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THEORY OF ELECTROMECHANICAL ENERGY CONVERSION

1.1. INTRODUCTION

The theory of electromechanical energy conversion allows us to establish expressions for torque in terms of machine electrical variables, generally the currents, and the displacement of the mechanical system. This theory, as well as the derivation of equivalent circuit representations of magnetically coupled circuits, is established in this chapter. In Chapter 2, we will discover that some of the inductances of the electric machine are functions of the rotor position. This establishes an awareness of the complexity of these voltage equations and sets the stage for the change of variables (Chapter 3) that reduces the complexity of the voltage equations by eliminating the rotor position dependent inductances and provides a more direct approach to establishing the expression for torque when we consider the individual electric machines.

1.2. MAGNETICALLY COUPLED CIRCUITS

Magnetically coupled electric circuits are central to the operation of transformers and electric machines. In the case of transformers, stationary circuits are magnetically

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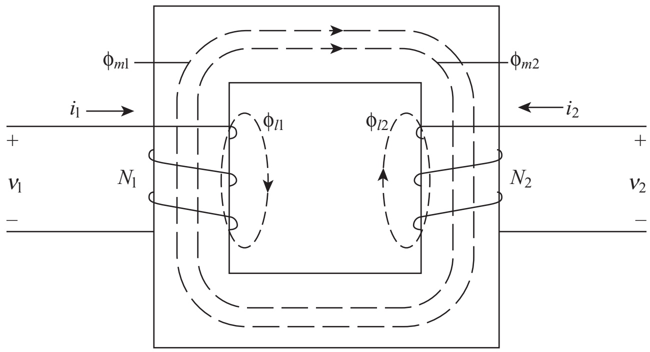

Figure 1.2-1. Magnetically coupled circuits.

coupled for the purpose of changing the voltage and current levels. In the case of electric machines, circuits in relative motion are magnetically coupled for the purpose of transferring energy between mechanical and electrical systems. Since magnetically coupled circuits play such an important role in power transmission and conversion, it is important to establish the equations that describe their behavior and to express these equations in a form convenient for analysis. These goals may be achieved by starting with two stationary electric circuits that are magnetically coupled as shown in Figure 1.2-1. The two coils consist of turns N_1 and N_2, respectively, and they are wound on a common core that is generally a ferromagnetic material with permeability large relative to that of air. The permeability of free space, μ_0, is $4\pi \times 10^{-7}$ H/m. The permeability of other materials is expressed as $\mu = \mu_r\mu_0$, where μ_r is the relative permeability. In the case of transformer steel, the relative permeability may be as high as 2000–4000.

In general, the flux produced by each coil can be separated into two components. A leakage component is denoted with an l subscript and a magnetizing component is denoted by an m subscript. Each of these components is depicted by a single streamline with the positive direction determined by applying the right-hand rule to the direction of current flow in the coil. Often, in transformer analysis, i_2 is selected positive out of the top of coil 2 and a dot placed at that terminal.

The flux linking each coil may be expressed

$$\Phi_1 = \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \tag{1.2-1}$$

$$\Phi_2 = \Phi_{l2} + \Phi_{m2} + \Phi_{m1} \tag{1.2-2}$$

The leakage flux Φ_{l1} is produced by current flowing in coil 1, and it links only the turns of coil 1. Likewise, the leakage flux Φ_{l2} is produced by current flowing in coil 2, and it links only the turns of coil 2. The magnetizing flux Φ_{m1} is produced by current flowing in coil 1, and it links all turns of coils 1 and 2. Similarly, the magnetizing flux Φ_{m2} is produced by current flowing in coil 2, and it also links all turns of coils 1 and 2. With the selected positive direction of current flow and the manner in that the coils are wound (Fig. 1.2-1), magnetizing flux produced by positive current in one coil adds to the magnetizing flux produced by positive current in the other coil. In other words, if both currents are flowing in the same direction, the magnetizing fluxes produced by each coil are in the same direction, making the total magnetizing flux or the total core flux the sum of the instantaneous magnitudes of the individual magnetizing fluxes. If the currents are in opposite directions, the magnetizing fluxes are in opposite directions. In this case, one coil is said to be magnetizing the core, the other demagnetizing.

