

## 1.1 INTRODUCTION

The purpose of this chapter is to provide an introduction to *compound interest* which is the foundation of investment mathematics. In particular, methods are developed for calculating the accumulated value and present value of an investment. Although we concentrate on financial investment in securities in this book, for completeness, the second half of this chapter is devoted to discussion of real investment in projects.

Readers unfamiliar with the concepts of *exponents* and *geometric series*, which occur frequently in compound interest calculations, should first read the Annex to this chapter.

## 1.2 ACCUMULATED VALUES

Suppose that a woman deposits £100 with a bank which pays a rate of interest of 6% per annum to its depositors and that she leaves any interest earned to accumulate within the account. At the start of the second year she would have  $£100 \times 1.06 = £106$  in the account, which would earn 6% interest during the second year. So in the second year the £106 grows at 6% to  $£106 \times 1.06$ , or alternatively  $£100(1.06)^2$ , which equals £112.36. The important point to grasp here is that interest is itself earning interest. This is the essence of what is known as *compound interest*. If interest were spent rather than left in the account, total interest received at the end of the second year would amount to only £12 rather than £12.36.

By the end of the third year, the £112.36 will grow to  $£112.36 \times 1.06$ , or alternatively  $£100(1.06)^3$ , which equals £119.10. Continuing the process further, the £100 grows to  $£100(1.06)^4$  after four years,  $£100(1.06)^5$  after 5 years and so on. In the general case, £100 grows to  $£100(1.06)^n$  after  $n$  years.

We may generalise the above *compounding* process further. Suppose that an amount  $A(0)$  is invested at time 0 at a compound interest rate of  $i$  per interval, where  $i$  is written as a decimal rather than a percentage. Then the accumulated amount  $A(n)$  after  $n$  intervals of time is given by the formula

$$A(n) = A(0)(1 + i)^n \quad (1.1)$$

This may be represented diagrammatically as shown in Figure 1.1. Note that the rate at which interest accrues has not been expressed as a rate *per annum* but as a rate *per interval*. A unit interval of time is not restricted to one year. It could be a half-year, a quarter of a year, an hour or any other interval of time.

Equation (1.1) is valid not only for an integral number of time intervals but also for fractions of time intervals, provided the rate of growth is constant. Consider the amount by which the investment has grown after  $1/m$  of a time interval (where  $m$  is an integer). If the sum has grown by a factor  $1 + y$ , we can say that

$$(1 + y)^m = 1 + i$$

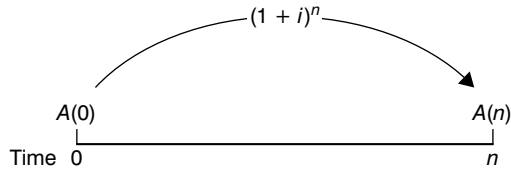


Figure 1.1

or

$$(1 + y) = (1 + i)^{1/m}$$

Thus, after  $n$  time intervals and  $r$  periods of  $1/m$  time intervals ( $r$  and  $m$  integers), the sum  $A(0)$  would grow to

$$A(0) \cdot (1 + i)^{n + (r/m)}$$

In other words, after any time period  $t$ ,  $A(0)$  would grow to  $A(0) \cdot (1 + i)^t$ .

### Example 1.1

What is the accumulated value of £300 invested at 5% per annum for (a) 6 years? (b)  $12\frac{1}{2}$  years?

#### Answer

- (a)  $A(6) = £300(1.05)^6 = \underline{£402.03}$ .  
 (b)  $A(12.5) = £300(1.05)^{12.5} = \underline{£552.06}$ .

### Example 1.2

What is the accumulated value of \$200 invested for  $8\frac{1}{4}$  years where the rate of interest is (a) 3% per half-year? (b) 2% per quarter?

#### Answer

- (a) The unit interval of time is a half-year so that  $n$  is  $8.25 \times 2 = 16.5$

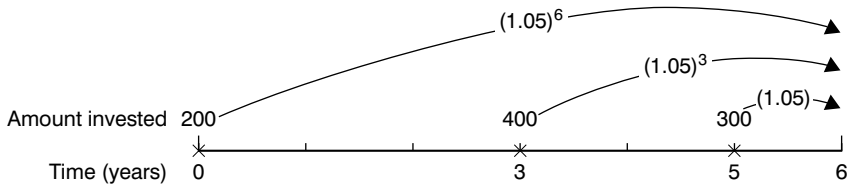
$$A(16.5) = \$200(1.03)^{16.5} = \underline{\$325.72}$$

- (b) The unit interval of time is a quarter so that  $n$  is  $8.25 \times 4 = 33$

$$A(33) = \$200(1.02)^{33} = \underline{\$384.45}$$

### Example 1.3

What is the accumulated value in 6 years' time of £200 invested now, £400 invested at the end of year 3 and £300 invested at the end of year 5, if the rate of interest is 5% per annum?

**Answer****Figure 1.2**

In this case there are a number of payments at different points in time. The solution is straightforward when it is realised that the accumulated values of the payments can be considered separately and then simply added together (see Figure 1.2).

