{X}$ denotes harmonic motion, the displacement, velocity, and acceleration can be expressed as

$$x(t) = \text{displacement} = \text{Re}[Ae^{i\omega t}] = A\cos\omega t \tag{1.16}$$

$$\dot{x}(t) = \text{velocity} = \text{Re}[i\omega Ae^{i\omega t}] = -\omega A\sin\omega t = \omega A\cos(\omega t + 90^\circ) \tag{1.17}$$

$$\ddot{x}(t) = \text{acceleration} = \text{Re}[-\omega^2 Ae^{i\omega t}] = -\omega^2 A\cos\omega t = \omega^2 A\cos(\omega t + 180^\circ) \tag{1.18}$$

where Re denotes the real part, or alternatively as

$$x(t) = \text{displacement} = \text{Im}[Ae^{i\omega t}] = A\sin\omega t \tag{1.19}$$

$$\dot{x}(t) = \text{velocity} = \text{Im}[i\omega Ae^{i\omega t}] = \omega A\cos\omega t = \omega A\sin(\omega t + 90^\circ) \tag{1.20}$$

$$\ddot{x}(t) = \text{acceleration} = \text{Im}[-\omega^2 Ae^{i\omega t}] = -\omega^2 A\sin\omega t = \omega^2 A\sin(\omega t + 180^\circ) \tag{1.21}$$

where Im denotes the imaginary part. Eqs. (1.16)–(1.21) are shown as rotating vectors in Fig. 1.9. It can be seen that the acceleration vector leads the velocity vector by 90°, and the velocity vector leads the displacement vector by 90°.

1.9.2 Definitions and Terminology

Several definitions and terminology are used to describe harmonic motion and other periodic functions. The motion of a vibrating body from its undisturbed or equilibrium position to its extreme position in one direction, then to the equilibrium position, then

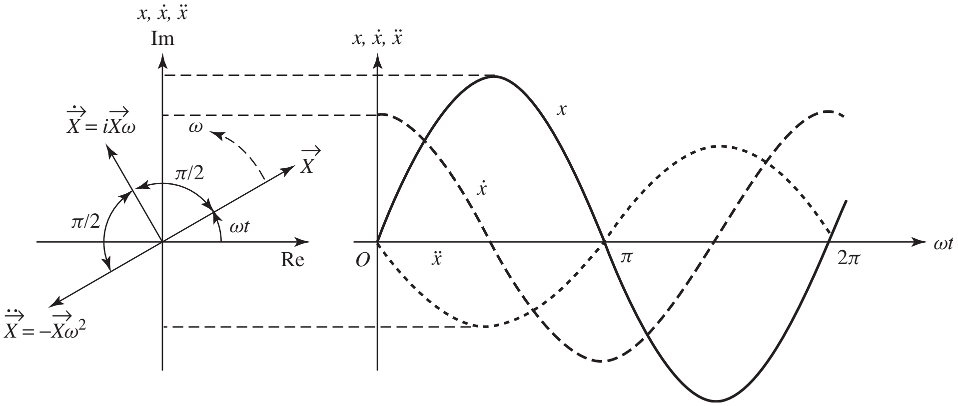


Figure 1.9 Displacement (x), velocity ($\dot{x}$), and acceleration ($\ddot{x}$) as rotating vectors.

to its extreme position in the other direction, and then back to the equilibrium position is called a *cycle* of vibration. One rotation or an angular displacement of 2π radians of pin P in the Scotch yoke mechanism of Fig. 1.7 or the vector $\vec{OP}$ in Fig. 1.8 represents a cycle.

