CHAPTER 1

FIRST-ORDER DIFFERENTIAL EQUATIONS

1 INTRODUCTION

A *differential equation* is an equation between specified derivatives of an unknown function, its values, and known quantities and functions. Many physical laws are most simply and naturally formulated as differential equations (or DEs, as we will write for short). For this reason, DEs have been studied by the greatest mathematicians and mathematical physicists since the time of Newton.

Ordinary differential equations are DEs whose unknowns are functions of a single variable; they arise most commonly in the study of dynamical systems and electrical networks. They are much easier to treat than *partial* differential equations, whose unknown functions depend on two or more independent variables.

Ordinary DEs are classified according to their order. The *order* of a DE is defined as the largest positive integer, n, for which an nth derivative occurs in the equation. Thus, an equation of the form

$$\phi(x,y,y') = 0$$

is said to be of the *first order*.

This chapter will deal with first-order DEs of the special form

$$M(x,y) + N(x,y)y' = 0 \tag{1}$$

A DE of the form (1) is often said to be of the *first degree*. This is because, considered as a polynomial in the derivative of highest order, y', it is of the first degree.

One might think that it would therefore be called "linear," but this name is reserved (within the class of first-order DEs) for DEs of the much more special form $a(x)y' + b(x)y + c(x) = 0$, which are *linear* in y and its derivatives. Such "linear" DEs will be taken up in §3, and we shall call first-order DEs of the more general form (1) *quasilinear*.

A primary aim of the study of differential equations is to find their *solutions*—that is, functions $y = f(x)$ which satisfy them. In this chapter, we will deal with the following special case of the problem of "solving" given DEs.

DEFINITION. A *solution* of (1) is a function $f(x)$ such that $M(x,f(x)) + N(x,f(x))f'(x) = 0$ for all x in the interval where $f(x)$ is defined.

The problem of solving (1) for given functions $M(x,y)$ and $N(x,y)$ is thus to determine all real functions $y = f(x)$ which satisfy (1), that is, all its solutions.

Example 1. Consider the first-order quasilinear DE

$$x + yy' = 0 \tag{2}$$

The solutions of (2) can be found by considering the formula $d(x^2 + y^2)/dx = 2(x + yy')$. Clearly, $y = f(x)$ is a solution of (2) if and only if $x^2 + y^2 = C$ is a constant.

The equation $x^2 + y^2 = C$ defines y *implicitly* as a two-valued function of x, for any positive constant C. Solving for y, we get for each positive constant C *two* solutions, the (single-valued)† functions $y = \pm\sqrt{C - x^2}$. The *graphs* of these solutions, the so-called *solution curves,* form two families of semicircles. These fill the upper half-plane $y > 0$ and the lower half-plane $y < 0$, respectively, in that there is one and only one such semicircle through each point in each half-plane.

Caution. Note that the functions $y = \pm\sqrt{C - x^2}$ are defined only in the interval $-\sqrt{C} \le x \le \sqrt{C}$, and that since y' does not exist (is "infinite") when $x = \pm\sqrt{C}$, these functions are solutions of (1) only on $-\sqrt{C} < x < \sqrt{C}$. Therefore, although the pairs of semicircles in Figure 1.1 *appear* to join together to form the full circle $x^2 + y^2 = C$, the latter is *not* a "solution curve" of (1). In fact, no solution curve of (2) can cross the x-axis (except possibly at the origin), because on the x-axis $y = 0$ the DE (2) implies $x = 0$ for *any* finite y'.

The preceding difficulty also arises if one tries to solve the DE (2) for y'. Dividing through by y, one gets $y' = -x/y$, an equation which cannot be satisfied if $y = 0$. The preceding difficulty is thus avoided if one restricts attention to regions where the DE (1) is normal, in the following sense.

DEFINITION. A *normal* first-order DE is one of the form

$$y' = F(x,y) \tag{3}$$

In the normal form $y' = -x/y$ of the DE (2), the function $F(x,y)$ is continuous in the upper half-plane $y > 0$ and in the lower half-plane where $y < 0$; it is undefined on the x-axis.

2 FUNDAMENTAL THEOREM OF THE CALCULUS

Although the importance of the theory of (ordinary) DEs stems primarily from its many applications to geometry, science, and engineering, a clear under-

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† In this book, the word "function" will always mean single-valued function, unless the contrary is expressly specified.

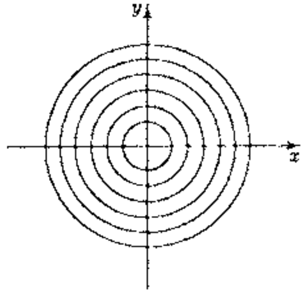

Figure 1.1 Integral curves of $x + yy' = 0$.

standing of its capabilities can only be achieved if its definitions and results are formulated precisely. Some of its most difficult results concern the *existence* and *uniqueness* of solutions. The nature of such existence and uniqueness theorems is well illustrated by the most familiar (and simplest!) class of ordinary DEs. These are the first-order DEs of the very special form

$$y' = g(x) \tag{4}$$

Such DEs are normal; their solutions are described by the fundamental theorem of the calculus, which reads as follows.

FUNDAMENTAL THEOREM OF THE CALCULUS. *Let the function $g(x)$ in the* DE (4) *be continuous in the interval $a \leq x \leq b$. Given a number c, there is one and only one solution $f(x)$ of the* DE (4) *in the interval such that $f(a) = c$. This solution is given by the definite integral*

$$f(x) = c + \int_a^x g(t)\,dt, \qquad c = f(a) \tag{5}$$

This basic result serves as a model of rigorous formulation in several respects. First, it specifies the region under consideration, as a vertical strip $a \leq x \leq b$ in the xy-plane. Second, it describes in precise terms the class of functions $g(x)$ considered. And third, it asserts the *existence* and *uniqueness* of a solution, given the "initial condition" $f(a) = c$.

We recall that the definite integral

$$\int_a^x g(t)\,dt = \lim_{\max \Delta t_k \to 0} \sum g(t_k)\,\Delta t_k, \qquad \Delta t_k = t_k - t_{k-1} \tag{5'}$$

is defined for each fixed x as a limit of Riemann sums; it is not necessary to find a formal expression for the indefinite integral $\int g(x)\,dx$ to give meaning to the definite integral $\int_a^x g(t)\,dt$, provided only that $g(t)$ is continuous. Such functions as the *error function* erf $x = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\, dt$ and the *sine integral function* Si $(x) = \int_0^x [(\sin t)/t]\, dt$ are indeed commonly *defined* as definite integrals; cf. Ch. 4, §1.

Quadrature. The preceding considerations enable one to solve DEs of the special form $y' = g(x)$ by inspection: for any a, one solution is the function $\int_a^x g(t)\, dt$; the others are obtained by adding an arbitrary constant C to this "particular" solution. Thus, the solutions of $y' = e^{-x^2}$ are the functions $y = \int e^{-x^2}\, dx = (\sqrt{\pi}/2)$ erf $x + C$; those of $xy' = \sin x$ are the functions $y =$ Si $(x) + C$; and so on. Note that from any one solution curve of $y' = g(x)$, the others are obtained by the vertical translations $(x,y) \mapsto (x, y + C)$.† Thus, they form a one-parameter family of curves, one for each value of the parameter C. This important geometrical fact is illustrated in Figure 1.2.

After $y' = f(x)$, the simplest type of DE is $y' = g(y)$. Any such DE is invariant under horizontal translation $(x,y) \mapsto (x + c, y)$. Hence, any horizontal line is cut by all solution curves at the same angle (such lines are called "isoclines"), and any horizontal translate $y = \phi(x + c)$ of any solution curve $y = \phi(x)$ is again a solution curve.

The DE $y' = y$ is the most familiar DE of this form. It can be solved by rewriting it as $dy/y = dx$; integrating, we get $x = \ln |y| + c$, or $y = \pm e^{x-c}$, where c is an arbitrary constant. Setting $k = \pm e^{-c}$, we get the general solution $y = ke^x$ but the solution $y = 0$ is "lost" until the last step.

Example 2. A similar procedure can be applied to any DE of the form $y' = g(y)$. Thus consider

$$y' = y^2 - 1 \tag{6}$$

Since $y^2 - 1 = (y + 1)(y - 1)$, the constant functions $y = -1$ and $y = 1$ are particular solutions of (6). Since $y^2 > 1$ if $|y| > 1$ whereas $y^2 < 1$ if $-1 < y$

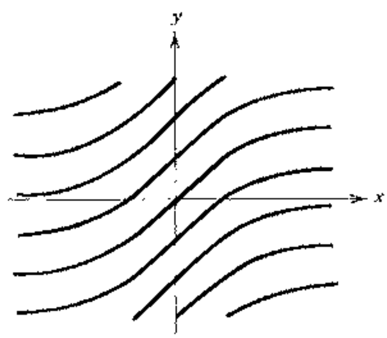

Figure 1.2 Solution curves of $y' = e^{-x^2}$.

† The symbol $\mapsto$ is to be read as "goes into".

< 1, all solutions are decreasing functions in the strip $|y| < 1$ and increasing functions outside it; see Figure 1.3.

Using the partial fraction decomposition $2/(y^2 - 1) = 1/(y - 1) - 1/(y + 1)$, one can rewrite (6) as $2\,dx = dy/(y - 1) - dy/(y + 1)$ from which we obtain, by integrating, $2(x - c) = \ln |(y - 1)/(y + 1)|$. Exponentiating both sides, we get $\pm e^{2(x-c)} = (y - 1)/(y + 1)$, which reduces after some manipulation to

(6′) $$y = \frac{1 \pm e^{2(x-c)}}{1 \mp e^{2(x-c)}} = \begin{Bmatrix} \tanh \\ \coth \end{Bmatrix} (c - x)$$

This procedure "loses" the special solutions $y = 1$ and $y = -1$, but gives all others. Note that if $y = f(x)$ is a solution of (6), then so is $1/y = 1/f(x)$, as can be directly verified from (6) (provided $y \neq 0$).

Example 3. A more complicated DE tractable by the same methods is $y' = y^3 - y$. Since $y^3 - y = y(y + 1)(y - 1)$, the constant functions $y = -1$, $y = 0$, and $y = 1$ are particular solutions. Since $y^3 > y$ if $-1 < y < 0$ or $1 < y$, whereas $y^3 < y$ if $y < -1$ or $0 < y < 1$, all solutions are increasing functions in the strips $-1 < y < 0$ and $y > 1$, and decreasing in the complementary strips.

To find the other solutions, we replace the DE $y' = dy/dx = y^3 - y$ by its reciprocal, $dx/dy = 1/(y^3 - y)$. We then use partial fractions to obtain the DE

(6″) $$\frac{dx}{dy} = \frac{1}{y^3 - y} = \frac{1}{2}\left\{\frac{1}{y + 1} + \frac{1}{y - 1} - \frac{2}{y}\right\}$$

The DE (6″) can be integrated termwise to give, after some manipulation, $x = \frac{1}{4}\ln |1 - y^{-2}| + C$, or $y = \pm[1 \mp \exp(2x - k)]^{-1/2}$, $k = 2c$.