£200 is invested for 6 years, £400 for 3 years and £300 for 1 year, so

$$\begin{aligned} \text{Accumulated value} &= \text{£}200(1.05)^6 + \text{£}400(1.05)^3 + \text{£}300(1.05) \\ &= \underline{\underline{\text{£}1046.07}} \end{aligned}$$

### 1.3 EFFECTIVE AND NOMINAL RATES OF INTEREST

Rates of interest only have meaning when they are related to a time interval. Thus, in Examples 1.1 and 1.3 above, the rate of interest was 5% *per annum* whereas in Example 1.2(a) the rate of interest was 3% *per half-year* and in Example 1.2(b) the rate of interest was 2% *per quarter*. Rates of interest expressed in this way give the actual rate of increase over the stated interval of time. They are known as *effective* rates of interest.

**Example 1.4**

If the effective annual rate of interest is 7%, what is the effective monthly rate of interest?

**Answer**

Let the effective monthly rate of interest be  $j$ . Then

$$\begin{aligned} (1 + j)^{12} &= 1.07 \\ 1 + j &= 1.00565 \\ j &= 0.565\% \end{aligned}$$

Where the effective rate of interest is expressed in terms of  $1/p$  of a year, it is often converted to an annual rate by simply multiplying by  $p$ . Thus, 3% per half-year would be quoted as being “6% per annum, convertible half-yearly”. Similarly, 2% per

quarter would be quoted as being “8% per annum, convertible quarterly”. Interest rates quoted in this way are known as *nominal* rates of interest. Quoted interest rates on savings products offered by banks are often nominal rates, e.g. convertible half-yearly.

Corresponding to a nominal rate of interest there always exists an effective annual rate of interest. For example, suppose £100 is invested for a year at a rate of interest of 8% per annum, convertible half-yearly (i.e. an effective rate of interest of 4% per half-year). This is not the same as an effective rate of interest of 8% per annum. The amount at the end of the year would be  $£100(1 + 0.08/2)^2$ . If the corresponding annual rate of interest is  $i$  per annum, the amount at the end of the year may also be written as  $£100(1 + i)$ . We therefore have

$$£100(1 + 0.08/2)^2 = £100(1 + i)$$

or

$$\begin{aligned} (1 + 0.08/2)^2 &= 1 + i \\ 1.0816 &= 1 + i \\ i &= 0.0816 \quad \text{or} \quad 8.16\% \end{aligned} \tag{1.2}$$

That is, a rate of interest expressed as 8% per annum convertible half-yearly is the same as an effective rate of interest of 8.16% per annum.

Equation (1.2) above may be generalised. Suppose we are given a rate of interest per annum, convertible  $p$  times a year (denoted by the symbol  $i^{(p)}$ ). Then the equivalent rate of interest per annum (denoted by the symbol  $i$ ) is given by

$$\left(1 + \frac{i^{(p)}}{p}\right)^p = 1 + i \tag{1.3}$$

If  $p$  is greater than 1 so that interest is convertible more frequently than once a year, interest itself earns interest within the year, and the effective rate of interest  $i$  exceeds the nominal rate of interest,  $i^{(p)}$ . As  $p$  increases so that intervals between additional interest become smaller and smaller, the margin between  $i$  and  $i^{(p)}$  widens at a decreasing rate, tending to a limit which corresponds to interest being *continuously compounded*. In this case (for which  $p$  is infinite), the resultant annual convertible rate of interest is referred to as the *force of interest* and is given the symbol  $\delta$ . The equivalent rate of interest per annum is given by

$$e^\delta = 1 + i$$

where

$$e = \lim_{p \rightarrow \infty} \left(1 + \frac{1}{p}\right)^p = 2.718$$

If the force of interest is quoted, then we have:

$$A(t) = A(0)e^{\delta t} \tag{1.4}$$

**Example 1.5**

If the effective annual rate of interest is 6%, what is (a) the annual rate of interest convertible half-yearly? (b) the force of interest?

**Answer**

(a) Let the annual rate of interest convertible half-yearly be  $i^{(2)}$ . Then

$$\left(1 + \frac{i^{(2)}}{2}\right)^2 = 1.06$$

$$1 + \frac{i^{(2)}}{2} = 1.029\ 56$$

$$\frac{i^{(2)}}{2} = 0.029\ 56$$

$$i^{(2)} = \underline{5.912\%}$$

(b) Let the force of interest be  $\delta$ . Then

$$e^\delta = 1.06$$

$$\delta = \log_e 1.06$$

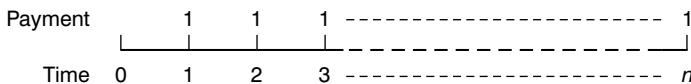
$$\delta = 0.0583$$

$$\delta = \underline{5.83\%}$$

**1.4 THE ACCUMULATED VALUE OF AN ANNUITY-CERTAIN**

An annuity-certain is a series of payments at fixed intervals of time for a fixed period of time. The payments may be of a constant amount or they may vary.

(1) *A series of payments of one per interval payable in arrears for n intervals*



**Figure 1.3**

The accumulated value of this series of payments is often denoted by the symbol  $s_{\overline{n}|}$ . Occasionally it is necessary to make clear the ruling rate of interest by placing it to the right, e.g.  $s_{\overline{n}|}10\%$ .

