CHAPTER 1

MATHEMATICAL PRELIMINARIES

Treatment of electrokinetic transport phenomena requires understanding of fluid mechanics, colloidal phenomena, and the interaction of charged particles, surfaces, and electrolytes with an external electrical field. Accordingly, dealing with electrokinetic transport processes requires familiarity with the units and dimensions of fundamental quantities from a diverse range of subjects. In this chapter, we outline the pertinent units and dimensions of the fundamental quantities encountered in electrokinetic transport processes.

Historically, the centimeter-gram-second (cgs) system of units was widely used in most colloid science and electrokinetics literature. However, with the popularity of the *Système Internationale d'Unités* (the SI system), most of the modern treatment of these subjects are based on SI units. Accordingly, most of the topics covered in this book are based on the SI system. To facilitate the conversion of other units into the SI system, the first few sections of this Chapter are devoted to definitions of the fundamental units and dimensions, description of the derived units in the SI system, values of the commonly encountered physical constants in various units, and conversion factors for different quantities from SI to non-SI units. The latter half of the Chapter outlines some of the mathematical fundamentals required to develop the theoretical treatments in the rest of the book, including a short primer of series functions, vector, and tensor operations.

Electrokinetic and Colloid Transport Phenomena, by Jacob H. Masliyah and Subir Bhattacharjee

1.1 UNITS

The fundamental quantities required in electrokinetic transport analysis are shown in Table 1.1, along with their SI units and symbols.

These fundamental quantities can be combined to yield different derived quantities, the units of which are combinations of the fundamental units. Table 1.2 provides some of the commonly used derived quantities and their SI units.

To provide a facile transition of the basic dimensions over large ranges, it is often convenient to use scale factors for the basic units. This is particularly important in terms of the length scales used to define the dimensions of extremely small colloidal particles. For instance, it is convenient to express particle sizes in terms of nanometer (nm) or micrometer (μm) instead of meter (m). Similarly, the colloidal forces are conveniently expressed in terms of nano-newtons (nN) or pico-newtons (pN) rather than newtons (N). Table 1.3 provides the commonly used scale factors for the basic units.

TABLE 1.1. Fundamental Quantities Used in Electrokinetic Transport Analysis, their SI Units and Symbols.

Quantity	Name of SI Unit	Symbol
Mass	kilogram	kg
Length	meter	m
Time	second	s
Temperature	Kelvin	K
Quantity of mass	mole	mol
Electric current	Ampere	A

TABLE 1.2. Derived Quantities and their SI Units.

Quantity	SI Unit Name	Symbol	Definition
Force	Newton	N	kg m s^{-2}
Pressure	Pascal	Pa	$\text{N m}^{-2} = \text{kg m}^{-1}\text{s}^{-2}$
Energy	Joule	J	$\text{N m} = \text{kg m}^{2}\text{s}^{-2}$
Power	Watt	W	$\text{J s}^{-1} = \text{kg m}^{2}\text{s}^{-3}$
Electric charge	Coulomb	C	A s
Electric potential	Volt	V	$\text{J C}^{-1} = \text{kg m}^{2}\text{s}^{-3}\text{A}^{-1}$
Electric resistance	Ohm	Ω	$\text{V A}^{-1} = \text{kg m}^{2}\text{s}^{-3}\text{A}^{-2}$
Electric conductance	Siemens	S	$\text{A V}^{-1} = \text{kg}^{-1}\text{m}^{-2}\text{s}^{3}\text{A}^{2}$
Electric capacitance	Farad	F	$\text{C V}^{-1} = \text{kg}^{-1}\text{m}^{-2}\text{s}^{4}\text{A}^{2}$
Frequency	Hertz	Hz	s^{-1}
Magnetic inductance	Henry	H	$\text{J A}^{-2} = \text{kg m}^{2}\text{s}^{-2}\text{A}^{-2}$
Dynamic viscosity			$\text{Pa s} = \text{N s m}^{-2} = \text{kg m}^{-1}\text{s}^{-1}$
Material density			kg m^{-3}

Source: Adapted from Russel *et al.* (1989) and Probstein (2003).

