CHAPTER **1**

MATHEMATICS USED IN ELECTROMAGNETISM

1.1 INTRODUCTION

Electromagnetics is the branch of physical science which deals with electric and magnetic fields and their interactions with each other and with physical objects. We normally think of electricity as the flow of electrons in a conductor, or as static electricity, or lightning. We think of magnetism as in permanent magnets, the earth's magnetism, compass needles, electromagnets, patterns in iron filings, and the like. The physical elements, electrons and charged particles on the one hand, and magnetic materials on the other, are only a portion of the phenomenon. More powerful are the invisible forces which draw objects together or repel them. These forces allow powerful electric machines, motors, and generators, to transform inchoate energy into useful form. They allow the wireless transmission of information, and thus energy, over distances great or small. Although not treated here, information is a form of energy, and thus intimately intertwined with all electromagnetic phenomena. Electromagnetic fields do not exist without physical manifestations, or the physical manifestations without the invisible fields. In order to understand these complex interactions, a special form of advanced mathematics had to be developed. More complex than the calculus used to determine the motion of physical bodies. This mathematics covers diffuse energy distributed over space, and within confined areas. Its first development was with the study of the distribution of heat within a solid, the flow of fluids in pipes, and steam in all its manifestations. When it was discovered that electricity was very similar to a fluid in its behavior, the mathematics was then applied to electrical currents. The distribution of heat inside a solid became the distribution of electric and magnetic fields and variations in current density.

The mathematics and symbolism used in electromagnetics are often unfamiliar and daunting to the uninitiated. With long partial differential equations, hard-to-visualize fields and vectors, and unusual symbols such as ∇, ∇^2, ∇, $\nabla \times \partial$, $\oint$, $\oiint$, $\oiiint$, etc., the subject is often prickly and difficult to approach. This chapter is intended to provide a summary of some of the fundamental mathematical concepts used in electromagnetic field theory and the symbolism used to convey them. In this way, the reader who has been exposed to them in the past will

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have a refresher, while the beginner will have a grasp of what is being discussed, and all readers will have a page to turn to for reference during the course of this book. This chapter is not intended to take the place of a mathematical textbook, but only as an introduction to the concepts which may be found therein.

1.2   NUMBERS

The fundamental basis on which the calculation of physical quantities rests is the concept of a number. Beginning with the numbers used for counting, many types of numbers have been discovered, leading to an infinite complexity and array of concepts. A number may be considered the object upon which the mathematical procedure operates. Numbers are objects in their own right, not just concepts used to count or measure physical objects. This is the reason that mathematics describes reality so well; it is also reality, and follows the same rules. The mathematics and physics of electrical theory is based on numbers. It is not possible to understand this theory in full without a foundation in real and complex numbers, vectors and scalars, coordinate systems, and other fundamental objects of mathematics. Indeed, the fundamental intuition necessary for safety relies on being able to judge the magnitude of a possible hazard, whether voltage, current, temperature, or force. While there is an extensive and well-developed science of number theory, for the purposes of this introduction, only a few significant concepts and definitions are needed. The numbers are diagrammed in Figure 1.1, the number line.

Natural numbers or counting numbers are the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. Natural numbers are based on the principles of similarity and multiplicity. Objects which are like each other can be grouped together and counted. Grouping together is recognizing a set of objects. Objects does not have to exist, either physically or mentally, to be counted. A number can be a number, in and of itself. The minimum degree of similarity is that the objects are identifiable. Multiplicity means that there is more than one object, thus counting is possible.

Integers are the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$. Adding zero to the set was one of the significant discoveries of ancient times, without which mathematics as we know it could not exist.

Positive integers are the same as the natural numbers. The search for clarity can lead to multiple definitions. Here the definition is descriptive, rather than functional, as in "counting."

Negative integers are the negatives of the natural numbers $\{-1, -2, -3, \ldots\}$. The opposite of positive is negative, leading to another descriptive category.

Nonnegative integers are natural numbers plus zero, the set $\{0, 1, 2, 3, \ldots\}$. And the opposite of negative may also be nonnegative, rather than positive. There can also be "non-positive" integers.

Rational numbers are the set $\mathbb{Q}$ of numbers which can be expressed as the ratio of two integers, a/b, where $b \neq 0$. When written as decimals, the digits either
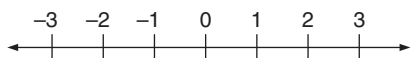

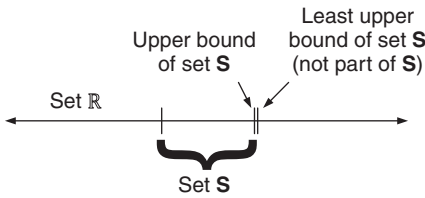Figure 1.1   The number line.

Figure 1.2   The real numbers on the number line.

terminate or are repeating. Rational numbers are the numbers used most often in computations. The longer decimals must be terminated somewhere when calculations are performed, resulting in truncation error. This is why it is best to keep ratios intact as long as possible when simplifying equations. Rational numbers are an example of the paradox of infinity. There are an infinite number of integers, extending to positive and negative infinity. Technically, one could say that there are twice infinity plus one (for zero) integers. The number of rational numbers would be determined from each integer divided by each other integer, except zero, resulting in a double infinite number of infinites.

Real numbers are the set $\mathbb{R}$ of numbers, such as the points on a line, where for every subset **S** which has an upper bound, there is a least upper bound (not a member of **S**) which is a member of $\mathbb{R}$. This means that for any set of numbers which does not increase to infinity, thus possessing an upper bound, there is one number which is not a member of the set but is larger than all members of the set, see Figure 1.2. This least upper bound is infinitely close to the largest number in the set. Thus the real numbers encompass all points on the number line.

Irrational numbers, such as $\pi, \varepsilon, \sqrt{2}$, are the set of all real numbers which are not rational numbers. They cannot be expressed as the ratio of two integers, a/b, where $b \neq 0$. When written as decimals, the digits do not terminate and do not repeat. Irrational numbers cannot be written as the quotient of two integers. In order to calculate using irrational numbers, they must be rounded to rational numbers. This is one reason why it is best to simplify an equation as much as possible before computation, so that the irrational numbers are approximated only in the final result. This results in greatest accuracy. Irrational numbers fill in the gaps on the number line between rational numbers. There is thus an infinite number of irrational numbers between each rational number. The number of infinities is basically beyond comprehension.

The *differential dx* of a number, x, is the smallest possible difference Δx as Δx is reduced toward zero. The concept of differential will be followed through different kinds of numbers, so that differentials in multiple dimensions can be defined.

Imaginary numbers are numbers whose square is a negative real number. An imaginary number is the product of a real number and the *imaginary unit*, usually called simply the "square root of minus one" expressed in mathematical notation as "i" where

$$i = \sqrt{-1} \tag{1.1}$$

A typical imaginary number would be bi, where b is a real number. Imaginary numbers are expressed as the product of a real number and the imaginary unit, not as a

single symbol, unless the real number is one. In mathematical notation "*i*" is always placed after the number it multiplies. In electrical engineering notation, the imaginary unit is represented by "*j*":

$$j = \sqrt{-1} \tag{1.2}$$

Unlike *i*, which is always placed after the number it multiplies, *j* is always placed before the number it multiplies. A typical imaginary number in electrical engineering format would be *jb*. When $b = 1$, the letter symbol alone is usually shown, as in $1i = i$ and $j1 = j$. The powers of *j* are important in many calculations:

$$j^0 = 1 \tag{1.3}$$

$$j^1 = \sqrt{-1} \tag{1.4}$$

$$j^2 = -1 \tag{1.5}$$

$$j^3 = -j \tag{1.6}$$

$$j^4 = j^0 \tag{1.7}$$

These powers then repeat in groups of four.

    Angles in geometry are formed by two rays (half lines) called the *sides* which meet at a point called the *vertex*. Angles are formally unitless, but in practice are measured in radians or degrees. Angles are measured in the counterclockwise direction from the positive real axis. Angle measurements and definitions are shown in Figure 1.3. One radian (rad) is the angle subtended by an arc having a length equal to one radius of the circle. Since the circumference of a circle of radius *r* is $2\pi r$, there are
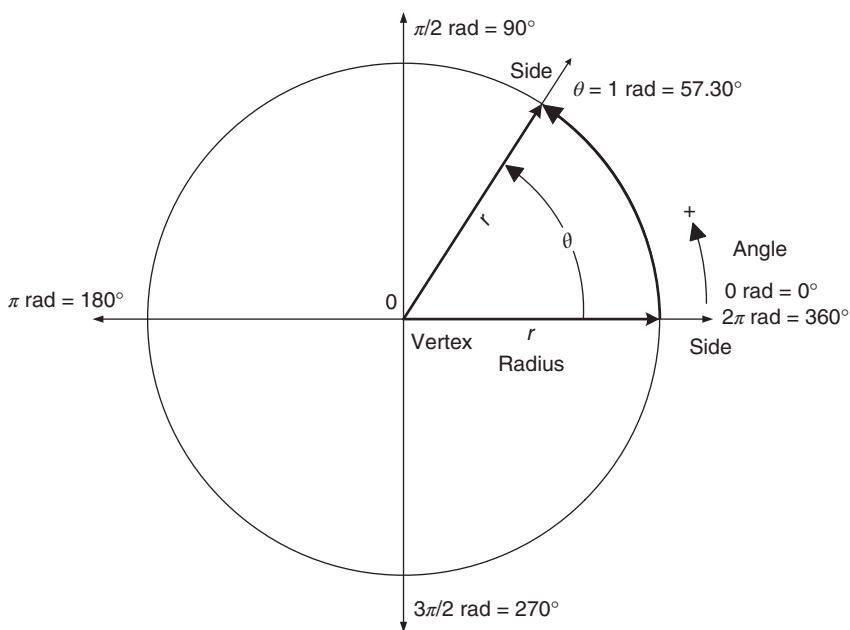

Figure 1.3   Angular measure in radians and degrees.

