1

Introduction

1.1 MATHEMATICAL INDUCTION

The principle of mathematical induction has been used for about 350 years. It was familiar to Fermat, in a disguised form, but the first clear statement seems to have been made by Pascal in proving results about the arrangement of numbers now known as Pascal's Triangle. This book contains many applications of inductive arguments and the aim here is to give some preliminary examples, illustrating why the method has become an indispensable tool in mathematics.

We begin with a general formulation of the principle. Let $p_1, p_2, p_3, \ldots$ be statements or propositions, each of which may be true or false.

The Principle of Induction

Suppose that (i) p_1 *is true*
and that, for $n \geq 1$ (ii) $p_n \Rightarrow p_{n+1}$,
then $p_1, p_2, p_3, \ldots$ *are all true.*

Perhaps the most familiar applications are concerned with proving statements like the following one.

Example 1.1

$$p_n : \qquad 1 + 2 + \cdots + n = \tfrac{1}{2} n\,(n+1)\,.$$

Proof The statement p_1 means that $1 = \frac{1}{2}1\,(1+1)$, which is true. Now suppose that p_n is true for some $n \geq 1$. Then, by adding $(n+1)$ to each side of the equation p_n, we obtain

$$1 + 2 + \cdots + n + (n+1) = \tfrac{1}{2} n\,(n+1) + (n+1) = \tfrac{1}{2}\,(n+1)\,(n+2)\,.$$

In other words, $p_n \Rightarrow p_{n+1}$. Then it follows from the principle that p_n is true for every $n \geq 1$. □

The next example is slightly harder, but the argument is very similar.

Example 1.2

$$p_n : \qquad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof In this case, p_1 reduces to the obvious statement that $1 = 1$. If p_n holds for some, $n \geq 1$, then we have

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)\{n(2n+1) + 6(n+1)\}}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

The last expression is equivalent to the right hand side of the equation p_{n+1}, so we have shown that $p_n \Rightarrow p_{n+1}$. □

1.2 HISTORICAL BACKGROUND

The historical material in this section is based on the book by Boyer [Boy68]. It is remarkable that Fermat published very little on the theory of numbers, but many of his penetrating ideas were noted in the margins of a 1621 edition of the *Arithmetica* of Diophantus. Some of his theorems were proved by an inductive method that he called *infinite descent* and he used it with great ingenuity. We can illustrate the method more simply by proving a well known classical result.

Example 1.3

$$\sqrt{2} \text{ is irrational.}$$

Proof We begin by assuming that

$$\sqrt{2} = \frac{k_1}{k_2},$$

where k_1 and k_2 are positive integers. This will lead to a contradiction which shows that $\sqrt{2}$ cannot be expressed as a ratio of integers.

Our assumption means that $k_1^2 = 2k_2^2$, so k_1 must be even and $k_1 > k_2$. Now write $k_1 = 2k_3$, so that $k_2^2 = 2k_3^2$ and we obtain