Suppose that the rate of interest is  $i$  per interval, Then

$$s_{\overline{n}|} = (1 + i)^{n-1} + (1 + i)^{n-2} + \dots + (1 + i) + 1$$

or, reversing the order of the terms on the right-hand side,

$$s_{\overline{n}|} = 1 + (1 + i) + \dots + (1 + i)^{n-2} + (1 + i)^{n-1}$$

The right-hand side of this equation is the sum of a geometric series with first term equal to 1 and common ratio equal to  $1 + i$ . There are  $n$  terms in the series. The sum of this geometric series is

$$\frac{1 \cdot (1 - (1 + i)^n)}{1 - (1 + i)} \quad \text{using equation (1.22)}$$

We therefore have the result:

$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i} \tag{1.5}$$

It is not usually necessary to calculate  $s_{\overline{n}|}$  as values can be obtained from compound interest tables (see back of this book).

(2) *A series of payments of one per interval payable in advance for n intervals*

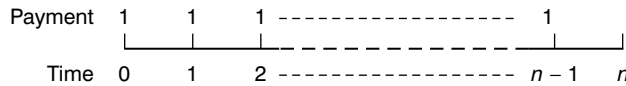


Figure 1.4

The accumulated value of this series of payments is often denoted by the symbol  $\ddot{s}_{\overline{n}|}$ . Note that this is the same as the series of payments in (1) above except that each payment accumulates for one additional interval. It therefore follows that

$$\ddot{s}_{\overline{n}|} = (1 + i)s_{\overline{n}|} \tag{1.6}$$

### 1.5 PRESENT VALUES

We have considered how to determine the accumulated value of a payment (or a number of payments) at some point in the future. We now consider the reverse problem of determining the amount which must be invested now to provide for a payment (or a number of payments) at some point (or points) in the future. In other words, we wish to determine the present value of amounts received at specified future points in time. As with accumulated values, present values depend on the ruling rate of interest but instead of *accumulating* we are *discounting*.

Suppose that we wish to make a payment of £600 in exactly 7 years from now and that the rate of interest is 9% per annum. How much must be invested at the present time to provide this amount? If the initial amount invested is  $A(0)$ , then using equation (1.1) derived earlier, we obtain

$$£600 = A(0)(1.09)^7$$

so that

$$\begin{aligned} A(0) &= \frac{\pounds 600}{(1.09)^7} \\ &= \pounds 328.22 \end{aligned}$$

In general, the amount  $A(0)$  which must be invested now to provide  $A(t)$  after  $t$  intervals of time is given by

$$A(0) = \frac{A(t)}{(1+i)^t} \quad (1.7)$$

where  $i$  is the rate of interest per interval, and  $t$  need not be an integer. Alternatively, if  $\delta$  is the force of interest,

$$A(0) = A(t)e^{-\delta t}$$

$A(0)$  is known as the *present value* of  $A(t)$ .

Equation (1.7) may be represented diagrammatically as in Figure 1.5.

$1/(1+i)$  is often given the symbol  $v$  and so equation (1.7) would then become

$$A(0) = A(t) \cdot v^t \quad (1.8)$$

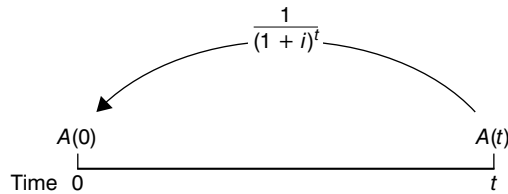


Figure 1.5

### Example 1.6

What is the present value of £100 due in exactly 8 years from now if the rate of interest is 7% per annum?

#### Answer

Here we have

$$t = 8 \quad A(t) = \pounds 100 \quad i = 0.07$$

So

$$\begin{aligned} \text{Present value} &= \frac{\pounds 100}{(1.07)^8} \\ &= \underline{\pounds 58.20} \end{aligned}$$

Note that the present value is less than the amount due in the future.

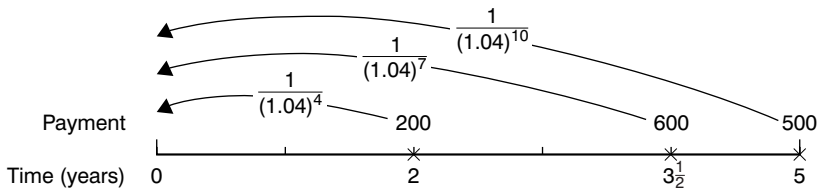
**Example 1.7**

Assuming a rate of interest of 4% per half-year, how much must be invested now to provide the following payments?

- £200 after 2 years
- plus £600 after 3½ years
- plus £500 after 5 years

**Answer**

The unit interval of time is a half-year in this case.



**Figure 1.6**

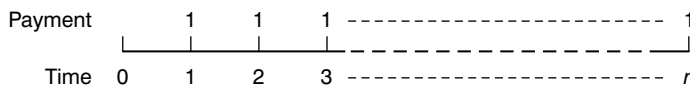
The amount which must be invested now is

$$\frac{£200}{(1.04)^4} + \frac{£600}{(1.04)^7} + \frac{£500}{(1.04)^{10}} = \underline{£964.69}$$

**1.6 THE PRESENT VALUE OF AN ANNUITY-CERTAIN**

The following types of annuity-certain are particularly important in investment and it is therefore useful to know the general formulae for their present values.

(1) A series of payments of one per interval payable in arrears for  $n$  intervals



**Figure 1.7**

The present value of this series of payments is often denoted by the symbol  $a_{\overline{n}|}$ . Suppose that the rate of interest is  $i$  per interval. Then

$$a_{\overline{n}|} = \frac{1}{(1+i)} + \frac{1}{(1+i)^2} + \dots + \frac{1}{(1+i)^n}$$

The right-hand side of this equation is the sum of a geometric series with first term equal to  $1/(1 + i)$  and common ratio equal to  $1/(1 + i)$ .