TABLE 1.3. Scale Factors for the Basic Units.

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{-1}	deci	d	10	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E

1.2 PHYSICAL CONSTANTS AND CONVERSION FACTORS

The commonly used physical constants and their values in SI units are listed in Table 1.4. The use of non-SI units for various quantities is still common in electrokinetics literature. Conversion factors between SI and other units are provided for some of these quantities in Table 1.5.

TABLE 1.4. Common Physical Constants and their Values in SI Units (Lide, 2001).

Quantity	Symbol	Value	SI Units
Avogadro number	N_A	6.022×10^{23}	mol^{-1}
Boltzmann constant	k_B	1.381×10^{-23}	$J\,K^{-1}$
Elementary charge	e	1.602×10^{-19}	C
Faraday constant	$\mathcal{F}$	9.649×10^{4}	$C\,mol^{-1}$
Magnetic permeability of vacuum	μ_0	1.2566×10^{-7}	NA^{-2}, $NC^{-2}s^{-2}$, or Hm^{-1}
Universal gas constant	R	8.314	$JK^{-1}mol^{-1}$
Permittivity of vacuum	ϵ_0	8.854×10^{-12}	$CV^{-1}m^{-1}$, $C^2N^{-1}m^{-2}$, or Fm^{-1}
Planck constant	h	6.626×10^{-34}	J s
Speed of light in vacuum	c	2.9979×10^{8}	$m\,s^{-1}$
Standard gravitational acceleration	g	9.8066	$m\,s^{-2}$
Standard atmospheric pressure (at sea level and 288.16 K)	p_0	1.01325×10^{5}	Pa
Zero of Celsius scale	T_0	273.15	K
1 liter	L	1.0000028×10^{-3}	m^3
k_BT/e at 298.16 K		25.69×10^{-3}	V
1 molar solution	M	1.0	mol/dm^3 or $kmol/m^3$

Source: Adapted from Hiemenz (1986), Russel *et al.* (1989), and Probstein (2003).

TABLE 1.5. Conversion Factors for Non-SI Units.

Unit	Abbreviation	Value
atmosphere	atm	101325 Pa (definition)
torr	torr	133.322 Pa = 1/760 atm
atomic mass unit	amu	1.6605×10^{-27} kg
bar	bar	1×10^{5} Pa
electron volt	eV	1.6022×10^{-19} J
poise	P	0.1 kg m^{-1} s^{-1}
liter	L	1×10^{-3} m^3 = 1 dm^3
Angstrom	Å	1×10^{-10} m
Debye	D	3.3356×10^{-30} C m
calorie	cal	4.184 J (definition)
inch	in	0.0254 m (definition)
pound	lbm	0.4536 kg

1.3 FREQUENTLY USED FUNCTIONS

Here we list some of the common series expansions and functions used frequently in this book. Excellent compilation of mathematical formulae is given by Jeffrey (1995).

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad -\infty < x < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \quad -1 < x \le 1$$

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \right) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \quad -1 \le x < 1$$

$$\sinh(x) = \frac{1}{2}[\exp(x) - \exp(-x)] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad -\infty < x < \infty$$

$$\cosh(x) = \frac{1}{2}[\exp(x) + \exp(-x)] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad -\infty < x < \infty$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \quad |x| < \pi/2$$

$$\cosh^2(x) = 1 + \sinh^2(x)$$

$$\sinh(x) \cong \cosh(x) \quad x \to \infty$$

$$\tanh(x) \longrightarrow 1 \quad x \to \infty$$

$$\sinh(x/2) = \pm\sqrt{\left(\frac{\cosh(x) - 1}{2}\right)} \quad [+ \text{ if } x > 0 \text{ and } - \text{ if } x < 0]$$