Symmetry. The labor of drawing solution curves of the preceding DEs is reduced not only by their invariance under horizontal translation, but by the use of other symmetries as well. Thus, the DEs $y' = y$ and $y' = y^3 - y$ are invariant under reflection in the x-axis [i.e., under $(x,y) \mapsto (x, -y)$]; hence, so are their solution curves. Likewise, the DEs $y' = 1 + y^2$ and $y' = y^2 - 1$ (and their solution curves) are invariant under $(x,y) \mapsto (-x, -y)$, i.e., under rotation through 180° about the origin. These symmetries are visible in Figures 1.3 and 1.4.

EXERCISES A

1. (a) Show that if $f(x)$ satisfies (6), then so do $1/f(x)$ and $-f(-x)$.
   (b) Explain how these facts relate to Figure 1.2.
2. Show that every solution curve (6′) of (6) is equivalent under horizontal translation and/or reflection in the x-axis to $y = (1 + e^{2x})/(1 - e^{2x})$ or to $y = (1 - e^{2x})/(1 + e^{2x})$.
3. (a) Show that if $y' = y^2 + 1$, then y is an increasing function and $x = \arctan y + c$.
   (b) Infer that no solution of $y' = y^2 + 1$ can be defined on an interval of length exceeding π.

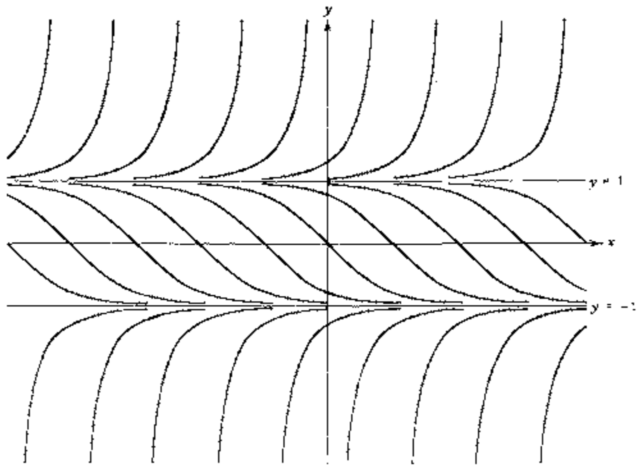

Figure 1.3 Solution curves of $y' = y^2 - 1$.

(c) Show that a nonhorizontal solution curve of $y' = y^2 \pm 1$ has a point of inflection on the x-axis and nowhere else.

4. Show that the solution curves of $y' = y^2$ are the x-axis and rectangular hyperbolas having this for one asymptote. [Hint: Rewrite $y' = y^2$ as $dy/y^2 = dx$.]

5. Sketch sample solution curves to indicate the qualitative behavior of the solutions of the following DEs: (a) $y' = 1 - y^3$, (b) $y' = \sin \pi y$, (c) $y' = \sin^2 y$.

6. Show that the solutions of $y' = g(y)$, for any continuous function g, are either all increasing functions or all decreasing functions in any strip $y_{i-1} < y < y_i$ between successive zeros of $g(y)$ [i.e., values y_i, such that $g(y_i) = 0$].

7. Show that the solutions of $y' = g(y)$ are convex up or convex down for given y according as $|g|$ is an increasing or decreasing function of y there.

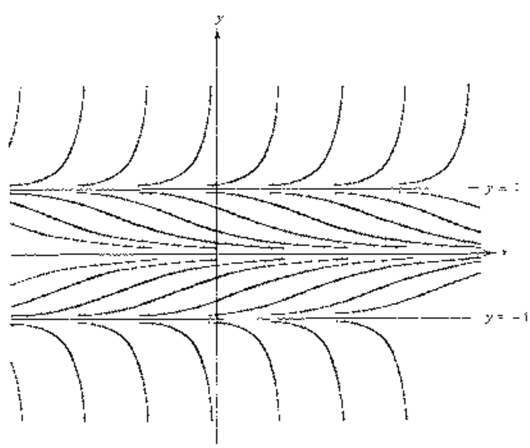

Figure 1.4 Solution curves of $y' = y^3 - y$.

*8. (a) Prove in detail that any nonconstant solution of (6) must satisfy

$$x = c - \tfrac{1}{2} \ln |(y - 1)/(y + 1)|$$

(b) Solve (6″) in detail, discussing the case $k = 0$ and the limiting case $k = \infty$ ($y = 0$).

*9. (a) Show that the choice $k < 0$ in (6′) gives solutions in the strip $-1 < y < 1$.
(b) Show that the choice $k = 1$ gives two solutions having the positive and negative y-axes for asymptotes, respectively.

3 FIRST-ORDER LINEAR EQUATIONS

In the next five sections, we will recall some very elementary, but extremely useful methods for solving important special families of first-order DEs. We begin with the first-order linear DE

$$a(x)y' + b(x)y + c(x) = 0 \tag{7}$$

It is called *homogeneous* if $c(x) \equiv 0$, and *inhomogeneous* otherwise.

Let the coefficient functions a, b, c be continuous. In any interval I where $a(x)$ does not vanish, the linear DE (7) can be reduced to the normal form

$$y' = -p(x)y + q(x) \tag{8}$$

with continuous coefficient functions $p = b/a$ and $q = c/a$.

The homogeneous linear case $y' = -p(x)y$ of (8) is solved easily, if not rigorously, as follows. We separate variables, $dy/y = -p(x)\,dx$; then we integrate (by quadratures), $\ln |y| = -\int p(x)\,dx + C$. Exponentiating both sides, we obtain $|y| = Ke^{-\int p(x)dx}$, where $K = e^C$ and any indefinite integral $P(x) = \int p(x)\,dx$ may be used.

This heuristic reasoning suggests that, if $P'(x) = p(x)$, then $ye^{P(x)}$ is a constant. Though this result was derived heuristically, it is easily verified rigorously:

$$d[ye^{P(x)}]/dx = y'e^{P(x)} + p(x)ye^{P(x)} = e^{P(x)}[y' + p(x)y] = 0$$

if and (since $e^{P(x)} \neq 0$) only if y satisfies (8). This proves the following result.

THEOREM 1. *If $P(x) = \int p(x)\,dx$ is an indefinite integral of the continuous function p, then the function $ce^{-P(x)} = ce^{-\int p(x)dx}$ is a solution of the DE $y' + p(x)y = 0$ for any constant c, and all solutions of the DE are of this form.*

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* The more difficult exercises in this book are starred.

We can treat the general case of (8) similarly. Differentiating the function $e^{P(x)}y$, where $P(x)$ is as before, we get

$$d[e^{P(x)}y]/dx = e^{P(x)}[y' + p(x)y] = -e^{P(x)}q(x)$$

It follows that, for some constant y_0, we must have $e^{P(x)}y = y_0 - \int_a^x e^{P(t)}q(t)\,dt$, whence

$$y = y_0 e^{-P(x)} - e^{-P(x)} \int_a^x e^{P(t)}q(t)\,dt \tag{8'}$$

Conversely, formula (8′) defines a solution of (8) with $y(a) = y_0$ for every y_0, by the Fundamental Theorem of the Calculus. This proves

THEOREM 2. *If $P(x)$ is as in Theorem* 1, *then the general solution of the* DE (8) *is given by* (8′). *Moreover, $y_0 = y(a)$ if and only if $P(x) = \int_a^x p(x)dx$.*

Quadrature. In the Fundamental Theorem of the Calculus, if the function g is nonnegative, the definite integral in (5) is the area under the curve $y = g(x)$ in the vertical strip between a and x. For this reason, the integration of (4) is called a *quadrature.* Formula (8′) reduces the solution of any first-order linear DE to the performance of a sequence of quadratures. Using Tables of Indefinite Integrals,† the solutions can therefore often be expressed explicitly, in terms of "elementary" functions whose numerical values have been tabulated ("tabulated functions").

Initial Value Problem. In general, the "initial value problem" for a first-order DE $y' = F(x,y)$ consists in finding a solution $y = g(x)$ that satisfies an initial condition $y(a) = y_0$, where a and y_0 are given constants. Theorem 2 states that the initial value problem always has one and only one solution for a *linear* DE (8), on any interval $a \leq x \leq b$ where $p(x)$ and $q(x)$ are defined and continuous.

Remark. There are often easier ways to solve linear DEs than substitution in (8′). This fact is illustrated by the following example.

Example 4. Consider the inhomogeneous linear DE

$$y' + y = x + 3 \tag{9}$$

Trying $y = ax + b$, one easily verifies that $x + 2$ is one solution of (9). On the other hand, if $y = f(x)$ is any other solution, then $z = y - (x + 2)$ must satisfy $z' + z = (y' + y) - (x + 3) = 0$, whence $z = ce^{-x}$ by Theorem 1. It follows that the general solution of (9) is the sum $ce^{-x} + x + 2$.

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† See the book by Dwight listed in the Bibliography. Kamke's book listed there contains an extremely useful catalog of solutions of DEs not of the form $y' = g(x)$. For a bibliography of function tables, see Fletcher, Miller, and Rosenhead.

4 SEPARABLE EQUATIONS

A differential equation that can be written in the form

(10) $$y' = g(x)h(y)$$

is said to be *separable.* Thus, the DEs $y' = y^2 - 1$ and $y' = y^3 - y$ of Examples 2 and 3 are obviously separable, with $g(x) = 1$. The DE $x + yy' = 0$ of Example 1, rewritten as $y' = (-x)(1/y)$ is separable except on the x-axis, where $1/y$ becomes infinite. As we have seen, the solutions $y = \pm\sqrt{C - x^2}$ of this DE cannot be expressed as single-valued functions of x on the x-axis, essentially for this reason.

A similar difficulty arises in general for DEs of the form

(11) $$M(x) + N(y)y' = 0$$

These can also be rewritten as

(11′) $$M(x)\,dx + N(y)\,dy = 0$$

or as $y' = -M(x)/N(y)$ and are therefore also said to be "separable." Whenever $N(y)$ vanishes, it is difficult or impossible to express y as a function of x.

It is easy to solve separable DEs formally. If $\phi(x) = \int M(x)\,dx$ and $\psi(y) = \int N(y)\,dy$ are any antiderivatives ("indefinite integrals") of $M(x)$ and $N(y)$, respectively, then the *level curves*

$$\phi(x) + \psi(y) = C$$

of the function $U(x,y) = \phi(x) + \psi(y)$ are solution curves of the DEs (11) and (11′). Moreover, the Fundamental Theorem of the Calculus assures us of the *existence* of such antiderivatives. Likewise, for any indefinite integrals $G(x) = \int g(x)\,dx$ and $H(y) = \int dy/h(y)$, the level curves of

$$G(x) - H(y) = C$$

may be expected to define solutions of (10), of the form

(11″) $$y = H^{-1}[C - G(x)]$$

However, the solutions defined in this way are only *local.* They are defined by the Inverse Function Theorem,† but only in *intervals of monotonicity* of $H(y)$ where $h(y)$ and hence $H'(y) = 1/h(y)$ has constant sign. Moreover, the range of $H(y)$ may be bounded, as in the case of the DE $y' = 1 + y^2$. In this case,

† This theorem states that if $H(y)$ is a strictly monotonic map of $[c,d]$ onto $[a,b]$, then $H^{-1}(y)$ is single-valued and monotonic from $[a,b]$ to $[c,d]$.