The *amplitude* of vibration denotes the maximum displacement of a vibrating body from its equilibrium position. The amplitude of vibration is shown as A in Figs. 1.7 and 1.8. The *period* of oscillation represents the time taken by the vibrating body to complete one cycle of motion. The period of oscillation is also known as the *time period* and is denoted by τ. In Fig. 1.8, the time period is equal to the time taken by the vector $\vec{OP}$ to rotate through an angle of 2π. This yields

$$\tau = \frac{2\pi}{\omega} \tag{1.22}$$

where ω is called the *circular frequency*. The frequency of oscillation or *linear frequency* (or simply the *frequency*) indicates the number of cycles per unit time. The frequency can be represented as

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi} \tag{1.23}$$

Note that ω is called the *circular frequency* and is measured in radians per second, whereas f is called the *linear frequency* and is measured in cycles per second (hertz). If the sine wave is not zero at time zero (i.e., at the instant we start measuring time), as shown in Fig. 1.10, it can be denoted as

$$y = A\sin(\omega t + \phi) \tag{1.24}$$

where $\omega t + \phi$ is called the *phase* of the motion and ϕ the *phase angle* or initial phase. Next, consider two harmonic motions denoted by

$$y_1 = A_1 \sin \omega t \tag{1.25}$$

$$y_2 = A_2 \sin(\omega t + \phi) \tag{1.26}$$

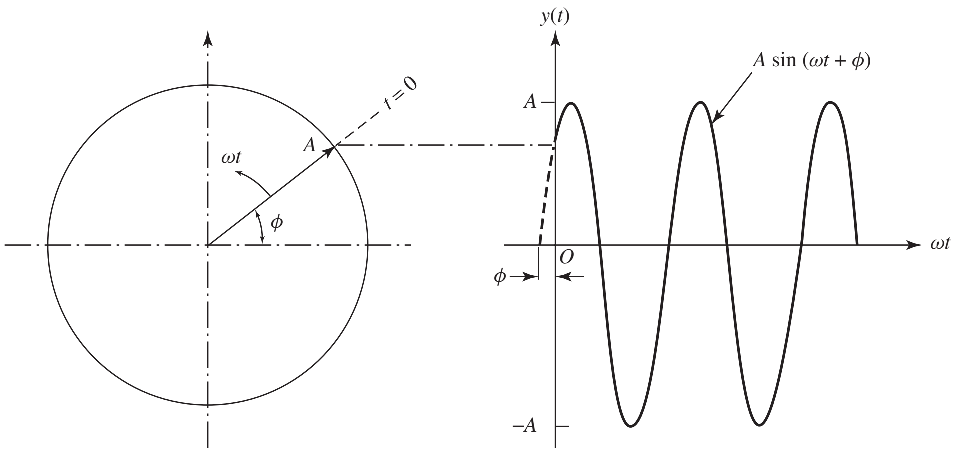


Figure 1.10 Significance of the phase angle ϕ.

Since the two vibratory motions given by Eqs. (1.25) and (1.26) have the same frequency ω, they are said to be *synchronous motions*. Two synchronous oscillations can have different amplitudes, and they can attain their maximum values at different times, separated by the time $t = \phi/\omega$, where ϕ is called the phase angle or *phase difference*. If a system (a single-degree-of-freedom system), after an initial disturbance, is left to vibrate on its own, the frequency with which it oscillates without external forces is known as its *natural frequency* of vibration. A discrete system having n degrees of freedom will have, in general, n distinct natural frequencies of vibration. A continuous system will have an infinite number of natural frequencies of vibration.

As indicated earlier, several harmonic motions can be combined to find the resulting motion. When two harmonic motions with frequencies close to one another are added or subtracted, the resulting motion exhibits a phenomenon known as *beats*. To see the phenomenon of beats, consider the difference of the motions given by

$$x_1(t) = X \sin \omega_1 t \equiv X \sin \omega t \tag{1.27}$$

$$x_2(t) = X \sin \omega_2 t \equiv X \sin(\omega - \delta)t \tag{1.28}$$

where δ is a small quantity. The difference of the two motions can be denoted as

$$x(t) = x_1(t) - x_2(t) = X[\sin \omega t - \sin(\omega - \delta)t] \tag{1.29}$$

Noting the relationship

$$\sin A - \sin B = 2 \sin \frac{A - B}{2} \cos \frac{A + B}{2} \tag{1.30}$$

the resulting motion $x(t)$ can be represented as

$$x(t) = 2X \sin \frac{\delta t}{2} \cos \left(\omega - \frac{\delta}{2} \right) t \tag{1.31}$$