$$\sqrt{2} = \frac{k_2}{k_3}.$$

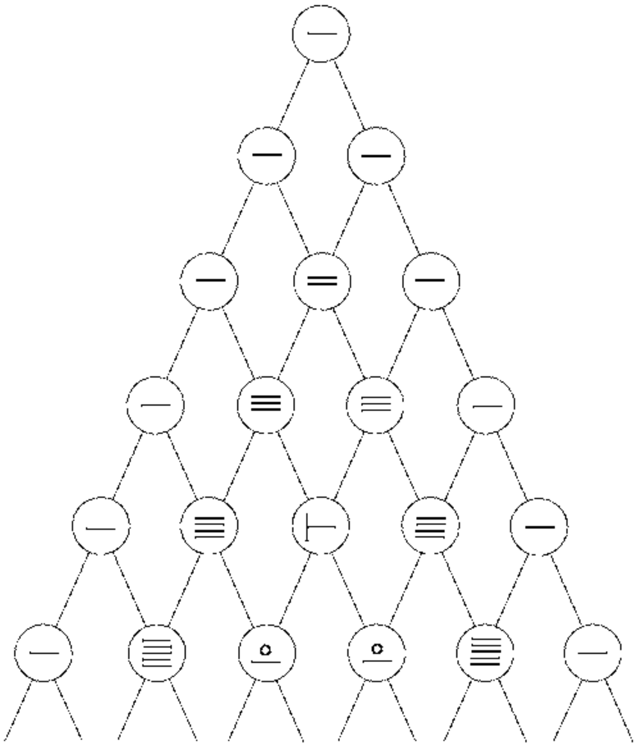

Figure 1. The Chinese triangle

By repeating this argument, we can obtain an equation

$$\sqrt{2} = \frac{k_n}{k_{n+1}},$$

for every $n = 1, 2, 3, \ldots$, where $k_1 > k_2 > k_3 \ldots$ This infinite descent is impossible, since the number of positive integers below k_1 is finite. $\square$

Figures 1 and 2 give two different sketches of what is misleadingly called Pascal's Triangle. The first is a Chinese version from a diagram that appeared in the *Ssu-yüan yü-chien (Precious Mirror of the Four Elements)* by Chu Shi-chieh in 1303. Chu disclaims credit for the triangle and it seems likely that it originated in China about 1100. Note the use of rod numerals and the zero symbol in Figure 1. It is also interesting that formulae for the summation of series, like those in our first two examples, also appeared without proof in the *Precious Mirror*. Of course, both figures represent the same mathematical object. The reason that the triangle is associated with Pascal is that, in 1654, he gave a clear explanation of the method of induction and used it to prove some new results about the triangle. In fact, the construction of this infinite

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\bullet & & \bullet & & \bullet & & \bullet & & \bullet & & \bullet
\end{array}
$$

Figure 2. Pascal's triangle

triangle is recursive so, with hindsight, inductive proofs now seem very natural.

Let the rth number in the nth row of Figure 2 be denoted by $\binom{n}{r}$ for $r = 0, 1, ..., n$ and $n = 0, 1, ...$. The first and last numbers in each row are both 1, so $\binom{n}{0} = \binom{n}{n} = 1$. The triangle is constructed by using the relation

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1} \tag{1.1}$$

for $r = 1, 2, ..., n-1$, to obtain the $(n+1)$th row from the nth. In other words, each entry is obtained by adding together the pair of numbers immediately above it in the previous row. It is a straightforward matter to establish the usual formula expressing $\binom{n}{r}$ in terms of factorials.

Example 1.4

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \tag{1.2}$$

Proof Let us take p_n to be the statement that equation (1.2) holds for $r = 0, 1, ..., n$. Note that p_0 and p_1 are both true, provided that 0! is interpreted as 1. Now suppose that p_n is true for some $n \geq 1$ and use the relation (1.1) to obtain

$$\begin{aligned}\binom{n+1}{r+1} &= \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r+1)!\,(n-r-1)!} \\ &= \frac{n!}{(r+1)!\,(n-r)!}\,\{r+1+n-r\} \\ &= \frac{(n+1)!}{(r+1)!\,(n+1-(r+1))!}\,.\end{aligned}$$

The calculation is valid for $r = 0, 1, ..., n-1$ and we already know that $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$. This completes the verification of p_{n+1}. □

A final historical point is that the Principle of Induction was included in 1889 as one of Peano's axioms for the natural numbers, thereby recognizing it as one of the foundations of arithmetic.

1.3 DYNAMIC PROGRAMMING

The term *dynamic programming* was coined by Bellman [Bel57] to describe the techniques which he brought together to study a class of optimization problems involving sequences of decisions. There have been many applications and further developments since that time. A central aim of this book is to describe the method at an elementary level and to illustrate the range of possible applications.

Our subject is sequential, or multi-stage decision problems, where the time variable is used to order the sequence. We shall begin with deterministic examples which do not involve any random quantities or unknown parameters. As we shall see later, random variation can be included by taking expectations. Statistical problems are usually more difficult to formulate and solve, but it is worth mentioning that some of the techniques of dynamic programming first emerged in the investigation of sequential decisions by Wald and others. A comprehensive account of these statistical developments can be found in the book by DeGroot [DeG70].