$\int_{-\infty}^{\infty} dy/(1+y^2) = \pi$. Therefore, *no* solution of the DE $y' = 1 + y^2$ can be continuously defined over an interval (a,b) of length exceeding π.

Example 5. Consider the DE $y' = (1 + y^2)e^{-x^2}$. Separating variables, we get $\int dy/(1 + y^2) = \int e^{-x^2}\,dx$, whose general solution is $\arctan y = (\sqrt{\pi}/2)\operatorname{erf} x + C$, or $y = \tan\{(\sqrt{\pi}/2)\operatorname{erf}(x) + C\}$.

The formal transformations (10′) and (10″) can be rigorously justified whenever $g(x)$ and $h(y)$ are *continuous* functions, in any interval in which $h(y)$ does not vanish. This is because the Fundamental Theorem of the Calculus again assures us that $\phi(x) = \int g(x)\,dx$ exists and is differentiable on any interval where $g(x)$ is defined and continuous, while $\psi(y) = \int dy/h(y)$ exists and is *strictly monotonic* in any interval (y_1,y_2) between successive zeros y_1 and y_2 of $h(y)$, which we also assume to be continuous. Hence, as in Example 2, the equation

$$\psi(y) - \phi(x) = \int \frac{dy}{g(y)} - \int h(x)\,dx = c$$

gives for each c a solution of $y' = g(x)h(y)$ in the strip $y_1 < y < y_2$. Near any x with $y_1 + c < \phi(x) < y_2 + c$, this solution is defined by the inverse function theorem, by the formula $y = \psi^{-1}(\phi(x) + c)$.

Orthogonal Trajectories. An *orthogonal trajectory* to a family of curves is a curve that cuts all the given curves at right angles. For example, consider the family of geometrically similar, coaxial ellipses $x^2 + my^2 = C$. These are integral curves of the DE $x + myy' = 0$, whose normal form $y' = -x/my$ has separable variables. The orthogonal trajectories of these ellipses have at each point a slope $y' = my/x$, which is the negative reciprocal of $-x/my$. Separating variables, we get $dy/y = m\,dx/x$, or $\ln|y| = m \ln|x|$, whence the orthogonal trajectories are given by $y = \pm|x|^m$.

More generally, the solution curves of any separable DE $y' = g(x)h(y)$ have as orthogonal trajectories the solution curves of the separable DE $y' = -1/g(x)h(y)$.

Critical Points. Points where $\partial u/\partial x = \partial u/\partial y = 0$ are called *critical points* of the function $u(x,y)$. Note that the directions of level lines and gradient lines may be very irregular near critical points; consider those of the functions $x^2 \pm y^2$ near their critical point $(0,0)$.

As will be explained in §5, the level curves of any function $u \in \mathscr{C}^1(D)$ satisfy the DE $\partial u/\partial x + y'\partial u/\partial y = 0$ in D, except at critical points of u. Clearly, their orthogonal trajectories are the solution curves of $\partial u/\partial y = y'\partial u/\partial x$, and so are everywhere tangent to the direction of $\nabla u = \operatorname{grad} u = (\partial u/\partial x, \partial u/\partial y)$. Curves having this property are called *gradient curves* of u. Hence the gradient curves of u are orthogonal trajectories of its level curves, except perhaps at critical points.

EXERCISES B

1. Find the solution of the DE $xy' + 3y = 0$ that satisfies the initial condition $f(1) = 1$.
2. Find equations describing all solutions of $y' = (x + y)^2$. [HINT: Set $u = x + y$.]
3. (a) Find all solutions of the DE $xy' + (1 - x)y = 0$
   (b) Same question for $xy' + (1 - x)y = 1$.
4. (a) Solve the DEs of Exercise 3 for the initial conditions $y(1) = 1$, $y(1) = 2$.
   (b) Do the same for $y(0) = 0$ and $y(0) = 1$, *or* prove that no solution exists.
5. (a) Find the general solution of the DE $y' + y = \sin 2t$.
   (b) For arbitrary (real) constants a, b, and $k \neq 0$, find a particular solution of

$$y' = ay + b \sin kt \tag{*}$$

   (c) What is the general solution of (*)?
6. (a) Find a polynomial solution of the DE

$$y' + 2y = x^2 + 4x + 7 \tag{**}$$

   (b) Find a solution of the DE (*) that satisfies the initial condition $y(0) = 0$.
7. Show that if k is a nonzero constant and $q(x)$ a polynomial of degree n, then the DE $xy' + y = q(x)$ has exactly one polynomial solution of degree n.

In Exs. 8 and 9, solve the DE shown and discuss its solutions qualitatively.

8. $dr/d\theta = r^2 \sin 1/r$ (polar coordinates).
9. $dr/d\theta = 2/\log r$.
10. (a) Show that the ellipses $5x^2 - 6xy + 5y^2 = C$ are integral curves of the DE

$$(5x + 3y) + (3x - 5y)y' = 0$$

   (b) What are its solution curves?

5 QUASILINEAR EQUATIONS; IMPLICIT SOLUTIONS

In this section and the next, we consider the general problem of solving *quasilinear* DEs (1), which we rewrite as

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{12}$$

to bring out the latent symmetry between the roles of x and y. Such DEs arise naturally if we consider the *level curves* of functions. If $G(x,y)$ is any continuously differentiable function, then the DE

$$\frac{\partial G}{\partial x}(x,y) + \frac{\partial G}{\partial y}(x,y)\,y' = 0 \tag{12'}$$

is satisfied on any level curve $G(x,y) = C$, at all points where $\partial G/\partial y \neq 0$. This DE is of the form (1), with $M(x,y) = \partial G/\partial x$ and $N(x,y) = \partial G/\partial y$.

For this reason, any function G which is related in the foregoing way to a quasilinear DE (1) or (12), or to a nonzero *multiple* of (12) of the form

$$\mu(x,y)[M(x,y)\,dx + N(x,y)\,dy] = 0, \qquad \mu \neq 0 \tag{12'}$$

is called an *implicit solution* of (12). Slightly more generally, an *integral* of (1) or (12) is defined as a function $G(x,y)$ of two variables that is constant on every solution curve of (1).

For example, the equation $x^4 - 6x^2y^2 + y^4 = C$ is an implicit solution of the quasilinear DE

$$(x^3 - 3xy^2) + (y^3 - 3x^2y)y' = 0$$

or

$$y' = \frac{(x^3 - 3xy^2)}{(3x^2y - y^3)}$$

The level curves of $x^4 - 6x^2y^2 + y^4$ have vertical tangents on the x-axis and the lines $y = \pm\sqrt{3}x$. Elsewhere, the DE displayed above is of the normal form $y' = F(x,y)$.

Critical Points. At points where $\partial\phi/\partial x = \partial\phi/\partial y = 0$, the directions of the gradient and level curves are undefined; such points are called "critical points" of ϕ. Thus, the function $x^2 + y^2$ has the origin for its only critical point, and the same is true of the function $x^4 - 6x^2y^2 + y^4$. (Can you prove it?) On the other hand, the function $\sin(x^2 + y^2)$ also has circles of critical points, occurring wherever r^2 is an odd integral multiple of $\pi/2$. Most functions have only *isolated* critical points, however, and in general we shall confine our attention to such functions.

We will now examine more carefully the connection between quasilinear DEs and level curves of functions, illustrated by the two preceding examples. To describe it accurately, we will need two more definitions. We first define a *domain*† as a nonempty *open connected set.* We call a function $\phi = \phi(x_1, \ldots, x_r)$ of *class* $\mathcal{C}^n$ in a domain D when all its derivatives $\partial\phi/\partial x_i$, $\partial^2\phi/\partial x_i\partial x_j$, ... of orders $1, \ldots, n$ exist and are continuous in D. We will write this condition in symbols as $\phi \in \mathcal{C}^n$ or $\phi \in \mathcal{C}^n(D)$. When ϕ is merely assumed to be continuous, we will write $\phi \subset \mathcal{C}$ or $\phi \subset \mathcal{C}(D)$.

To make the connection between level curves and quasilinear DEs rigorous, we will also need to assume the following basic theorem.

† See Apostol, Vol. 1, p. 252. Here and later, page references to authors refer to the books listed in the selected bibliography.

IMPLICIT FUNCTION THEOREM.† *Let $u(x,y)$ be a function of class $\mathscr{C}^n$ ($n \geq 1$) in a domain containing (x_0,y_0); let u_0 denote $u(x_0,y_0)$, and let $u_y(x_0,y_0) \neq 0$. Then there exists positive numbers ϵ and η such that for each $x \in (x_0 - \epsilon, x_0 + \epsilon)$ and $C \in (u_0 - \epsilon, u_0 + \epsilon)$, the equation $u(x,y) = C$ has a unique solution $y = f(x,C)$* in the interval $(y_0 - \eta, y_0 + \eta)$. *Moreover, the function f so defined is also of class $\mathscr{C}^n$.*

It follows that if $u \in \mathscr{C}^n(D)$, $n \geq 1$, the level curves of u are graphs of functions $y = f(x,c)$, also of class $\mathscr{C}^n$, except where $\partial u/\partial y = 0$. In Example 1, $u = x^2 + y^2$ and there is one such curve, the x-axis $y = 0$; this divides the plane into two subdomains, the half-planes $y > 0$ and $y < 0$. Moreover, the locus (set) where $\partial u/\partial y = 0$ consists of the points where the circles $u =$ const have vertical tangents and the "critical point" $(0,0)$ where $\partial u/\partial x = \partial u/\partial y = 0$—that is, where the surface $z = u(x,y)$ has a horizontal tangent plane.

This situation is typical: for most functions $u(x,y)$, the partial derivative $\partial u/\partial y$ vanishes on isolated curves that divide the (x,y)-plane into a number of regions where $\partial u/\partial y \neq 0$ has constant sign, and hence in which the Implicit Function Theorem applies.

THEOREM 3. *In any domain where $\partial u/\partial y \neq 0$, the level curves of any function $u \in \mathscr{C}^1$ are solution curves of the quasilinear* DE

$$\phi(x,y,y') = M(x,y) + N(x,y)y' = 0 \tag{13}$$

where $M(x,y) = \partial u/\partial x$ and $N(x,y) = \partial u/\partial y$.

Proof. By the Chain Rule, $du/dx = \partial u/\partial x + (\partial u/\partial y)y'$ along any curve $y = f(x)$. Hence, such a curve is a level curve of u if and only if

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}y' = 0$$

By the Implicit Function Theorem, the level curves of u, being graphs of functions $y = f(x)$ in domains where $\partial f/\partial y \neq 0$, are therefore solution curves of the quasilinear DE (13). In the normal form $y' = F(x,y)$ of this DE, therefore, $F(x,y) = -(\partial u/\partial x)/(\partial u/\partial y)$ becomes infinite precisely when $\partial u/\partial y = 0$.