The sum of this geometric series is

$$\frac{\frac{1}{(1 + i)} \left( 1 - \frac{1}{(1 + i)^n} \right)}{1 - \frac{1}{(1 + i)}}$$

Multiplying the numerator and denominator by  $1 + i$ , we obtain

$$\frac{1 - \frac{1}{(1 + i)^n}}{1 + i - 1} = \frac{1 - \frac{1}{(1 + i)^n}}{i}$$

We therefore have the result:

$$a_{\overline{n}|} = \frac{1 - \frac{1}{(1 + i)^n}}{i} \tag{1.9}$$

Values of  $a_{\overline{n}|}$  are commonly found in compound interest tables (see back of book).

(2) *A series of payments of one per interval payable in advance for n intervals*

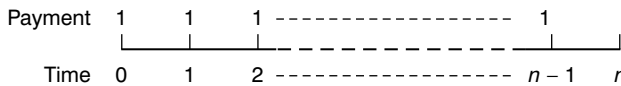


Figure 1.8

The present value of this series of payments is often denoted by the symbol  $\ddot{a}_{\overline{n}|}$ . This is the same as the series of payments in (1) above except that each payment is discounted for one interval fewer. It therefore follows that

$$\ddot{a}_{\overline{n}|} = (1 + i)a_{\overline{n}|} \tag{1.10}$$

(3) *An infinite series of payments made at the end of each interval; the first payment is  $d_1$  and each subsequent payment is  $(1 + g)$  times the previous payment*

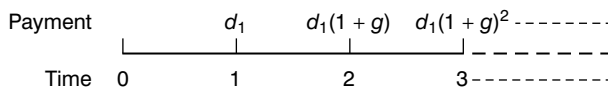


Figure 1.9

Suppose again that the rate of interest is  $i$  per interval. Then

$$\text{Present value} = \frac{d_1}{(1+i)} + \frac{d_1(1+g)}{(1+i)^2} + \frac{d_1(1+g)^2}{(1+i)^3} + \dots$$

The right-hand side of this equation is an infinite geometric series with first term equal to  $d_1/(1+i)$  and common ratio equal to  $(1+g)/(1+i)$ . If  $i$  is greater than  $g$ , the series is convergent and the sum of the series is

$$\frac{\frac{d_1}{(1+i)}}{1 - \frac{(1+g)}{(1+i)}}$$

Multiplying the numerator and denominator by  $1+i$  we obtain

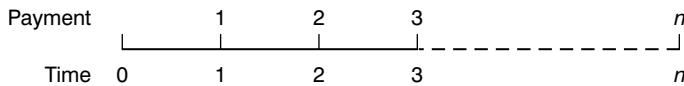
$$\frac{d_1}{(1+i) - (1+g)}$$

so that the present value equals

$$\frac{d_1}{i-g} \tag{1.11}$$

Note that when there is zero growth, i.e.  $g = 0$ , we have an infinite series of payments (known as a *perpetuity*) of  $d_1$  per interval, payable in arrears with present value equal to  $d_1/i$ . The same result can be obtained by multiplying equation (1.9) by  $d_1$  and setting  $n = \infty$ .

(4) An increasing annuity in which the first payment is one after one interval, the second payment is two after two intervals, and so on, with a final payment of  $n$  after  $n$  intervals



**Figure 1.10**

The present value of this series of payments is often denoted by the symbol  $(Ia)_{\overline{n}|}$ . It is convenient to use  $v$  for  $1/(1+i)$ . Then

$$(Ia)_{\overline{n}|} = v + 2v^2 + \dots + nv^n \tag{1.12}$$

$$(1+i)(Ia)_{\overline{n}|} = 1 + 2v + 3v^2 + \dots + nv^{n-1} \tag{1.13}$$

Subtracting equation (1.12) from equation (1.13),

$$\begin{aligned} i(Ia)_{\overline{n}|} &= 1 + v + v^2 + \dots + v^{n-1} - nv^n \\ (Ia)_{\overline{n}|} &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \end{aligned} \tag{1.14}$$

(5) A series of payments of one per interval payable  $p$  times per interval in arrears for  $n$  intervals

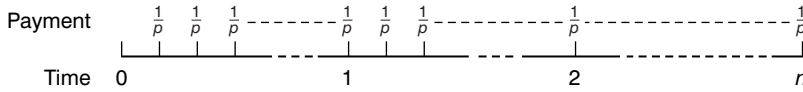


Figure 1.11

The present value of this series of payments is often denoted by the symbol  $a_{\overline{n}|}^{(p)}$ . Again, it is convenient to use  $v$  for  $1/(1+i)$ . We then have

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} v^{1/p} + \frac{1}{p} v^{2/p} + \dots + \frac{1}{p} v^{np/p}$$