2π radians in a complete circle. One degree (°) is the angle subtended by one 360th part of the circumference, thus there are 360° in a complete circle. The conversion factor between degrees and radians is

$$1\,\text{rad} = \frac{1}{2\pi} = 0.159 \quad \text{circumference} = 0.159 \times 360° = 57.30° \qquad (1.8)$$

Complex numbers are the set $\mathbb{C}$ of numbers defined as the sum of one real and one imaginary number:

$$Z = a + jb \qquad (1.9)$$

When expressed in this format, complex numbers are said to be in *rectangular form*. The process of definition as the sum of a real and an imaginary may be somewhat circular. A complex number is an object which *may* be expressed as the sum of a real and an imaginary number. This does not mean that it must be, or even needs to be. The two parts of a complex number may be extracted by the functions "Re" for real and "Im" for imaginary:

$$a = \text{Re}(Z) \qquad (1.10)$$
$$b = \text{Im}(Z) \qquad (1.11)$$

Complex numbers may be plotted in the *complex plane* where the horizontal axis is the real axis and the vertical axis is the imaginary axis. The *rectangular form* of a complex number $a + jb$ is shown graphically as a point on the complex plane. Figure 1.4 shows the complex number $3 + j4$ plotted in the complex plane.

The object which is a complex number can also be expressed radially, based on a circle instead of a rectangle. The *polar form* of a complex number is represented
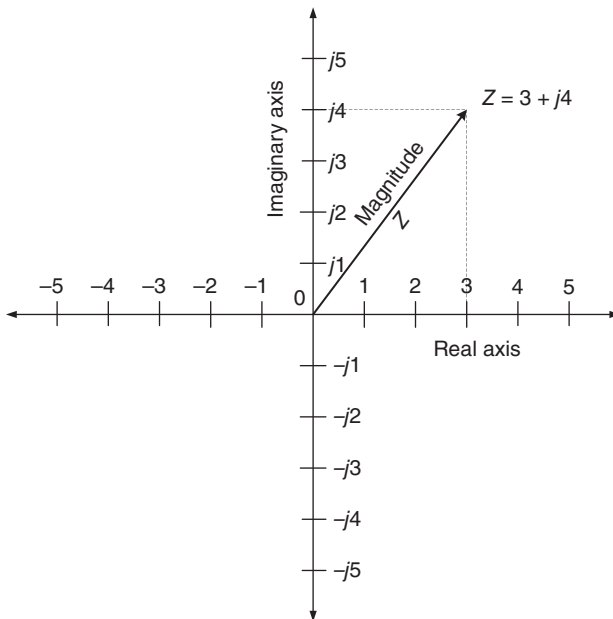

Figure 1.4   Complex numbers in rectangular form.

by a magnitude and an angle. The *magnitude* (Mag) of a complex number Z is the length of line OZ:

$$\text{Mag}(Z) = |Z| = \sqrt{a^2 + b^2} \qquad (1.12)$$

The vertical bar expression is a shorthand for magnitude. This equation is the first of two equations which change the expression of a complex number from rectangular to polar form. The second one extracts the angle. The *argument* (Arg) of a complex number is the angle between the real axis and the line OZ:

$$\text{Arg}(Z) = \theta = \tan^{-1}\frac{b}{a} \qquad (1.13)$$

For the example in Figure 1.4,

$$|Z| = \sqrt{3^2 + 4^2} = 5$$

and

$$\theta = \tan^{-1}\frac{4}{3} = 53.13°$$

The polar form of a complex number is written as

$$Z = |Z|\angle\theta = |Z|(\cos\theta + j\sin\theta) \qquad (1.14)$$

Figure 1.5 shows the complex number $3 + j4$ in polar form as $5\angle 53.13°$.

The *exponential form* of complex numbers is a variation on the polar form. It is derived from the rectangular form using Euler's formula:

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad (1.15)$$
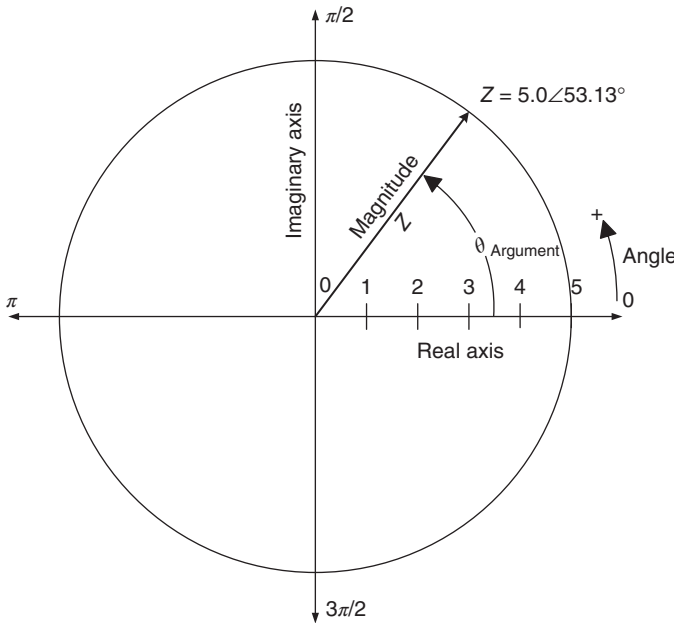
where θ is the argument in radians.

Figure 1.5   Complex numbers in polar form.

The irrational number $e = 2.718281828\ldots$ is Euler's number, the basis of the natural logarithms, named in honor of the Swiss mathematician Leonhard Euler. The expression $e^{j\theta}$ describes a circle with a radius of 1. A circular trajectory is followed as θ is increased from 0 to 2π. The natural logarithms are so called because this number is observed in nature in many forms. And yet, one cannot *measure* e to the degree of accuracy, to which it is known. The number e is its own entity, and can be calculated to any degree of precision, regardless of what is known about nature.

Then

$$z\, e^{j\theta} = a + jb \tag{1.16}$$

and

$$a = z \cos \theta \tag{1.17}$$
$$b = z \sin \theta \tag{1.18}$$

Vectors are geometric quantities which possess both magnitude and direction. Vectors are primarily used for the mathematical representation of forces. A mechanical force will have a certain magnitude and direction. The gravitational force vector points to the center of mass of the object, with the magnitude of weight. The vector of a hammer's force will be down onto the head of the nail. The vector of wind velocity will have a direction by the compass (N–S–E–W) and a magnitude of speed. Electromagnetic vectors are the same. An electric field vector will have a magnitude of volts per meter and direction to (or from) the point charge. A magnetic field vector will go from north to south with a magnitude of tesla. An electron or moving charge will have a force vector perpendicular to both the electric and magnetic fields, with a magnitude proportional to their product. This is what produces generator or motor action.

Vectors belonging to the set $\mathbb{R}^n$ exist in n dimensions. The *unit vector* of the one-dimensional rectangular coordinate system $\mathbb{R}^1$ is the vector $\hat{\mathbf{i}}$ (designated in lowercase with a cap) of magnitude 1 in the direction of the x-axis, shown in Figure 1.6. Vectors are designated as uppercase bold face letters. A one-dimensional vector can be written as

$$\mathbf{A} = a\hat{\mathbf{i}} \tag{1.19}$$

While complex numbers exhibit characteristics also belonging to vectors, they are not, strictly speaking, the type of vectors we are talking about. This is because geometrical vectors do not contain imaginary numbers. In the wider mathematical sense, one can, and does, have vectors containing imaginary numbers. *Phasors* are the vector representations of complex numbers. A phasor is drawn as a directed line on the complex plane from $0 + j0$ to the complex number Z at point $a + jb$. The phasor representation is shown as the arrow in Figures 1.4 and 1.5. Phasors are used to represent the magnitude and phase angle of sinusoidal waveforms, such as voltage and currents. The angle θ in $Z = z\angle\theta$ is replaced by the angle ωt, where $\omega = 2\pi f$, where f is the frequency of the sinusoidal wave.
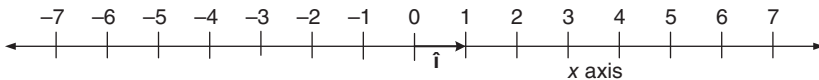

Figure 1.6   One-dimensional vectors.

Scalar is another name for a real number, especially as in contrast to a vector.

The *norm* $\|\mathbf{A}\|$ of the vector $\mathbf{A}$ in the set $\mathbb{R}^1$ is defined as the magnitude or length of the vector. For the one-dimensional vector $\mathbf{A}$, the norm is a. This appears to be redundant, saying that the magnitude of a number is the number itself. In mathematics, the definitions must be precise and even apply to the cases of the obvious.

The *unit vectors* of the two-dimensional rectangular coordinate system $\mathbb{R}^2$ are the vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ (designated in lowercase with a cap) of magnitude 1 in the direction of the axes x and y. Rectangular coordinates are often called *Cartesian coordinates* in honor of their discoverer René Descartes. A two-dimensional vector can be written as the ordered pair (a, b) or as the sum of vectors $a\hat{\mathbf{i}} + b\hat{\mathbf{j}}$. Thus the unit vectors can be written as:

$$\hat{\mathbf{i}} = (1, 0) \tag{1.20}$$

$$\hat{\mathbf{j}} = (0, 1) \tag{1.21}$$

The *origin* of a two-dimensional Cartesian coordinate system is the point $(0, 0)$. The origin of an n-dimensional Cartesian coordinate system is the n-dimensional point $(0, 0, \dots, 0)$. The origin may seem trivial, but the placement of the origin in a logical location will greatly simplify the solution of many a geometrical problem.

The *norm* $\|\mathbf{A}\|$ of the vector $\mathbf{A}$ in the set $\mathbb{R}^2$ is defined as the magnitude or length of the vector. This is comparable to the magnitude of a complex number. The magnitude of vector (a, b) is

$$\mathbf{A} = \|\mathbf{A}\| = \sqrt{a^2 + b^2} \tag{1.22}$$

The magnitude of vector $(4, 5)$ is calculated as

$$\mathbf{A} = \sqrt{4^2 + 5^2} = 6.40$$

Figure 1.7 shows the vector $\mathbf{A} = (4, 5)$. In the rectangular plane, a *differential length vector* $\mathbf{dl}$ may be derived from the differential lengths of the axes dx and dy as

$$\mathbf{dl} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} \tag{1.23}$$

In the rectangular plane, a *differential area vector* $\mathbf{dS}$ may be derived from the differential lengths dx and dy as

$$\mathbf{dS} = dx\hat{\mathbf{i}}\,dy\hat{\mathbf{j}} \tag{1.24}$$

The differential area vector of dx and dy will point in the vertical, or z-direction.