To describe the relationship between the DE (13) and the function u, we need a new notion.

DEFINITION. An *integral* of a first-order quasilinear DE (1) is a function of two variables, $u(x,y)$, which is constant on every solution curve of (1).

Thus, the function $u(x,y) = x^2 + y^2$ is an integral of the DE $x + yy' = 0$

† Courant and John, Vol. 2, p. 218. We will reconsider the Implicit Function Theorem in greater depth in §12.

because, upon replacing the variable y by any function $\pm\sqrt{C - x^2}$, we obtain $u(x,y) = C$. This integral is most easily found by rewriting $x + y\,dy/dx = 0$ in differential form, as $x\,dx + y\,dy = 0$, and recognizing that $x\,dx + y\,dy = \frac{1}{2}d(x^2 + y^2)$ is an "exact" differential (see §6).

Level curves of an integral of a quasilinear DE are called *integral curves* of the DE; thus, the circles $x^2 + y^2 = C$ are integral curves of the DE $x + yy' = 0$, although not solution curves.

Example 6. From the DE $yy' = x$, rewritten as $y\,dy/dx = x$, we get the equation $y\,dy - x\,dx = 0$. Since $y\,dy - x\,dx = \frac{1}{2}d(y^2 - x^2)$, we see that the integral curves of the DE are the branches of the hyperbolas $y^2 = x^2 + C$ and the asymptotes $y = \pm x$, as shown in Figure 1.5. The branches $y = \pm\sqrt{x^2 + k^2}$ are solution curves, but each level curve $y = \pm\sqrt{x^2 - k^2}$ has *four* branches separated by the x-axis (the line where the integral curves have vertical tangents).

Note that, where the level curves $y = x$ and $y = -x$ of $y^2 - x^2$ cross, the *gradient* $(\partial F/\partial x, \partial F/\partial y)$ of the integral $F(x,y) = y^2 - x^2$ vanishes: $(\partial F/\partial x, \partial F/\partial x) = (0,0)$.

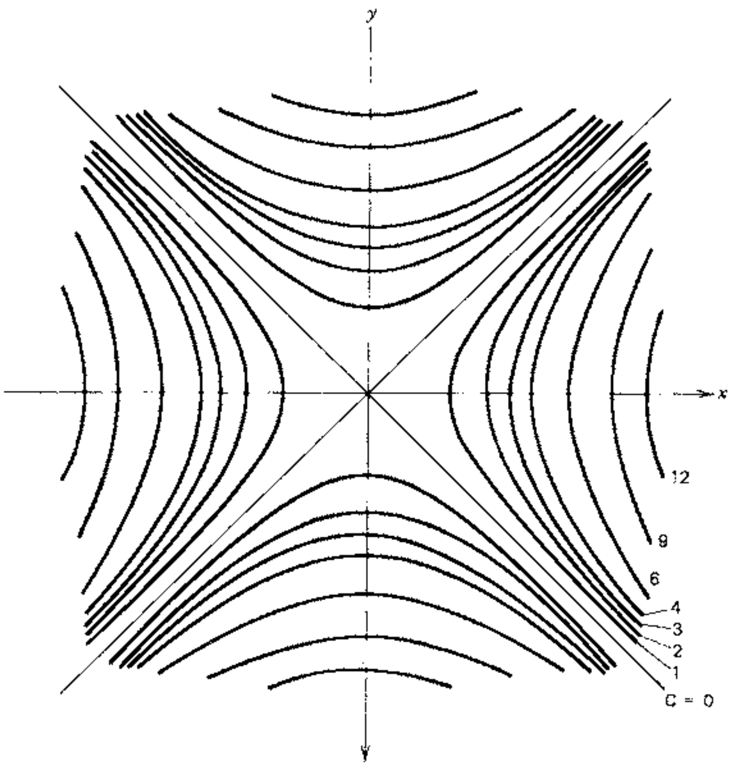


Figure 1.5 Level curves $c = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12$ of $y^2 - x^2$.

6 EXACT DIFFERENTIALS; INTEGRATING FACTORS

A considerably larger class of "implicit solutions" of quasinormal DEs can be found by examining more closely the condition that $M(x,y)\,dx + N(x,y)\,dy$ be an "exact differential" dU, and by looking for an "integrating factor" $\mu(x,y)$ that will convert the equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{14}$$

into one involving a "total" or "exact" differential

$$\mu dU = \mu(x,y)[M(x,y)\,dx + N(x,y)\,dy] = 0$$

whose (implicit) solutions are the level curves of U.

In general, the quasinormal DE (1) or

$$M(x,y) + N(x,y)y' = 0 \tag{14'}$$

is said to be *exact* when there exists a function $U(x,y)$ of which it is the 'total differential', so that $\partial U/\partial x = M(x,y)$ and $\partial U/\partial y = N(x,y)$, or equivalently

$$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy = M(x,y)\,dx + N(x,y)\,dy \tag{14''}$$

Since $dU = 0$ on any solution curve of the DE (14), we see that solution curves of (14) must lie on level curves of U, just as in the "separable variable" case.

Since $\partial^2U/\partial x\partial y = \partial^2U/\partial y\partial x$, clearly a necessary condition for (14′) to be an exact differential is that $\partial N/\partial x = \partial M/\partial y$. It is shown in the calculus that the converse is also true locally. More precisely, the following result is true.

THEOREM 4. *If $M(x,y)$ and $N(x,y)$ are continuously differentiable functions in a simply connected domain, then (14′) is an exact differential if and only if $\partial N/\partial x = \partial M/\partial y$.*

The function $U = U(P)$ *for* (14) is constructed as the *line integral* $\int_0^P [M(x,y)dx + N(x,y)\,dy]$ from a fixed point 0 in the domain [perhaps $0 = (0,0)$] to a variable point $P = (x,y)$. Thus, for the DE $x + yy' = 0$ of Example 1, this procedure gives $\int_0^P (x\,dx + y\,dy) = (x^2 + y^2)/2$, showing again that the solution curves of $x + yy' = 0$ lie on the circles $x^2 + y^2 = C$ with center $(0,0)$. More generally, in the separable equation case of $g(x)\,dx + dy/h(y)$, we have $\partial[g(x)]/\partial y = 0 = \partial[1/h(y)]/\partial x$, giving $G(x) + H(y) = C$ as in §5.

Even when the differential $M\,dx + N\,dy$ is not exact, one can often find a function $\mu(x,y)$ such that the product

$$(\mu M)\,dx + (\mu N)\,dy = du$$

is an exact differential. The contour lines $u(x,y) = C$ will then again be integral curves of the DE $M(x,y) + N(x,y)y' = 0$ because $du/dx = \mu(M + Ny') = 0$; and segments of these contour lines between points of vertical tangency will be solution curves. Such a function μ is called an integrating factor.

DEFINITION. An *integrating factor* for a differential $M(x,y)\,dx + N(x,y)\,dy$ is a nonvanishing function $\mu(x,y)$ such that the product $(\mu M)\,dx + (\mu N)\,dy$ is an exact differential.

Thus, as we saw in §3, for any indefinite integral $P(x) = \int p(x)\,dx$ of $p(x)$, the function $\exp\{P(x)\}$ is an integrating factor for the linear DE (8). Likewise, the function $1/h(x)$ is an integrating factor for the separable DE (11).

The differential $x\,dy - y\,dx$ furnishes another interesting example. It has an integrating factor in the right half-plane $x > 0$ of the form $\mu(x) = 1/x^2$, since $dy/x - y\,dx/x^2 = d(y/x)$; cf. Ex. C11. A more interesting integrating factor is $1/(x^2 + y^2)$. Indeed, the function

$$\theta(x,y) = \int_{(1,0)}^{(x,y)} \frac{(x\,dy - y\,dx)}{(x^2 + y^2)}$$

is the angle made with the positive x-axis by the vector (x,y). That is, it is just the polar angle θ when the point (x,y) is expressed in polar coordinates. Therefore, the integral curves of $xy' = y$ in the domain $x > 0$ are the radii $\theta = C$, where $-\pi/2 < \theta < \pi/2$; the solution curves are the same.

Note that the differential $(x\,dy - y\,dx)/(x^2 + y^2)$ is not exact in the punctured plane, consisting of the x,y-plane with the origin deleted. For θ changes by 2π in going around the origin. This is possible, even though $\partial[x/(x^2 - y^2)]/\partial x = \partial[-y/(x^2 + y^2)]/\partial y$, because the punctured plane is not a simply connected domain.

Still another integrating factor of $x\,dy - y\,dx$ is $1/xy$, which replaces $x\,dy - y\,dx = 0$ by $dy/y = dx/x$, or $\ln|y| = \ln|x| + C$ in the interior of each of the four quadrants into which the coordinate axes divide the (x,y)-plane. Exponentiating both sides, we get $y = kx$.

A less simple example concerns the DE $x(x^3 - 2y^3)y' = (2x^3 - y^3)y$. Here an integrating factor is $1/x^2y^2$. If we divide the given DE by x^2y^2, we get

$$\frac{d}{dx}\left(\frac{x^2}{y} + \frac{y^2}{x}\right) = \frac{2x^3y - x^4y' - y^4 + 2xy^3y'}{x^2y^2}$$

Hence the solution curves of the DE are $(x^2/y) + (y^2/x) = C$, or $x^3 + y^3 = Cxy$.

Parametric Solutions. Besides "explicit" solutions $y = f(x)$ and "implicit" solutions $U(x,y) = C$, quasinormal DEs (14) can have "parametric" solutions. Here by a *parametric solution* is meant a parametric curve $x = g(t)$, $y = h(t)$ along which the line integral $\int M(x,y)\,dx + N(x,y)\,dy$, defined as

$$\int [M(g(t),h(t))g'(t) + N(g(t),h(t))h'(t)]\,dt \tag{15}$$

vanishes. Thus, the curves $x = A\cos t$, $y = A\sin t$ are parametric solutions of $x + yy' = 0$. They are also solutions of the *system* of two first-order DEs $dx/dt = -y$, $dy/dt = x$, and will be studied from this standpoint in Chapter 5.

EXERCISES C

1. Find an integral of the DE $y' = y^2/x^2$, and plot its integral curves. Locate its critical points, if any.
2. Sketch the level curves and gradient lines of the function $x^3 + 3x^2y + y^3$. What are its critical points?
3. Same question as Exercise 2 for $x^3 - 3x^2y + y^3$.
4. Find equations describing all solutions of

$$y^2 = \frac{1}{2x + y}$$

5. For what pairs of positive integers n,r is the function $|x|^n$ of class $\mathcal{C}^r$?
6. Solve the DE $xy' + y = 0$ by the method of separation of variables. Discuss its solution curves, integral curves, and critical points.
7. (a) Reduce the *Bernoulli* DE $y' + p(x)y = q(x)y^n$, $n \neq 1$, to a linear first-order DE by the substitution $u = y^{1-n}$.
   (b) Express its general solution in terms of indefinite integrals.