Before proceeding, it is appropriate to point out that this is an idealization of the actual magnetic system. Clearly, all of the leakage flux may not link all the turns of the coil producing it. Likewise, all of the magnetizing flux of one coil may not link all of the turns of the other coil. To acknowledge this practical aspect of the magnetic system, the number of turns is considered to be an equivalent number rather than the actual number. This fact should cause us little concern since the inductances of the electric circuit resulting from the magnetic coupling are generally determined from tests.

The voltage equations may be expressed in matrix form as

$$\mathbf{v} = \mathbf{ri} + \frac{d\boldsymbol{\lambda}}{dt} \tag{1.2-3}$$

where $\mathbf{r} = \text{diag}[r_1 \; r_2]$, is a diagonal matrix and

$$(\mathbf{f})^T = [f_1 \;\; f_2] \tag{1.2-4}$$

where f represents voltage, current, or flux linkage. The resistances r_1 and r_2 and the flux linkages λ_1 and λ_2 are related to coils 1 and 2, respectively. Since it is assumed that Φ_1 links the equivalent turns of coil 1 and Φ_2 links the equivalent turns of coil 2, the flux linkages may be written

$$\lambda_1 = \mathrm{N}_1\Phi_1 \tag{1.2-5}$$

$$\lambda_2 = \mathrm{N}_2\Phi_2 \tag{1.2-6}$$

where Φ_1 and Φ_2 are given by (1.2-1) and (1.2-2), respectively.

Linear Magnetic System

If saturation is neglected, the system is linear and the fluxes may be expressed as

$$\Phi_{l1} = \frac{N_1 i_1}{\mathcal{R}_{l1}} \tag{1.2-7}$$

$$\Phi_{m1} = \frac{N_1 i_1}{\mathcal{R}_m} \tag{1.2-8}$$

$$\Phi_{l2} = \frac{N_2 i_2}{\mathcal{R}_{l2}} \tag{1.2-9}$$

$$\Phi_{m2} = \frac{N_2 i_2}{\mathcal{R}_m} \tag{1.2-10}$$

where $\mathcal{R}_{l1}$ and $\mathcal{R}_{l2}$ are the reluctances of the leakage paths and $\mathcal{R}_m$ is the reluctance of the path of the magnetizing fluxes. The product of N times i (ampere-turns) is the magnetomotive force (MMF), which is determined by the application of Ampere's law. The reluctance of the leakage paths is difficult to express and measure. A unique determination of the inductances associated with the leakage flux is typically either calculated or approximated from design considerations. The reluctance of the magnetizing path of the core shown in Figure 1.2-1 may be computed with sufficient accuracy from the well-known relationship

$$\mathcal{R} = \frac{l}{\mu A} \tag{1.2-11}$$

where l is the mean or equivalent length of the magnetic path, A the cross-section area, and μ the permeability.

Substituting (1.2-7)–(1.2-10) into (1.2-1) and (1.2-2) yields

$$\Phi_1 = \frac{N_1 i_1}{\mathcal{R}_{l1}} + \frac{N_1 i_1}{\mathcal{R}_m} + \frac{N_2 i_2}{\mathcal{R}_m} \tag{1.2-12}$$

$$\Phi_2 = \frac{N_2 i_2}{\mathcal{R}_{l2}} + \frac{N_2 i_2}{\mathcal{R}_m} + \frac{N_1 i_1}{\mathcal{R}_m} \tag{1.2-13}$$

Substituting (1.2-12) and (1.2-13) into (1.2-5) and (1.2-6) yields

$$\lambda_1 = \frac{N_1^2}{\mathcal{R}_{l1}} i_1 + \frac{N_1^2}{\mathcal{R}_m} i_1 + \frac{N_1 N_2}{\mathcal{R}_m} i_2 \tag{1.2-14}$$

$$\lambda_2 = \frac{N_2^2}{\mathcal{R}_{l2}} i_2 + \frac{N_2^2}{\mathcal{R}_m} i_2 + \frac{N_2 N_1}{\mathcal{R}_m} i_1 \tag{1.2-15}$$