The graph of $x(t)$ given by Eq. (1.31) is shown in Fig. 1.11. It can be observed that the motion, $x(t)$, denotes a cosine wave with frequency $(\omega_1 + \omega_2)/2 = \omega - \delta/2$, which is approximately equal to ω, and with a slowly varying amplitude of

$$2X \sin \frac{\omega_1 - \omega_2}{2} t = 2X \sin \frac{\delta t}{2}$$

Whenever the amplitude reaches a maximum, it is called a *beat*. The frequency δ at which the amplitude builds up and dies down between 0 and $2X$ is known as the *beat frequency*. The phenomenon of beats is often observed in machines, structures, and electric power houses. For example, in machines and structures, the beating phenomenon occurs when the forcing frequency is close to one of the natural frequencies of the system.

Example 1.1 Find the difference of the following harmonic functions and plot the resulting function for $A = 3$ and $\omega = 40$ rad/s: $x_1(t) = A \sin \omega t$, $x_2(t) = A \sin 0.95\omega t$.

SOLUTION The resulting function can be expressed as

$$\begin{aligned} x(t) &= x_1(t) - x_2(t) = A \sin \omega t - A \sin 0.95\omega t \\ &= 2A \sin 0.025\omega t \cos 0.975\omega t \end{aligned} \tag{E1.1.1}$$

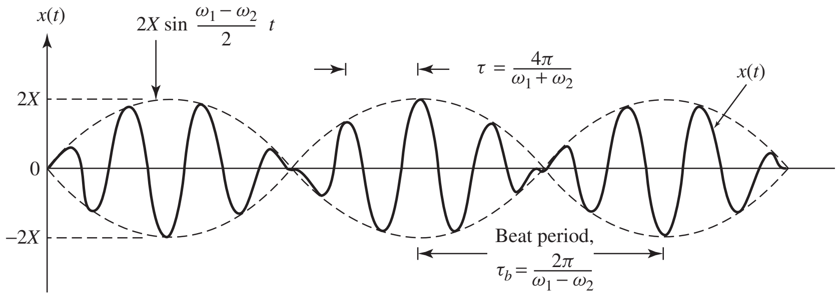


Figure 1.11 Beating phenomenon.

The plot of the function $x(t)$ is shown in Fig. 1.11. It can be seen that the function exhibits the phenomenon of beats with a beat frequency of $\omega_b = 1.00\omega - 0.95\omega = 0.05\omega = 2$ rad/s.

1.10 PERIODIC FUNCTIONS AND FOURIER SERIES

Although harmonic motion is the simplest to handle, the motion of many vibratory systems is not harmonic. However, in many cases the vibrations are periodic, as indicated, for example, in Fig. 1.1(*b*). Any periodic function of time can be represented as an infinite sum of sine and cosine terms using Fourier series. The process of representing a periodic function as a sum of harmonic functions (i.e., sine and cosine functions) is called *harmonic analysis*. The use of Fourier series as a means of describing periodic motion and/or periodic excitation is important in the study of vibration. Also, a familiarity with Fourier series helps in understanding the significance of experimentally determined frequency spectrums. If $x(t)$ is a periodic function with period τ, its Fourier series representation is given by

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots + b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \tag{1.32}$$

where $\omega = 2\pi/\tau$ is called the *fundamental frequency* and $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are constant coefficients. To determine the coefficients a_n and b_n, we multiply Eq. (1.32) by $\cos n\omega t$ and $\sin n\omega t$, respectively, and integrate over one period $\tau = 2\pi/\omega$: for example, from 0 to $2\pi/\omega$. This leads to

$$a_0 = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t)\, dt = \frac{2}{\tau} \int_0^{\tau} x(t)\, dt \tag{1.33}$$

$$a_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \cos n\omega t\, dt = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t\, dt \tag{1.34}$$

$$b_n = \frac{\omega}{\pi} \int_0^{2\pi/\omega} x(t) \sin n\omega t\, dt = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t\, dt \tag{1.35}$$

Equation (1.32) shows that any periodic function can be represented as a sum of harmonic functions. Although the series in Eq. (1.32) is an infinite sum, we can approximate most periodic functions with the help of only a first few harmonic functions.