The right-hand side is a geometric series with first term  $(1/p)v^{1/p}$  and the common ratio  $v^{1/p}$ . There are  $np$  terms. So

$$\begin{aligned} a_{\overline{n}|}^{(p)} &= \frac{1}{p} v^{1/p} \left\{ \frac{1 - v^n}{1 - v^{1/p}} \right\} \\ &= \frac{1 - v^n}{p(1+i)^{1/p}(1 - v^{1/p})} \\ &= \frac{1 - v^n}{p\{(1+i)^{1/p} - 1\}} \\ &= \frac{i a_{\overline{n}|}}{p\{(1+i)^{1/p} - 1\}} \quad \text{using equation (1.9)} \\ &= \frac{i}{i^{(p)}} a_{\overline{n}|} \quad \text{using equation (1.3)} \end{aligned}$$

We therefore have the result

$$a_{\overline{n}|}^{(p)} = \frac{i}{i^{(p)}} a_{\overline{n}|} \tag{1.15}$$

(6) A series of payments of one per interval payable  $p$  times per interval in advance for  $n$  intervals

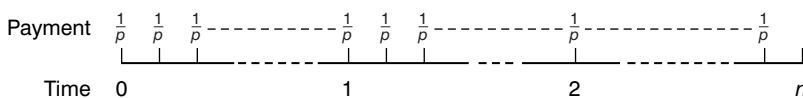


Figure 1.12

The present value of this series of payments is often denoted by the symbol  $\ddot{a}_{\overline{n}|}^{(p)}$ . The problem is the same as in (5) above except that each payment must be discounted by  $1/p$  of an interval less. So

$$\begin{aligned}\ddot{a}_{\overline{n}|}^{(p)} &= (1+i)^{1/p} a_{\overline{n}|}^{(p)} \\ &= (1+i)^{1/p} \left( \frac{i}{i^{(p)}} \right) a_{\overline{n}|} && \text{using equation (1.15)} \\ &= \left( 1 + \frac{i^{(p)}}{p} \right) \left( \frac{i}{i^{(p)}} \right) a_{\overline{n}|} && \text{using equation (1.3)}\end{aligned}$$

or

$$\ddot{a}_{\overline{n}|}^{(p)} = \left( \frac{i}{i^{(p)}} + \frac{i}{p} \right) a_{\overline{n}|} \quad (1.16)$$

### Excel Application 1.1

Excel contains several functions useful for financial calculations.

The PV function, for example, can be used for calculating several present value calculations, including simple annuities.

The PV function can take 5 parameters: the rate of interest per period (rate), the number of periods (nper), the (constant) amount paid each period (pmt), any additional amount paid in the future (fv) and an indicator as to whether payments are made at the beginning or end of each period (0 for the end, 1 for the beginning).

Example 1: To calculate the present value of \$100 paid in 10 years, when the rate of interest is 10% p.a., you could enter in any cell on the spreadsheet:

$$=PV(10\%,10,0,100,0)$$

This returns a value of  $-38.55$ . The negative sign is a convention in Excel – if the future value is positive, then the present value is represented as a negative.

Example 2: To calculate the present value of an annuity of 10 per month (paid in arrear) for 10 years at a rate of interest of 12% p.a. (convertible monthly), you would enter (in any cell in the spreadsheet):

$$=PV(12\%/12,120,10,0,0)$$

This returns  $-697.01$  as present value for the annuity.

The function FV is very similar to PV, but produces the future value rather than the present value. For example, the accumulated value at the end of the 10 years is given by:

$$=FV(12\%/12,120,10,0,0)$$

This produces a number of  $-2300.39$ .

## 1.7 INVESTMENT PROJECT ANALYSIS

The term “investment” has a double meaning. It can mean real investment in projects or it can mean financial investment in equities, bonds or other securities. For the rest of this chapter we will confine our attention to the former. More precisely, we will consider the appraisal of investment projects by companies using discounted cash flow (DCF) techniques. These techniques are based on compound interest theory. After this chapter, however, the emphasis will be on financial investment in securities.

The application of DCF techniques is a straightforward aspect of project appraisal. The difficult part is estimating the inputs of the model, namely the expected cash flows and the appropriate rate of interest.<sup>1</sup> Many investment projects have intangible or unmeasurable benefits or costs, including synergies with the rest of the business; the measurement of such costs or benefits may involve a degree of subjectivity. Furthermore, all project appraisal decisions involve a choice, accept or reject, so they should be defined in terms of *incremental* cash flow by comparing the project with the “base case”, i.e. the status quo.

We will assume that all cash flows are discrete, thus ignoring the possibility of continuous payments. In other words, we assume that any continuous payments can be approximated by a set of discrete payments. The *net cash flow* is simply the cash inflow minus the cash outflow. For most projects encountered in practice, there is a *conventional* net cash flow pattern. This means that there are net cash outflows for one or more years followed by net cash inflows for all subsequent years. *Unconventional* net cash flow patterns have more than one change of sign in the sequence of net cash flows.

## 1.8 NET PRESENT VALUE

Let  $C_t$  be the net cash flow at time  $t$  ( $j = 1, 2, \dots, n$ ). The *net present value* (NPV) of an investment project at a rate of interest  $i$  is given by:

$$\text{NPV} = \sum_{j=1}^n C_j(1+i)^{-t_j} \quad (1.17)$$

In other words, we take all the net cash flows and discount them to the start of the project. The NPV measures the added value to the company of undertaking the project. It should be calculated using a rate of interest equal to the rate of return that the company would expect to pay to finance the project (the “cost of capital”). This could be the return shareholders expect on equity capital or the return that has to be paid on debt capital, or a weighted average of the two. However, account should be taken of the risk of the project. The higher the risk, the higher the volatility of shareholders’ returns and hence the higher the rate of interest that should be employed to discount payments.