$$\sinh(x) = 2\sinh(x/2)\cosh(x/2)$$

$$\sinh^2(x) = \frac{1}{2}[\cosh(2x) - 1]$$

$$\sinh(2x) = 2\sinh(x)\cosh(x)$$

$$\cosh(x/2) = \left(\frac{1+\cosh(x)}{2}\right)^{1/2}$$

$$\cosh(x) = \cosh^2(x/2) + \sinh^2(x/2)$$

$$\cosh^2(x) = \frac{1}{2}[1 + \cosh(2x)]$$

$$\cosh(2x) = 2\cosh^2(x) - 1$$

$$\tanh(x/2) = \frac{\sinh(x)}{1+\cosh(x)} = \frac{\cosh(x)-1}{\sinh(x)} = \frac{1-\exp(-x)}{1+\exp(-x)}$$

$$1 - \tanh(x/2) = \frac{2\exp(-x)}{1+\exp(-x)}$$

$$\tanh(x) = \frac{2\tanh(x/2)}{1+\tanh^2(x/2)} = \frac{2\coth(x/2)}{\operatorname{csch}^2(x/2)+2}$$

$$\tanh^2(x) = \frac{\cosh(2x)-1}{1+\cosh(2x)}$$

$$\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)}$$

Some commonly used integrals are provided next. These being indefinite integrals, one should remember to add an integration constant to each result.

$$\int \frac{dx}{\sinh^2(x)} = -\coth(x)$$

$$\int \frac{dx}{\cosh^2(x)} = \tanh(x)$$

$$\int \frac{dx}{1+\cosh(x)} = \tanh(x/2)$$

$$\int \frac{dx}{1-\cosh(x)} = \coth(x/2)$$

$$\int \tanh(kx)dx = \frac{1}{k}\ln[\cosh(kx)]$$

$$\int \tanh^2(kx)dx = x - \frac{1}{k}\tanh(kx)$$

$$\int \coth(kx)dx = \frac{1}{k}\ln|\sinh(kx)|$$

1.4 VECTOR OPERATIONS

A scalar quantity is defined by a single real number. Temperature and mass are good examples of scalar quantities. A vector quantity is defined by a magnitude and a direction. Velocity of a projectile is a vector quantity. The magnitude of a vector **u** is given by $|\mathbf{u}|$ or simply u.

Addition and subtraction of two vectors, **u** and **v**, are illustrated in Figure 1.1.

Multiplication of a vector **u** by a scalar quantity s results in changing the magnitude of the vector to $s|\mathbf{u}|$ or simply su. The vector direction remains same.

A vector can be multiplied with another vector in several ways. Scalar or dot product of two vectors, **u** and **v**, is given by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} = uv \cos\phi$$

where ϕ is the angle formed between the vectors **u** and **v**. Here, u and v are the magnitudes of the vectors **u** and **v**, respectively. The scalar product rules are

Commutative:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$

Not associative:

$$(\mathbf{u} \cdot \mathbf{v})\,\mathbf{w} \neq \mathbf{u}\,(\mathbf{v} \cdot \mathbf{w})$$

Distributive:

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$$

Vector product or cross product of two vectors **u** and **v** is given by another vector defined by

$$\mathbf{u} \times \mathbf{v} = uv \sin\phi\, \mathbf{n}$$

where ϕ is the angle between the two vectors and **n** is a vector of unit length (magnitude) normal to both the vectors **u** and **v** in the sense in which a right-handed screw

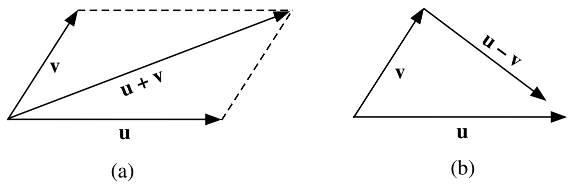

Figure 1.1. (a) Addition and (b) subtraction of two vectors.