Fourier series can also be represented by the sum of sine terms only or cosine terms only. For example, any periodic function $x(t)$ can be expressed using cosine terms only as

$$x(t) = d_0 + d_1 \cos(\omega t - \phi_1) + d_2 \cos(2\omega t - \phi_2) + \cdots \tag{1.36}$$

where

$$d_0 = \frac{a_0}{2} \tag{1.37}$$

$$d_n = (a_n^2 + b_n^2)^{1/2} \tag{1.38}$$

$$\phi_n = \tan^{-1} \frac{b_n}{a_n} \tag{1.39}$$

The Fourier series, Eq. (1.32), can also be represented in terms of complex numbers as

$$\begin{aligned} x(t) &= e^{i(0)\omega t}\left(\frac{a_0}{2} - \frac{ib_0}{2}\right) \\ &\quad + \sum_{n=1}^{\infty}\left[e^{in\omega t}\left(\frac{a_n}{2} - \frac{ib_n}{2}\right) + e^{-in\omega t}\left(\frac{a_n}{2} + \frac{ib_n}{2}\right)\right] \end{aligned} \tag{1.40}$$

where $b_0 = 0$. By defining the complex Fourier coefficients c_n and c_{-n} as

$$c_n = \frac{a_n - ib_n}{2} \tag{1.41}$$

$$c_{-n} = \frac{a_n + ib_n}{2} \tag{1.42}$$

Eq. (1.40) can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \tag{1.43}$$

The Fourier coefficients c_n can be determined, using Eqs. (1.33)–(1.35), as

$$\begin{aligned} c_n = \frac{a_n - ib_n}{2} &= \frac{1}{\tau}\int_0^{\tau} x(t)(\cos n\omega t - i \sin n\omega t)\, dt \\ &= \frac{1}{\tau}\int_0^{\tau} x(t) e^{-in\omega t}\, dt \end{aligned} \tag{1.44}$$

The harmonic functions $a_n \cos n\omega t$ or $b_n \sin n\omega t$ in Eq. (1.32) are called the *harmonics of order* n of the periodic function $x(t)$. A harmonic of order n has a period τ/n. These harmonics can be plotted as vertical lines on a diagram of amplitude (a_n and b_n or d_n and ϕ_n) versus frequency ($n\omega$), called the *frequency spectrum* or *spectral diagram*.

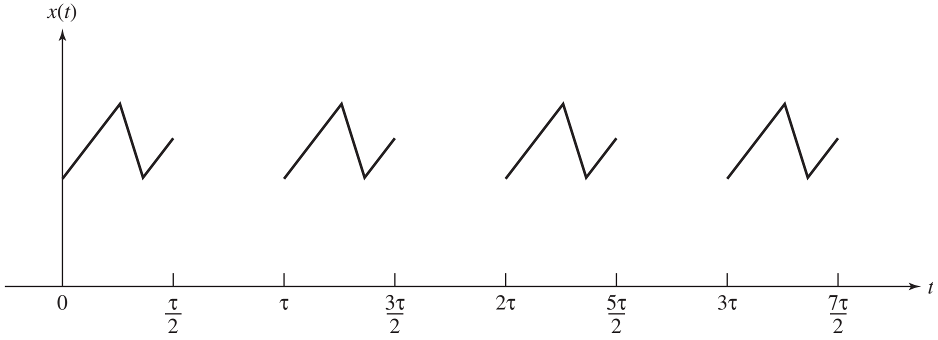


Figure 1.12 Typical periodic function.

1.11 NONPERIODIC FUNCTIONS AND FOURIER INTEGRALS

As shown in Eqs. (1.32), (1.36), and (1.43), any periodic function can be represented by a Fourier series. If the period τ of a periodic function increases indefinitely, the function $x(t)$ becomes nonperiodic. In such a case, the Fourier integral representation can be used as indicated below.