Cash outflows must not include interest payments as these are taken into account implicitly in the discounting process. For example, if cash inflows occur later than cash outflows, the former will be discounted more in the NPV calculation.

In a sense, NPV measures the overall profit of the project, expressed at the beginning of the project. This concept becomes more obvious if we first look at the *accumulated*

<sup>1</sup> In addition, there may be some optionality in that projects that are not feasible now may become so later.

profit at the end of the project. If the project ends  $T$  years from now, the accumulated profit ( $A$ ) is given by:

$$A = \sum_{j=1}^n C_{t_j}(1+i)^{T-t_j} \quad (1.18)$$

If we then multiply equation (1.18) by  $(1+i)^{-T}$ , we obtain the NPV equation (1.17). This demonstrates the point that the NPV is just the present value of the accumulated profit.

## 1.9 INTERNAL RATE OF RETURN

The *internal rate of return* (IRR) of an investment project is the rate of interest at which the NPV (or, alternatively, the accumulated profit)<sup>2</sup> of the investment project is equal to zero. It is called the *internal* rate of return because it is the return earned by the project itself. No external market rate of interest is involved in the calculation.

Imagine a hypothetical bank account that represents the project. The amounts invested can be thought of as being paid out of the bank account and cash inflows received from the project can be thought of as being paid into the bank account. A fixed rate of interest is charged on negative balances (“overdrafts”) and the same fixed rate is paid on positive cash balances (“deposits”). Interest is left to accumulate within the account. Then if balances in the bank account are accumulated at the IRR, the final balance of the bank account will be exactly zero.

For project appraisal purposes, the IRR must be compared with the cost of capital. In simple terms, if the internal rate of return is higher than the cost of capital, the project is worthwhile because it will provide a higher rate of return than that which the company will have to pay to finance the project.

If we are appraising a single investment project with a conventional net cash flow pattern, NPV and IRR lead to consistent decisions. Let  $i_0$  be the IRR and  $i_1$  be the cost of capital. If there is a conventional net cash flow pattern, the NPV of a project will be a continuously decreasing function of the rate of interest. Therefore because  $\text{NPV}(i_0) = 0$ , we have  $\text{NPV}(i_1) > 0$  if and only if  $i_0 > i_1$ .

If, on the other hand, we are appraising a project with an unconventional net cash flow pattern, there may be more than one possible value for the IRR.<sup>3</sup> For such projects, decision-making using the IRR can be problematic.

### Example 1.8

A company intends to put \$2,238,600 into an investment project which provides \$1.2m, \$1.5m and \$2m at the end of 3 years, 8 years and 12 years respectively. Show that the internal rate of return is 10% per annum.

<sup>2</sup> We can set up an *equation of value* at any time. The important point to remember is that all cash inflows and all cash outflows must be discounted (or accumulated) to the *same* point in time. It is not sensible to compare cash flows discounted to different points in time.

<sup>3</sup> This corresponds to negative cash balances arising in the hypothetical bank account at some stage in the life of the project.

**Answer**

The NPV of the project (in \$m) is given by:

$$\text{NPV} = -2.2386 + \frac{1.2}{(1+i)^3} + \frac{1.5}{(1+i)^8} + \frac{2}{(1+i)^{12}} \quad (1.19)$$

Substituting  $i = 0.1$  we see that the right-hand side of equation (1.19) is equal to zero. Thus the internal rate of return is 10% per annum.

## 1.10 DISCOUNTED PAYBACK PERIOD

The *discounted payback period* of a project is the time it takes for the discounted incoming cash flows to recover the discounted outgoing cash flows. In other words, at the end of the discounted payback period, all incoming cash flows have repaid all outgoing cash flows together with any interest due on loans necessary to finance outgoing cash flows. It can be regarded as the time it takes for the project to become profitable.

Knowledge of discounted payback period may be useful if capital is scarce. A project which becomes profitable earlier may be considered to be better than one which becomes profitable later because it will allow capital to be repaid and investment in another project to go ahead. However, it does not tell the investor about the overall profitability of a project. A project which carries on for longer but which pays back capital more slowly might have a higher NPV. If the rate of interest used in the NPV calculation is the appropriate one, it should not matter when the project becomes profitable. Later cash flows are discounted more. However, an investor's view of risk might be such that it is appropriate to use a higher interest rate for discounting cash flows from projects that are "longer", and this would be a better way to take account of time preferences than using discounted payback period. Nevertheless, despite these drawbacks, it may still be considered useful to know the discounted payback period for the purposes of budgeting and financial planning.

The *payback period* is the same as the discounted payback period except that the cash flows are not discounted. It is the period after which the sum of incoming cash flows is first not less than the sum of outgoing cash flows. It is difficult to find a good reason to use the payback period in investment project appraisal. It is one way of ensuring that the preference for early cash flows over late cash flows is taken into account but it is better to do this using an NPV approach.

### Example 1.9

A machine can be purchased for £5000, which will give rise to the cash inflows and costs shown in Table 1.1.

The machine can be sold after 6 years for £1000. All cash flows are at the end of the year. The cost of capital is 10% per annum effective. Calculate

- (i) the net present value of the project;
- (ii) the discounted payback period.

