Section One

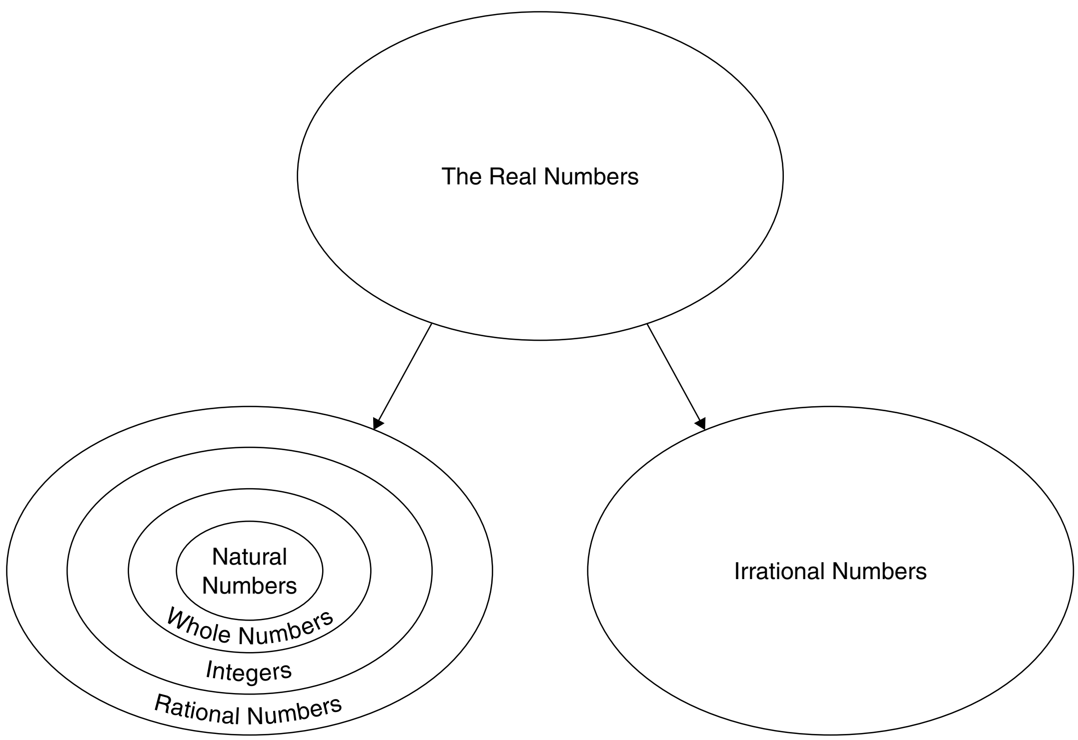
Numbers: Theory and Operation

LIST 1

The Real Numbers

Numbers and the operations we can perform with them are the basis of our numerical system. Each set of numbers listed below is a part or subset of the *Real Numbers,* which is also known as the *Real Number System* or the *Continuum of Real Numbers.*

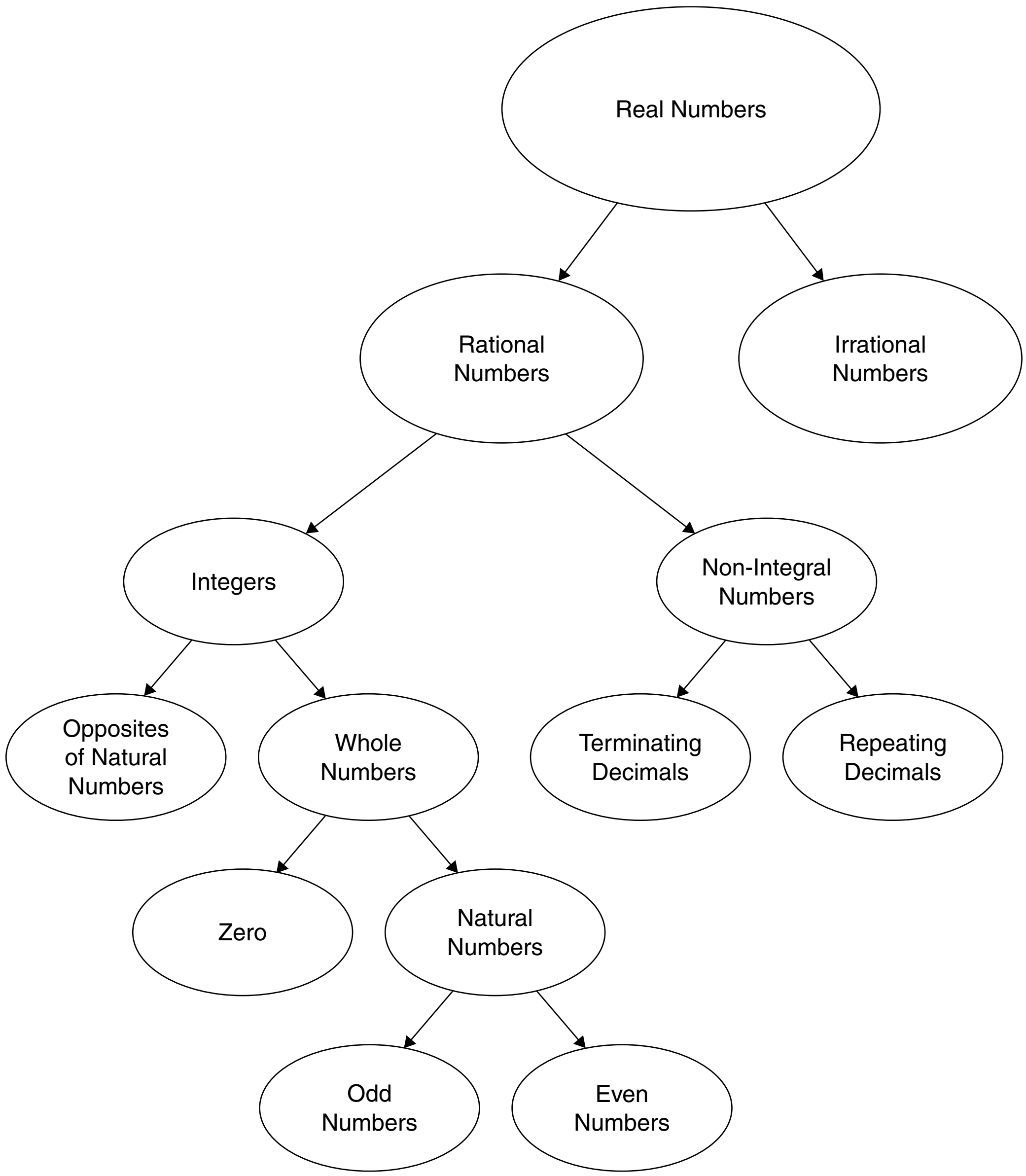
- *Natural Numbers*—the set of the counting numbers. They can be classified as odd or even. {1, 2, 3, 4, 5 . . .}
- *Whole Numbers*—the set of the natural numbers and zero. {0, 1, 2, 3, 4, 5 . . .}
- *Integers*—the set of the natural numbers, their opposites, and zero. {. . . –3, –2, –1, 0, 1, 2, 3 . . .}
- *Rational Numbers*—the set of all the numbers that can be expressed in the form *a*/*b* where *a* and *b* are integers, $b \neq 0$. *Examples:* integers, finite decimals and their opposites, and repeating decimals and their opposites.
- *Irrational Numbers*—the set of the numbers that cannot be written as terminating or repeating decimals. *Examples:* $\sqrt{2}$, $\sqrt{3}$, π, and *e.*



LIST 2

Classification of Real Numbers

In life, it is common to classify things that have common characteristics. In math, we can classify real numbers as shown by the following chart.



LIST 3

Cardinal and Ordinal Numbers

When we count, or specify a total number of items, we are using *cardinal numbers.* For example, *ten* is a cardinal number. *Ordinal numbers* are used to show order. *Tenth* is an ordinal number. Several examples of cardinal and ordinal numbers appear below. The shortened forms of ordinal numbers are also shown.

Cardinal Number	*Ordinal Number*	*Shortened Form*
1	First	1st
2	Second	2nd
3	Third	3rd
4	Fourth	4th
5	Fifth	5th
6	Sixth	6th
7	Seventh	7th
8	Eighth	8th
9	Ninth	9th
10	Tenth	10th
11	Eleventh	11th
12	Twelfth	12th
13	Thirteenth	13th
14	Fourteenth	14th
15	Fifteenth	15th
16	Sixteenth	16th
17	Seventeenth	17th
18	Eighteenth	18th
19	Nineteenth	19th
20	Twentieth	20th
21	Twenty-first	21st
22	Twenty-second	22nd
23	Twenty-third	23rd
24	Twenty-fourth	24th
25	Twenty-fifth	25th
26	Twenty-sixth	26th
27	Twenty-seventh	27th
28	Twenty-eighth	28th
29	Twenty-ninth	29th
30	Thirtieth	30th

Successive numbers repeat in the same manner:

100	One hundredth	100th
101	One hundred first	101st
102	One hundred second	102nd
103	One hundred third	103rd
104	One hundred fourth	104th
105	One hundred fifth	105th

LIST 4

Prime Numbers

A *prime number* is an integer greater than 1 whose only whole number factors are itself and 1. For example, 2 is a prime number because its only factors are 2 and 1. The number 383 is prime for the same reason. Its only factors are 383 and 1. Following is a list of the prime numbers less than 1,000:

2	127	283	467	661	877
3	131	293	479	673	881
5	137	307	487	677	883
7	139	311	491	683	887
11	149	313	499	691	907
13	151	317	503	701	911
17	157	331	509	709	919
19	163	337	521	719	929
23	167	347	523	727	937
29	173	349	541	733	941
31	179	353	547	739	947
37	181	359	557	743	953
41	191	367	563	751	967
43	193	373	569	757	971
47	197	379	571	761	977
53	199	383	577	769	983
59	211	389	587	773	991
61	223	397	593	787	997
67	227	401	599	797	
71	229	409	601	809	
73	233	419	607	811	
79	239	421	613	821	
83	241	431	617	823	
89	251	433	619	827	
97	257	439	631	829	
101	263	443	641	839	
103	269	449	643	853	
107	271	457	647	857	
109	277	461	653	859	
113	281	463	659	863	

LIST 5

Types of Prime Numbers

Prime numbers are *natural numbers* that have only two factors: 1 and the number. Certain types of primes have common characteristics. These primes are grouped together and have special names.