Let the typical periodic function shown in Fig. 1.12 be represented by a complex Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \qquad \omega = \frac{2\pi}{\tau} \tag{1.45}$$

where

$$c_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t) e^{-in\omega t}\, dt \tag{1.46}$$

Introducing the relations

$$n\,\omega = \omega_n \tag{1.47}$$

$$(n+1)\omega - n\,\omega = \omega = \frac{2\,\pi}{\tau} = \Delta\omega_n \tag{1.48}$$

Eqs. (1.45) and (1.46) can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\tau} (\tau c_n) e^{i\omega_n t} = \frac{1}{2\,\pi} \sum_{n=-\infty}^{\infty} (\tau c_n) e^{i\omega_n t}\, \Delta\omega_n \tag{1.49}$$

$$\tau\, c_n = \int_{-\tau/2}^{\tau/2} x(t) e^{-i\omega_n t}\, dt \tag{1.50}$$

As $\tau \to \infty$, we drop the subscript n on ω, replace the summation by integration, and write Eqs. (1.49) and (1.50) as

$$x(t) = \lim_{\substack{\tau\to\infty \\ \Delta\omega_n\to 0}} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (\tau c_n) e^{i\omega_n t} \Delta\omega_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t}\, d\omega \tag{1.51}$$

$$X(\omega) = \lim_{\substack{\tau\to\infty \\ \Delta\omega_n\to 0}} (\tau c_n) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t}\, dt \tag{1.52}$$

Equation (1.51) denotes the Fourier integral representation of $x(t)$ and Eq. (1.52) is called the *Fourier transform* of $x(t)$. Together, Eqs. (1.51) and (1.52) denote a *Fourier transform pair*. If $x(t)$ denotes excitation, the function $X(\omega)$ can be considered as the spectral density of excitation with $X(\omega)\, d\omega$ denoting the contribution of the harmonics in the frequency range ω to $\omega + d\omega$ to the excitation $x(t)$.

Example 1.2 Consider the nonperiodic rectangular pulse load $f(t)$, with magnitude f_0 and duration s, shown in Fig. 1.13(a). Determine its Fourier transform and plot the amplitude spectrum for $f_0 = 200$ lb, $s = 1$ sec, and $t_0 = 4$ sec.

SOLUTION The load can be represented in the time domain as

$$f(t) = \begin{cases} f_0, & t_0 < t < t_0 + s \\ 0, & t_0 > t > t_0 + s \end{cases} \tag{E1.2.1}$$

The Fourier transform of $f(t)$ is given by, using Eq. (1.52),

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = \int_{t_0}^{t_0+s} f_0 e^{-i\omega t}\, dt \\ &= f_0 \frac{i}{\omega} (e^{-i\omega(t_0+s)} - e^{-i\omega t_0}) \\ &= \frac{f_0}{\omega} \{[\sin\,\omega(t_0+s) - \sin\,\omega t_0] + i[\cos\,\omega(t_0+s) - \cos\,\omega t_0]\} \end{aligned} \tag{E1.2.2}$$

The amplitude spectrum is the modulus of $F(\omega)$:

$$|F(\omega)| = |F(\omega)F^*(\omega)|^{1/2} \tag{E1.2.3}$$

where $F^*(\omega)$ is the complex conjugate of $F(\omega)$:

$$F^*(\omega) = \frac{f_0}{\omega} \{[\sin\,\omega(t_0+s) - \sin\,\omega t_0] - i[\cos(\omega t_0 + s) - \cos\,\omega t_0]\} \tag{E1.2.4}$$

By substituting Eqs. (E1.2.2) and (E1.2.4) into Eq. (E1.2.3), we can obtain the amplitude spectrum as

$$|F(\omega)| = \frac{f_0}{|\omega|} (2 - 2\cos\omega s)^{1/2} \tag{E1.2.5}$$

or

$$\frac{|F(\omega)|}{f_0} = \frac{1}{|\omega|} (2 - 2\cos\omega)^{1/2} \tag{E1.2.6}$$

The plot of Eq. (E1.2.6) is shown in Fig. 1.13(b).