Bellman deserves the credit for giving a clear statement of the principles of dynamic programming and for demonstrating a wide range of applications, but it would be misleading to suggest that the method is new since it relies heavily on mathematical induction. It turns out that it is natural to treat a sequence of decisions by reversing the order and, for this reason, the analysis is called *backwards induction.*

A deterministic model which covers many applications and a statement of Bellman's *principle of optimality* will be given in Chapter 2. It will be helpful to consider first some simple examples.

Example 1.5 The entries in the matrix

$$\begin{pmatrix} 2 & 5 & 3 & 8 & 6 \\ 4 & 2 & 9 & 4 & 1 \\ 5 & 3 & 2 & 6 & 9 \\ 0 & 3 & 8 & 5 & 0 \end{pmatrix}$$

represent costs associated with the positions in the rectangle. It is required to find an optimal route from the top left hand corner to the bottom right-hand corner which consists of steps, either to the right or downwards, at each stage. The cost of following any particular route is the sum of all the entries encountered on the way. For example, the cost of moving down the first column and along the bottom row is $2 + 4 + 5 + 0 + 3 + 8 + 5 + 0 = 27$. An optimal route is one for which the total cost is a minimum.

In order to find such a route, we construct another matrix in which each entry represents the minimum cost, for that position, of reaching the bottom right-hand corner. The complete minimum-cost matrix is

$$\left(\begin{array}{lllllllll} 24 & & 23 & & 25 & \rightarrow & 22 & & 16 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 22 & \rightarrow & 18 & & 22 & & 14 & \rightarrow & 10 \\ & & \downarrow & & \downarrow & & & & \downarrow \\ 21 & \rightarrow & 16 & \rightarrow & 13 & \rightarrow & 11 & & 9 \\ \downarrow & & & & & & \downarrow & & \downarrow \\ 16 & \rightarrow & 16 & \rightarrow & 13 & \rightarrow & 5 & \rightarrow & 0 \end{array}\right)$$

and the construction proceeds by examining the columns in reverse order. Thus, for the position in row 1, column 5, there is only one admissible path, as indicated by the arrows, and the minimum cost is $6 + 1 + 9 + 0 = 16$. We can now deal with positions in column 4, starting at the bottom where the minimum cost is $5 + 0 = 5$. The next entry in row 3, column 4, is obtained by noting that it is preferable to make the initial move downwards, giving a total cost $6 + 5 = 11$, rather than moving to the right. It does not take long to work backwards through the matrix in this way: the minimum cost for any position can be found by comparing the two neighbouring entries on the right and below it. The last entry to be obtained is that in the top left-hand corner and this is based on the fact that the smallest total of $2 + 22 = 24$ is achieved by making the first move downwards. Notice that the optimal route through the matrix is determined by following the arrows and, finally, we can verify the minimum total cost by adding up the appropriate entries in the original matrix: $2 + 4 + 2 + 3 + 2 + 6 + 5 + 0 = 24$.

A similar problem is that of finding the shortest path between two vertices in a network. The theory required for constructing such paths and a related technique called critical path analysis will be considered in Chapter 3. In order to introduce the basic ideas, suppose we have a network with vertices labelled