Twin primes are a set of two consecutive odd primes, which differ by only two. The twin primes less than 100 follow:

3,5	29,31
5,7	41,43
11,13	59,61
17,19	71,73

Symmetric primes, also called *Euler primes,* are a pair of prime numbers that are the same distance from a given number on a number line. There are no symmetric primes for 1, 2, or 3. It has not been proven if all natural numbers greater than 3 have symmetric primes. The following list shows the symmetric primes for the numbers 1 through 25.

Number	**Symmetric Primes**
1	None
2	None
3	None
4	3,5
5	3,7
6	5,7
7	3,11
8	5,11; 3,13
9	7,11; 5,13
10	7,13; 3,17
11	5,17; 3,19
12	11,13; 7,17; 5,19
13	7,19; 3,23
14	11,17; 5,23
15	13,17; 11,19; 7,23
16	15,17; 13,19; 3,29
17	11,23; 5,29; 3,31
18	17,19; 13,23; 7,29; 5,31
19	9,29; 7,31
20	17,23; 11,29; 3,37
21	19,23; 13,29; 11,31; 5,37
22	13,31; 7,37; 3,41
23	17,29; 13,33; 5,41; 3,43
24	19,29; 17,31; 11,37; 7,41; 5,43
25	19,31; 13,37; 7,43; 3,47

LIST 5

(Continued)

An *emirp* is a prime number that remains a prime when its digits are reversed. *Emirp,* of course, is *prime* spelled backward. Following are the emirps less than 500:

11	13	17	31	37	71	73	79	97	101
107	113	131	149	151	157	167	179	181	191
199	311	313	337	347	353	359	373	383	389

Relatively prime numbers are numbers whose greatest common factor is 1. If two numbers are relatively prime, they are said to be relatively prime in pairs. Notice that the numbers of the following list are not limited to primes. Numbers that are relatively prime do not have to be prime numbers. They must not have any common factors other than 1. Below is a list of the numbers from 1 through 10 that are relatively prime in pairs:

1,2	2,3	3,4	4,5	5,6
1,3	2,5	3,5	4,7	5,7
1,4	2,7	3,7	4,9	5,8
1,5	2,9	3,8		5,9
1,6		3,10		
1,7				
1,8				
1,9				
1,10				

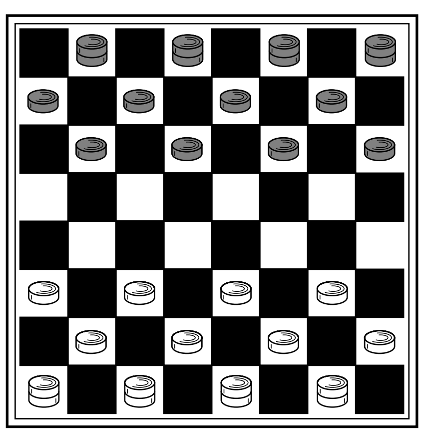
6,7	7,8	8,9	9,10
	7,9		
	7,10		

LIST 6

Composite Numbers

Composite numbers are positive integers that have more than two positive whole number factors. The number 6 is a composite number because its factors are 1, 2, 3, and 6. Note that 1 is the only natural number that is neither prime nor composite. The first 100 composite numbers follow:

4	39	72	104
6	40	74	105
8	42	75	106
9	44	76	108
10	45	77	110
12	46	78	111
14	48	80	112
15	49	81	114
16	50	82	115
18	51	84	116
20	52	85	117
21	54	86	118
22	55	87	119
24	56	88	120
25	57	90	121
26	58	91	122
27	60	92	123
28	62	93	124
30	63	94	125
32	64	95	126
33	65	96	128
34	66	98	129
35	68	99	130
36	69	100	132
38	70	102	133



LIST 7

Perfect Squares and Cubes

The *square* of a number results when a number is multiplied by itself—for example, $4 \times 4 = 16$. Thus we say 4 squared is 16.

The *cube* of a number results when a number is multiplied by itself twice—for example, $4 \times 4 \times 4 = 64$. Thus, we say that 4 cubed is 64.

The following list contains the squares and cubes of numbers 1 to 25:

Number	***Squared***	***Cubed***
1	1	1
2	4	8
3	9	27
4	16	64
5	25	125
6	36	216
7	49	343
8	64	512
9	81	729
10	100	1,000
11	121	1,331
12	144	1,728
13	169	2,197
14	196	2,744
15	225	3,375
16	256	4,096
17	289	4,913
18	324	5,832
19	361	6,859
20	400	8,000
21	441	9,261
22	484	10,648
23	529	12,167
24	576	13,824
25	625	15,625

LIST 8

Abundant, Deficient, and Perfect Numbers

The ancient Greeks were thinkers of the first order. They enjoyed mathematics and categorized all the natural numbers as being abundant, deficient, or perfect:

Abundant—a number that is less than the sum of its factors, excluding itself

Deficient—a number that is greater than the sum of its factors, excluding itself

Perfect—a number that is equal to the sum of its factors, excluding itself

The first list that follows groups the first 50 numbers as the ancient Greeks might have. The second list contains perfect numbers.

Number	*Factors Excluding Itself*	*Sum*	*Type*
1		0	Deficient
2	1	1	Deficient
3	1	1	Deficient
4	1,2	3	Deficient
5	1	1	Deficient
6	1,2,3	6	Perfect
7	1	1	Deficient
8	1,2,4	7	Deficient
9	1,3	4	Deficient
10	1,2,5	8	Deficient
11	1	1	Deficient
12	1,2,3,4,6	16	Abundant
13	1	1	Deficient
14	1,2,7	10	Deficient
15	1,3,5	9	Deficient
16	1,2,4,8	15	Deficient
17	1	1	Deficient
18	1,2,3,6,9	21	Abundant
19	1	1	Deficient
20	1,2,4,5,10	22	Abundant
21	1,3,7	11	Deficient
22	1,2,11	14	Deficient
23	1	1	Deficient
24	1,2,3,4,6,8,12	36	Abundant
25	1,5	6	Deficient
26	1,2,13	16	Deficient
27	1,3,9	13	Deficient
28	1,2,4,7,14	28	Perfect
29	1	1	Deficient
30	1,2,3,5,6,10,15	42	Abundant

LIST 8

(Continued)

Number	*Factors Excluding Itself*	*Sum*	*Type*
31	1	1	Deficient
32	1,2,4,8,16	31	Deficient
33	1,3,11	15	Deficient
34	1,2,17	20	Deficient
35	1,5,7	13	Deficient
36	1,2,3,4,6,9,12,18	55	Abundant
37	1	1	Deficient
38	1,2,19	22	Deficient
39	1,3,13	17	Deficient
40	1,2,4,5,8,10,20	50	Abundant
41	1	1	Deficient
42	1,2,3,6,7,14,21	54	Abundant
43	1	1	Deficient
44	1,2,4,11,22	40	Deficient
45	1,3,5,9,15	33	Deficient
46	1,2,23	26	Deficient
47	1	1	Deficient
48	1,2,3,4,6,8,12,16,24	76	Abundant
49	1,7	8	Deficient
50	1,2,5,10,25	43	Deficient

Perfect Numbers

Perfect numbers are mathematical rarities that have no practical use. Still, mathematicians have found them challenging and have even worked out a formula to find them: $2^{p-1}(2^p - 1)$ where p and $(2^p - 1)$ are prime numbers. No one has found an odd perfect number, and no one has proven whether odd perfect numbers exist. A list of the first eight perfect numbers and their formulas follows. After that the numbers simply become too large to write.

Perfect Number	*Formula*
6	$2^1(2^2 - 1)$
28	$2^2(2^3 - 1)$
496	$2^4(2^5 - 1)$
8,128	$2^6(2^7 - 1)$
33,550,336	$2^{12}(2^{13} - 1)$
8,589,869,056	$2^{16}(2^{17} - 1)$
137,438,691,328	$2^{18}(2^{19} - 1)$
2,305,843,008,139,952,128	$2^{30}(2^{31} - 1)$

LIST 9

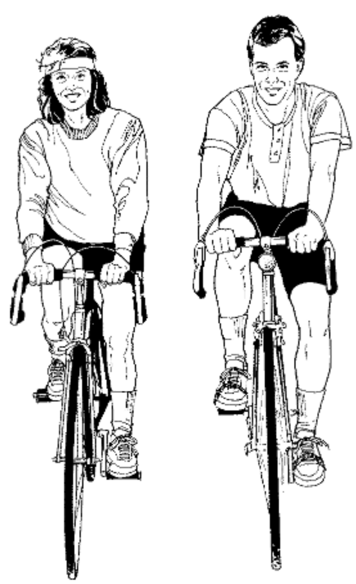
Amicable Numbers

Amicable numbers come in pairs. They are quite special, because each number of the pair has factors that (excluding itself) add up to equal the other number of the pair.

The first pair of amicable numbers is 220 and 284. The factors of 220, excluding itself, are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, and 110. Their sum equals 284. The factors of 284, excluding itself, are 1, 2, 4, 71, and 142, which add up to 220.

More than 1,000 pairs of amicable numbers have been found. This list offers the ten smallest pairs:

220 and 284
1,184 and 1,210
2,620 and 2,924
5,020 and 5,564
6,232 and 6,368
10,744 and 10,856
12,285 and 14,595
17,296 and 18,416
63,020 and 76,084
66,928 and 66,992



LIST 10

Some Lesser-Known Types of Numbers

Mathematicians often group numbers according to their properties. While most students of math are familiar with the types of numbers found in List 1, "The Real Numbers," there are many others that are not so well known.