In Exs. 8 and 9, solve the DE exhibited, sketch its solution curves, and describe them qualitatively:

8. $y' = y/x - x^2$. 9. $y' = y/x - \dfrac{1}{\ln|x|}$.

10. Find all solutions of the DE $|x| + |y|y' = 0$. In which regions of the plane is the differential on the left side exact?

*11. Show that the reciprocal of any homogeneous quadratic function $Q(x) = Ax^2 + 2Bxy + Cy^2$ is an integrating factor of $x\,dy - y\,dx$.

*12. Show that if u and v are both integrals of the DE $M(x,y) + N(x,y)y' = 0$, then so are $u + v$, uv except where $v = 0$, $\lambda u + \mu v$ for any constants λ and μ, and $g(u)$ for any single-valued function g.

*13. (a) What are the level lines and critical points of $\sin(x + y)$?
   (b) Show that for $u = \sin(x + y)$, $(x_0,y_0) = (0,0)$, and $\delta = \epsilon = \frac{1}{4}$, $f(x,c)$ in the Implicit Function Theorem need not exist if $\eta < \frac{1}{4}$ while it may not be unique if $\eta > 4$.

7 LINEAR FRACTIONAL EQUATIONS

An important first-order DE is the *linear fractional equation*

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by}, \qquad ad \neq bc \tag{16}$$

which is the normal form of

$$(ax + by)y' - (cx + dy) = 0 \tag{16'}$$

It is understood that the coefficients a, b, c, d are constants.

The integration of the DE (16) can be reduced to a quadrature by the substitution $y = vx$. This substitution replaces (16) by the DE

$$xv' + v = \frac{c + dv}{a + bv}$$

in which the variables x and v can be separated. Transposing v, we are led to the separation of variables

$$\frac{(a + bv)\,dv}{bv^2 + (a - d)v - c} + \frac{dx}{x} = 0$$

Since the integrands are rational functions, this can be integrated in terms of elementary functions. Thus, x can be expressed as a function of $v = y/x$: we have $x = kG(y/x)$, where

$$G(v) = \exp\left\{-\int\left[\frac{a + bv}{bv^2 + (a - d)v - c}\right]dv\right\}$$

More generally, any DE of the form $y' = F(y/x)$ can be treated similarly. Setting $v = y/x$ and differentiating $y = xv$, we get $xv' + v = F(v)$. This is clearly equivalent to the separable DE

$$\frac{dv}{F(v) - v} = \frac{dx}{x} = d\,(\ln x)$$

whence $x = K \exp\{\int dv/[F(v) - v]\}$.

Alternatively, we can introduce polar coordinates, setting $x = r\cos\theta$ and $y = r\sin\theta$. If $\psi = \gamma - \theta$ is the angle between the tangent direction γ and the radial direction θ, then

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\psi = \frac{\cot\gamma\cot\theta + 1}{\cot\theta - \cot\gamma}$$

Since $\tan\gamma = y' = F(y/x) = F(\tan\theta)$, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1 + \tan\gamma\tan\theta}{\tan\gamma - \tan\theta} = \frac{1 + (\tan\theta)F(\tan\theta)}{F(\tan\theta) - \tan\theta} = Q(\theta) \tag{17}$$

This can evidently be integrated by a quadrature:

$$r(\theta) = r(0) \exp \int_0^\theta Q(\theta)\, d\theta \qquad (17')$$

The function on the right is well-defined, by the Fundamental Theorem of the Calculus, as long as $\tan \gamma \neq \tan \theta$, that is, as long as $y' \neq y/x$.

Invariant Radii. The radii along which the denominator of $Q(\theta)$ vanishes are those where (16) is equivalent to $d\theta/dr = 0$. Hence, these radii are particular solution curves of (16); they are called *invariant radii*. They are the solutions $y = \tau x$, for constant $\tau = \tan \theta$. Therefore, they are the radii $y = \tau x$ for which $y' = \tau = (c + d\tau)/(a + b\tau)$, by (16), and so their slopes τ are the roots of the quadratic equation

$$b\tau^2 + (a - d)\tau = c \qquad (18)$$

If $b \neq 0$, Eq. (18) has zero, one, or two real roots according as its discriminant is negative, zero, or positive. This discriminant is

$$\Delta = (a - d)^2 + 4bc = (a + d)^2 - 4(ad - bc) \qquad (18')$$

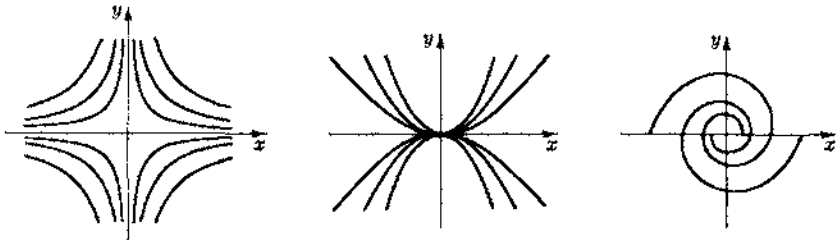
In the sectors between adjacent invariant radii, $d\theta/dr$ has constant sign; this fact facilitates the sketching of solution curves. Together with the invariant radii, the solution curves (17′) form a regular curve family in the *punctured plane*, consisting of the xy-plane with the origin deleted.

Similarity Property. Each solution of the linear fractional DE (16) is transformed into another solution when x and y are both multiplied by the same nonzero constant k. The reason is, that both $y' = dy/dx$ and y/x are unchanged by the transformation $(x,y) \to (kx,ky)$. In polar coordinates, if $r = f(\theta)$ is a solution of (17), then so is $r = kf(\theta)$. Since the transformation $(x,y) \to (kx,ky)$ is a similarity transformation of the xy-plane for any fixed k, it follows that the solution curves in the sector between any two adjacent invariant radii are all *geometrically similar* (and similarly placed). This fact is apparent in the drawings of Figure 1.6.

Note also that the hyperbolas in Figure 1.6*a* are the orthogonal trajectories of those of Figure 1.5. This is because they are integral curves of $yy' = x$ and $xy' = -y$, respectively, and x/y is the negative reciprocal of $-y/x$.

EXERCISES D

1. Sketch the integral curves of the DEs in Exs. C8 and C9 in the neighborhood of the origin of coordinates.
2. Express in closed form all solutions of the following DEs:
   (a) $y' = (x^2 - y^2)/(x^2 + y^2)$ (b) $y' = \sin (y/x)$



(a) $xy' + y = 0$ (b) $xy' = 2y$ (c) $y' = (3x + y)/(x - 3y)$

Figure 1.6 Integral curves.

3. (a) Show that the inhomogeneous linear fractional DE

$$(cx + dy + e)\,dx - (ax + by + f)\,dy = 0, \qquad ad \neq bc$$

can be reduced to the form (16) by a translation of coordinates.

(b) Using this idea, integrate $(x + y + 1)\,dx = (2x - y - 1)\,dy$.

(c) For what sets of constants a, b, c, d, e, f, is the displayed DE exact?

4. Find all integral curves of $(x^n + y^n)y' - x^{n-1}y = 0$. [Hint: Set $u = y/x$.]

5. Prove in detail that the solutions of any homogeneous DE $y' = g(y/x)$ have the Similarity Property described in §7.

6. Show that the solution curves of $y' = G(x,y)$ cut those of $y' = F(x,y)$ at a constant angle β if and only if $G = (\tau + F)/(1 - \tau F)$, where $\tau = \tan \beta$.

7. Let A, B, C be constants, and K a parameter. Show that the coaxial conics $Ax^2 + 2Bxy + Cy^2 = K$, satisfy the DE $y' = -(Ax + By)/(Bx + Cy)$.

8. (a) Show that the differential $(ax + by)\,dy - (cx + dy)\,dx$ is exact if and only if $a + d = 0$, and that in this case the integral curves form a family of coaxial conics.

(b) Using Exs. 6 and 7, show that if $\tan \beta = (a + d)/(c - b)$, the curves cutting the solution curves of the linear fractional DE $y' = (cx + dy)/(ax + by)$ at an angle β form a family of coaxial conics.

9. For the linear fractional DE (16) show that

$$y'' = (ad - bc)[cx^2 - (a - d)xy - by^2]/(ax + by)^3$$

Discuss the domains of convexity and concavity of solutions.

10. Find an integrating factor for $y' + (2y/x) = a$, and integrate the DE by quadratures.

8 GRAPHICAL AND NUMERICAL INTEGRATION

The simplest way to sketch approximate solution curves of a given first-order normal DE $y' = F(x,y)$ proceeds as follows. Draw a short segment with slope $\lambda_i = F(x_i,y_i) = \tan \theta_i$ through each point (x_i,y_i) of a set of sample points sprinkled fairly densely over the domain of interest. Then draw smooth curves so as to have at every point a slope y' approximately equal to the average of the $F(x_i,y_i)$

at nearby points, weighting the nearest points most heavily (i.e., using graphical interpolation). Methods of doing this systematically are called schemes of *graphical integration.*

The preceding construction also gives a graphical representation of the direction field associated with a given normal first-order DE. This is defined as follows.

DEFINITION. A *direction field* in a region D of the plane is a function that assigns to every point (x,y) in D a *direction.* Two directions are considered the same if they differ by an integral multiple of 180°, or π radians.

With every quasinormal DE $M(x,y) + N(x,y)y' = 0$, there is associated a direction field. This associates with each point (x_k,y_k) not a *critical point* where $M = N = 0$, a short segment parallel to the vector $(N(x_k,y_k), -M(x_k,y_k))$. Such segments can be vertical whereas this is impossible for normal DEs.

It is very easy to integrate graphically the linear fractional equation (16) because solution curves have the same slope along each radius $y = vx$, $v =$ constant: each radius $y = kx$ is an isocline. We need only draw segments having the right direction fairly densely on radii spaced at intervals of, say, 30°. After tracing one approximate integral curve through the direction field by the graphical method described above, we can construct others by taking advantage of the Similarity Property stated in §7.

Numerical Integration. With modern computers, it is easy to construct accurate numerical tables of the solutions of initial value problems, where they exist, for most reasonably well-behaved functions $F(x,y)$. Solutions may exist only locally. Thus, to solve the initial value problem for $y' = 1 + y^2$ for the initial value $y(0) = 0$ on $[0,1.6]$ is impossible, since the solution $\tan x$ becomes infinite when $y = \pi/2 = 1.57086. \ldots$ We will now describe three very simple methods (or "algorithms") for computing such tables; the numerical solution of ordinary DEs will be taken up systematically in Chapters 7 and 8.

Simplest is the so-called *Euler method,* whose convergence to the exact solution (for $F \in \mathscr{C}^1$) was first proved by Cauchy around 1840 (see Chapter 7, §2). One starts with the given initial value, $y(a) = y_0 = c$, setting $X_0 = a$ and $Y_0 = y_0$, and then for a suitable step-size h computes recursively

$$X_{n+1} = X_n + h, \qquad Y_{n+1} = Y_n + hF(X_n,Y_n) \tag{19}$$

A reasonably accurate table can usually be obtained in this way, by letting $h = .001$ (say), and printing out every tenth value of Y_n.