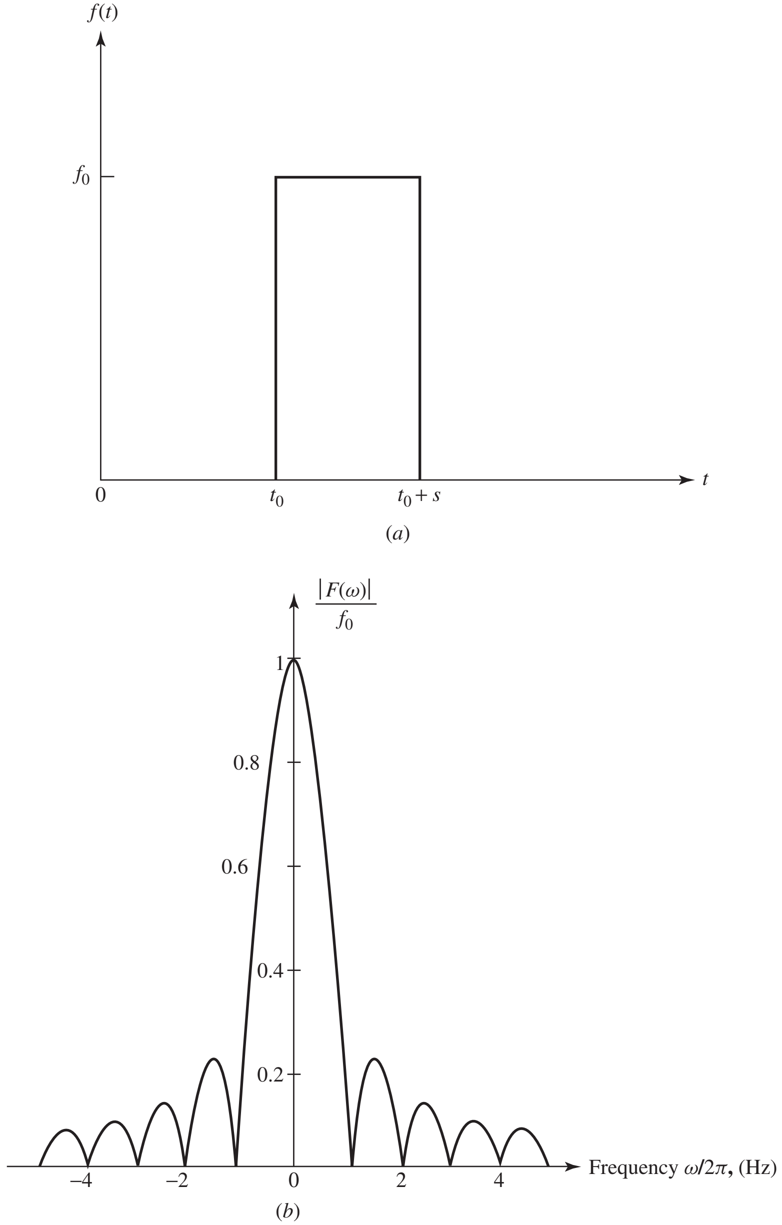


Figure 1.13 Fourier transform of a nonperiodic function: (*a*) rectangular pulse; (*b*) amplitude spectrum.

1.12 LITERATURE ON VIBRATION OF CONTINUOUS SYSTEMS

Several textbooks, monographs, handbooks, encyclopedia, vibration standards, books dealing with computer programs for vibration analysis, vibration formulas, and specialized topics as well as journals and periodicals are available in the general area of vibration of continuous systems. Among the large number of textbooks written on the subject of vibrations, the books by Magrab [10], Fryba [11], Nowacki [12], Meirovitch [13], and Clark [14] are devoted specifically to the vibration of continuous systems. Monographs by Leissa on the vibration of plates and shells [15, 16] summarize the results available in the literature on these topics. A handbook edited by Harris and Piersol [17] gives a comprehensive survey of all aspects of vibration and shock. A handbook on viscoelastic damping [18] describes the damping characteristics of polymeric materials, including rubber, adhesives, and plastics, in the context of design of machines and structures. An encyclopedia edited by Braun et al. [19] presents the current state of knowledge in areas covering all aspects of vibration along with references for further reading.