Algebraic Numbers—numbers that are the solution of an algebraic equation. *Example:* $2 + x = 9$. The answer is 7, which is an algebraic number.

Almost Perfect Numbers—numbers that are one more or one less than the sum of all of their factors, except themselves. *Example:* 4 is almost perfect because $1 + 2 = 3$, which is one less than 4. (All powers of 2 are almost perfect.)

Automorphic Numbers—numbers that when raised to a power end in the original number. *Example:* $5^2 = 25$; $5^3 = 125$; $6^2 = 36$; $25^2 = 625$.

Complex Numbers—numbers in the form $a + bi$ where a and b stand for real numbers and $i^2 = -1$ or ($i = \sqrt{-1}$). *Example:* $-2 + \sqrt{-2}$ can be written as $-2 + i\sqrt{2}$.

Crowd—a chain of three sociable numbers. No one has yet found such a chain, but neither has anyone proven that such chains do not exist.

Cute Numbers—numbers that have exactly four factors. *Examples:* 6, whose factors are 1, 2, 3, and 6; 8, whose factors are 1, 2, 4, and 8.

Cyclic Numbers—an integer of n digits with the following characteristic: When multiplied by a number from 1 to n, the product has the same digits as the original number, in the same cycle. *Example:* 142,857 is the smallest cyclic number, other than 1.

$$1 \times 142{,}857 = 142{,}857$$
$$2 \times 142{,}857 = 285{,}714$$
$$3 \times 142{,}857 = 428{,}571$$
$$4 \times 142{,}857 = 571{,}428$$
$$5 \times 142{,}857 = 714{,}285$$
$$6 \times 142{,}857 = 857{,}142$$

Denominate Numbers—numbers whose unit represents a unit of measure. *Examples:* 3 pounds, 7 inches, 2 quarts.

Fibonacci Numbers—numbers of the sequence 1, 1, 2, 3, 5, etc., . . . where successive numbers are the sum of the two preceding numbers. *Examples:* $1 + 1 = 2$; $1 + 2 = 3$; $2 + 3 = 5$.

Imaginary Numbers—numbers involving the imaginary unit i, where $i^2 = -1$ or ($i = \sqrt{-1}$). *Example:* $\sqrt{-2} = i\sqrt{2}$.

Lucas Numbers—numbers of the sequence 1, 3, 4, 7, 11, 18, . . . where successive numbers are the sum of the two preceding numbers. *Examples:* $1 + 3 = 4$; $3 + 4 = 7$; $4 + 7 = 11$; $7 + 11 = 18$; etc.

Mersenne Primes—primes of the form $2^p - 1$ where p is prime. *Examples:* 3, 7, 31, 127.

LIST 10

(Continued)

Random Numbers—numbers that are obtained without any pattern and can be described only by listing the digits. *Example:* Picking numbers from a hat, such as 8, 3, 7, 0, 1, 2, 9, 9, 8, 1, 2, and 6.

Repunit Numbers—an abbreviated form of "repeated unit" of the digit 1, but excluding the number 1. *Examples:* 11, 111, 1111, etc.

Sociable Numbers—one of a chain of numbers whose factors add up to the next number in the chain. *Example:* a five-link chain: 12496, 14288, 15472, 14536, 14264.

The sum of the factors of 12496 = 14288.
The sum of the factors of 14288 = 15472.
The sum of the factors of 15472 = 14536.
The sum of the factors of 14536 = 14264.
The sum of the factors of 14264 = 12496.

Note that the chain is completed with the last number.

Surds—algebraic numbers that cannot be written as an exact ratio of two integers. A surd is one type of irrational number. (The other type of irrational number is the transcendental number.) *Examples:* $\sqrt{2}, \sqrt{3}$.

Transcendental Numbers—any irrational numbers that are not algebraic numbers. *Examples:* π and *e*.

Unit Number—the number 1.

Untouchable Numbers—numbers that are never the sum of the factors of any other number. *Examples:* 2, 5, 52, 88, 96, and 120.

Weird Numbers—a special type of abundant number that does not represent the sum of any of its factors. *Examples:* 70; 836; 4,030; 5,830; and 7,192.

LIST 11

Multiplication Table

Even in this day of high-speed computers and calculators, a basic knowledge of multiplication facts is important to understand some of the relationships between numbers. Knowing the multiplication tables is necessary for any problem that requires multiplying or dividing, especially when a calculator is not handy. Here is one example.

×	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36	40	44	48
5	5	10	15	20	25	30	35	40	45	50	55	60
6	6	12	18	24	30	36	42	48	54	60	66	72
7	7	14	21	28	35	42	49	56	63	70	77	84
8	8	16	24	32	40	48	56	64	72	80	88	96
9	9	18	27	36	45	54	63	72	81	90	99	108
10	10	20	30	40	50	60	70	80	90	100	110	120
11	11	22	33	44	55	66	77	88	99	110	121	132
12	12	24	36	48	60	72	84	96	108	120	132	144

LIST 12

Rules for Finding Divisibility

Understanding divisibility is useful in many mathematical applications. Two of the most obvious are reducing fractions and finding common denominators. In advanced mathematics, divisibility tests and common factors are useful in factoring polynomials.

For a number to be divisible by

2, the number must be an even number, ending in 2, 4, 6, 8, 0.

3, the sum of the digits of the number must be divisible by 3.

4, the number must be even and the last two digits of the number are divisible by 4.

5, the number must end in 0 or 5.

6, the number must be even and the sum of its digits must be divisible by 3.

7, you must be able to drop the ones digit, and subtract two times the ones digit from the remaining number. If that answer can be divided by 7, the original number can be divided by 7.

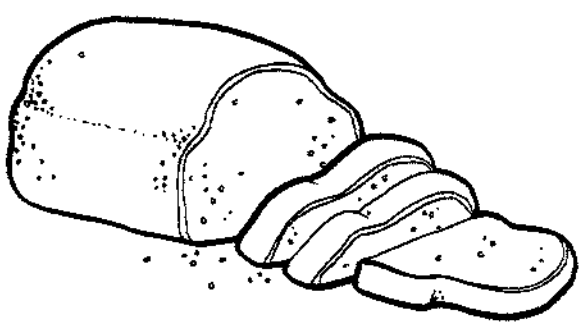
8, the number formed by the last 3 digits of the number can be divided by 8.

9, the sum of the digits must be divisible by 9.

10, the number must end in 0.

11, you must first add the alternate digits, beginning with the first digit. Next you must add the alternate digits, beginning with the second digit. Subtract the smaller sum from the larger. If the difference is divisible by 11, the original number is divisible by 11.

12, the number must be divisible by both 3 and 4.



LIST 13

Rules for Finding the Greatest Common Factor (GCF)

A *factor* is a number that divides into a larger number evenly. The *greatest common factor* (GCF) is the largest number that divides into two or more numbers evenly. Being able to find the greatest common factor is an important skill for reducing fractions. It is also helpful in algebra for factoring polynomials. The GCF of two or more numbers can be found in three ways.

Listing Factors

- List all the numbers that divide evenly into the first given number. These are its factors.
- List all the factors of the second given number.
- Circle the largest factor that appears in both lists. This is the greatest common factor.

Find the greatest common factor of 24 and 36.

The factors of 24 are 1, 2, 3, 4, 6, 8, (12), 24.

The factors of 36 are 1, 2, 3, 4, 6, 9, (12), 18, 36.

12 is the greatest common factor of 24 and 36.

Expressing Each Number as the Product of Primes

- Factor each number into its primes.
- Circle those factors (by pairs) common to each.
- The greatest common factor is the product of the numbers that are circled.

Find the greatest common factor of 24 and 36.

$$24 = (2) \times (2) \times 2 \times (3)$$
$$36 = (2) \times (2) \times 3 \times (3)$$
$$2 \times 2 \times 3 = 12$$

12 is the greatest common factor of 24 and 36.

Using Euclid's Method

- Divide the larger number by the smaller, expressing the quotient with a remainder.
- Divide the previous divisor by the previous remainder.
- Continue this process until the remainder is 0.
- The last divisor is the greatest common factor of the number.

Find the greatest common factor of 27 and 90.

$$90 \div 27 = 3\,r\,9 \quad (27 \times 3 = 81;\ 90 - 81 = 9)$$
$$27 \div 9 = 3\,r\,0 \quad (9 \times 3 = 27;\ 27 - 27 = 0)$$

9 is the greatest common factor of 27 and 90.

LIST 14

Rules for Finding the Least Common Multiple (LCM)

A *common multiple* is a number that two other numbers will divide into evenly. The *least common multiple* (LCM) is the lowest multiple of two numbers. It is most useful for finding common denominators. The LCM of two numbers can be found in three ways.

Strategy One

▶ Start with the bigger number.	Find the least common multiple of 10 and 15.
▶ List its multiples by multiplying the number by 1, 2, 3, 4, 5, etc.	15 is the larger number. Multiples of 15: 15, 30, . . .
▶ After each multiplication, check to see if the multiple of the larger number is also a multiple of the smaller number. If it is, you have found the least common multiple.	Is 15 a multiple of 10? No. Is 30 a multiple of 10? Yes. 30 is the least common multiple of 10 and 15.

Strategy Two

▶ Find the product of the two numbers.	Find the least common multiple of 10 and 15.
▶ Divide this product by the greatest common factor of the numbers.	The product of 10 and 15 is 150. The greatest common factor of 10 and 15 is 5. $150 \div 5 = 30$. 30 is the least common multiple of 10 and 15.

Strategy Three

▶ Factor each number into its primes.	Find the least common multiple of 24 and 36.
▶ Write each product using exponents.	$24 = 2 \times 2 \times 2 \times 3$ $36 = 2 \times 2 \times 3 \times 3$
▶ Write each base.	$24 = 2^3 \times 3$ $36 = 2^2 \times 3^2$
▶ If the base is a factor of only one number, write the base and the exponent in exponential form.	The bases are 2 and 3.
▶ If the base is a factor of more than one number, write the base in exponential form using the larger (or largest) exponents of the bases. If the exponents of a given base are the same, write the base and exponent in exponential form.	$2^3 \times 3^2$
▶ The least common multiple is the product of these numbers.	$2^3 \times 3^2 = 72$ The least common multiple of 24 and 36 is 72.