While logical and easy to analyze, rectangular formulations are inconvenient for many geometrical arrangements. In order to make calculations easier, or even possible, coordinate systems often need to be adjusted to suit the physical arrangement being investigated. Much geometry is circular, such as the cross-section of a wire or of a rotating machine, motor, or generator. It is therefore much simpler to consider the fields and vectors based on radius and angle than in x and y. The *polar form* of a two-dimensional vector which would be (a, b) in Cartesian coordinates is defined as
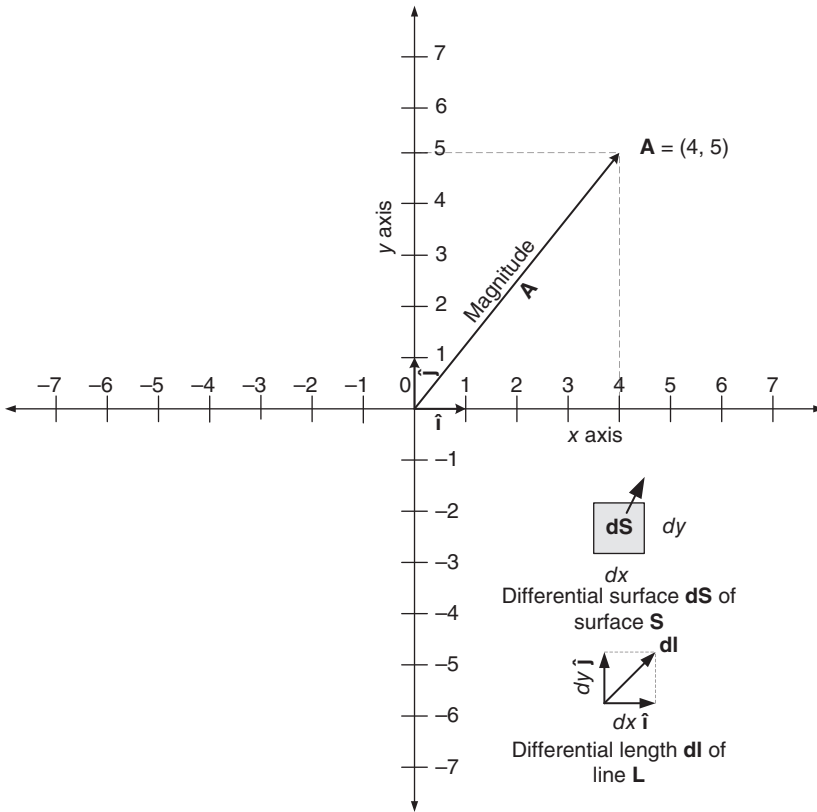
$$\mathbf{A} = (r, \phi) \tag{1.25}$$

Figure 1.7   Two-dimensional vectors in rectangular form.

where r is the radius and ϕ is the angle of the vector relative to the positive x-axis. Angles are measured in the counterclockwise direction from the positive x-axis. The polar coordinate system is shown in Figure 1.8.

The unit vectors of the polar coordinate system are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$, but unlike the Cartesian unit vectors, they are not on fixed axes, but are located at the endpoint of a vector $\mathbf{A}$ whose starting point is the origin. Thus their location changes with magnitude $\mathbf{A}$ and angle ϕ. The radial unit vector $\hat{\mathbf{r}}$ touches the end point of the vector $\mathbf{A}$ and is normal (perpendicular) to a circle C with its center in the origin. The angular unit vector $\hat{\boldsymbol{\phi}}$ is normal to the vector $\mathbf{A}$, with its starting point at the end of $\mathbf{A}$ and is tangential to the circle C, in the direction of increasing angle. The polar unit vectors can be expressed in terms of the Cartesian unit vectors:

$$\hat{\mathbf{r}} = \cos \phi \hat{\mathbf{i}} + \sin \phi \hat{\mathbf{j}} \qquad (1.26)$$

$$\hat{\boldsymbol{\phi}} = -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}} \qquad (1.27)$$
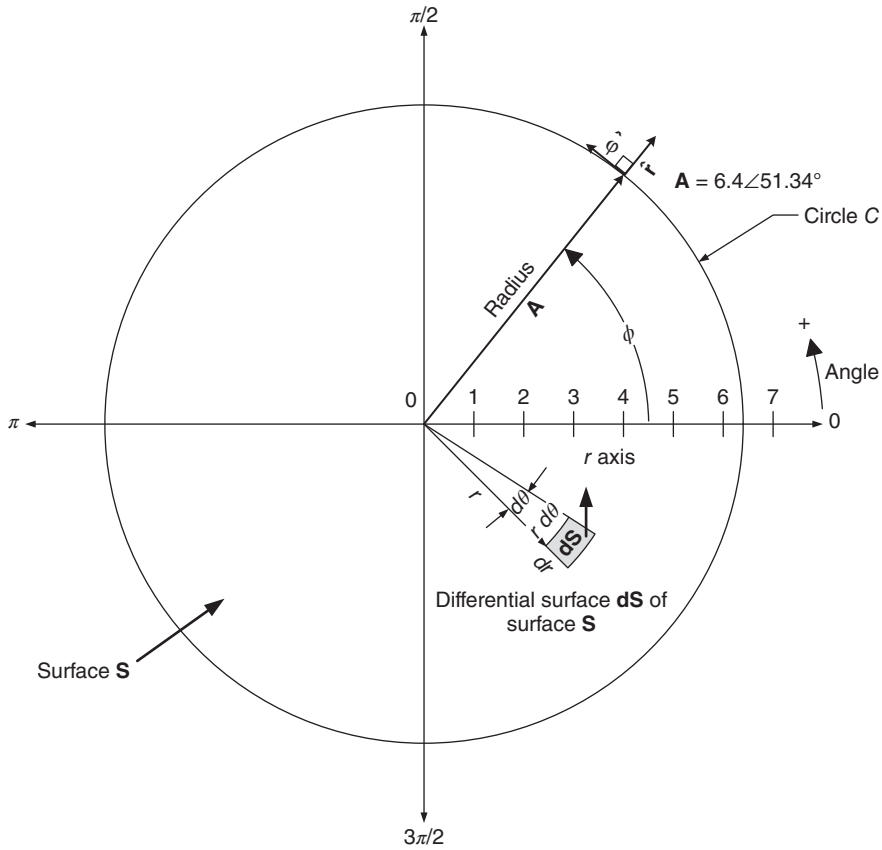
Figure 1.8   Two-dimensional vectors in polar form.

The components of the vector (a, b) converted to polar coordinates (r, ϕ) are

$$r = \sqrt{a^2 + b^2} \tag{1.28}$$

$$\phi = \tan^{-1}\frac{b}{a} \tag{1.29}$$

The Cartesian unit vectors are expressed in polar form as

$$\hat{\mathbf{i}} = \cos\phi\hat{\mathbf{r}} - \sin\phi\hat{\boldsymbol{\phi}} \tag{1.30}$$

$$\hat{\mathbf{j}} = \sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}} \tag{1.31}$$

The components of the vector (r, ϕ) converted to rectangular coordinates (a, b) are

$$a = r\cos\phi \tag{1.32}$$

$$b = r\sin\phi \tag{1.33}$$

The vector $\mathbf{A} = (4, 5)$, which has been plotted in Cartesian coordinates in Figure 1.7, has the following polar coordinates:

$$r = \sqrt{4^2 + 5^2} = 6.40$$
$$\phi = \tan^{-1}\frac{5}{4} = 51.34°$$

A plot of the vector $\mathbf{A} = (6.40, 51.34°)$ in polar coordinates is shown in Figure 1.8.

In the polar plane, *differential area* $\mathbf{dS}$ may be defined from the differential length dr and the differential angle $d\theta$ as

$$\mathbf{dS} = r dr d\theta \tag{1.34}$$

As with the rectangular formulation, $\mathbf{dS}$ is a unit vector in the vertical or z-direction.

Cartesian coordinates in three dimensions are members of the set $\mathbb{R}^3$ and are similar to those in two dimensions, with the addition of the vertical z-axis and the unit vector $\hat{\mathbf{k}}$ in the positive z-direction. We have already entered into the third dimension with the differential surface vectors. The unit vectors can be written as

$$\hat{\mathbf{i}} = (1, \ 0, \ 0) \tag{1.35}$$
$$\hat{\mathbf{j}} = (0, \ 1, \ 0) \tag{1.36}$$
$$\hat{\mathbf{k}} = (0, \ 1, \ 0) \tag{1.37}$$