If greater accuracy is desired, one can reduce h to $.0001$, printing out $Y_0, Y_{100}, Y_{200}, Y_{300}, \ldots$, and "formatting" the results so that values are easy to look up.

Improved Euler Method. The preceding algorithm, however, is very wasteful, as Euler realized. As he observed, one can obtain much more accurate results with roughly the same computational effort by replacing (19) with the following "improved" Euler algorithm

$$\begin{aligned}(20)\qquad Z_{n+1} &= Y_n + hF(X_n,Y_n)\\ Y_{n+1} &= Y_n + \frac{h}{2}\left[F(X_n,Y_n) + F(X_{n+1},Z_{n+1})\right]\end{aligned}$$

With $h = .001$, this "improved" Euler method gives 5-digit accuracy in most cases, while requiring only about twice as much arithmetic per time step. Whereas with Euler's method, to use 10 times as many mesh points ordinarily gives only one more digit of accuracy, the same mesh refinement typically gives two more digits of accuracy with the improved Euler method.

As will be explained in Chapter 8, when truly accurate results are wanted, it is better to use other, more sophisticated methods that give *four* additional digits of accuracy each time h is divided by 10. In the special case of *quadrature*—that is, of DEs of the form $y' = g(x)$ (see §2)—to do this is simple. It suffices to replace (19) by *Simpson's Rule*.

$$(21)\qquad Y_{n+1} = Y_n + \frac{h}{6}\left[g(x_n) + 4g\left(\frac{x_n + h}{2}\right) + g(x_n + h)\right]$$

For example, one can compute the natural logarithm of 2,

$$y(2) = \ln 2 = \int_1^2 dx/x = .69314718\ldots$$

with 8-digit accuracy by choosing $n = 25$ and using the formula

$$\ln 2 = \frac{1}{150}\sum_{k=1}^{25}\left[\frac{50}{48 + 2k} + \frac{50}{49 + 2k} + \frac{50}{50 + 2k}\right]$$

Caution. To achieve 8-digit accuracy in summing 25 terms, one must use a computer arithmetic having at least 9-digit accuracy. Many computers have only 7-digit accuracy!

Taylor Series Method. A third scheme of numerical integration is obtained by truncating the Taylor series formula after the term in y_n'', and writing

$$Y_{n+1} = Y(x_n + h) = Y_n + hY_n' + h^2Y_n''/2 + 0(h^3)$$

For the DE $y' = y$, since $y_n' = y_n'' = y_n$, this method gives $Y_{n+1} = (1 + h + h^2/2)Y_n$, and so it is equivalent to the improved Euler method.

For the DE $y' = 1 + y^2$, since $y'' = 2yy' = 2y(1 + y^2)$, the method gives

$$Y_{n+1} = Y_n + h(1 + Y_n^2) + h^2(Y_n + Y_n^3)$$

This differs from the result given by Euler's improved method. In general, since $d[F(x,y)]/dx = \partial F/\partial x + (\partial F/\partial y)\, dy/dx$, $Y''_n = (F_x + FF_y)_n$. This makes the method easy to apply.

The error per step, like that of the improved Euler method, is roughly proportional to the *cube* of h. Since the number of steps is proportional to h^{-1}, the *cumulative* error of both methods is roughly proportional to h^2. Thus, one can obtain two more digits of accuracy with it by using 10 times as many mesh points.

As will be explained in Chapter 8, when truly accurate results are wanted, one should use other, more sophisticated methods that give *four* additional digits of accuracy when 10 times as many mesh points are used.

Constructing Function Tables. Many functions are most simply defined as solutions of initial value problems. Thus e^x is the solution of $y' = y$ that satisfies the initial condition $e^0 = 1$, and $\tan x$ is the solution of $y' = 1 + y^2$ that satisfies $\tan 0 = 0$. Reciprocally, $\ln x$ is the solution of $y' = 1/y$ that satisfies $\ln 0 = 1$, while $\arctan x$ is the solution of $y' = 1/(1 + y^2)$ that satisfies $\arctan 0 = 0$.

It is instructive and enjoyable (using modern computers) to try to construct tables of numerical values of such functions, using the methods described in this section, and other methods to be discussed in Chapters 7 and 8. The accuracy of the computer output, for different methods and choices of the mesh length h, can be determined by comparison with standard tables.† One can often use simple recursion formulas instead, like

$$e^{x+h} = e^h e^x \quad \text{and} \quad \tan(x + h) = \frac{\tan x + \tan h}{1 - \tan x \tan h},$$

after evaluating $e^h = 1.01005167$, and also by its Taylor series $\tan x = x + x^3/3 + 2x^5/15 + \cdots$ $\tan(.01) = 0.0100003335\ldots$ Such comparisons will often reveal the limited accuracy of machine computations (perhaps six digits).

EXERCISES E

1. For each of the following initial value problems, make a table of the approximate numerical solution computed by the Euler method, over the interval and for the mesh lengths specified:
   (a) $y' = y$ with $y(0) = 1$, on $[0,1]$, for $h = 0.1$ and 0.02.
   (b) $y' = 1 + y^2$ with $y(0) = 0$, on $[0,1.6]$, for $h = 0.1$, 0.05, and 0.02.
2. Knowing that the exact solutions of the preceding initial value problems are e^x and $\tan x$:
   (a) Evaluate the *errors* $E_n = Y_n - y(X_n)$ for the examples of Exercise 1.
   (b) Tabulate the ratios E_i/hx, verifying when it is true that they are roughly independent of h and x.

† See for example Abramowitz and Stegun, which contains also a wealth of relevant material.

3. Compute approximate solutions of the initial value problems of Exercise 1 by the *improved* Euler method.
4. Find the errors of the approximate values computed in Exercise 3, and analyze the ratios Y_i/h^2x (cf. Ex. 2).
5. Use Simpson's Rule to compute a table of approximate values of the natural logarithm function $\ln x = \int_1^x dt/t$, on the interval $[1,2]$.
6. Construct a table of the function $\arctan x = \int_0^x dt/(1 + t^2)$ on the interval $[0,1]$ by Simpson's Rule, and compare the computed value of arctan 1 with $\pi/4$.

*7. In selected cases, test how well your tables agree with the identities $\arctan(\tan x) = x$ and $\ln(e^x) = x$.

*8. Let e_n be the approximate value of e obtained using Euler's method to solve $y' = y$ for the initial condition $y(0) = 1$ on $[0,1]$, on a uniform mesh with mesh length $h = 1/n$.
   (a) Show that $\ln e_n = n \ln(1 + h)$.
   (b) Infer that $\ln e_n = 1 - h/2 + h^2/3 - \cdots$.
   (c) From this, derive the formula

$$(*) \qquad -\ln(e_n/e) = h/2 - h^2/3 + \cdots$$

   (d) From formula (*) show that, as $h \downarrow 0$, $e - e_n = (he/2)[1 - (h/6) + O(h^2)]$.

9 THE INITIAL VALUE PROBLEM

For any normal first-order differential equation $y' = F(x,y)$ and any "initial" x_0 (think of x as time), the *initial value problem* consists in finding the solution or solutions of the DE, for $x \geq x_0$, which also satisfy $f(x_0) = c$. In geometric language, this amounts to finding the solution curve or curves that issue from the point (x_0,c) to the right in the (x,y)-plane. As we have just seen, most initial value problems are easy to solve on modern computers, if one is satisfied with *approximate* solutions accurate to (say) 3–5 decimal digits.

However, there is also a basic *theoretical* problem of proving the *uniqueness* of this solution.

When $F(x,y) = g(x)$ depends on x alone, this theoretical problem is solved by the Fundamental Theorem of the Calculus (§2). Given $x_0 = a$ and $y_0 = c$, the initial value problem for the DE $y' = g(x)$ has one and only one solution, given by the definite integral (5′).

The initial value problem is said to be *well-posed* in a domain D when there is one and only one solution $y = f(x,c)$ in D of the given DE for each given $(x_0,c) \in D$, and when this solution varies continuously with c. To show that the initial value problem is well-posed, therefore, requires proving theorems of *existence* (there is a solution), *uniqueness* (there is only one solution), and *continuity* (the solution depends continuously on the initial value). The concept of a well-posed initial value problem gives a precise mathematical interpretation of the physical concept of *determinism* (cf. Ch. 6, §5). As was pointed out by Hadamard, solutions which do not have the properties specified are useless physically because no physical measurement is exact.

It is fairly easy to show that the initial value problems discussed so far are well-posed. Thus, using formula (8′), one can show that the initial value problem is well-posed for the linear DE $y' + p(x)y = q(x)$ in any vertical strip $a < x < b$ where p and q are continuous. The initial value problem is also well-posed for the linear fractional DE (16) in each of the half-planes $ax + by > 0$ and $ax + by < 0$.

Actually, for the initial value problem for $y' = F(x,y)$ to be well-posed in a domain D, it is sufficient that $F \in \mathcal{C}^1$ in D. But it is not sufficient that $F \in \mathcal{C}$: though the continuity of F implies the existence of at least one solution through every point (cf. Ch. 6,§13), it does not necessarily imply uniqueness, as the following example shows.

Example 7. Consider the curve family $y = (x - C)^3$, sketched in Figure 1.7. For fixed C, we have

$$y' = \frac{\partial y}{\partial x} = 3(x - C)^2 = 3y^{2/3} \tag{22}$$

a DE whose right side is a continuous function of position (x,y). Through every point (x_0,c) of the plane passes just one curve $y = (x - C)^3$ of the family, for which $C = x_0 - c^{1/3}$ depends continuously on (x_0,c). Hence, the initial value problem for the DE (22) always has one and only one solution of the form $y = (x - C)^3$. But there are also other solutions.

Thus, the function $y = 0$ also satisfies (22). Its graph is the envelope of the curves $y = (x - C)^3$. In addition, for any $\alpha < \beta$, the function defined by the three equations

$$y = \begin{cases} (x - \alpha)^3, & x < \alpha \\ 0 & \alpha \leq x \leq \beta \\ (x - \beta)^3 & x > \beta \end{cases} \tag{22′}$$

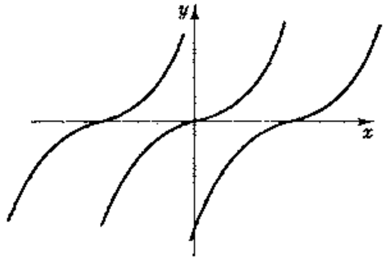


Figure 1.7 Solution curves of $y' = 3y^{2/3}$.

is a solution of (22). Hence, the first-order DE $y' = 3y^{2/3}$ has a *two-parameter* family of solutions, depending on the parameters α and β.

*10 UNIQUENESS AND CONTINUITY

The rest of this chapter will discuss existence, uniqueness, and continuity theorems for initial value problems concerning *normal* first-order DEs $y' = F(x,y)$. Readers who are primarily interested in applications are advised to skip to Chapter 2.