LIST 15

Types of Fractions

A *fraction* is a part of a whole. There are many types of fractions.

Simple Fraction—a fraction in which the numerator and denominator are both integers. Also known as a *common fraction.*

Examples: $\frac{2}{3}$, $\frac{7}{3}$, $-\frac{6}{7}$, $\frac{5}{1}$

Proper Fraction—a fraction in which the numerator is less than the denominator.

Examples: $\frac{1}{4}$, $\frac{2}{7}$, $-\frac{1}{8}$

Improper Fraction—a fraction in which the numerator is equal to or greater than the denominator. Improper fractions are usually changed to whole or mixed numbers.

Examples: $\frac{5}{3}$, $\frac{7}{7}$, $-\frac{11}{8}$

Simplified Fraction—a fraction whose numerator and denominator are integers and their greatest common factor is 1.

Examples: $\frac{1}{2}$, $\frac{4}{7}$, $-\frac{8}{11}$

Mixed Number—a number that is a combination of an integer and a proper fraction. Thus, it is "mixed."

Examples: $2\frac{2}{3}$, $5\frac{7}{8}$, $-2\frac{1}{2}$

Unit Fraction—a fraction in which the numerator is 1.

Examples: $\frac{1}{5}$, $\frac{1}{14}$

An Integer Represented as a Fraction—a fraction in which the denominator is 1.

Examples: $\frac{2}{1}$, $-\frac{3}{1}$

Complex Fraction—a fraction in which the numerator or the denominator, or both numerator and denominator, are fractions.

Examples: $\frac{\frac{3}{5}}{\frac{7}{8}}$, $\frac{\frac{7}{9}}{4}$, $\frac{5}{\frac{1}{3}}$

Reciprocal—the fraction that results from interchanging the numerator and denominator.

Example: 4 is the reciprocal of $\frac{1}{4}$.

LIST 15

(Continued)

Similar Fractions—two or more simple fractions that have the same denominator.

Examples: $\frac{3}{7}$, $\frac{4}{7}$, $\frac{6}{7}$

Zero Fraction—a fraction in which the numerator is zero. A zero fraction equals zero.

Example: $\frac{0}{3} = 0$

Undefined Fraction—a fraction with a denominator of zero. ($\frac{7}{0}$ means 7 divided by 0, which is an impossibility because nothing can be divided by 0. Therefore, the fraction remains undefined.)

Indeterminate Form—an expression having no quantitative meaning.

Example: $\frac{0}{0}$

LIST 16

Rules for Simplifying Fractions

You can simplify fractions by dividing numerators and denominators by common factors.

- Find the largest number that will divide into the numerator and denominator of the fraction evenly. This number is the greatest common factor between the numerator and denominator.

  Simplify $\frac{18}{24}$

  6 is the greatest common factor of 18 and 24.

- Divide both numerator and denominator by the greatest common factor. (If you did not find the greatest common factor the first time, you can simplify further by finding another common factor.)

  $\frac{18 \div 6}{24 \div 6} = \frac{3}{4}$

  or

  $\frac{18 \div 2}{24 \div 2} = \frac{9}{12}$ $\frac{9 \div 3}{12 \div 3} = \frac{3}{4}$

  $\frac{18}{24} = \frac{3}{4}$

For Renaming Improper Fractions as Mixed Numbers

- Divide the numerator by the denominator.

  Simplify $\frac{7}{5}$ $\frac{7}{5} = 1\frac{2}{5}$

- Write the mixed number. If there is no remainder, you will write a whole number.

  Simplify $\frac{8}{4}$ $\frac{8}{4} = 2$

- Simplify the remaining fraction according to the rules above.

  Simplify $\frac{18}{4}$ $\frac{18}{4} = 4\frac{2}{4} = 4\frac{1}{2}$

LIST 17

Rules for Operations with Fractions

The following rules apply to addition, subtraction, multiplication, and division of fractions.

Adding Fractions (Like Denominators)

- Add the numerators.
- Write the sum over the common denominator. (Do not add the denominators.)
- Simplify if possible.

$$\frac{3}{4} + \frac{2}{4} = \frac{5}{4} = 1\frac{1}{4}$$

Subtracting Fractions (Like Denominators)

- Subtract the numerators.
- Write the difference over the common denominator. (Do not subtract the denominators.)
- Simplify if possible.

$$\frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

Adding Fractions (Unlike Denominators)

- Find the least common denominator by finding the least common multiple of the denominators. In the example to the right, the least common multiple of the denominators is 20. Therefore, 20 is the least common denominator.

$$\frac{4}{5} = \frac{}{20} \qquad + \frac{3}{4} = \frac{}{20}$$

- Write equivalent fractions with the common denominator.

$$\frac{4}{5} \times \frac{4}{4} = \frac{16}{20} \qquad + \frac{3}{4} \times \frac{5}{5} = \frac{15}{20}$$

- Add the numerators. (Do not add the denominators.)
- Simplify if possible.

$$\frac{31}{20} = 1\frac{11}{20}$$

LIST 17
(Continued)

Subtracting Fractions (Unlike Denominators)

- Find the least common denominator by finding the least common multiple of the denominators.
- Write equivalent fractions with the common denominator.
- Subtract the numerators. (Do not subtract the denominators.)
- Simplify if possible.

$$\frac{4}{5} = \frac{\quad}{20}$$
$$-\frac{3}{4} = \frac{\quad}{20}$$

$$\frac{4}{5} \times \frac{4}{4} = \frac{16}{20}$$
$$-\frac{3}{4} \times \frac{5}{5} = \frac{15}{20}$$
$$\frac{1}{20}$$

Multiplying Fractions

- Multiply the numerators.
- Multiply the denominators.
- Simplify if possible.
- For some fractions you may be able to simplify before multiplying; then follow the previous steps.

$$\frac{1}{3} \times \frac{3}{4} = \frac{3}{12} = \frac{1}{4}$$

$$\frac{\cancel{3}^{1}}{\cancel{4}_{2}} \times \frac{\cancel{2}^{1}}{\cancel{3}_{1}} = \frac{1}{2}$$

Dividing Fractions

- Write the first fraction.
- Change the division sign to a multiplication sign.
- Write the reciprocal of the divisor. (This is the fraction to the right of the division sign.)
- Multiply the numerators.
- Multiply the denominators.
- Simplify if possible.

$$\frac{3}{4} \div \frac{2}{3}$$

The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}$$

LIST 18

Rules for Operations with Mixed Numbers

A *mixed number* is just what its name implies: a whole number combined with a fraction. The two are mixed together to express a value that lies somewhere between two whole numbers. Although working with mixed numbers is much like working with fractions, there are a few additional skills you will need to know.

Adding Mixed Numbers (Like Denominators)

- Add the numerators of the fractions. (Do not add the denominators.)
- Add the whole numbers.
- Simplify if possible.

$$\begin{array}{r} 3\frac{1}{5} \\ +2\frac{3}{5} \\ \hline 5\frac{4}{5} \end{array}$$

Subtracting Mixed Numbers (Like Denominators) without Regrouping

- If the numerator of the fraction following the subtraction sign is smaller than the numerator of the first fraction, subtract the numerators. (Do not subtract the denominators.)
- Subtract the whole numbers.
- Simplify if possible.

$$\begin{array}{r} 5\frac{6}{7} \\ -3\frac{2}{7} \\ \hline 2\frac{4}{7} \end{array}$$

Subtracting Mixed Numbers (Like Denominators) with Regrouping

- If the numerator following the subtraction sign is larger than the first numerator, you must regroup "1" from the whole number of the first fraction. Rewrite the "1" as a fraction with the same denominator, and add it to the first fraction.
- Subtract the numerators. (Do not subtract the denominators.)
- Subtract the whole numbers.
- Simplify if possible.

$$\begin{array}{r} 8\frac{1}{4} \\ -3\frac{3}{4} \\ \hline \end{array} \qquad 8 = 7\frac{4}{4}$$

$$8\frac{1}{4} = 7\frac{4}{4} + \frac{1}{4} = 7\frac{5}{4}$$
$$-3\frac{3}{4} \qquad\qquad = 3\frac{3}{4}$$
$$4\frac{2}{4} = 4\frac{1}{2}$$

Adding Mixed Numbers (Unlike Denominators)

- Write equivalent fractions with the same denominators.
- Add the numerators. (Do not add the denominators.)
- Add the whole numbers.
- Simplify if possible.

$$\begin{array}{r} 3\frac{1}{3} = 3\frac{5}{15} \\ +2\frac{4}{5} = 2\frac{12}{15} \\ \hline 5\frac{17}{15} = 6\frac{2}{15} \end{array}$$

LIST 18

(Continued)

Subtracting Mixed Numbers (Unlike Denominators) Without Regrouping

- Write equivalent fractions with the same denominators.
- Subtract the numerators. (Do not subtract the denominators.)
- Subtract the whole numbers.
- Simplify if possible.

$$\begin{array}{r} 3\frac{4}{5} = 3\frac{8}{10} \\ -2\frac{1}{10} = 2\frac{1}{10} \\ \hline 1\frac{7}{10} \end{array}$$

Subtracting Mixed Numbers (Unlike Denominators) with Regrouping

- Write equivalent fractions with the same denominators.
- If necessary, rename "1" from the whole number of the first fraction and rewrite it as a fraction with the same denominator. Add it to the first fraction.
- Subtract the numerators. (Do not subtract the denominators.)
- Subtract the whole numbers.
- Simplify if possible.