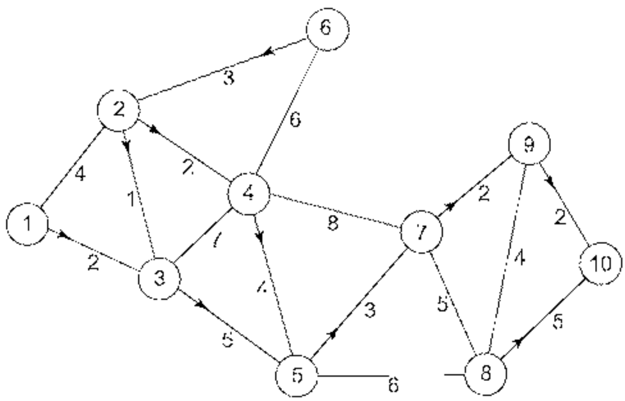

Figure 3. Network for Example 1.6

$1, 2, ..., n$, where the aim is to find the shortest path between vertex 1 and vertex n. Some, but not all, pairs of vertices are directly linked by an arc and the distance d_{ij} between i and j is given for these pairs. For simplicity, let us assume that there is at least one path between any two vertices: its length is obtained by adding the d_{ij} over the corresponding sequence of arcs. For any vertex i, the number of alternative paths from i to n is finite. Let f_i be the length of the shortest of these paths. Thus, $f_n = 0$ and it is not difficult to see that, for each $i < n$,

$$f_i = \min_j \{d_{ij} + f_j\}. \tag{1.3}$$

The minimization on the right of this equation is equivalent to choosing a direction of departure from vertex i: the index j runs over all vertices directly linked to i by an arc.

Suppose that all the given distances d_{ij} are positive. Then it can be shown that the system of equations (1.3) has a unique solution for the shortest lengths, $f_1, f_2, ..., f_{n-1}$, with $f_n = 0$. If we can determine this solution, then an optimal path from 1 to n can be found by following directions that attain the minimum in (1.3) at every vertex on the way. The solution is constructed by backwards induction: more precisely, the f_i are determined in increasing order. This is best demonstrated by looking at a particular case.

Example 1.6 Vertices $1, 2, ..., 10$ are shown in Figure 3 and each arc is marked with its length d_{ij}. The solution, giving the length of the shortest path to vertex 10 for each i, is found by working from right to left on the network:

$i =$	10	9	7	8	5	4	3	2	1	6
$f_i =$	0	2	4	5	7	11	12	13	14	16

The f_i satisfy equation (1.3) and, for each vertex i, the solution indicates a direction as shown by the arrows. Thus, the shortest path from 1 to 10 is

$$1 \to 3 \to 5 \to 7 \to 9 \to 10$$

and its length is given by

$$f_1 = 2 + 5 + 3 + 2 + 2 = 14.$$

1.4 THE EXECUTIONER'S TALE

We end this introduction with a cautionary tale. Inductive arguments are not always straightforward and the following anecdote contains one that is plausible, but false.

Many years ago, one Friday in court, a prisoner was convicted of a crime and sentenced to death. The executioner visited him in his cell and offered a hope of freedom. 'As it happens', he said, 'I am allowed some discretion in my work and I enjoy an occasional gamble. In your case, the execution is scheduled for next week and I have written the day: Monday, Tuesday, ... or Saturday, on a paper sealed in this envelope. I will visit you here early on Monday and then on the following days, if necessary, and ask whether you know the day of your execution. If you answer correctly at the first attempt, then you can go free, but otherwise I must do my job.'

The following Monday when the executioner arrived to ask his question, the prisoner replied immediately, 'Yes, it must be today.' 'What makes you say that?' said the executioner and this was the prisoner's argument. 'Consider the situation on Saturday morning. If you arrive then, I shall be certain that it is the appointed day, so it must be earlier in the week. Now consider Friday. If you ask me then, I will be sure of the answer because we have eliminated Saturday. Having excluded both the last two days, we can repeat the same argument for Thursday, and so on. By proceeding backwards in time we can eliminate the days until Monday is left as the only possibility.' The prisoner seemed quite satisfied with his conclusion and, to be fair, the executioner did not betray any emotion as he handed over the envelope to be opened — it was Wednesday!

There are several confusing features in the above argument; more than enough to invalidate the conclusion.

1.5 SUMMARY

Part I of this book is concerned with deterministic dynamic programming and the basic theory is described in Chapter 2. The mathematical model constructed there is in discrete time and it involves motion in a state space where the changes of state are controlled by a sequence of actions or decisions. Many sequential decision problems can be specified in this way, but the general model is introduced mainly to clarify the underlying ideas. There are many other problems that do not fit conveniently into this structure, but they can be investigated by using the same principles. In particular, the language and notation used to describe networks and critical path analysis in Chapter 3 are quite different, but the main ideas are similar. The following chapter also gives applications of discrete dynamic programming, each of which requires a slightly different approach. The method of backwards induction is developed further in Chapter 5 by showing how the special properties of convex and concave functions can be used to simplify the analysis of sequential decisions.