The Cartesian coordinates of a three-dimensional vector (a, b, c) are defined as $B = (a, b, c)$ where a and b are the x and y dimensions of the projection of the vector $\mathbf{B}$ on the x–y plane and c is the height of the end of the vector $\mathbf{B}$ above the x–y plane. The vector $\mathbf{B} = (4, 5, 6)$ is shown in Figure 1.9, the vector $\mathbf{A} = (4, 5)$ is the projection of the vector $\mathbf{B}$ on the x–y plane. The magnitude of a vector $\mathbf{B} = (a, b, c)$ in three-dimensional Cartesian coordinates is

$$B = \|\mathbf{B}\| = \sqrt{a^2 + b^2 + c^2} \tag{1.38}$$

The magnitude of the vector $\mathbf{B} = (4, 5, 6)$ is

$$B = \sqrt{4^2 + 5^2 + 6^2} = 8.775$$

In the three-dimensional Cartesian coordinate system, a *differential length vector* $\mathbf{dl}$ may be defined from the differential lengths dx, dy, and dz as

$$\mathbf{dl} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \tag{1.39}$$

In the three-dimensional Cartesian coordinate system, *three differential area vectors* $\mathbf{dS}_x$, $\mathbf{dS}_y$, and $\mathbf{dS}_z$, may be defined from the differential lengths dx, dy, and dz as

$$\left.\begin{array}{l} \mathbf{dS}_x = dy\,dz \\ \mathbf{dS}_y = dx\,dz \\ \mathbf{dS}_z = dx\,dy \end{array}\right\} \tag{1.40}$$
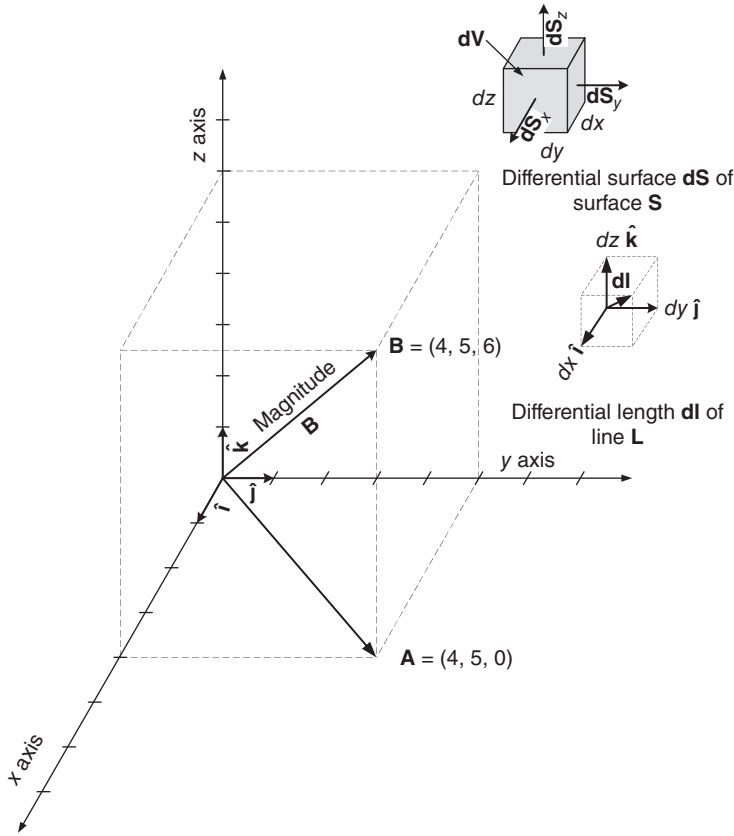
Figure 1.9   Three-dimensional vectors in rectangular form.

Each surface vector will point in the subscripted dimension, exactly as the differential surface vectors in a plane. A differential volume $\mathbf{dV}$ may be defined as

$$\mathbf{dV} = dx\, dy\, dz \tag{1.41}$$

There are two versions of the polar format for three dimensions, cylindrical and spherical.

The *cylindrical coordinate* system is simply a vertical extension of the polar coordinate system into three dimensions. The cylindrical coordinates of a three-dimensional vector $\mathbf{B} = (a, b, c)$ in Cartesian coordinates are defined as $\mathbf{B} = (r, \phi, k)$ where r is the magnitude of $\mathbf{B}$, ϕ is the angle of the projection of the vector on the x–y plane relative to the positive x-axis and k is the height of the end of the vector above the x–y plane. It can be seen that r and ϕ are the same as the two dimensions of polar coordinates. Angles are measured in the counterclockwise direction from the positive x-axis, looking downward from the positive z-axis. The cylindrical coordinate system is shown in Figure 1.10.

The unit vectors of the cylindrical coordinate system are $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{k}}$, but, except for $\hat{\mathbf{k}}$, which is identical to the Cartesian unit vector $\hat{\mathbf{k}}$, they are not on fixed axes but

Differential surface **dS** of
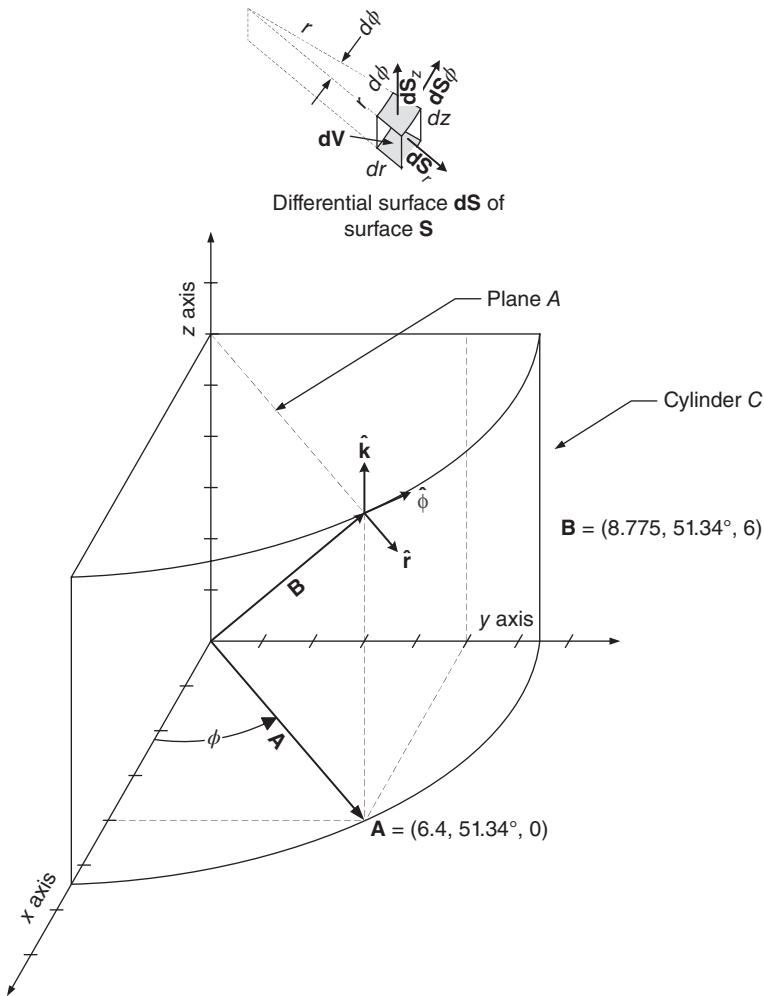surface **S**

Figure 1.10   Three-dimensional vectors in cylindrical form.

are located at the endpoint of a vector whose starting point is the origin. Thus their location changes with radius r, angle ϕ, and height k. The radial unit vector $\hat{\mathbf{r}}$ is parallel to the x–y plane and normal to the surface of cylinder C and normal to and pointing away from the z-axis. The angular unit vector $\hat{\boldsymbol{\phi}}$ is normal to the plane A, defined by the vector **B** and the vertical z-axis. The angular unit vector $\hat{\boldsymbol{\phi}}$ is tangential to the cylinder C, in the direction of increasing angle. The cylindrical unit vectors can be expressed in terms of the Cartesian unit vectors:

$$\hat{\mathbf{r}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \tag{1.42}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \tag{1.43}$$

$$\hat{\mathbf{k}} = \hat{\mathbf{k}} \tag{1.44}$$

The components of the vector (a, b, c) converted to cylindrical coordinates (r, ϕ, k) are

$$r = \sqrt{a^2 + b^2 + c^2} \tag{1.45}$$

$$\phi = \tan^{-1}\frac{b}{a} \tag{1.46}$$

$$k = c \tag{1.47}$$

The cylindrical coordinates of the example vector of $\mathbf{B} = (4, 5, 6)$, shown in Figure 1.10 are

$$r = \sqrt{4^2 + 5^2 + 6^2} = 8.775$$

$$\phi = \tan^{-1}\frac{5}{4} = 51.34°$$

$$k = 6$$

The Cartesian unit vectors are expressed in cylindrical form as

$$\widehat{\mathbf{i}} = \cos \phi \widehat{\mathbf{r}} - \sin \phi \widehat{\boldsymbol{\phi}} \tag{1.48}$$

$$\widehat{\mathbf{j}} = \sin \phi \widehat{\mathbf{r}} + \cos \phi \widehat{\boldsymbol{\phi}} \tag{1.49}$$

$$\widehat{\mathbf{k}} = \widehat{\mathbf{k}} \tag{1.50}$$

The components of the vector $\mathbf{B} = (r, \phi, k)$ converted to Cartesian coordinates (a, b, c) are

$$a = r \cos \phi \tag{1.51}$$

$$b = r \sin \phi \tag{1.52}$$

$$c = k \tag{1.53}$$

In the three-dimensional cylindrical coordinate system, a *differential length vector* **dl** may be defined from the differential lengths dr, $d\phi$, and dz as

$$\mathbf{dl} = dr\widehat{\mathbf{r}} + rd\phi\widehat{\boldsymbol{\phi}} + dz\widehat{\mathbf{k}} \tag{1.54}$$

In the three-dimensional cylindrical coordinate system, *three differential area vectors* $\mathbf{dS}_r$, $\mathbf{dS}_z$, and $\mathbf{dS}_\phi$, may be defined from the differential lengths dr, dz, and $d\phi$ as

$$\left.\begin{array}{l} \mathbf{dS}_r = r\, d\phi\, dz \\ \mathbf{dS}_z = r\, d\phi\, dr \\ \mathbf{dS}_\phi = dr\, dz \end{array}\right\} \tag{1.55}$$

Each surface vector will point in the subscripted dimension, exactly as the differential surface vectors in a plane. A differential volume **dV** may be defined as

$$\mathbf{dV} = rdrd\phi dz \tag{1.56}$$

The *spherical coordinates* of a three-dimensional vector $\mathbf{B} = (a, b, c)$ in Cartesian coordinates are defined as $\mathbf{B} = (r, \theta, \phi)$ where r is the magnitude of $\mathbf{B}$, θ is the angle of the vector on the r–z plane relative to the positive z-axis and ϕ is the angle of the projection of the vector on the x–y plane relative to the positive x-axis. Angles are measured in the counterclockwise direction from the positive x-axis.