$$\begin{array}{r} 4\frac{1}{8} = 4\frac{3}{24} \\ -2\frac{2}{3} = 2\frac{16}{24} \\ \hline \end{array}$$

$$4 = 3\frac{24}{24}$$

$$\begin{array}{r} 4\frac{3}{24} = 3\frac{24}{24} + \frac{3}{24} = 3\frac{27}{24} \\ -2\frac{16}{24} \qquad\qquad = 2\frac{16}{24} \\ \hline 1\frac{11}{24} \end{array}$$

Multiplying Mixed Numbers

- Change the mixed numbers to improper fractions. (To change a mixed number to an improper fraction, multiply the denominator by the whole number and add the numerator. Write this number over the original denominator.)
- If possible, simplify the fractions before multiplying.
- Multiply the numerators.
- Multiply the denominators.
- Simplify if possible.

$$3\frac{1}{2} \times 2\frac{4}{7} = \frac{7}{2} \times \frac{18}{7} =$$

$$\frac{\cancel{7}^{1}}{\cancel{2}_{1}} \times \frac{\cancel{18}^{9}}{\cancel{7}_{1}} = \frac{9}{1} = 9$$

LIST 18

(Continued)

Dividing Mixed Numbers

- Change the mixed numbers to improper fractions. (To change a mixed number to an improper fraction, multiply the denominator by the whole number and add the numerator. Write this number over the original denominator.)

$$4\frac{2}{3} \div 1\frac{1}{3} = \frac{14}{3} \div \frac{4}{3} =$$

- Change the divisor to its reciprocal, and rewrite the division sign as multiplication.

$$\frac{14}{3} \times \frac{3}{4} =$$

- If possible, simplify the fractions before multiplying.
- Multiply the numerators.
- Multiply the denominators.
- Simplify if possible.

$$\frac{\cancel{14}^{7}}{\cancel{3}_{1}} \times \frac{\cancel{3}^{1}}{\cancel{4}_{2}} = \frac{7}{2} = 3\frac{1}{2}$$

LIST 19

Place Value Chart

Understanding place value causes some students (and their teachers) many headaches. Since the value of any digit depends on its "place," understanding place value is an essential skill. In the example below, 5 represents 5 ten thousands and also 5 ten-millionths. The digits are the same, but the values are quite different.

trillions	hundred billions	ten billions	billions	hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones		tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths	millionths	ten-millionths
3,	2	8	7,	3	8	4,	6	5	1,	2	9	6	.	3	7	8	2	6	1	5

LIST 20

Types of Decimals

In the broadest sense, a *decimal* is any numeral in the base ten number system. Following are several types of decimals.

Decimal Fraction—a number that has no digits other than zeros to the left of the decimal point.

Examples: 0.349, 0.84, 0.3001

Mixed Decimal—an integer and a decimal fraction.

Examples: 8.341, 27.1, 341.07

Similar Decimals—decimals that have the same number of places to the right of the decimal point.

Examples: 3.87 and 0.12; 14.015 and 3.396

Decimal Equivalent of a Proper Fraction—the decimal fraction that equals the proper fraction.

Examples: $0.25 = \frac{1}{4}$, $0.3 = \frac{3}{10}$

Finite (or Terminating) Decimal—a decimal that has a finite number of digits.

Examples: 0.3, 0.2765, 5.38412

Infinite (or Nonterminating) Decimal—a decimal that has an unending number of digits to the right of the decimal point.

Examples: π, $\sqrt{3}$, $0.\overline{33}$, $0.\overline{37}$, 34.12794…

Repeating (or Periodic) Decimal—a nonterminating decimal in which the same digit or group of digits repeats. A bar is used to show that a digit or group of digits repeats. The repeating set is called the period or repetend.

Examples: $0.\overline{3}$, $0.\overline{37}$

Nonrepeating (or Nonperiodic) Decimal—a decimal that is nonterminating and nonrepeating. Such decimals are irrational numbers.

Examples: π, $\sqrt{3}$

LIST 21

Rules for Operations with Decimals

Adding, subtracting, multiplying, and dividing decimals is not so hard as it may seem. Use the following guides.

Adding Decimals

- Line up the numbers according to the decimal points before adding. Keep columns straight and the digits in their proper places.
- Add zeros for placeholders if necessary.
- After setting up the problem, bring the decimal point straight down.
- Remember that a whole number is placed to the left of the decimal point. For example, the whole number 5 is written as 5.0.
- Add as you would with whole numbers.
- If you carry, carry to the next place.

$2.73 + 0.145 =$

$$\begin{array}{r} 2.73 \\ +0.145 \\ \hline \end{array}$$

$$\begin{array}{r} 2.730 \\ +0.145 \\ \hline 2.875 \end{array}$$

$7.4 + 5 =$

$$\begin{array}{r} 7.4 \\ +5.0 \\ \hline 12.4 \end{array}$$

Subtracting Decimals

- Line up the numbers according to decimal points before subtracting. Keep the columns straight and the digits in their proper places.
- Add zeros for placeholders if necessary.
- After setting up the problem, bring the decimal point straight down.
- Remember that a whole number is placed to the left of the decimal point. The whole number 8 is written as 8.0.
- Regroup as you would with whole numbers.
- Subtract as you would with whole numbers.

$2.75 - 0.042 =$

$$\begin{array}{r} 2.75 \\ -0.042 \\ \hline \end{array}$$

$$\begin{array}{r} 2.750 \\ -0.042 \\ \hline 2.708 \end{array}$$

$9.34 - 8 =$

$$\begin{array}{r} 9.34 \\ -8.00 \\ \hline 1.34 \end{array}$$

Multiplying Decimals

- Line up the numbers by columns, not according to decimal points.
- Multiply as you would with whole numbers.
- Count the places held by digits to the right of the decimal points in the numbers you multiplied.
- Start at the right of your answer, and count the same number of places to the left. Mark the decimal point there.

$4.32 \times 0.7 =$

$$\begin{array}{rl} 4.32 & (2 \text{ places}) \\ \times\ 0.7 & +(1 \text{ place}) \\ \hline 3.024 & (3 \text{ places}) \end{array}$$

LIST 21

(Continued)

Dividing a Decimal by a Whole Number

- Bring the decimal point straight up.
- Divide as you would with whole numbers.

```
   0.41
8)3.28
  32
    8
    8
```

Dividing a Decimal by a Decimal

- Move the decimal point to the right of the divisor, making the divisor a whole number. (This is the same as multiplying a decimal by a power of 10.)
- Move the decimal point in the dividend to the right the same number of places.
- Bring the decimal point straight up.
- Divide as you would with whole numbers.
- If necessary, add a zero or zeros to the dividend to finish dividing. (The problem might work out evenly, or you might need to round off your answer.)

```
    4.1
.8)3.28

   410.     Add zero as a placeholder.
.8)328.0
   32
    8
    8
     0
     0

   2.925    Add zeros to finish dividing.
.8)2.3400
   16
    74
    72
     20
     16
      40
      40
```

LIST 22

Rules for Changing Decimals to Fractions

Decimals can easily be converted to fractions. The key is understanding the place value of the decimal.

- ▶ Read the decimal. Here are some examples:
  - One place to the right of the decimal point is tenths.
  - Two places to the right of the decimal point are hundredths.
  - Three places to the right of the decimal point are thousandths.
  - Four places to the right of the decimal point are ten-thousandths.
- ▶ Write the decimal as a fraction with a denominator that is the same value of the decimal.
- ▶ Simplify if possible.

Change each decimal listed below to a fraction.

$$0.5 = \frac{5}{10} = \frac{1}{2}$$

$$0.23 = \frac{23}{100}$$

$$0.145 = \frac{145}{1000} = \frac{29}{200}$$

$$0.7625 = \frac{7625}{10000} = \frac{1525}{2000} = \frac{305}{400} = \frac{61}{80}$$

If the Decimal Is a Mixed Decimal

- ▶ Multiply by $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$, etc. Here are some examples:
  - Multiply by $\frac{1}{10}$ if the decimal has only one digit to the right of the decimal point.
  - Multiply by $\frac{1}{100}$ if the decimal has two digits to the right of the decimal point.
  - Multiply by $\frac{1}{1000}$ if the decimal has three digits to the right of the decimal point.
- ▶ Rewrite the mixed number as an improper fraction.
- ▶ If possible, simplify, either before or after you multiply.

Change each mixed decimal listed below to a fraction.

$$0.3\frac{1}{3} = 3\frac{1}{3} \times \frac{1}{10} = \frac{\cancel{10}^{1}}{3} \times \frac{1}{\cancel{10}_{1}} = \frac{1}{3}$$

$$0.87\frac{1}{2} = 87\frac{1}{2} \times \frac{1}{100} = \frac{\cancel{175}^{7}}{2} \times \frac{1}{\cancel{100}_{4}} = \frac{7}{8}$$

$$0.666\frac{2}{3} = \frac{\cancel{2000}^{2}}{3} \times \frac{1}{\cancel{1000}_{1}} = \frac{2}{3}$$

LIST 22

(Continued)

If the Decimal Repeats

▶ Multiply by 10, 100, 1,000, etc. Here are some examples:

Change each repeating decimal to a fraction. $0.\overline{3},\ 0.\overline{34},\ 0.\overline{371}$

- Multiply by 10 if one digit repeats. $n = 0.\overline{3}$ so $10n = 3.\overline{3}$
- Multiply by 100 if two digits repeat. $n = 0.\overline{34}$ so $100n = 34.\overline{34}$
- Multiply by 1000 if three digits repeat. $n = 0.\overline{371}$ so $1000n = 371.\overline{371}$

▶ Subtract.

$$\begin{array}{rl} 10n = & 3.\overline{3} \\ -n = & -0.\overline{3} \end{array} \qquad \begin{array}{rl} 100n = & 34.\overline{34} \\ -\ n = & -0.\overline{34} \end{array} \qquad \begin{array}{rl} 1000n = & 371.\overline{371} \\ -\ n = & -0.\overline{371} \end{array}$$

▶ Divide by the coefficient of variable.

$$\frac{9n}{9} = \frac{3}{9} \qquad \frac{99n}{99} = \frac{34}{99} \qquad \frac{999n}{999} = \frac{371}{999}$$

▶ Simplify if possible.

$$n = \frac{1}{3} \qquad n = \frac{34}{99} \qquad n = \frac{371}{999}$$

Therefore: $0.\overline{3} = \frac{1}{3},\ 0.\overline{34} = \frac{34}{99},\ 0.\overline{371} = \frac{371}{999}$

LIST 23

Rules for Changing Fractions to Decimals

There are two methods for changing fractions to decimals. The first is to rewrite the fraction to make the denominator a decimal equivalent in the form of tenths, hundredths, or thousandths. The second is to divide the denominator of the fraction into its numerator. Both methods are detailed below.