Some of the most interesting applications of dynamic programming involve models which include random variation. If all the random variables have known distributions, the model is called stochastic but, if there are unknown distributions, it is statistical. Part II of the book, consisting of Chapters 6, 7 and 8, extends the principles of dynamic programming to stochastic models. It is a straightforward matter to extend the model of Chapter 2 by introducing random variables and expectations. The aim is to minimize the total expected cost over the period of interest, since we cannot predict the exact costs associated with different decisions. Similarly, in other cases, it is appropriate to maximize expected rewards. The general principles are not seriously affected, but we now have a much wider range of possible applications. In Part II, we shall restrict attention to simple forms of decision procedure. Chapter 7 is concerned with optimal stopping problems, where the essential choice at each stage is whether to stop the underlying random process or allow it to continue. The special problems discussed in Chapter 8 also involve optimal stopping in various different settings. For example, in the marriage problem, sometimes called the secretary problem, the aim is to find an optimal policy for selecting the best from a sequence of candidates arranged in random order.

Part III is an introduction to more advanced topics. The theory of Markov decision processes was developed to deal with sequential decisions in a general model for random processes with discrete time and space variables. Chapters 9 and 10 describe methods for constructing policies that are optimal in the long run. This is done first by discounting future costs so that the total expectation over an infinite period of time is finite. However, discounting is not always appropriate and, in Chapter 10, another approach is described which leads to the minimization of the average cost over an infinite future. A full account of the theory and applications of Markov decision processes is contained in the recent book by Puterman [Put94].

The final chapter describes a statistical decision problem. Wald's fundamental work in sequential analysis [Wal47] led to the emergence of ideas very similar to the dynamic programming techniques being developed independently by Bellman. A key result in sequential hypothesis testing is the optimality of Wald's sequential probability ratio test and the proof obtained by Arrow, Blackwell and Girshick [ABG49] illustrates this similarity of ideas. The book by DeGroot [DeG70] covers the statistical background and gives a more detailed description of this and many other decision problems in statistics.

A comprehensive treatment of dynamic programming and its applications is contained in the two volumes by Whittle [Whi82], [Whi83]. They present a wide range of sequential decision problems in deterministic and stochastic control, including Markov decision processes and some statistical problems.

EXERCISES

1.1 Show that, for any positive integer n,

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots+\frac{1}{n\,(n+1)}=\frac{n}{n+1}.$$

1.2 Prove that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n.$$

1.3 Use the method of infinite descent to prove that $\sqrt{3}$ is irrational.

1.4 It is required to start at the $(1,1)$ position in the matrix below and proceed to the opposite corner, moving one step to the right or one step down at each stage, in such a way as to minimize the sum of the elements encountered. Find the optimal paths for this matrix.

$$\begin{pmatrix} 1 & 0 & 4 & -5 & 2 \\ 3 & 2 & 1 & 4 & -3 \\ 4 & 3 & -2 & 7 & 5 \\ 6 & -4 & -1 & 2 & 0 \end{pmatrix}$$

1.5 Sketch the network having 10 vertices and 15 arcs with lengths d_{ij} specified as follows:

$$\begin{array}{lllll} d_{12}=3, & d_{25}=4, & d_{46}=6, & d_{67}=\ 4, & d_{7,10}=14, \\ d_{13}=4, & d_{36}=3, & d_{47}=8, & d_{69}=10, & d_{8,10}=\ 6, \\ d_{14}=5, & d_{45}=3, & d_{57}=5, & d_{78}=\ 7, & d_{9,10}=\ 8. \end{array}$$

Find the shortest path from vertex 1 to vertex 10.

1.6 Invent your own network and construct the shortest paths on it.