Example 9 shows that the mere continuity of $F(x,y)$ does not suffice to ensure the uniqueness of solutions $y = f(x)$ of $y' = F(x,y)$ with given $f(a) = c$. However, it is sufficient that $F \in \mathscr{C}^1(D)$. We shall prove this and continuity at the same time, using for much of the proof the following generalization of the standard Lipschitz condition.

DEFINITION. A function $F(x,y)$ satisfies a *one-sided* Lipschitz condition in a domain D when, for some finite constant L

$$y_2 > y_1 \qquad \text{implies} \qquad F(x,y_2) - F(x,y_1) \leq L(y_2 - y_1) \tag{23}$$

identically in D. It satisfies a *Lipschitz condition*† in D when, for some nonnegative constant L (Lipschitz constant), it satisfies the inequality

$$|F(x,y) - F(x,z)| \leq L|y - z| \tag{23'}$$

for all point pairs (x,y) and (x,z) in D having the same x-coordinate.

The same function F may satisfy Lipschitz conditions with different Lipschitz constants, or no Lipschitz conditions at all, as the domain D under consideration varies. For example, the function $F(x,y) = 3y^{2/3}$ of the DE in Example 9 satisfies a Lipschitz condition in any half-plane $y \geq \epsilon$, $\epsilon > 0$, with $L = 2\epsilon^{-1/3}$, but no Lipschitz condition in the half-plane $y > 0$. More generally, one can prove the following.

LEMMA 1. *Let F be continuously differentiable in a bounded closed convex‡ domain D. Then it satisfies a Lipschitz condition there, with $L = \sup_D |\partial F/\partial y|$.*

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* In this book, starred sections may be omitted without loss of continuity.

† R. Lipschitz, *Bull. Sci. Math.* 10 (1876), p. 149; the idea of the proof is due to Cauchy (1839). See Ince, p. 76, for a historical discussion.

‡ A set of points is called *convex* when it contains, with any two points, the line segment joining them.

Proof. The domain being convex, it contains the entire vertical segment joining (x,y) with (x,z). Applying the Law of the Mean to $F(x,\eta)$ on this segment, considered as a function of η, we have

$$|F(x,y) - F(x,z)| = |y - z| \left|\frac{\partial F(x,\eta)}{\partial \eta}\right|$$

for some η between y and z. The inequality (23′), with $L = \sup_D |\partial F/\partial y|$, follows. A similar argument shows that (23) holds with $L = \max_D \partial F/\partial y$.

The case $F(x,y) = g(x)$ of ordinary integration, or "quadrature," is easily identified as the case when $L = 0$ in (23′). A Lipschitz condition is satisfied even if $g(x)$ is discontinuous.

LEMMA 2. *Let σ be a differentiable function satisfying the differential inequality*

$$\sigma'(x) \leq K\sigma(x), \qquad a \leq x \leq b \tag{24}$$

where K is a constant. Then

$$\sigma(x) \leq \sigma(a)e^{K(x-a)}, \qquad \text{for} \qquad a \leq x \leq b \tag{24'}$$

Proof. Multiply both sides of (24) by e^{-Kx} and transpose, getting

$$0 \geq e^{-Kx}[\sigma'(x) - K\sigma(x)] = \frac{d}{dx}\{\sigma(x)e^{-Kx}\}$$

The function $\sigma(x)e^{-Kx}$ thus has a negative or zero derivative and so is nonincreasing for $a \leq x \leq b$. Therefore, $\sigma(x)e^{-Kx} \leq \sigma(a)e^{-Ka}$, q.e.d.

LEMMA 3. *The one-sided Lipschitz condition (23) implies that*

$$[g(x) - f(x)][g'(x) - f'(x)] \leq L[g(x) - f(x)]^2$$

for any two solutions $f(x)$ and $g(x)$ of $y' = F(x,y)$.

Proof. Setting $f(x) = y_1$, $g(x) = y_2$, we have

$$[g(x) - f(x)][g'(x) - f'(x)] = (y_2 - y_1)[F(x,y_2) - F(x,y_1)]$$

from the DE. If $y_2 > y_1$, then, by (23′), the right side of this equation has the upper bound $L(y_2 - y_1)^2$. Since all expressions are unaltered when y_1 and y_2 are interchanged, we see that the inequality of Lemma 3 is true in any case.

We now prove that solutions of $y' = F(x,y)$ depend continuously (and hence uniquely) on their initial values, provided that a one-sided Lipschitz condition holds.

THEOREM 5. *Let $f(x)$ and $g(x)$ be any two solutions of the first-order normal* DE $y' = F(x,y)$ *in a domain D where F satisfies the one-sided Lipschitz condition* (23). *Then*

$$|f(x) - g(x)| \leq e^{L(x-a)}|f(a) - g(a)|, \qquad \text{if} \qquad x \geq a \tag{25}$$

Proof. Consider the function

$$\sigma(x) = [g(x) - f(x)]^2$$

Computing the derivative by elementary formulas, we have

$$\sigma'(x) = 2[g(x) - f(x)] \cdot [g'(x) - f'(x)]$$

By Lemma 3, this implies that $\sigma'(x) \leq 2L\sigma(x)$; and by Lemma 2, this implies $\sigma(x) \leq e^{2L(x-a)}\sigma(a)$. Taking the square root of both sides of this inequality (which are nonnegative), we get (25), completing the proof.

As the special case $f(a) = g(a)$ of Theorem 5, we get uniqueness for the initial value problem: in any domain where F satisfies the one-sided Lipschitz condition (23), at most one solution of $y' = F(x,y)$ for $x \geq a$, satisfies $f(a) = c$. However, we do not get uniqueness or continuity for *decreasing* x. We now prove that we have uniqueness and continuity in both directions when the Lipschitz condition (23′) holds.

THEOREM 6. *If* (23′) *holds in Theorem* 5, *then*

$$|f(x) - g(x)| \leq e^{L|x-a|}|f(a) - g(a)| \tag{26}$$

In particular, the DE $y' = F(x,y)$ *has at most one solution curve passing through any point* $(a,c) \in D$.

Proof. Since (23′) implies (23), we know that the inequality (23) holds; from Theorem 5, this gives (26) for $x \geq a$. Since (23′) also implies (23) when x goes to $-x$, we also have by Theorem 5

$$|f(x) - g(x)| \leq e^{L(a-x)}|f(a) - g(a)| = e^{L|x-a|}|f(a) - g(a)|$$

giving (26) also for $x \leq a$, and completing the proof.

EXERCISES F

1. In which domains do the following functions satisfy a Lipschitz condition?
   (a) $F(x,y) = 1 + x^2$ (b) $F(x,y) = 1 + y^2$
   (c) $F(x,y) = y/(1 + x^2)$ (d) $F(x,y) = x/(1 + y^2)$
2. Find all solutions of $y' = |xy|$.
3. Show that the DE $xu' - 2u + x = 0$ has a two-parameter family of solutions. [Hint: Join together solutions satisfying $u(0) = 0$ in each half-plane separately.]

4. Let f and g be solutions of $y' = F(x,y)$, where F is a continuous function. Show that the functions m and M, defined as $m(x) = \min(f(x), g(x))$ and $M(x) = \max(f(x), g(x))$, satisfy the same DE. [HINT: Discuss separately the cases $f(x) = g(x)$, $f(x) < g(x)$, and $f(x) > g(x)$.]

5. Let $\sigma(t)$, positive and of class $\mathcal{C}^1$ for $a \leq t \leq a + \epsilon$, satisfy the differential inequality $\sigma'(t) \leq K\sigma(t) \log \sigma(t)$. Show that $\sigma(t) \leq \sigma(a) \exp [K(t - a)]$.

6. Let $F(x,y) = y \log (1/y)$ for $0 < y < 1$, $F(y) = 0$ for $y = 0$. Show that $y' = F(x,y)$ has at most one solution satisfying $f(0) = c$, even though F does not satisfy a Lipschitz condition.

7. *(Peano uniqueness theorem).* For each fixed x, let $F(x,y)$ be a nonincreasing function of y. Show that, if $f(x)$ and $g(x)$ are two solutions of $y' = F(x,y)$, and $b > a$, then $|f(b) - g(b)| \leq |f(a) - g(a)|$. Infer a uniqueness theorem.

8. Discuss uniqueness and nonuniqueness for solutions of the DE $y' = -y^{1/3}$. [HINT: Use Ex. 7.]

9. (a) Prove a uniqueness theorem for $y' = xy$ on $-\infty < x,y < +\infty$.
   *(b) Prove the same result for $y' = y^{2/3} + 1$.

10. *(Generalized Lipschitz condition.)* Let $F \in \mathcal{C}$ satisfy

$$|F(x,y) - F(x,z)| \leq k(x)|y - z|$$

identically on the strip $0 < x < a$. Show that, if the improper integral $\int_0^a k(x)\,dx$ is finite, then $y' = F(x,y)$ has at most one solution satisfying $y(0) = 0$.

*11. Let F be continuous and satisfy

$$|F(x,y) - F(x,z)| \leq K|y - z| \log (|y - z|^{-1}), \qquad \text{for} \qquad |y - z| < 1$$

Show that the solutions of $y' = F(x,y)$ are unique.

*11 A COMPARISON THEOREM

Since most DEs cannot be solved in terms of elementary functions, it is important to be able to compare the unknown solutions of one DE with the known solutions of another. It is also often useful to compare functions satisfying the *differential inequality*

$$f'(x) \leq F(x,f(x)) \tag{27}$$

with exact solutions of the DE (3). The following theorem gives such a comparison.

THEOREM 7. *Let F satisfy a Lipschitz condition for $x \geq a$. If the function f satisfies the differential inequality* (27) *for $x \geq a$, and if g is a solution of $y' = F(x,y)$ satisfying the initial condition $g(a) = f(a)$, then $f(x) \leq g(x)$ for all $x \geq a$.*

Proof. Suppose that $f(x_1) > g(x_1)$ for some x_1 in the given interval, and define x_0 to be the largest x in the interval $a \leq x \leq x_1$ such that $f(x) \leq g(x)$. Then

$f(x_0) = g(x_0)$. Letting $\sigma(x) = f(x) - g(x)$, we have $\sigma(x) \geq 0$ for $x_0 \leq x \leq x_1$; and, also for $x_0 \leq x \leq x_1$,

$$\sigma'(x) = f'(x) - g'(x) \leq F(x,f(x)) - F(x,g(x)) \leq L(f(x) - g(x)) = L\sigma(x)$$

where L is the Lipschitz constant for the function F. That is, the function σ satisfies the hypothesis of Lemma 2 of §10 on $x_0 \leq x \leq x_1$, with $K = L$. Hence $\sigma(x) \leq \sigma(x_0)e^{L(x-x_0)} = 0$ and so σ, being nonnegative, vanishes identically. But this contradicts the hypothesis $f(x_1) > g(x_1)$. We conclude that $f(x) \leq g(x)$ for all x in the given interval, q.e.d.