The unit vectors of the spherical coordinate system are $\widehat{\mathbf{r}}$, $\widehat{\boldsymbol{\theta}}$, and $\widehat{\boldsymbol{\phi}}$, but unlike the Cartesian unit vectors, they are not on fixed axes, but are located at the endpoint

of a vector whose starting point is the origin. Thus their location changes with radius r, angle θ, and angle ϕ. The radial unit vector $\hat{\mathbf{r}}$ is always radially pointing away from the origin and normal to the sphere S. The angular unit vector $\hat{\boldsymbol{\theta}}$ is normal to the cone C, generated by sweeping the vector $\mathbf{B}$ around the vertical z-axis, and pointing away from the vertical z-axis. The angular unit vector $\hat{\boldsymbol{\theta}}$ is normal to the vector $\mathbf{B}$. The angular unit vector $\hat{\boldsymbol{\phi}}$ is identical to the same vector in the cylindrical coordinate system, normal to the plane A, defined by the vector $\mathbf{B}$ and the vertical z-axis. The angular unit vector $\hat{\boldsymbol{\phi}}$ is tangential to the sphere S, in the direction of increasing angle, and normal to the other two unit vectors. The spherical coordinate system is shown in Figure 1.11. The spherical unit vectors can be expressed in terms of the Cartesian unit vectors:

$$\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \tag{1.57}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \tag{1.58}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \tag{1.59}$$

The components of the vector (a, b, c) converted to spherical coordinates (r, θ, ϕ) are

$$r = \sqrt{a^2 + b^2 + c^2} \tag{1.60}$$

$$\theta = \cos^{-1}\frac{c}{r} \tag{1.61}$$

$$\phi = \tan^{-1}\frac{b}{a} \tag{1.62}$$

The cylindrical coordinates of the example vector of $\mathbf{B} = (4, 5, 6)$ shown in Figure 1.11 are

$$r = \sqrt{4^2 + 5^2 + 6^2} = 8.775$$

$$\theta = \cos^{-1}\frac{6}{8.775} = 46.86°$$

$$\phi = \tan^{-1}\frac{5}{4} = 51.34°$$

The Cartesian unit vectors are expressed in spherical form as

$$\hat{\mathbf{i}} = \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}} \tag{1.63}$$

$$\hat{\mathbf{j}} = \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}} \tag{1.64}$$

$$\hat{\mathbf{k}} = \cos\theta\hat{\mathbf{i}} - \sin\theta\hat{\mathbf{j}} \tag{1.65}$$

The components of the vector (r, θ, ϕ) converted to rectangular coordinates (a, b, c) are

$$a = r\sin\theta\cos\phi \tag{1.66}$$

$$b = r\sin\theta\sin\phi \tag{1.67}$$

$$c = r\cos\theta \tag{1.68}$$

In the three-dimensional spherical coordinate system, a *differential vector* $\mathbf{dl}$ may be defined from the differential lengths dr, $d\theta$, and $d\phi$ as

$$\mathbf{dl} = dr\,\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}} \tag{1.69}$$
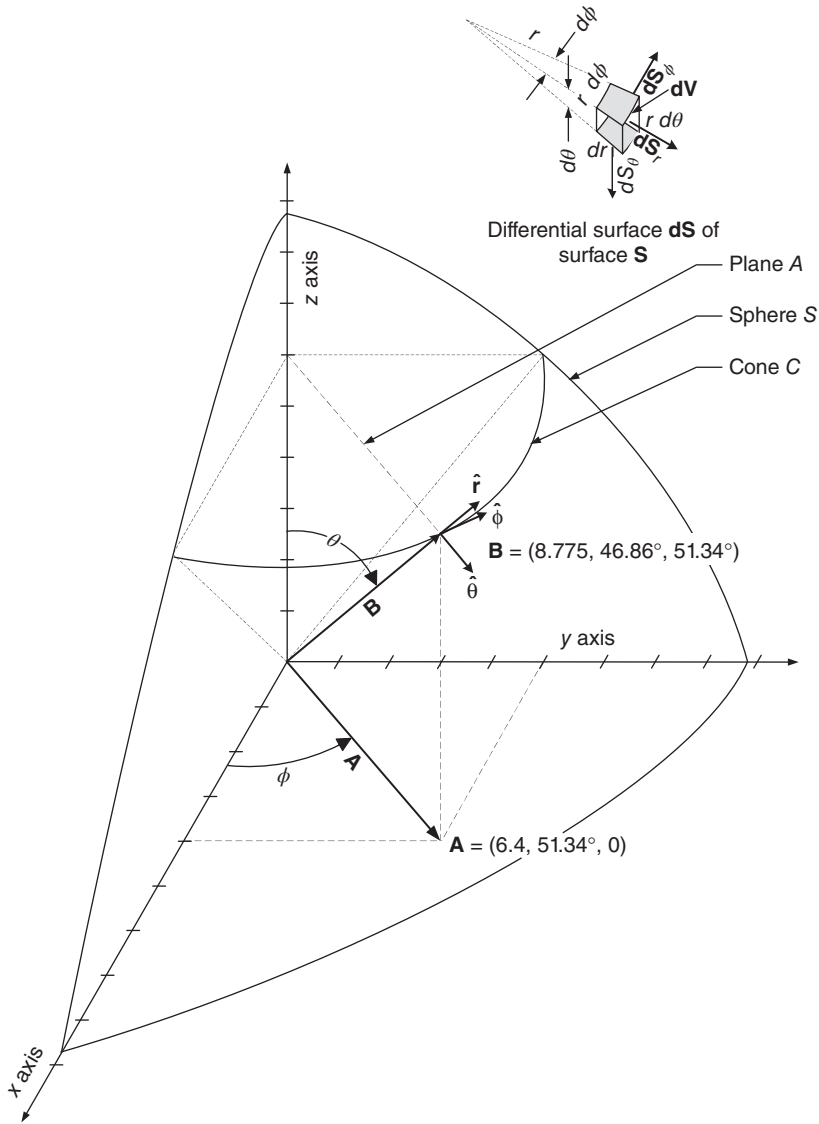
Figure 1.11   Three-dimensional vectors in spherical form.

In the three-dimensional spherical coordinate system, *three differential area vectors* $\mathbf{dS}_r$, $\mathbf{dS}_\theta$, and $\mathbf{dS}_\phi$, may be defined from the differential lengths dr, $d\theta$, and $d\phi$ as

$$\left.\begin{aligned}
\mathbf{dS}_r &= r^2 \sin\theta\, d\theta\, d\phi \\
\mathbf{dS}_\theta &= r \sin\theta\, dr\, d\phi \\
\mathbf{dS}_\phi &= r\, dr\, d\theta
\end{aligned}\right\} \tag{1.70}$$

Each surface vector will point in the subscripted dimension, exactly as the differential surface vectors in a plane. A differential volume $\mathbf{dV}$ may be defined as

$$\mathbf{dV} = r^2 \sin\theta dr d\theta d\phi \tag{1.71}$$

A *matrix* is an extension of the concept of a vector, where the elements can be arranged in rows and columns in two or more dimensions. Matrices are used to simplify the notation when a calculation is made with several nearly identical equations, such as one for each dimension in a system. Matrices are also a type of number in their own right. The 3×3 matrix $\mathbf{M}$ is written as

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{1.72}$$

The magnitude of a matrix is the *determinant*. For a 3×3 matrix,

$$|\mathbf{M}| = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}$$

$$= m_{11}(m_{22}m_{33} - m_{23}m_{32}) + m_{12}(m_{21}m_{33} - m_{23}m_{31}) + m_{13}(m_{21}m_{32} - m_{22}m_{31}) \tag{1.73}$$

1.3   MATHEMATICAL OPERATIONS WITH VECTORS

Vector *addition* and *subtraction* are carried out term by term in the rectangular coordinate system. For vectors $\mathbf{A} = (a, b, c)$ and $\mathbf{B} = (d, e, f)$:

$$\mathbf{A} + \mathbf{B} = (a + d,\ b + e, c + f) \tag{1.74}$$
$$\mathbf{A} - \mathbf{B} = (a - d,\ b - e, c - f) \tag{1.75}$$

The product of a *vector* $\mathbf{A}$ and a *scalar* α is the product of the scalar and each element of the vector:

$$\alpha\mathbf{A} = (\alpha a,\ \alpha b, \alpha c) \tag{1.76}$$

The scalar may be positive or negative. If the scalar is negative, the direction of the vector will be reversed. Note that while the magnitude of a vector is always positive, the vector may have any direction.

The *dot product* or *inner product* of two vectors $\mathbf{A} = (a, b, c)$ and $\mathbf{B} = (d, e, f)$ separated by angle α is a scalar quantity defined as

$$\mathbf{A} \cdot \mathbf{B} = ad + be + cf \tag{1.77}$$

or

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\alpha \tag{1.78}$$

A two-dimensional example of a dot product is shown in Figure 1.12. The dot product is commutative:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1.79}$$

The *cross product* or *outer product* of two vectors $\mathbf{A}$ and $\mathbf{B}$ separated by angle α is a vector defined as

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{i}}_{\mathbf{n}} AB\sin\alpha \tag{1.80}$$
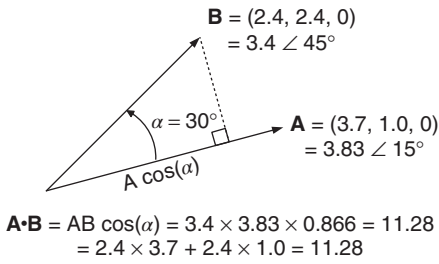
$\mathbf{B} = (2.4, 2.4, 0)$
$= 3.4 \angle 45°$

$\alpha = 30°$

$\mathbf{A} = (3.7, 1.0, 0)$
$= 3.83 \angle 15°$

$A \cos(\alpha)$

$\mathbf{A \cdot B} = AB \cos(\alpha) = 3.4 \times 3.83 \times 0.866 = 11.28$
$= 2.4 \times 3.7 + 2.4 \times 1.0 = 11.28$

Figure 1.12   Dot product of two vectors.