Pretlove [20], gives some computer programs in BASIC for simple analyses, and Rao [9] gives computer programs in Matlab, C++, and Fortran for the vibration analysis of a variety of systems and problems. Reference [21] gives international standards for acoustics, mechanical vibration, and shock. References [22–24] basically provide all the known formulas and solutions for a large variety of vibration problems, including those related to beams, frames, and arches. Several books have been written on the vibration of specific systems, such as spacecraft [25], flow-induced vibration [26], dynamics and control [27], foundations [28], and gears [29]. The practical aspects of vibration testing, measurement, and diagnostics of instruments, machinery, and structures are discussed in Refs. [30–32].

The most widely circulated journals that publish papers relating to vibrations are the *Journal of Sound and Vibration, ASME Journal of Vibration and Acoustics, ASME Journal of Applied Mechanics, AIAA Journal, ASCE Journal of Engineering Mechanics, Earthquake Engineering and Structural Dynamics, Computers and Structures, International Journal for Numerical Methods in Engineering, Journal of the Acoustical Society of America, Bulletin of the Japan Society of Mechanical Engineers, Mechanical Systems and Signal Processing, International Journal of Analytical and Experimental Modal Analysis, JSME International Journal Series III, Vibration Control Engineering, Vehicle System Dynamics,* and *Sound and Vibration.* In addition, *the Shock and Vibration Digest, Noise and Vibration Worldwide, and Applied Mechanics Reviews* are abstract journals that publish brief discussions of recently published vibration papers.

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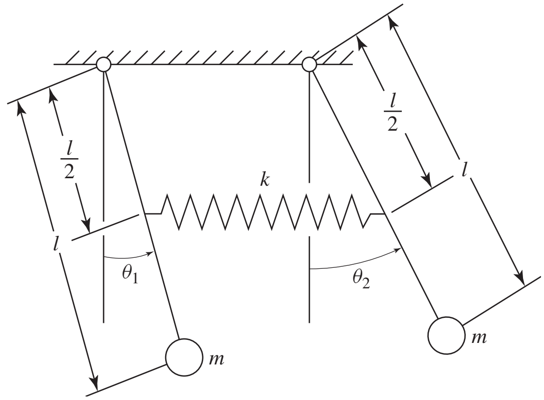

Figure 1.14 Two simple pendulums connected by a spring.

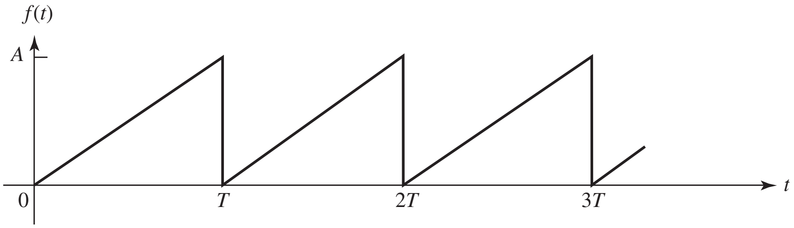

Figure 1.15 Sawtooth function.

PROBLEMS

1.1 Express the following function as a sum of sine and cosine functions:

$$f(t) = 5\sin(10t - 2.5)$$

1.2 Consider the following harmonic functions:

$$x_1(t) = 5\sin 20t \quad \text{and} \quad x_2(t) = 8\cos\left(20t + \frac{\pi}{3}\right)$$

Express the function $x(t) = x_1(t) + x_2(t)$ as (**a**) a cosine function with a phase angle, and (**b**) a sine function with a phase angle.

1.3 Find the difference of the harmonic functions $x_1(t) = 6\sin 30t$ and $x_2(t) = 4\cos(30t + \pi/4)$ (**a**) as a sine function with a phase angle, and (**b**) as a cosine function with a phase angle.

1.4 Find the sum of the harmonic functions $x_1(t) = 5\cos\omega t$ and $x_2(t) = 10\cos(\omega t + 1)$ using (**a**) trigonometric relations, (**b**) vectors, and (**c**) complex numbers.