THEOREM 8 (Comparison Theorem). *Let f and g be solutions of the DEs*

$$y' = F(x,y), \qquad z' = G(x,z) \tag{28}$$

respectively, where $F(x,y) \leq G(x,y)$ in the strip $a \leq x \leq b$ and F or G satisfies a Lipschitz condition. Let also $f(a) = g(a)$. Then $f(x) \leq g(x)$ for all $x \in [a,b]$.

Proof. Let G satisfy a Lipschitz condition. Since $y' = F(x,y) \leq G(x,y)$, the functions f and g satisfy the conditions of Theorem 7 with G in place of F. Therefore, the inequality $f(x) \leq g(x)$ for $x \geq a$ follows immediately.

If F satisfies a Lipschitz condition, the functions $u = -f(x)$ and $v = -g(x)$ satisfy the DEs $u' = -F(x, -u)$ and

$$v' = -G(x, -v) \leq -F(x, -v)$$

Theorem 6, applied to the functions v, u and $H(u,v) = -F(x, -v)$ now yields the inequality $v(x) \leq u(x)$ for $x \geq a$, or $g(x) \geq f(x)$, as asserted.

The inequality $f(x) \leq g(x)$ in this Comparison Theorem can often be replaced by a strict inequality. Either f and g are identically equal for $a \leq x \leq x_1$, or else $f(x_0) < g(x_0)$ for some x_0 in the interval (a, x_1). By the Comparison Theorem, the function $\sigma_1(x) = g(x) - f(x)$ is nonnegative for $a \leq x \leq x_1$, and moreover $\sigma_1(x_0) > 0$. Much as in the preceding proof

$$\sigma_1'(x) = G(x,g(x)) - F(x,f(x)) \geq G(x,g(x)) - G(x,f(x)) \geq -L\sigma_1$$

Hence $[e^{Lx}\sigma_1(x)]' = e^{Lx}[\sigma_1' + L\sigma_1] \geq 0$; from this expression $e^{Lx}\sigma_1(x)$ is a nondecreasing function on $a \leq x \leq x_1$. Consequently, we have

$$\sigma_1(x) \geq \sigma_1(x_0)e^{-L(x-x_0)} > 0$$

which gives a strict inequality. This proves

COROLLARY 1. *In Theorem 6, for any $x_1 > a$, either $f(x_1) < g(x_1)$, or $f(x) \equiv g(x)$ for all $x \in [a,x_1]$.*

Theorem 7 can also be sharpened in another way, as follows.

COROLLARY 2. *In Theorem 7, assume that F, as well as G, satisfies a Lipschitz condition and, instead of $f(a) = g(a)$, that $f(a) < g(a)$. Then $f(x) < g(x)$ for $x > a$.*

Proof. The proof will be by contradiction. If we had $f(x) \geq g(x)$ for some $x > a$, there would be a first $x = x_1 > a$ where $f(x) \geq g(x)$. The two functions $y = \phi(x) = f(-x)$ and $z = \psi(x) = g(-x)$ satisfy the DEs $y' = -F(-x,y)$ and $z' = -G(-x,z)$ as well as the respective initial conditions $\phi(-x_1) = \psi(-x_1)$. Since $-F(-x,y) \geq -G(-x,y)$, we can apply Theorem 7 in the interval $[-x_1, -a]$, knowing that the function $-F(-x,y)$ satisfies a Lipschitz condition. We conclude that $\phi(-a) \geq \psi(-a)$, that is, that $f(a) \geq g(a)$, a contradiction.

*12 REGULAR AND NORMAL CURVE FAMILIES

In this chapter, we have analyzed many methods for solving first-order DEs of the related forms $y' = F(x,y), M(x,y) + N(x,y)y' = 0$, and $M(x,y)\,dx + N(x,y)\,dy = 0$, describing conditions under which their "solution curves" and/or "integral curves" constitute "one-parameter families" filling up appropriate domains of the (x,y)-plane. In this concluding section, we will try to clarify further the relationship between such first-order DEs and one-parameter curve families.

A key role is played by the Implicit Function Theorem, which shows† that the level curves $u = C$ of any function $u \in \mathcal{C}^1(D)$ have the following properties in any domain D not containing any critical point: (i) one and only one curve of the family passes through each point of D, (ii) each curve of the family has a tangent at every point, and (iii) the tangent direction is a continuous function of position. Thus, they constitute a regular curve family in the sense of the following definition.

DEFINITION. A *regular* curve family is a curve family that satisfies conditions (i) through (iii).

Thus, the circles $x^2 + y^2 = C$ $(C > 0)$ form a regular curve family; they are the integral curves of $x + yy' = 0$, the DE of Example 1. Concerning the DE $y' = y^3 - y$ of Example 2, even though it is harder to integrate, we can say more: its solution curves form a normal curve family in the following sense.

DEFINITION. A regular curve family is *normal* when no curve of the family has a vertical tangent anywhere.

Almost by definition, the curves of any normal curve family are solution curves of the normal DE $y' = F(x,y)$, where $F(x,y)$ is the slope at (x,y) of the curve passing through it. Moreover, by Theorem 5′, if $F \in \mathcal{C}^1$, there are no other solution curves.

The question naturally arises: do the solution curves of $y' = F(x,y)$ always form a normal curve family in any domain where $F \in \mathcal{C}^1$? They always do *locally*, but the precise formulation and proof of a theorem to this effect are very dif-

† Where $\partial u/\partial y = 0$ but $\partial u/\partial x \neq 0$, we can set $x = g(y)$ locally on the curve; see below.

ficult, and will be deferred to Chapter 6. There we will establish the simpler result that the initial value problem is locally *well-posed* for such DEs, after treating (in Chapter 4) the case that F is *analytic* (i.e., the sum of a convergent power series).

In the remaining paragraphs of this chapter, we will simply try to clarify further what the Implicit Function theorem does and does not assert about "level curves."

Parametrizing Curve Families. Although the name "level curve" suggests that for each C the set of points where $F(x,y) = C$ is always a single curve, this is not so. Thus, consider the level curves of the function $F(x,y) = (x^2 + y^2)^2 - 2x^2 + 2y^2$. The level curve $F = 0$ is the lemniscate $r^2 = 2\cos 2\theta$, and is divided by the critical point at the origin into two pieces. Inside each lobe of this lemniscate is one piece of the level curve $F = C$ for $-1 < C < 0$, while the "level curve" $F = -1$ consists of the other two critical points $(\pm 1, 0)$.

Similarly, in the infinite horizontal strip $-1 < y < 1$, every solution curve $y = \sin x + C$ of the DE $y' = \cos x$ consists of an infinite number of pieces. The same is true of the interval curves of the DE $\cos x\, dx = \sin x\, dy$, which are the level curves of $e^{-y}\sin x$. (These can also be viewed as the graphs of the functions $y = y = \ln|\sin x| + C$ and the vertical lines $y = \pm n\pi$.) In general, one cannot parametrize the level curves of $F(x,y)$ globally by the parameter C.

However, one can parametrize the level curves of any function $u \in \mathcal{C}^1$ *locally*, in some neighborhood of any point (x_0,y_0) where $\partial u/\partial y \neq 0$. For, by the Implicit Function Theorem, there exist positive ϵ and η such that for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ and $c \in (u_0 - \epsilon, u_0 + \epsilon)$, there is exactly one $y \in (y_0 - \eta, y_0 + \eta)$ such that $u(x,y) = c$. This defines a function $y(x,c)$ locally, in a rectangle of the (x,u)-plane. The parameter c parametrizes the level curves of $u(x,y)$ in the corresponding neighborhood of (x_0,y_0) in the (x,y)-plane; cf. Figure 1.8.

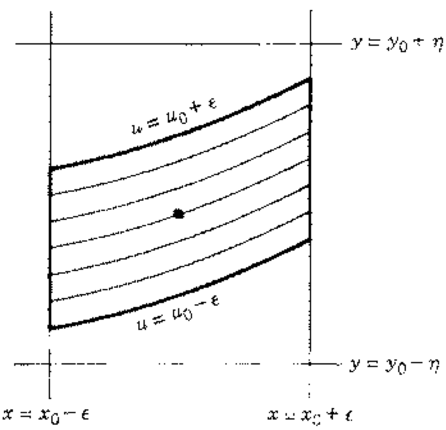

Figure 1.8

EXERCISES G

1. Let $f(u)$ be continuous and $a + bf(u) \neq 0$ for $p \leq u \leq q$. Show that the DE $y' = f(ax + by + c)$ (a,b,c are constants) has a solution passing through every point of the strip $p < ax + by + c < q$.
2. Find all solutions of the DE $y' = |x^3y^3|$.
3. Show that, if M and N are homogeneous functions of the same degree, then (1′) has the integrating factor $(xM + yN)^{-1}$ in any simply connected domain where $xM + yN$ does not vanish.
4. Show that if $g(y)$ satisfies a Lipschitz condition, the solutions of $y' = g(y)$ form a normal curve family in the (x,y)-plane. [Hint: Apply the Inverse Function Theorem to $x = \int dy/g(y) + C$.]
5. Let $g(x)$ be continuous for $0 \leq x < \infty$, $\lim_{x\to\infty} g(x) = b$ and $a > 0$. Show that, for every solution $y = f(x)$ of $y' + ay = g(x)$, we have $\lim_{x\to\infty} f(x) = b/a$.
6. Show that if $a < 0$ in Ex. 5, then there exists one and only one solution of the DE such that $\lim_{x\to\infty} f(x) = b/a$.

*7. *(Osgood's Uniqueness Theorem.)* Suppose that $\phi(u)$ is a continuous increasing function defined and positive for $u > 0$, such that $\int_\epsilon^1 du/\phi(u) \to \infty$ as $\epsilon \to 0$. If $|F(x,y) - F(x,z)| < \phi(|y - z|)$, then the solutions of the DE (3) are unique. [Hint: Use Ex. E4.]

8. Let F, G, f, g be as in Theorem 8, and $F(x,y) < G(x,y)$. Show that $f(x) < g(x)$ for $x > a$, without assuming that F or G satisfies a Lipschitz condition.
9. Show that the conditions $dx/dt = |x|^{1/2}$ and $x(0) = -1$ define a well-posed initial value problem on $[0,a)$ if $a \leq 1$, but not if $a > 1$.
10. (a) Find the critical points of the DE $x\,dy = y\,dx$.
    (b) Show that in the punctured plane (the x,y-plane with the origin deleted), the integral curves of $xy' = y$ are the lines $\theta = c$, where θ is a *periodic* angular variable only determined up to integral multiples of 2π.
    (c) What are its solution curves?
    (d) Show that the *real* variables $x/r = \cos\theta$ and $y/r = \sin\theta$ are integrals of $xy' = y$, and describe carefully their level curves.

*11. (a) Prove that there is no *real*-valued function $u \in \mathcal{C}^1$ in the punctured plane of Ex. 10 whose level curves are the integral curves of $xy' = y$.
    (b) Show that the integral curves of $y' = (x + y)/(x - y)$ are the equiangular spirals $r = ke^\theta = e^{(\theta - c)}$, $k \neq 0$.
    (b) Prove that there is no real-valued function $u \in \mathcal{C}^1$ whose level curves are these spirals.