12

CCW

A

B

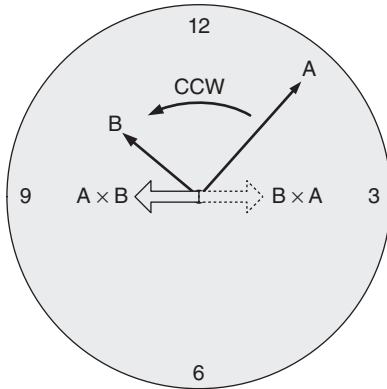9 $A \times B$ ⇐ ⇨ $B \times A$ 3

6

Figure 1.13   Right-hand rule for the cross product of two vectors.

The unit vector $\widehat{\mathbf{i}_n}$ is normal (perpendicular) to the two vectors **A** and **B** in the direction indicated by the "right-hand rule." The right-hand rule can be visualized as a clock face with two hands A and B, Figure 1.13, where A is to the right of B. A vector emerging perpendicular to the face of the clock toward the observer represents the direction of the cross product. An example of a cross product is shown in Figure 1.14. Because of the right-hand rule, the cross product is not commutative:

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B} \qquad (1.81)$$

The vector $\mathbf{B} \times \mathbf{A}$ is also shown in the figures, in the direction opposite to $\mathbf{A} \times \mathbf{B}$. The cross product, like the dot product, may also be calculated using the elements of each vector:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ a & b & c \\ d & e & f \end{vmatrix} = \widehat{\mathbf{i}}(bf - ce) + \widehat{\mathbf{j}}(af - cd) + \widehat{k}(ae - bd) \qquad (1.82)$$

1.4   CALCULUS WITH VECTORS — THE GRADIENT

A *scalar field* is a function which assigns a scalar to each point within a region of a space. A scalar field may be written as $f(x, y, z)$ in a three-dimensional space. Examples of a scalar field are gravitational fields and electric fields.

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{i}}_n \, AB \sin(\alpha) = \hat{\mathbf{k}} \, 3.4 \times 3.83 \times 0.5$$
$$= \hat{\mathbf{i}}(1.0 \times 0 - 2.4 \times 0) + \hat{\mathbf{j}}(3.7 \times 0 - 1.0 \times 0)$$
$$+ \, \hat{\mathbf{k}}(3.7 \times 2.4 - 1.0 \times 2.4)$$
$$= (0, \, 0, \, 6.5)$$

$\mathbf{A} \times \mathbf{B}$

$\mathbf{B} = (2.4, \, 2.4, \, 0)$
$= 3.4 \angle 45°$

$A \sin(\alpha)$

$\alpha = 30°$

$\mathbf{A} = (3.7, \, 1.0, \, 0)$
$= 3.83 \angle 15°$

$$\mathbf{B} \times \mathbf{A} = \hat{\mathbf{i}}_n \, AB \sin(-\alpha) = \hat{\mathbf{k}} \, 3.4 \times 3.83 \times (-0.5)$$
$$= \hat{\mathbf{i}}(2.4 \times 0 - 1.0 \times 0) + \hat{\mathbf{j}}(2.4 \times 0 - 3.7 \times 0)$$
$$+ \, \hat{\mathbf{k}}(2.4 \times 1.0 - 3.7 \times 2.4)$$
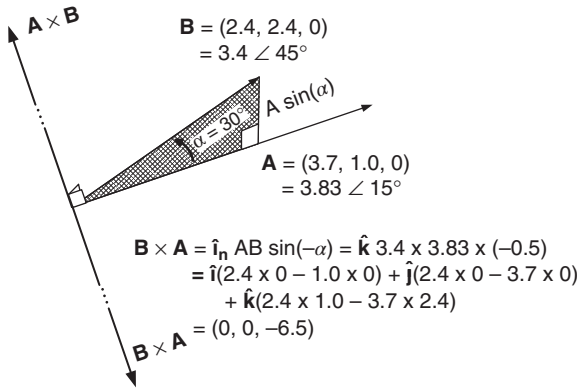$$= (0, \, 0, \, -6.5)$$

$\mathbf{B} \times \mathbf{A}$

Figure 1.14   Cross product of two vectors.

The *differential* of a scalar field is

$$df(x, y, z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \qquad (1.83)$$

The *del operator* (∇) is used to denote the taking of partial derivatives in each direction:

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \qquad (1.84)$$

The del operator is the three-dimensional equivalent to the symbol for taking the derivative with no function specified, $\frac{d}{dx}$. The del operator applied to a three-dimensional scalar field is called the *gradient* (grad) of the field:

$$\mathbf{grad}\, f(x, y, z) = \nabla f(x, y, z) = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad (1.85)$$

The gradient in cylindrical coordinates is

$$\nabla f(r, \phi, z) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}} \qquad (1.86)$$

The gradient in spherical coordinate is

$$\nabla f(r, \theta, \phi) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} \qquad (1.87)$$

The *scalar field* of a set of points on the *r*-axis may be written as $f(r)$. An example of a scalar field is the electric potential $V(r)$ of a point charge q:

$$V(r) = \frac{q}{4\pi\varepsilon_0 r}\text{V} \qquad (1.88)$$

where,

> $V(r) =$ is the potential in volts (V) at a point r meters (m) from the point charge,

> $q =$ is the charge in coulombs (C), and

$\varepsilon_0 = \frac{1}{4\pi \times 10^{-7}} =$ is the permittivity of free space in farad per meter (F/m).

Work is required to move a charge in an electric field. If a charge of $1\,\mathrm{C}$ is moved across a potential difference of $1\,\mathrm{V}$, $1\,\mathrm{J}$ (joule) of work will be expended. If there are two charges of opposite signs which are naturally attracted to each other, work is required to separate them. If the two charges are of the same polarity, the force will be required to move them toward each other. The force in newtons (N) exerted by the charge q on the charge q' (and by the charge q' on the charge q) at a distance x along the direction of the vector $\hat{\mathbf{i}}$ is given by *Coulomb's law*:

$$\mathbf{F} = \frac{qq'}{4\pi\varepsilon_0 r^2}\hat{\mathbf{i}}\,N \tag{1.89}$$

A *vector field* differs from the scalar field in that a vector exists at each point in the Euclidean space. An example of this is the *electric field* (V/m). The electric field established by a charge q is defined where the charge q' (at distance x from the charge q) becomes infinitely small, so that only the charge q is being considered.

$$\mathbf{E} = \frac{\mathbf{F}}{q'} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{i}}\,\frac{V}{m} \tag{1.90}$$

The electric field vector is the gradient of the scalar electric field.

$$\mathbf{E} = -\nabla V \tag{1.91}$$

The electric field of a point charge calculated by the gradient definition:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} = -\frac{\partial}{\partial r}\frac{q}{4\pi\varepsilon_0 r}\hat{\mathbf{r}} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \tag{1.92}$$

Figure 1.15 shows the field produced by a charge of 1 nanocoulomb (nC or 10^{-9} C) over the range of 2–$5\,\mathrm{m}$. At point a, $2\,\mathrm{m}$, the field is $4.5\,\mathrm{V}$, and at point b, $5\,\mathrm{m}$, the field is $1.8\,\mathrm{V}$. Thus, if the $1\,\mathrm{nC}$ charge q at the origin is positive, and the charge q' at point a is $-1\,\mathrm{nC}$, then it will take $2.7\,\mathrm{nJ}$ to move this charge to point b. This can be calculated more formally is we consider the differential distance $\mathbf{dl}$ along the path a to b, where

$$\mathbf{dl} = dr\hat{\mathbf{r}} \tag{1.93}$$

and

$$\mathbf{dV} = \nabla V \cdot \mathbf{dl} \tag{1.94}$$

If we integrate from a to b,

$$\int_a^b \mathbf{dV} = \int_a^b \nabla V \cdot \mathbf{dl} = V(b) - V(a) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \tag{1.95}$$

For example, the scalar field

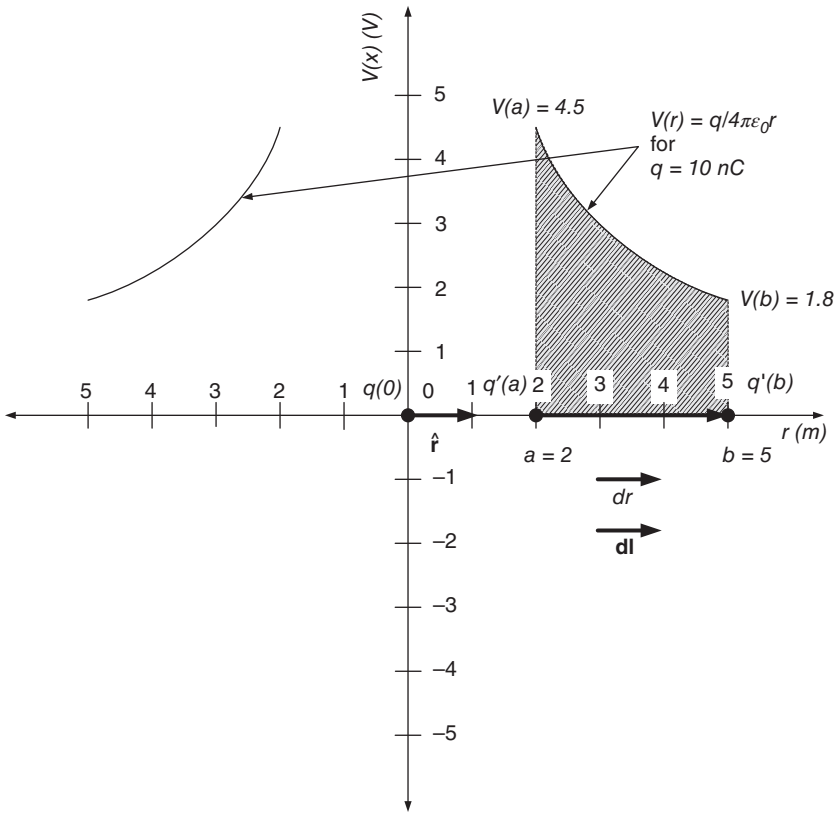$$f(x, y) = 0.2x^2 + 0.5y$$

Figure 1.15   Scalar field in one dimension.