would advance if rotated from **u** to **v**. A convenient form of a cross product is given by

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$$

where $\mathbf{u} = u_1\mathbf{i}_1 + u_2\mathbf{i}_2 + u_3\mathbf{i}_3$ and $\mathbf{v} = v_1\mathbf{i}_1 + v_2\mathbf{i}_2 + v_3\mathbf{i}_3$. Here, $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ are orthogonal unit vectors, with u_1, u_2, and u_3 being the magnitudes of vector **u** in the $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ directions, respectively. Similarly, v_1, v_2, and v_3 are the magnitudes of the vector **v** in the $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ directions, respectively. The magnitude of a vector **u** is given by

$$|\mathbf{u}| = u = \sqrt{\sum_{i=1}^{3} u_i^2}$$

One can also multiply two vectors to obtain a tensor or a dyadic product. The dyadic product of two vectors **u** and **v** is given by **uv**. We will discuss dyadic products in the next section.

A compilation of useful vector identities and vector operations is given by Bird *et al.* (2002).

Some commonly used differential vector operations in different orthogonal coordinate systems are given below. Here, ψ is used for a scalar and **u** is used for a vector.

Cartesian Coordinates (x, y, z)

The orthogonal curvilinear coordinates, in the case of Cartesian coordinates, are defined by the unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ directed along the x, y, and z coordinates, respectively, and the vector **u** is given by

$$\mathbf{u} = u_x\mathbf{i}_x + u_y\mathbf{i}_y + u_z\mathbf{i}_z$$

The differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial x}\mathbf{i}_x + \frac{\partial}{\partial y}\mathbf{i}_y + \frac{\partial}{\partial z}\mathbf{i}_z$$

Some useful differential operations are given by

$$\nabla\psi = \frac{\partial\psi}{\partial x}\mathbf{i}_x + \frac{\partial\psi}{\partial y}\mathbf{i}_y + \frac{\partial\psi}{\partial z}\mathbf{i}_z$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

$$\begin{aligned}\nabla \times \mathbf{u} &= \begin{pmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_x & u_y & u_z \end{pmatrix} \\ &= \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\mathbf{i}_x + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\mathbf{i}_y + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathbf{i}_z\end{aligned}$$

Normally, $\nabla\psi$ is referred to as the gradient of the scalar ψ, $\nabla \cdot \mathbf{u}$ is known as the divergence of the vector $\mathbf{u}$, and $\nabla \times \mathbf{u}$ is known as the curl of vector $\mathbf{u}$. Here, ∇^2 is a Laplacian operator.

Cylindrical Coordinates (r, θ, z)

The cylindrical orthogonal coordinate system is defined by the three orthogonal unit vectors given by $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_z$ acting along the r, θ, and z directions, respectively. A vector $\mathbf{u}$ is given by

$$\mathbf{u} = u_r\mathbf{i}_r + u_\theta\mathbf{i}_\theta + u_z\mathbf{i}_z$$

The differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{i}_\theta + \frac{\partial}{\partial z}\mathbf{i}_z$$

Some useful differential operations are given by

$$\begin{aligned}\nabla\psi &= \frac{\partial \psi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\mathbf{i}_\theta + \frac{\partial \psi}{\partial z}\mathbf{i}_z \\ \nabla^2\psi &= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \\ \nabla \cdot \mathbf{u} &= \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial \theta}u_\theta + \frac{\partial}{\partial z}u_z\end{aligned}$$

$$\begin{aligned}\nabla \times \mathbf{u} &= \frac{1}{r}\begin{pmatrix} \mathbf{i}_r & r\,\mathbf{i}_\theta & \mathbf{i}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ u_r & ru_\theta & u_z \end{pmatrix} \\ &= \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{i}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{i}_\theta + \left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\mathbf{i}_z\end{aligned}$$