**Table 1.1**

Year	Inflows from sales (£)	Maintenance costs (£)
1	1000	50
2	1500	50
3	2000	100
4	1500	100
5	1000	150
6	500	200

**Answer**

(1) Time	(2) Net cash flow (£)	(3) Discount factor	(4) DCF (£)	(5) $\sum$ DCF (£)
0	-5000	1.000000	-5000	-5000
1	950	0.909091	863.6	-4136.4
2	1450	0.826446	1198.3	-2938.1
3	1900	0.751315	1427.5	-1510.6
4	1400	0.683013	956.2	-554.4
5	850	0.620921	527.8	-26.6
6	1300	0.564474	733.8	+707.2

(i)  $NPV = \sum \text{Column (4)} = +£707.2$ .

(ii) From Column (5), discounted payback period is 6 years.

**Excel Application 1.2**

Another very useful pair of functions in Excel are XNPV and XIRR, which can be used to calculate net present values and internal rates of return, respectively.

The parameters for XNPV are, first of all, a rate of interest (rate) and then two ranges of numbers, the amounts of the payments and the dates of the payments.

For example, to calculate the net present value in example 1.9, enter in cells A1 to A7, 1 Jan 2002, 31 Dec 2002, 31 Dec 2003, 31 Dec 2004, 31 Dec 2005, 31 Dec 2006, 31 Dec 2007 (i.e. arbitrary dates representing the times “now” and then 6 dates at annual intervals) and enter in cells B1 to B7, -5000, 950, 1450, 1900, 1400, 850, 1300 (i.e. the amounts corresponding to the dates).

If you enter the formula  $=XNPV(10\%,B1:B7,A1:A7)$ , it returns the answer 707.84. This is slightly different from the answer in the solution to example 1.9, because the XNPV calculation is based on days and it is assumed that there are exactly 365 days in the year.

If this function is not available, run the Setup program to install the Analysis ToolPak. After you install the Analysis ToolPak, you must enable it by using the **Add-Ins** command on the **Tools** menu.

The function XIRR returns the internal rate of return for an arbitrary set of dates and cash flows. It requires the dates and amounts of the cash flows to be entered as well as an initial guess as to what the answer might be.

Example 1.8 can be replicated in Excel by entering the following dates in cells A1 to A4, 1 Jan 2002, 31 Dec 2004, 31 Dec 2009, 31 Dec 2013 and the following amounts in cells B1 to B4, -2.2386, 1.2, 1.5 and 2.0. If the formula =XIRR(B1:B4,A1:A4,5%), it returns the answer 0.09996, which is extremely close to 10%.

## 1.11 ANALYSIS OF DECISION CRITERIA

We have already rejected as inadequate the payback method of appraisal. Cash flows after the payback period are ignored, the time value of money is ignored and the method cannot cope with unconventional cash flow patterns. The discounted payback method can provide some useful information but suffers from the same drawbacks as the payback method apart from taking the time value of money into account.

How does the IRR criterion compare with the NPV criterion? We have established that they will normally lead to the same conclusion if they are applied to a single project. But the IRR and NPV criteria can conflict if we are comparing two projects. Consider two projects that are mutually exclusive. One gives the higher NPV and the other the higher IRR. Which project is better? The most common reason for this situation is that the projects differ in scale, that is, one project (the one with the higher NPV) is longer or requires more capital than the other. If the cost of capital is 10%, is an IRR of 100% on an initial outlay of £1 better than an IRR of 25% on an initial outlay of £1,000,000? And which is better, a return of 100% per annum for a day or a return of 25% per annum for a year, with the same initial outlay? If two projects are mutually exclusive, the one with the greater NPV should be chosen, as this will maximise shareholders' wealth.

## 1.12 SENSITIVITY ANALYSIS

NPV is an important concept because it is the ultimate measure of profitability of a project. However, the input variables are generally uncertain. For this reason it is often useful to conduct a sensitivity analysis. This involves varying the key parameters of the project, one at a time, to assess the sensitivity of the project NPV (or IRR) to variation in each parameter. For example, we could set upper and lower bounds for revenues, costs and the starting date for incoming cash flows. The NPV is then calculated with one of the variables at the upper or lower bound, while each of the other variables is set equal to the best estimate. The NPV calculation is repeated to test the sensitivity of the project NPV to each of the variables. This process can be carried out easily nowadays with the use of spreadsheets.

Sensitivity analysis disaggregates the problem. By identifying the factors causing most concern, it indicates what further information would be useful and how the project might be re-designed to reduce the risk.

### Annex 1.1 Exponents

An *exponent* (also known as an *index* or a *power*) is a number placed at the right of and above a symbol, number or expression. If the exponent is a positive integer it indicates that the symbol is to be multiplied by itself as many times as there are units in the integer. For example,

$$3^2 = 3 \times 3 = 9$$

or

$$y^3 = y \times y \times y$$

Suppose that the exponent of  $y$  is of the form  $1/q$  where  $q$  is an integer. Then  $y^{1/q}$  is a number which when raised to the power  $q$  equals  $y$ . Thus

$$12^{1/5} = z \quad \text{where} \quad z^5 = 12$$

In practice, expressions such as  $12^{1/5}$  may be calculated on a hand-held calculator using the  $y^{1/x}$  button (with  $y = 12$  and  $x = 5$ ) or the  $y^x$  button (with  $y = 12$  and  $x = 0.2$ ). Both methods give an answer of 1.64375 for  $12^{1/5}$ .