represents a specific numerical value at each point in the x, y space. The differential of the field is

$$df(x, y) = 0.4xdx + 0.5dy$$

and the gradient of the field is

$$\nabla f = 0.4x\hat{\mathbf{i}} + 0.5\hat{\mathbf{j}}$$

The angles of the gradient are

$$\phi_g = \tan^{-1}\left[\frac{0.5}{0.4\,x}\right] = \tan^{-1}\left[\frac{1.25}{x}\right]$$

$$\theta_g = \tan^{-1}\left[\frac{\sqrt{0.16\,x^2 + 0.25}}{0}\right] = \frac{\pi}{2}$$

The gradient has a magnitude representing the slope of the field in the x- and y-directions, pointing in the direction of the greatest change in slope from its origin at a point (x, y). Recalling the distance vector **dl**, the differential of the scalar field

may be expressed as

$$df(x, y, z) = \left(\frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \right) \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
(1.96)

since $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1$. The angles of **dl** are

$$\left. \begin{aligned} \phi_{dl} &= \tan^{-1}\left[\frac{dy}{dx} \right] \\[2mm] \theta_{dl} &= \tan^{-1}\left[\frac{\sqrt{dx^2 + dy^2}}{dz} \right] \end{aligned} \right\}$$
(1.97)

Then

$$df = \nabla f \cdot \mathbf{dl} = |\nabla f| \, |\mathbf{dl}| \, \cos\theta = |\nabla f|\mathbf{dl}\cos\theta_{df}$$
(1.98)

Where θ_{df} is the angle between ∇f and **dl**. In the example, the angles of **dl** are (with $dz = 0$)

$$\left. \begin{aligned} \phi_{dl} &= \tan^{-1}\left[\frac{dy}{dx} \right] = \tan^{-1}\left[2\frac{df}{dx} - 0.8x \right] = \tan^{-1}0 = 0 \\[2mm] \theta_{dl} &= \tan^{-1}\left[\frac{\sqrt{dx^2 + dy^2}}{0} \right] = \frac{\pi}{2} \end{aligned} \right\}$$

Since the example is in two dimensions, $\theta_g = \theta_{dl} = \pi/2$. Because of the definition of the function $f(x, y) = 0.2x^2 + 0.5y$, $\phi_{dl} = 0$. Then $\theta_{df} = \phi_g$ and $df(x, y)$ may be calculated as

$$df(x, y) = \sqrt{0.16x^2 + 0.25} \, \mathbf{dl} \cos\left[\tan^{-1}\left(\frac{1.25}{x} \right) \right]$$

It is apparent that this is equal to the same quantity calculated previously:

$$df(x, y) = (\hat{\mathbf{i}}0.4x)(dx\,\hat{\mathbf{i}}) + (\hat{\mathbf{j}}0.5)(dy\,\hat{\mathbf{j}}) = 0.4xdx + 0.5dy$$

The *gradient theorem may* be defined for a line between two points A and B:

$$\int_A^B df = \int_A^B \nabla f \cdot \mathbf{dl} = f(B) - f(A)$$
(1.99)

The line integral is dependent only on the end points and the function f, and does not change if the length of the line is longer or shorter, or if the line takes different routes from A to B. The gradient theorem is also called the *fundamental theorem of calculus for line integrals*. In Figure 1.16, the starting point $A = (5, 1)$ and the end point $B = (2, 3)$ making $f(A) = 5.2$ and $f(B) = 2.3$. Three different paths are shown in the figure, all with the same line integral. Thus:

$$\int_A^B df = f(B) - f(A) = 2.3 - 5.2 = -3.2$$

A closed line integral, where $A = B$, is symbolized with a circle over the integral sign:
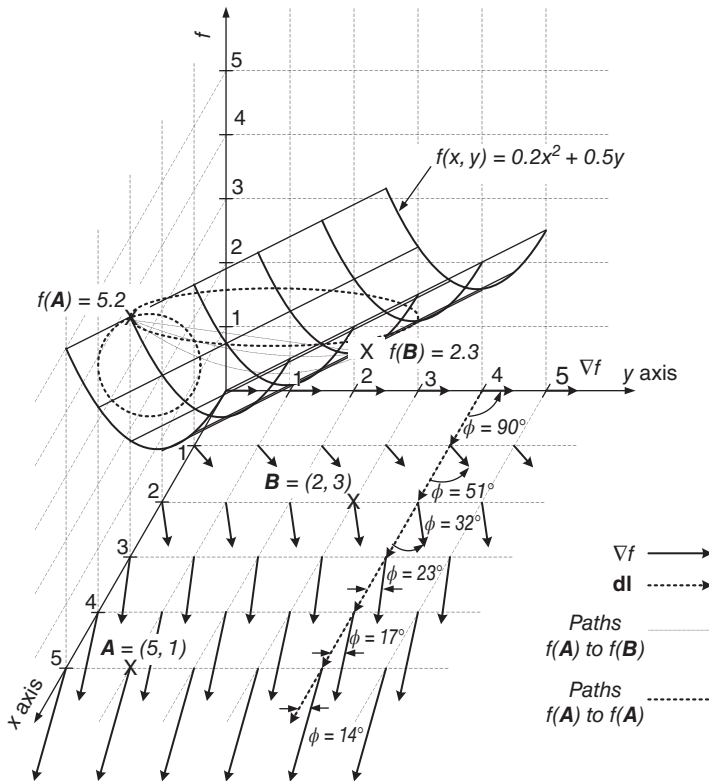
$$\oint df = f(A) - f(A) = 0$$
(1.100)

Figure 1.16   Two-dimensional scalar field $f(x,y) = 2x^2 + 5y$ and its gradient grad f.

A corollary to the gradient theorem is that *the line integral of a gradient around a closed loop is always zero*. In Figure 1.16, lines $f(A)$ to $f(A)$ have a line integral of zero.

1.5   DIVERGENCE, CURL, AND STOKES' THEOREM

When a boundary is established by a closed surface **S** in a three-dimensional space which contains a vector field **A**, the net amount of flow through the space is called the *flux*.

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} \tag{1.101}$$

Because the flux will also be different at each point in space, the derivative of the flux at any point in space is defined as a quantity called the *divergence*:

$$\nabla \cdot \mathbf{A} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \right) \cdot (a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} \tag{1.102}$$
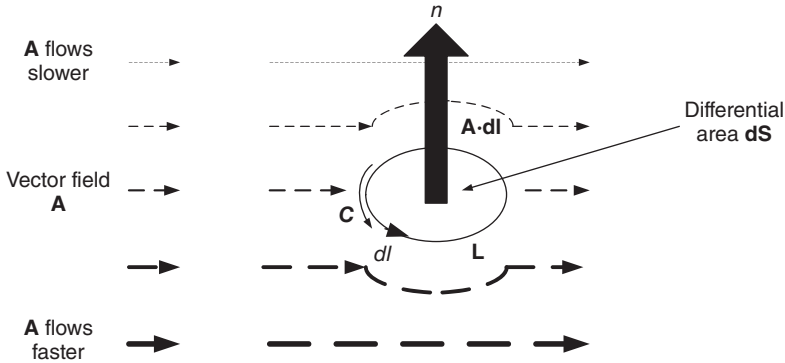
Figure 1.17   Closed path **L** of circulation C with normal vector **n** defining the direction of **A·dl** in accordance with the right-hand rule.

The *divergence theorem* relates the flux within a volume **V** composed of many infinitesimal differential volumes **dV** with the surface **S** composed of many infinitesimal differential areas d**S**.

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{A} \, \mathbf{dV} \qquad (1.103)$$

The circulation C of a fluid in a closed path **L** against a vector field **A** is shown in Figure 1.17. The normal vector **n**, established by the right-hand rule, establishes the direction of **A·dl**. The circulation is expressed by

$$C = \oint_L \mathbf{A} \cdot \mathbf{dl} \qquad (1.104)$$

The *curl* is a vector operation defined as

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a & b & c \\ d & e & f \end{vmatrix} = \hat{\mathbf{i}} \left(\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z} \right) + \hat{\mathbf{j}} \left(\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x} \right) + \hat{\mathbf{k}} \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} \right)$$

$$(1.105)$$

The circulation for an infinitesimally small loop **L**, shown in Figure 1.17 as an eddy in flowing water, in differential form is:

$$C = (\nabla \times \mathbf{A}) \cdot \mathbf{dS} \qquad (1.106)$$

Stokes' theorem links together these results for a finite-sized open surface **S** enclosed by a line **L**:

$$\oint_L \mathbf{A} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} \qquad (1.107)$$

Recalling the gradient theorem, it is clear that the line integral around a closed loop must be zero, then using Stokes' theorem:

$$\oint_L \nabla f \cdot \mathbf{dl} = \int_S (\nabla \times \nabla f) \cdot \mathbf{dS} = 0 \qquad (1.108)$$

Using the definitions of divergence and curl, a useful identity for vector analysis is

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{1.109}$$

1.6   MAXWELL'S EQUATIONS

The Scottish physicist James Clerk Maxwell (1831–1879) codified the empirically derived physical laws discovered by Faraday, Ampère, and Gauss along with some necessary corrections into a set of 20 equations which define electromagnetic field theory. The original versions of these equations have been updated by Oliver Heaviside (1850–1925) using the concepts of vector calculus outlined above into four equations, which are today generally known as "Maxwell's Equations."