Spherical Coordinates (r, θ, ϕ)

The spherical orthogonal coordinate system is defined by the three orthogonal unit vectors given by $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ acting along the r, θ, and ϕ directions, respectively. A vector $\mathbf{u}$ is given by

$$\mathbf{u} = u_r\mathbf{i}_r + u_\theta\mathbf{i}_\theta + u_\phi\mathbf{i}_\phi$$

The differential operator ∇ is given by

$$\nabla = \frac{\partial}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\mathbf{i}_\phi$$

Some useful differential operations are given by

$$\nabla\psi = \frac{\partial\psi}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{i}_\phi$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}$$

$$\begin{aligned}\nabla\times\mathbf{u} &= \frac{1}{r^2\sin\theta}\begin{pmatrix}\mathbf{i}_r & r\,\mathbf{i}_\theta & r\sin\theta\,\mathbf{i}_\phi\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ u_r & ru_\theta & r\sin\theta u_\phi\end{pmatrix}\\ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\phi) - \frac{\partial u_\theta}{\partial\phi}\right]\mathbf{i}_r\\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(ru_\phi)\right]\mathbf{i}_\theta\\ &\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right]\mathbf{i}_\phi\end{aligned}$$

In any orthogonal curvilinear system of coordinates, the cross product of a gradient, *i.e.*, $\nabla\psi$, is zero. In other words

$$\nabla\times\nabla\psi = 0$$

Also, for a vector, **u**, one can write

$$\nabla\cdot(\nabla\times\mathbf{u}) = 0$$

1.5 TENSOR OPERATIONS

A second order tensor is a quantity that has nine components that are associated with three orthogonal directions and normal planes. The components of a tensor quantity $\bar{\bar{T}}$ is given by

$$\bar{\bar{T}} = \begin{pmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{pmatrix}$$

For $\bar{\bar{T}}$ to be a stress tensor, each term T_{ij} represents the stress on the i^{th} plane in the j-direction. In general, $T_{ij} \neq T_{ji}$ unless the second order tensor is *symmetric*, where $T_{ij} = T_{ji}$.

A second order tensor can also be given as

$$\bar{\bar{T}} = \sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{i}_i\mathbf{i}_j T_{ij}$$

and

$$\bar{\bar{T}} = \mathbf{i}_1\mathbf{i}_1 T_{11} + \mathbf{i}_1\mathbf{i}_2 T_{12} + \mathbf{i}_1\mathbf{i}_3 T_{13} + \mathbf{i}_2\mathbf{i}_1 T_{21} + \mathbf{i}_2\mathbf{i}_2 T_{22} + \mathbf{i}_2\mathbf{i}_3 T_{23} + \mathbf{i}_3\mathbf{i}_1 T_{31} + \mathbf{i}_3\mathbf{i}_2 T_{32} + \mathbf{i}_3\mathbf{i}_3 T_{33}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ are unit vectors. Here, $\mathbf{i}_i\mathbf{i}_j$ is called the unit dyad, which follows certain rules when associated with a vector operation such as:

$$\mathbf{i}_i\mathbf{i}_j \cdot \mathbf{i}_k = \mathbf{i}_i(\mathbf{i}_j \cdot \mathbf{i}_k) = \mathbf{i}_i\delta_{jk}$$

$$\mathbf{i}_i \cdot \mathbf{i}_j\mathbf{i}_k = (\mathbf{i}_i \cdot \mathbf{i}_j)\mathbf{i}_k = \mathbf{i}_k\delta_{ij}$$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = 1 \quad \text{if} \quad i = j$$
$$\delta_{ij} = 0 \quad \text{if} \quad i \neq j$$

When the components of a second order tensor are formed from components of two vectors **u** and **v**, the resulting product is called a dyadic product of **u** and **v**, given by **uv**, where

$$\mathbf{uv} = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{i}_i\mathbf{i}_j u_i v_j$$