Rewriting the Fraction

- Multiply the numerator and the denominator of the fraction so that the denominator is equal to tenths, hundredths, or thousandths.
- A simple way to see if this method will work is to divide the denominator into tenths, hundredths, or thousandths. If the denominator divides evenly, multiply the numerator and denominator by the same number to find the equivalent fraction.
- Change the fraction to an equivalent fraction.

Change $\frac{1}{2}$ to a decimal.

$\frac{1}{2} \times \frac{5}{5} = \frac{5}{10} = 0.5$

$\frac{1}{2} = 0.5$

Change $\frac{3}{4}$ to a decimal.

$\frac{3}{4} \times \frac{25}{25} = \frac{75}{100} = 0.75$

$\frac{3}{4} = 0.75$

Dividing the Numerator by the Denominator

- For fractions whose denominators are not equivalent to tenths, hundredths, or thousandths, divide the numerator by the denominator.
- Add a decimal point after the numerator, and add two zeros. (Add more zeros if you are trying to find repeating decimals. For repeating decimals, be sure to indicate the digits that repeat by putting a bar over them. In this case omit the next step.)
- Write the remainder as a fraction.

Change $\frac{1}{3}$ to a decimal.

$3\overline{)1}$

$0.33\frac{1}{3} = 0.\overline{3}$

$$\begin{array}{r} 3\overline{)1.00} \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

$\frac{1}{3} = 0.\overline{3}$

LIST 24

Rules for Changing Decimals to Percents

Percent means hundredths and is denoted by the % symbol. Changing a decimal to a percent requires that you change the decimal to hundredths first. You may do this in one of two ways: (1) change the decimal to an equivalent fraction or (2) change the decimal to a percent directly. Both methods are shown below.

Changing the Decimal to a Fraction, Then to a Percent

Change 0.7 to a percent.

- Write the decimal as a fraction. $0.7 = \frac{7}{10}$
- If necessary, change the fraction to an equivalent fraction with a denominator of 100.
- Change the fraction to a percent. $\frac{7}{10} = \frac{70}{100} = 70\%$

Changing the Decimal Directly to a Percent

Change 0.7 to a percent.

- Move the decimal point two places to the right and include the percent symbol. (Writing the % sign indicates that the decimal was multiplied by 100.) $0.70 = 70\%$

LIST 25

Rules for Changing Percents to Decimals

Since *percent* means hundredths, percents can be converted directly to decimals. You can also convert percents to decimals by writing equivalent fractions. Both methods are shown below.

Changing the Percent Directly to a Decimal

- Change the percent directly to a decimal by moving the decimal point two places to the left. (This is the same as dividing by 100.)

Change each percent listed below to a decimal.

$58\% = 0.58$
$125\% = 1.25$
$2\% = 0.02$
$33\frac{1}{3}\% = 0.33\frac{1}{3}$

Changing the Percent Directly to a Decimal by Writing an Equivalent Fraction

- Write the percent as a fraction with a denominator of 100.
- Express the fraction as a decimal.

Change each percent listed below to a decimal.

$29\% = \frac{29}{100} = 0.29$

$150\% = \frac{150}{100} = 1.5$

$33\frac{1}{3}\% = \frac{33\frac{1}{3}}{100} = 0.33\frac{1}{3}$

LIST 26

Rules for Changing Fractions to Percents

There are two methods for changing fractions to percents. The first is to change the fraction to an equivalent fraction with a denominator of 100. The second, used for fractions that cannot be changed to equivalent fractions with denominators of 100, is to change the fraction first to a decimal, and then change the decimal to a percent.

Changing Fractions to Percents Using Equivalent Fractions

- ▶ If the denominator of the fraction is a factor of 100, change the fraction to an equivalent fraction with a denominator of 100. Do that by multiplying the numerator and denominator by the same number.
- ▶ Change the new fraction to a percent.

Change each fraction to a percent.

$$\frac{2}{5} \times \frac{20}{20} = \frac{40}{100} = 40\%$$

$$\frac{3}{4} \times \frac{25}{25} = \frac{75}{100} = 75\%$$

Changing Fractions to Percents Using the Decimal Method

- ▶ For fractions whose denominators are not factors of 100, divide the numerator of the fraction by its denominator.
- ▶ Add a decimal point and two zeros. (The two zeros are necessary to change the fraction to a percent, because percent means hundredths.)
- ▶ Divide. Write any remainder as a fraction.
- ▶ Change the decimal to a percent.

Change $\frac{4}{9}$ to a percent.

$$9\overline{)4.00}$$

$$0.44\frac{4}{9} = 44\frac{4}{9}\%$$

$$\begin{array}{r} 0.44\frac{4}{9} \\ 9\overline{)4.00} \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

LIST 27

Rules for Changing Percents to Fractions

The word *percent* means hundredths. $n\%$ means $n \times \frac{1}{100}$ or $n \times 0.01$. When changing percents to fractions, use the meaning of percent expressed as a fraction.

Percents can easily be changed to fractions, as the following two methods show.

Changing the Percent Directly to a Fraction

- Change the percent directly to a fraction with a denominator of 100. (The number of the percent becomes the numerator of the fraction.)
- Simplify if possible.

$$50\% = \frac{50}{100} = \frac{1}{2}$$

$$125\% = \frac{125}{100} = 1\frac{25}{100} = 1\frac{1}{4}$$

Changing the Percent When It Is a Mixed Number

- Multiply the percent by $\frac{1}{100}$.
- Simply if possible, either before you multiply or after.

Change $87\frac{1}{2}\%$ to a fraction.

$$87\frac{1}{2}\% = 87\frac{1}{2} \times \frac{1}{100} = \frac{175}{2} \times \frac{1}{100}$$

$$\frac{\cancel{175}^{7}}{2} \times \frac{1}{\cancel{100}_{4}} = \frac{7}{8}$$

$$87\frac{1}{2}\% = \frac{7}{8}$$

LIST 28

Percent Equivalents

The following chart shows the relationships between fractions, decimals, and percents.

Word Name	*Fraction*	*Decimal*	*Percent*
One-half	$\frac{1}{2}$	0.50	50%
One-fourth	$\frac{1}{4}$	0.25	25%
Three-fourths	$\frac{3}{4}$	0.75	75%
One-third	$\frac{1}{3}$	$0.33\frac{1}{3}$ or $0.\overline{3}$	$33\frac{1}{3}\%$ or $33.\overline{3}\%$
Two-thirds	$\frac{2}{3}$	$0.66\frac{2}{3}$ or $0.\overline{6}$	$66\frac{2}{3}\%$ or $66.\overline{6}\%$
One-fifth	$\frac{1}{5}$	0.20	20%
Two-fifths	$\frac{2}{5}$	0.40	40%
Three-fifths	$\frac{3}{5}$	0.60	60%
Four-fifths	$\frac{4}{5}$	0.80	80%
One-sixth	$\frac{1}{6}$	$0.16\frac{2}{3}$ or $0.1\overline{6}$	$16\frac{2}{3}\%$ or $16.\overline{6}\%$
Five-sixths	$\frac{5}{6}$	$0.83\frac{1}{3}$ or $0.8\overline{3}$	$83\frac{1}{3}\%$ or $83.\overline{3}\%$
One-eighth	$\frac{1}{8}$	$0.12\frac{1}{2}$ or 0.125	$12\frac{1}{2}\%$ or 12.5%
Three-eighths	$\frac{3}{8}$	$0.37\frac{1}{2}$ or 0.375	$37\frac{1}{2}\%$ or 37.5%
Five-eighths	$\frac{5}{8}$	$0.62\frac{1}{2}$ or 0.625	$62\frac{1}{2}\%$ or 62.5%
Seven-eighths	$\frac{7}{8}$	$0.87\frac{1}{2}$ or 0.875	$87\frac{1}{2}\%$ or 87.5%
One-ninth	$\frac{1}{9}$	$0.11\frac{1}{9}$ or $0.\overline{1}$	$11\frac{1}{9}\%$ or $11.\overline{1}\%$
Two-ninths	$\frac{2}{9}$	$0.22\frac{2}{9}$ or $0.\overline{2}$	$22\frac{2}{9}\%$ or $22.\overline{2}\%$
Four-ninths	$\frac{4}{9}$	$0.44\frac{4}{9}$ or $0.\overline{4}$	$44\frac{4}{9}\%$ or $44.\overline{4}\%$
Five-ninths	$\frac{5}{9}$	$0.55\frac{5}{9}$ or $0.\overline{5}$	$55\frac{5}{9}\%$ or $55.\overline{5}\%$

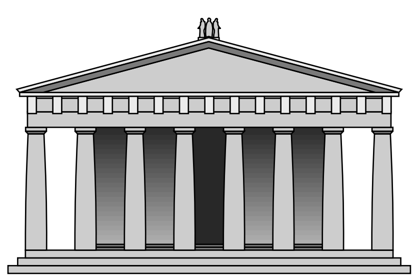
LIST 28
(Continued)

Seven-ninths	$\frac{7}{9}$	$0.77\frac{7}{9}$ or $0.\overline{7}$	$77\frac{7}{9}\%$ or $77.\overline{7}\%$
Eight-ninths	$\frac{8}{9}$	$0.88\frac{8}{9}$ or $0.\overline{8}$	$88\frac{8}{9}\%$ or $88.\overline{8}\%$
One-tenth	$\frac{1}{10}$	0.10	10%
Three-tenths	$\frac{3}{10}$	0.30	30%
Seven-tenths	$\frac{7}{10}$	0.70	70%
Nine-tenths	$\frac{9}{10}$	0.90	90%
One	1	1.00	100%

LIST 29
Rules for Solving Proportions

A *proportion* is a statement that two ratios are equal. Proportions can be helpful in solving word problems, particularly those involving percents.