Faraday's law, states that a changing magnetic field induces a corresponding electric field:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.110}$$

This can be thought of as an electric circuit composed of a loop of wire in combination with a time-varying magnetic field produced by a spinning magnet, the combination being an electric generator, Figure 1.18. In integral form, the moving magnetic field provides the work to move an electric charge along a specific path.

$$\oint_{\mathbf{L}} \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt} \int_{\mathbf{S}} \mathbf{D} \cdot \mathbf{dS} \tag{1.111}$$
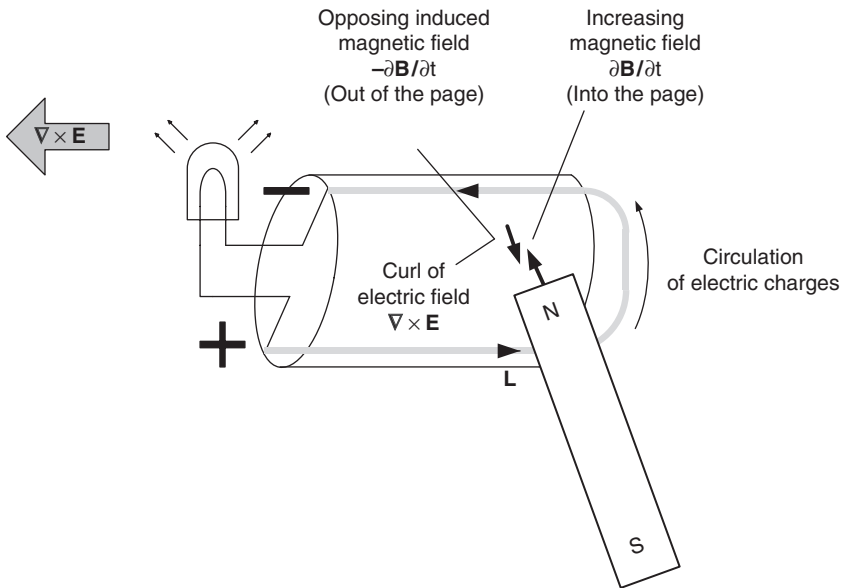

Figure 1.18   Faraday's law: the moving magnetic field induces an electric field which produces a current that induces a magnetic field opposing that which caused it.
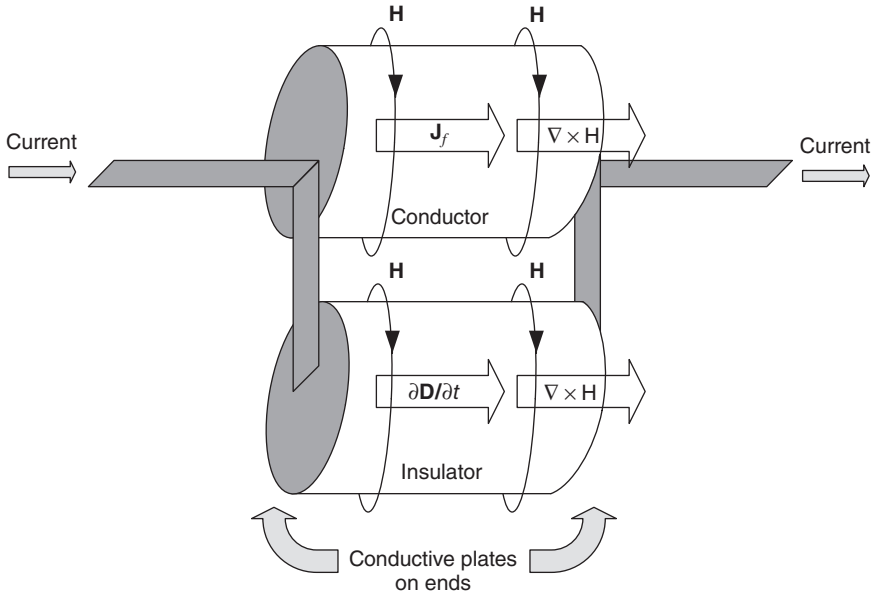
Figure 1.19   Illustration of Ampère's law for a resistor and capacitor.

Ampère's law with Maxwell's displacement current correction states that a changing electric field produces a corresponding magnetic field, as in Figure 1.19,

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{1.112}$$

In point form, you imagine a moving electric charge producing a magnetic field according to the right-hand rule. In integral form, the work required for a magnet to move along a path **L** is provided by electric charges moving across the boundaries of a volume.

$$\oint_{\mathbf{L}} \mathbf{H} \cdot \mathbf{dl} = \int_{\mathbf{S}} \mathbf{J}_f \cdot \mathbf{dS} + \frac{d}{dt} \int_{\mathbf{S}} \mathbf{D} \cdot \mathbf{dS} \tag{1.113}$$

Gauss's law for electric fields in volume form is, "The electric flux through any closed surface is proportional to the enclosed electric charge." A "Gaussian surface" is an arbitrary closed three-dimensional surface (such as a sphere) which is chosen to ease the computation of volume integrals. This is illustrated in Figure 1.20, where the electric field diverges from the positive point charge and converges on the negative point charge.

$$\nabla \cdot \mathbf{D} = \rho_f \tag{1.114}$$

If Gauss's law for electric fields is expressed in integral form, it becomes

$$\oint_{\mathbf{S}} \mathbf{D} \cdot \mathbf{dS} = \int_{\mathbf{V}} \rho_f d\mathbf{V} \tag{1.115}$$

Where the surface encloses some electrical charges as in the figure, and if you make the surface such that all field lines cross it perpendicularly, then the dot product becomes multiplication and the math is easier.
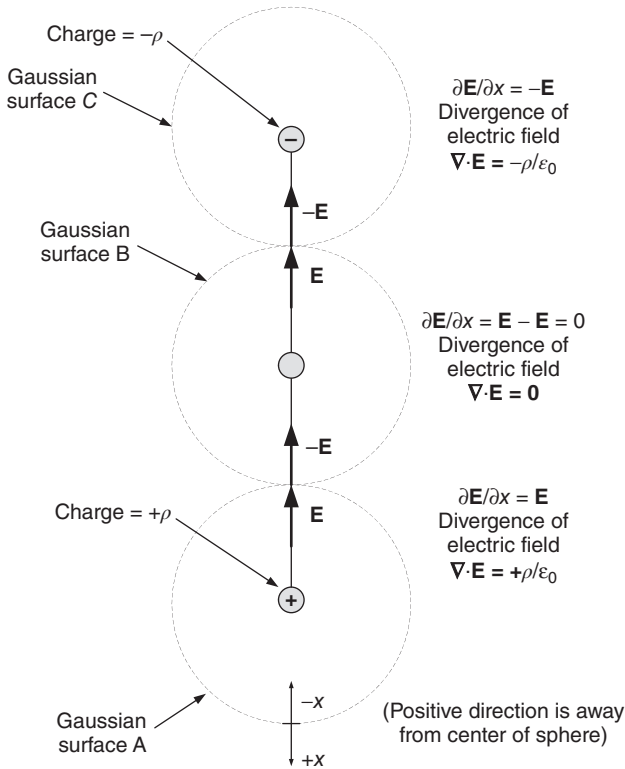
Figure 1.20   Illustration of Gauss's law for the field around a point charge.

Gauss's law for magnetic fields implies that there is no magnetic current point charge:

$$\nabla \cdot \mathbf{B} = 0 \tag{1.116}$$

All magnetic sources are dipoles, having both a north and a south pole. Unlike electric fields, this, while they may have dipoles, may also have both independent positive and independent negative charges as sources. This can be seen in Figure 1.21, where no surface can enclose a net divergence or convergence of lines of flux. The integral form is

$$\oint_{\mathbf{S}} \mathbf{B} \cdot \mathbf{dS} = 0 \tag{1.117}$$

This means that whatever three-dimensional enclosure you construct, it will have as many magnetic field lines entering as leaving, as can be seen in the figure.

Maxwell's equations are written in both derivative and integral forms, with many variations based on the physical situation which they are being used to evaluate. The derivative forms have more simplicity and compactness, but the integral form may be easier to visualize as a three-dimensional object is involved, even though it is only a fictitious Gaussian surface. Whether visualizing points or surfaces, the three-dimensional characteristics of invisible, and usually moving, force fields are

$\partial \mathbf{B}/\partial x = 0$
Divergence of
magnetic field
$\nabla \cdot \mathbf{B} = 0$

$-x$

Gaussian
surface

(Positive direction is away
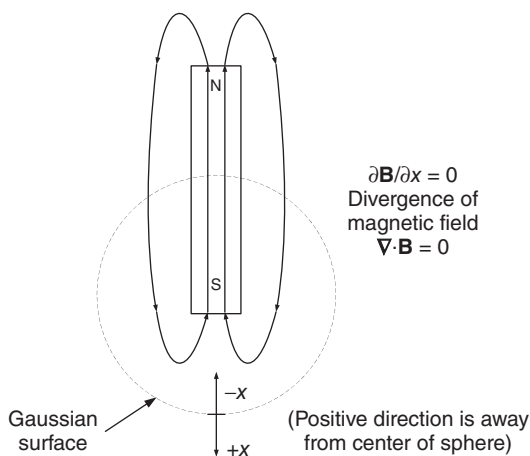from center of sphere)

$+x$

Figure 1.21   Illustration of
Gauss's law for magnetic fields.

more difficult to picture in mathematical than in physical terms. The terms used here
are listed for reference:

$\mathbf{E} =$ electric field intensity (volts per meter)

$\mathbf{B} =$ magnetic field density (tesla or webers per square meter)

$\mathbf{H} =$ magnetic field intensity (amperes per meter)

$\mathbf{J}_f =$ free current density (in a conductor) (amperes per square meter)

$\mathbf{D} =$ electricdisplacement field (coulombs per square meter)

Additionally, Maxwell's equations include the conservation of charge, which states
that charge can neither be created nor destroyed:

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \qquad (1.118)$$

The additional variable is

$\rho_f =$ free charge density (in a conductor) (coulombs per cubic meter) (not to be
confused with electrical resistivity also designated by ρ)

Maxwell's equations are supplemented by the constitutive laws, which are nec-
essary for their application to physical materials:

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P} \qquad (1.119)$$

$$\mathbf{B} = \mu \mathbf{H} \qquad (1.120)$$

The additional variables are

$\varepsilon =$ permittivity (farad per meter)

$\mu =$ permeability (henry per meter)

$\mathbf{P} =$ portion of the electric field due to polarization in matter

These equations define how much electric field strength it requires to impose a charge
on a material, and how much magnetizing intensity it takes to magnetize a material.

Ohm's law for fixed and moving conductors is also a necessary part of Maxwell's equations:

$$J_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1.121}$$

The additional variables are

σ = conductivity (siemens)

$\mathbf{v}$ = velocity (meters per second)

The current density in a material includes both current imposed by an electric field dependent on the conductivity of the material and current induced by movement of the material in a magnetic field.