If the exponent of  $y$  is a fraction  $p/q$  then  $y^{p/q}$  is defined as  $(y^{1/q})^p$ . Thus  $12^{4/5}$  should be interpreted as  $(12^{1/5})^4$ . In practice, expressions of the form  $y^{p/q}$  may be calculated using the  $y^x$  button of a hand-held calculator. For the expression  $12^{4/5}$ , we enter  $y = 12$  and  $x = 4/5 = 0.8$  to obtain 7.30037.

A negative exponent indicates that, in addition to the operations indicated by the numerical value of the exponent, the quantity is to be reciprocated. It makes no difference whether the reciprocation is performed before or after the other exponential operations. For example,

$$3^{-2} = (3^2)^{-1} = (9)^{-1} = \frac{1}{9}$$

or

$$3^{-2} = (3^{-1})^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

If  $a$  and  $b$  are positive numbers, the following *laws of exponents* are valid:

- (i)  $a^n \cdot a^m = a^{n+m}$
- (ii)  $a^m / a^n = a^{m-n}$
- (iii)  $(a^m)^n = a^{mn}$
- (iv)  $(ab)^n = a^n \cdot b^n$
- (v)  $(a/b)^n = a^n / b^n$

where  $m$  and  $n$  are any numbers.

**Annex 1.2 Geometric series**

The summation of the general form of a finite geometric series is

$$a + aR + aR^2 + aR^3 + \dots + aR^{n-1}$$

where  $a$  and  $R$  are constants. There are  $n$  terms in the series. Note that each term of the series can be obtained by multiplying the previous term by a common ratio,  $R$ . For example,

$$\begin{aligned} \text{Third term} &= \text{Second term} \times R \\ &= aR \times R \\ &= aR^2 \end{aligned}$$

Examples of a finite geometric series are

(i)  $2 + 10 + 50 + 250$

In this case, the first term is 2, the common ratio is 5 and there are four terms in the series.

(ii)  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

In this case, the first term is 1, the common ratio is  $\frac{1}{2}$  and there are five terms in the series.

The above series may be evaluated by simply adding up the terms. Thus, the sum of series (i) is 312 and the sum of series (ii) is  $1\frac{15}{16}$ . However, adding up the terms in a geometric series is an extremely laborious process when there are a large number of terms. In such a situation we use the following formula, the proof of which is given below.

$$S_n = \frac{a(1 - R^n)}{1 - R}$$

where  $n$  = the number of terms in the series

$a$  = the first term of the series

$R$  = the common ratio

$S_n$  = the sum of the series.

*Proof*

$$S_n = a + aR + aR^2 + \dots + aR^{n-1} \quad (1.20)$$

Multiplying equation (1.20) by  $R$  we obtain

$$RS_n = aR + aR^2 + aR^3 + \dots + aR^n \quad (1.21)$$

Subtracting equation (1.21) from equation (1.20) gives

$$S_n - RS_n = a - aR^n$$

$$S_n(1 - R) = a(1 - R^n)$$

so that

$$S_n = \frac{a(1 - R^n)}{1 - R} \quad (1.22)$$

Consider again the series (i) and (ii) above to confirm the validity of the formula:

(i)  $2 + 10 + 50 + 250$ . Thus  $a = 2$ ,  $R = 5$  and  $n = 4$ . So the sum of the series is

$$\begin{aligned} \frac{2(1 - 5^4)}{1 - 5} &= \frac{2(-624)}{-4} \\ &= \underline{312} \text{ as before.} \end{aligned}$$

(ii)  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ . Thus  $a = 1$ ,  $R = \frac{1}{2}$  and  $n = 5$ . So the sum of the series is

$$\begin{aligned} \frac{1(1 - (\frac{1}{2})^5)}{1 - \frac{1}{2}} &= \frac{\frac{31}{32}}{\frac{1}{2}} \\ &= \underline{1\frac{15}{16}} \text{ as before} \end{aligned}$$

An infinite geometric series has an infinite number of terms and is therefore of the form

$$a + aR + aR^2 + aR^3 + \dots$$

where the symbol ... indicates that the series extends indefinitely. If  $R$  is numerically greater than 1, the series does not have a finite sum and is said to be *divergent*. If  $R$  is numerically less than 1, the sum of the series moves closer to a finite limit as more terms are added. Such a series is said to be *convergent* and its sum,  $S$ , is given by the following formula:

$$S = \frac{a}{1 - R} \quad (\text{where } -1 < R < +1) \quad (1.23)$$

Consider the series  $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$ .

The sum of the first three terms is  $1\frac{3}{4}$

The sum of the first four terms is  $1\frac{7}{8}$

The sum of the first five terms is  $1\frac{15}{16}$

Intuitively, we see that as we add more and more terms, the sum gets closer and closer to the limit 2. In this case,  $a = 1$  and  $R = \frac{1}{2}$ , and substituting these values into equation (1.23), we see that the sum of the series is in fact

$$\frac{1}{1 - \frac{1}{2}} = 2$$

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**REFERENCES**

- Booth, P.M., Chadburn, R., Cooper, D., Haberman, S. and James, D. (1999), *Modern Actuarial Theory and Practice*, Chapman and Hall.
- Donald, D.W.A. (1975), *Compound Interest and Annuities-Certain*, Heinemann.
- McCutcheon, J.J. and Scott, W.F. (1991), *An Introduction to the Mathematics of Finance*, Heinemann.