	Solve for *n*.
▶ Write the proportion.	$\frac{n}{4} = \frac{15}{6}$
▶ Show the cross products of the proportion.	$4 \times 15 = 6 \times n$
▶ Find the products.	$60 = 6n$
▶ Divide both sides of the equation by the coefficient of *n*.	$\frac{60}{6} = \frac{6n}{6}$
	$10 = n$



LIST 30

Rules for Finding Percents of Numbers

The percent of a number can be found in one of three ways. The three are detailed below.

The Decimal Method

- ▶ Change the percent to an equivalent decimal.
- ▶ Multiply the decimal by the number.

Find 20% of 80.

20% = 0.2

$$\begin{array}{r} 80 \\ \underline{\times 0.2} \\ 16.0 \end{array}$$

20% of 80 = 16

The Fraction Method

- ▶ Change the percent to an equivalent fraction.
- ▶ Simplify if possible.
- ▶ Multiply the fraction by the number.
- ▶ Simplify if possible.

Find 20% of 80.

$$20\% = \frac{20}{100} = \frac{1}{5}$$

$$\frac{1}{\cancel{5}_1} \times \frac{\cancel{80}^{16}}{1} = \frac{16}{1} = 16$$

20% of 80 = 16

The Proportion Method

- ▶ Write a proportion using this form:

$$\frac{Part}{Base} = \frac{Percent}{100}$$

Note that in the expression, the number after the word *of* is the base.

- ▶ Show the cross products of the equation.
- ▶ Find the products.
- ▶ Divide both sides of the equation by the coefficient of n.

Find 20% of 80.

$$\frac{n}{80} = \frac{20}{100}$$

$$80 \times 20 = 100 \times n$$

$$1600 = 100n$$

$$\frac{1600}{100} = \frac{100n}{100}$$

$$16 = n$$

20% of 80 = 16

LIST 31

Rules for Finding Percent and the Base

Finding what percent a number is of another number and finding a number when a certain percent of it is given are confusing operations for many students. The following lists make the steps of each process clear.

Finding the Percent

What percent of 64 is 16?

- Write a proportion using this form:

$$\frac{Part}{Base} = \frac{Percent}{100}$$

$$\frac{16}{64} = \frac{n}{100}$$

Note that in the expression, the number after the word *of* is the base.

- Show the cross products of the proportion.

$16 \times 100 = 64 \times n$

- Find the products.

$1600 = 64n$

- Divide both sides by the coefficient of *n*.

$$\frac{1600}{64} = \frac{64n}{64}$$

$25 = n$

25% of 64 = 16

Finding the Base

15 is 25% of what number?

- Write a proportion using this form:

$$\frac{Part}{Base} = \frac{Percent}{100}$$

$$\frac{15}{n} = \frac{25}{100}$$

Note that the phrase "what number" after the word *of* is the base.

- Show the cross products of the proportion.

$15 \times 100 = 25 \times n$

- Find the products.

$1500 = 25n$

- Divide both sides by the coefficient of *n*.

$$\frac{1500}{25} = \frac{25n}{25}$$

$60 = n$

15 = 25% of 60

LIST 32

Rules for Operations with Integers

Integers include all positive and negative whole numbers, and zero. Because negative numbers are the opposites of positives, special rules are needed for working with them.

Adding Two Integers

- When the integers are positive, add them, and the sign remains positive.
  $^{+}4 + {}^{+}3 = {}^{+}7$
- When the integers are negative, add the absolute values. The sign remains negative. (The absolute value of any integer is its distance from 0 on the number line. The absolute value of both $^{+}4$ and $^{-}4$ is 4.)
  $^{-}4 + {}^{-}3 =$
  $|{}^{-}4| + |{}^{-}3| = 4 + 3 = 7$
  $^{-}4 + {}^{-}3 = {}^{-}7$
- When the signs of the integers are different, subtract the absolute values (the smaller from the larger), and keep the sign of the integer with the greater absolute value.
  $^{-}4 + {}^{+}9 =$
  $|9| - |{}^{-}4| = 9 - 4 = 5$
  $^{-}4 + {}^{+}9 = 5$
- When the integers are opposites, their sum is 0.
  $^{-}4 + {}^{+}4 = 0$

Adding More than Two Integers

- *Method One:* Work from left to right and add integers two at a time, following the rules above.
  $^{-}4 + ({}^{+}6) + ({}^{-}7) + ({}^{+}2) =$
  $^{+}2 + ({}^{-}7) + ({}^{+}2) =$
  $^{-}5 + ({}^{+}2) = {}^{-}3$
- *Method Two:* Add all positive integers, and then add all the negative integers, following the rules above. Find the sum of your answers.
  $^{-}4 + ({}^{+}6) + ({}^{-}7) + ({}^{+}2) =$
  $^{+}6 + ({}^{+}2) + ({}^{-}4) + ({}^{-}7) =$
  $^{+}8 + ({}^{-}11) = {}^{-}3$
  $^{-}4 + ({}^{+}6) + ({}^{-}7) + ({}^{+}2) = {}^{-}3$

LIST 32
(Continued)

Subtracting Integers

▶ Rewrite the problem by using the definition of subtraction:	$^{+}3-(^{+}6)={}^{+}3+(^{-}6)={}^{-}3$
$(a-b)=a+(^{-}b)$	$^{-}2-(^{-}5)={}^{-}2+(^{+}5)={}^{+}3$
▶ Follow the rules for adding integers.	$^{+}12-(^{-}4)={}^{+}12+{}^{+}4={}^{+}16$

Multiplying Two Integers

▶ Find the product of the absolute values of the numbers.	$\lvert 8\rvert=\lvert{}^{-}8\rvert=8$ $\quad\lvert 7\rvert=\lvert{}^{-}7\rvert=7$
▶ Use the correct sign:	
▪ If both integers are positive, the answer is positive.	$8\times 7=56$
▪ If both integers are negative, the answer is positive.	$^{-}8\times(^{-}7)=56$
▪ If one number you multiplied is positive and the other is negative, the answer is negative.	$^{-}8\times 7={}^{-}56$
▪ If one of the numbers is 0, the answer is 0.	$^{-}8\times 0=0$

Multiplying More Than Two Integers

▶ *Method One:* Working from left to right, multiply the integers two at a time, following the rules above.	$^{-}3\times(^{-}7)\times 2=$ $21\times 2=42$
▶ *Method Two:* Find the product of the absolute values. If all the numbers you multiply are positive, the answer is positive. If there is an odd number of negative factors, the answer is negative. If there is an even number of negative factors, the answer is positive. If any one of the factors is 0, your answer is 0.	$\lvert 3\rvert=\lvert{}^{-}3\rvert=3$ $\lvert 7\rvert=\lvert{}^{-}7\rvert=7$ $\lvert 2\rvert=\lvert{}^{-}2\rvert=2$ $^{-}3\times(^{-}7)\times 2=42$ (There are two negative factors.) $^{-}3\times 7\times 2={}^{-}42$ (There is one negative factor.) $^{-}3\times(^{-}7)\times(^{-}2)={}^{-}42$ (There are three negative factors.) $^{-}3\times(^{-}7)\times 0=0$

LIST 32

(Continued)

Dividing Integers

- Find the quotient of the absolute values. $|3| = |{}^{-}3| = 3 \qquad |21| = |{}^{-}21| = 21$
- Use the correct sign:
  - If both integers are positive, the quotient is positive. $21 \div 3 = 7$
  - If both integers are negative, the quotient is positive. ${}^{-}21 \div ({}^{-}3) = 7$
  - If the integers have different signs, the quotient is negative. ${}^{-}21 \div 3 = {}^{-}7$
- If 0 is divided by an integer, the quotient is 0. $0 \div ({}^{-}7) = 0$
- If an integer is divided by 0, the quotient is undefined. ${}^{-}7 \div 0 = \emptyset$

LIST 33

Properties of Integers

Integers have special properties. Understanding those properties can make computation easier. The Commutative Property, for example, allows you to change the order of adding or multiplying integers. The Associative Property allows you to change grouping.

In the chart below, *a, b,* and *c* are integers.

	Addition	***Multiplication***
Closure Property	$a + b$ is an integer	ab is an integer
Commutative Property	$a + b = b + a$	$ab = ba$
Associative Property	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity Property	$a + 0 = a$	$1(a) = a$
Inverse Property	$a + {}^{-}a = 0$	
Multiplication Property of Zero		$a(0) = 0$
Distributive Property	$a(b + c) = ab + bc$	

LIST 34

Rules for Finding the Average (Mean)

An *average,* also referred to as the *mean,* is a number that represents a set of numbers. Averages are useful for comparing data. There are many kinds of averages, including batting averages in baseball, average incomes, and the average grade you maintain in your math class.

A *weighted average* is an average in which one item (or items) is given more importance, or "weight," than the others. For example, a unit test might count for 50% of a student's test grade, while two chapter tests count for 25% each. In this case, the unit test would be counted twice in finding the student's test average.

Steps to finding both an average and weighted average follow.

Finding an Average

- Add all the items you need to average.
- Divide the sum by the total number of items you added.
- If necessary, add a decimal point and zeros. (It is usually not necessary to work out problems past the hundredths place. Round off to the nearest tenth.)

Find the average of 92, 84, and 87.

$92 + 84 + 87 = 263$

$$3\overline{)263.00} = 87.66 \approx 87.7$$

Finding a Weighted Average

- Multiply each item by its weight. For example, if an item is to be counted twice, it will be multiplied by 2. If an item is to be counted three times, it is multiplied by 3. (An item that is not weighted is multiplied by one.)
- Find the sum of the items and weighted items.
  - Find the sum of the weights.
  - Divide the sum of the items by the sum of the weights.
  - If necessary, add a decimal point and zeros, and round your answer to the desired place.