A unit tensor is defined as

$$\bar{\bar{I}} = \sum_i \sum_j \mathbf{i}_i\mathbf{i}_j\delta_{ij}$$

or

$$\bar{\bar{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The magnitude of a tensor is given by

$$|\bar{\bar{T}}| = \sqrt{\frac{1}{2}\sum_i \sum_j T_{ij}^2}$$

The addition of tensors $\bar{\bar{T}}$ and $\bar{\bar{A}}$ or dyadic products simply follows

$$\bar{\bar{T}} + \bar{\bar{A}} = \sum_i \sum_j (T_{ij} + A_{ij})$$

The multiplication of a tensor or a dyadic product by a scalar gives

$$s\bar{\bar{T}} = \sum_i \sum_j \mathbf{i}_i \mathbf{i}_j (s T_{ij})$$

The vector product (or dot product) of a tensor with a vector is given by

$$(\bar{\bar{T}} \cdot \mathbf{u}) = \sum_i \mathbf{i}_i \left\{ \sum_j T_{ij} u_j \right\}$$

The vector product (or dot product) of a vector with a tensor is given by

$$(\mathbf{u} \cdot \bar{\bar{T}}) = \sum_i \mathbf{i}_i \left\{ \sum_j u_j T_{ji} \right\}$$

In general $\bar{\bar{T}} \cdot \mathbf{u} \neq \mathbf{u} \cdot \bar{\bar{T}}$ unless the tensor $\bar{\bar{T}}$ is symmetric where $T_{ij} = T_{ji}$. In expanded form one can write for $\bar{\bar{T}} \cdot \mathbf{u}$ in Cartesian coordinates

$$\begin{aligned}\bar{\bar{T}} \cdot \mathbf{u} = {} & (T_{xx}u_x + T_{xy}u_y + T_{xz}u_z)\mathbf{i}_x \\ & + (T_{yx}u_x + T_{yy}u_y + T_{yz}u_z)\mathbf{i}_y \\ & + (T_{zx}u_x + T_{zy}u_y + T_{zz}u_z)\mathbf{i}_z\end{aligned}$$

In other orthogonal coordinates, one can simply replace (x, y, z) by the respective new coordinates *e.g.*, with (r, θ, ϕ) for a spherical coordinate system.

1.6 VECTOR AND TENSOR INTEGRAL THEOREMS

1.6.1 The Divergence and Gradient Theorems

If a volume V is enclosed by a surface S, then

$$\int_V (\nabla \cdot \mathbf{u}) dV = \oint_S (\mathbf{u} \cdot \mathbf{n}) dS$$

where $\mathbf{n}$ is the outwardly directed unit normal vector. As $\mathbf{u} \cdot \mathbf{n} = \mathbf{n} \cdot \mathbf{u}$, one can write

$$\int_V (\nabla \cdot \mathbf{u}) dV = \oint_S (\mathbf{n} \cdot \mathbf{u}) dS$$

Two related theorems for scalars, ψ, and tensors, $\bar{\bar{T}}$, can be written as

$$\int_V \nabla \psi dV = \oint_S \mathbf{n} \psi dS$$

and

$$\int_V (\nabla \cdot \bar{\bar{T}}) dV = \oint_S (\mathbf{n} \cdot \bar{\bar{T}}) dS$$

Clearly, the tensor $\bar{\bar{T}}$ can be replaced by a dyadic product.

1.6.2 The Stokes Theorem

If a surface S is bounded by a closed curve C, then

$$\int_S \mathbf{n} \cdot (\nabla \times \mathbf{u}) dS = \oint_C (\mathbf{t} \cdot \mathbf{u}) dC$$

where $\mathbf{t}$ is a unit tangential vector in the direction of integration along path C, $\mathbf{n}$ is the unit normal vector to the surface S in the direction that a right-hand screw would move if its head were twisted in the direction of integration along contour C (Bird *et al.* 2002).

1.7 REFERENCES

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