1.5 The angular motions of two simple pendulums connected by a soft spring of stiffness k are described by (Fig. 1.14)

$$\theta_1(t) = A\cos\omega_1 t\cos\omega_2 t, \qquad \theta_2(t) = A\sin\omega_1 t\sin\omega_2 t$$

where A is the amplitude of angular motion and ω_1 and ω_2 are given by

$$\omega_1 = \frac{k}{8\,m}\sqrt{\frac{l}{g}}, \qquad \omega_2 = \sqrt{\frac{g}{l}} + \omega_1$$

Plot the functions $\theta_1(t)$ and $\theta_2(t)$ for $0 \le t \le 13.12$ s and discuss the resulting motions for the following data: $k = 1$ N/m, $m = 0.1$ kg, $l = 1$ m, and $g = 9.81$ m/s^2.

1.6 Find the Fourier cosine and sine series expansion of the function shown in Fig. 1.15 for $A = 2$ and $T = 1$.

1.7 Find the Fourier cosine and sine series representation of a series of half-wave rectified sine pulses shown in Fig. 1.16 for $A = \pi$ and $T = 2$.

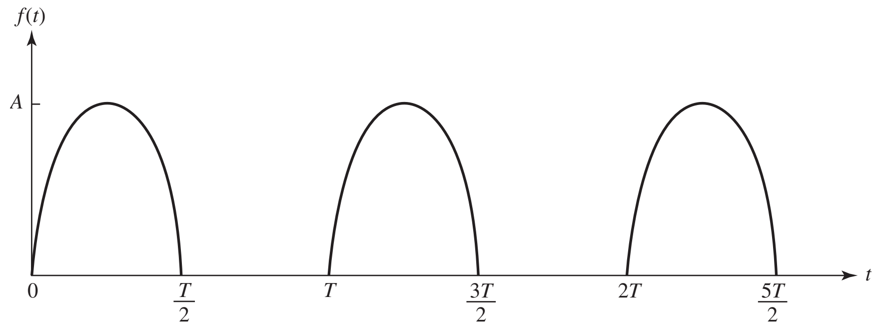

Figure 1.16 Half sine pulses.

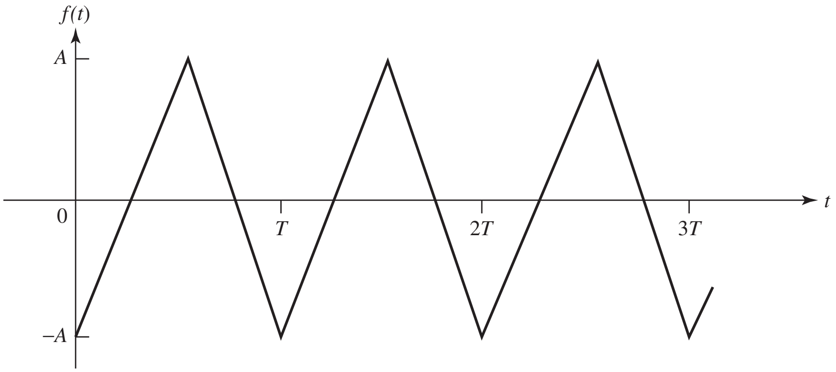

Figure 1.17 Triangular wave.

1.8 Find the complex Fourier series expansion of the sawtooth function shown in Fig. 1.15.

1.9 Find the Fourier series expansion of the triangular wave shown in Fig. 1.17.

1.10 Find the complex Fourier series representation of the function $f(t) = e^{-2t}$, $-\pi < t < \pi$.

1.11 Consider a transient load, $f(t)$, given by

$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-t}, & t \geq 0 \end{cases}$$

Find the Fourier transform of $f(t)$.

1.12 The Fourier sine transform of a function $f(t)$, denoted by $F_s(\omega)$, is defined as

$$F_s(\omega) = \int_0^\infty f(t) \sin \omega t \, dt, \qquad \omega > 0$$

and the inverse of the transform $F_s(\omega)$ is defined by

$$f(t) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin \omega t \, d\omega, \qquad t > 0$$

Using these definitions, find the Fourier sine transform of the function $f(t) = e^{-at}$, $a > 0$.

1.13 Find the Fourier sine transform of the function $f(t) = te^{-t}$, $t \geq 0$.

1.14 Find the Fourier transform of the function

$$f(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$