Find the average of 92, 84, and 87, if 87 is weighted twice.

$92 \times 1 = 92$
$84 \times 1 = 84$
$87 \times 2 = 174$

Sum of the numbers = 350
Sum of the weights = 4

$$4\overline{)350.0} = 87.5$$

LIST 35

Rules for Rounding Numbers

Rounding is an important estimation skill. When you go to the grocery store, for example, it can be helpful to round off and estimate the cost of the items you are buying before you actually buy them.

Steps for Rounding Up

- Circle the digit that is to be rounded.
- If the digit to the right is 5 or greater, round the "circled" digit up by adding 1 to it.
- Change all digits to the right of the rounded digit to zeros.
- Delete any zeros that are not placeholders.

Round 3854 to the nearest thousand.

③854 ≈ 4000

Round 2.874 to the nearest tenth.

2.⑧74 ≈ 2.9~~00~~

≈ 2.9

Steps for Rounding Down

- Circle the digit that is to be rounded.
- If the digit to the right is less than 5, the "circled" digit stays the same.
- Change all digits to the right of the rounded digit to zeros.
- Delete any zeros that are not placeholders.

Round 3512 to the nearest hundred.

3⑤12 ≈ 3500

Round 2.874 to the nearest hundredth.

2.8⑦4 ≈ 2.87~~0~~

≈ 2.87

When 9 Is in the Place You Are Rounding

- Circle the digit that is to be rounded, which in this case is 9.
- If the digit to the right is 5 or greater, round the "circled" 9 up by adding 1 to it. If the digit to the right is less than 5, follow the steps for rounding down.
- Since 9 + 1 = 10, write 0 in place of the 9 and add 1 to the digit to the left.
- Change all numbers to the right of the rounded number to zeros.
- Delete any zeros that are not placeholders.

Round 3985 to the nearest hundred.

3⑨85 ≈ 4000

Round 2.897 to the nearest hundredth.

2.8⑨7 ≈ 2.90~~0~~

≈ 2.90

LIST 36

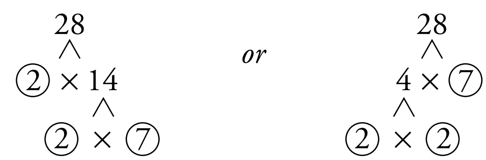
Rules for Finding Prime Factorizations

The *prime factorization* of a number means expressing the number as a product of prime numbers. The following lists are helpful for finding prime factorizations.

Finding the Prime Factorization through a Factor Tree

- ▶ Find any pair of factors of the number.
- ▶ Circle the prime factor(s).
- ▶ Find any other factors.
- ▶ Circle the prime factor(s).
- ▶ When you have found all the prime factors, write them out as a product.
- ▶ Write the product using exponents.

Find the prime factorization of 28.



$2 \times 2 \times 7$ $\qquad$ $2 \times 2 \times 7$

$2^2 \times 7$ $\qquad$ $2^2 \times 7$

$2^2 \times 7$ is the prime factorization of 28.

Finding the Prime Factorization by Dividing by Primes

- ▶ Divide by 2 if possible, until the quotient is no longer divisible by 2.
- ▶ Divide by 3 if possible, until the quotient is no longer divisible by 3.
- ▶ Divide by 5 if possible, until the quotient is no longer divisible by 5.
- ▶ Continue this pattern, dividing by prime numbers only, until the quotient is prime.
- ▶ Write the product of the divisors and quotient as a product.
- ▶ Write the product of the divisors and quotient using exponents.

Find the prime factorization of 140.

$2\overline{)140}$ = 70 $\quad$ Divide by 2.

$2\overline{)70}$ = 35 $\quad$ Divide by 2.

$5\overline{)35}$ = 7 $\quad$ Cannot divide by 3; so divide by 5. 7 is prime.

$2 \times 2 \times 5 \times 7$

$2^2 \times 5 \times 7$ is the prime factorization of 140.

LIST 37

Scientific Notation

Scientific notation is used to express very large or very small numbers. For example, the mean distance of Pluto from the sun is about 3,670,000,000 miles. That is a jawbreaker to say and even worse to write. Using scientific notation, the number can be expressed as 3.67×10^9.

In writing scientific notation for large numbers, follow these rules:

- ▶ The first factor must be greater than or equal to 1 and less than 10.
- ▶ The second factor must be a power of 10 expressed in exponential form.
- ▶ To write numbers with exponents, count the number of places to the right of the first nonzero number in standard form. Use the number of places as the exponent. In the case of 3,670,000,000, the first nonzero number is 3. Since 9 digits are to the right of the 3, 9 is the exponent. Therefore, $3,670,000,000 = 3.67 \times 10^9$.

In writing very small numbers, follow these rules:

- ▶ The first factor must be greater than or equal to 1 and less than 10.
- ▶ The second factor must be a negative power of 10 expressed in exponential form.
- ▶ To write numbers with a negative exponent, count the number of places to the right of the decimal point, up to and *including* the first nonzero number. Use the number of places as the negative exponent. For example, $0.00079 = 7.9 \times 10^{-4}$.

LIST 38

Bases

The *base* of any number system is the number of different symbols used. The Arabic system, which we use, is a base ten system because ten symbols are used in writing numerals: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. It is thought that the base ten system reflects our eight fingers and two thumbs. Primitive people found it easier to count that way.

Number systems can be based on any amount of symbols, however. The binary system (base two), for example, has only two digits, 0 and 1. Computers perform their calculations in binary codes. Pulses of electrical energy representing 0 and 1 turn tiny switches on and off in microcircuits.

Following is a comparison of the numerals of the base ten system with those of base two, base five, and base eight.

Base Ten	***Base Two***	***Base Five***	***Base Eight***
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	10	5
6	110	11	6
7	111	12	7
8	1,000	13	10
9	1,001	14	11
10	1,010	20	12
11	1,011	21	13
12	1,100	22	14
13	1,101	23	15
14	1,110	24	16
15	1,111	30	17
16	10,000	31	20
17	10,001	32	21
18	10,010	33	22
19	10,011	34	23
20	10,100	40	24

LIST 39

Big Numbers

Most of us can comprehend numbers up to around a hundred thousand. That is about how many people a big football stadium can seat. If we use our imaginations and envision five stadiums that large side by side, all filled, we can grasp a half-million. Ten such filled stadiums add up to 1 million, but after that, the numbers soon become too large to imagine. The following list shows numbers up to a googolplex, which is big even for big numbers. To eliminate the appearance of a page covered with zeros, we use exponents to express numbers after 1 decillion.

Word Name	***Digits***
One million	1,000,000
One billion	1,000,000,000
One trillion	1,000,000,000,000
One quadrillion	1,000,000,000,000,000
One quintillion	1,000,000,000,000,000,000
One sextillion	1,000,000,000,000,000,000,000
One septillion	1,000,000,000,000,000,000,000,000
One octillion	1,000,000,000,000,000,000,000,000,000
One nonillion	1,000,000,000,000,000,000,000,000,000,000
One decillion	1,000,000,000,000,000,000,000,000,000,000,000
One undecillion	10^{36}
One duodecillion	10^{39}
One tredecillion	10^{42}
One quattuordillion	10^{45}
One quindecillion	10^{48}
One sexdecillion	10^{51}
One septendecillion	10^{54}
One octodecillion	10^{57}
One novemdecillion	10^{60}
One vigintillion	10^{63}
One googol	10^{100}
One googolplex	10^{googol}

Are numbers infinite? A simple test suggests that they are. Try imagining the biggest number you can. No matter how big it is, it is not the biggest. You can always add at least 1 to it.

LIST 40

Mathematical Signs and Symbols

The following list provides the signs and symbols used most often in mathematics.

$+$	addition, plus, positive (if the sign is on the upper left-hand side of a number)
$-$	subtraction, minus, opposite of, negative (if the sign is on the upper left-hand side of a number)
$\times$	multiplication, multiply by, times
$\bullet$	multiplication, multiply by, times
$*$	multiplication, multiply by, times
$\div$	division, divide by
$\frac{x}{y}$	division
/	division
$=$	equals, is equal to
$\approx$	is approximately equal to
$\equiv$	equivalence
$\neq$	is not equal to
$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to
$\therefore$	therefore
∞	infinity
$	dollar sign
¢	cents sign
#	number or pounds
%	percent
:	is to
Δ	triangle
$\square$	square
$\angle ABC$	angle *ABC*
$m\angle ABC$	the measure of angle *ABC*
⌉	right angle
$\overset{\frown}{AB}$	arc *AB*
$\parallel$	is parallel to
$\perp$	is perpendicular to

LIST 40

(Continued)

$\overrightarrow{AB}$	ray AB
$\overline{AB}$	segment AB
$m\angle AB$	measure of line segment AB
$\overleftrightarrow{AB}$	line AB
π	pi, which is about 3.14
$\cong$	is congruent to
$\sim$	is similar to
$\propto$	is proportional to
( )	parentheses, grouping symbol
[ ]	braces, grouping symbol
$\pm$	plus or minus
$\|n\|$	absolute value of n
(x,y)	ordered pair
x^a	x to the a power
$\sqrt{\ }$	positive square root
$-\sqrt{\ }$	negative square root
$f(x)$	f of x, the value of f at x
{ }	indicates a set, or is used as a grouping symbol
ϕ	empty set
$\in$	is an element of
$\cap$	intersection
$\cup$	union
$P(E)$	probability of event E
$n!$	n factorial
nPr	number of permutations of n items, taken r at a time
nCr	number of combinations of n items, taken r at a time
θ	theta, an angle in standard position
Sin θ	sine of θ
Cos θ	cosine of θ
Tan θ	tangent of θ
Cot θ	cotangent of θ
Sec θ	secant of θ
Csc θ	cosecant of θ