When the magnetic system is linear, the flux linkages are generally expressed in terms of inductances and currents. We see that the coefficients of the first two terms on the right-hand side of (1.2-14) depend upon the turns of coil 1 and the reluctance of the magnetic system, independent of the existence of coil 2. An analogous statement may be made regarding (1.2-15). Hence, the self-inductances are defined as

$$\begin{aligned} L_{11} &= \frac{N_1^2}{\mathcal{R}_{l1}} + \frac{N_1^2}{\mathcal{R}_m} \\ &= L_{l1} + L_{m1} \end{aligned} \tag{1.2-16}$$

$$L_{22} = \frac{N_2^2}{\Re_{l2}} + \frac{N_2^2}{\Re_m}$$
$$= L_{l2} + L_{m2} \tag{1.2-17}$$

where L_{l1} and L_{l2} are the leakage inductances and L_{m1} and L_{m2} the magnetizing inductances of coils 1 and 2, respectively. From (1.2-16) and (1.2-17), it follows that the magnetizing inductances may be related as

$$\frac{L_{m2}}{N_2^2} = \frac{L_{m1}}{N_1^2} \tag{1.2-18}$$

The mutual inductances are defined as the coefficient of the third term of (1.2-14) and (1.2-15).

$$L_{12} = \frac{N_1 N_2}{\Re_m} \tag{1.2-19}$$

$$L_{21} = \frac{N_2 N_1}{\Re_m} \tag{1.2-20}$$

Obviously, $L_{12} = L_{21}$. The mutual inductances may be related to the magnetizing inductances. In particular,

$$L_{12} = \frac{N_2}{N_1} L_{m1}$$
$$= \frac{N_1}{N_2} L_{m2} \tag{1.2-21}$$

The flux linkages may now be written as

$$\boldsymbol{\lambda} = \mathbf{Li}, \tag{1.2-22}$$

where

$$\mathbf{L} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_{l1} + L_{m1} & \dfrac{N_2}{N_1} L_{m1} \\ \dfrac{N_1}{N_2} L_{m2} & L_{l2} + L_{m2} \end{bmatrix} \tag{1.2-23}$$

Although the voltage equations with the inductance matrix **L** incorporated may be used for purposes of analysis, it is customary to perform a change of variables that yields the well-known equivalent T circuit of two magnetically coupled coils. To set the stage for this derivation, let us express the flux linkages from (1.2-22) as

$$\lambda_1 = L_{l1}i_1 + L_{m1}\left(i_1 + \frac{N_2}{N_1}i_2\right) \tag{1.2-24}$$

$$\lambda_2 = L_{l2}i_2 + L_{m2}\left(\frac{N_1}{N_2}i_1 + i_2\right) \tag{1.2-25}$$

Now we have two choices. We can use a substitute variable for $(N_2/N_1)i_2$ or for $(N_1/N_2)i_1$. Let us consider the first of these choices

$$N_1 i_2' = N_2 i_2 \tag{1.2-26}$$

whereupon we are using the substitute variable i_2' that, when flowing through coil 1, produces the same MMF as the actual i_2 flowing through coil 2. This is said to be referring the current in coil 2 to coil 1, whereupon coil 1 becomes the reference coil. On the other hand, if we use the second choice, then

$$N_2 i_1' = N_1 i_1 \tag{1.2-27}$$

Here, i_1' is the substitute variable that produces the same MMF when flowing through coil 2 as i_1 does when flowing in coil 1. This change of variables is said to refer the current of coil 1 to coil 2.

We will derive the equivalent T circuit by referring the current of coil 2 to coil 1; thus from (1.2-26)

$$i_2' = \frac{N_2}{N_1}i_2 \tag{1.2-28}$$

Power is to be unchanged by this substitution of variables. Therefore,

$$v_2' = \frac{N_1}{N_2}v_2 \tag{1.2-29}$$

whereupon $v_2 i_2 = v_2' i_2'$. Flux linkages, which have the units of volt-second, are related to the substitute flux linkages in the same way as voltages. In particular,

$$\lambda_2' = \frac{N_1}{N_2}\lambda_2 \tag{1.2-30}$$

Substituting (1.2-28) into (1.2-24) and (1.2-25) and then multiplying (1.2-25) by N_1/N_2 to obtain λ_2', and if we further substitute $(N_2^2/N_1^2)L_{m1}$ for L_{m2} into (1.2-25), then

$$\lambda_1 = L_{l1}i_1 + L_{m1}(i_1 + i_2') \tag{1.2-31}$$

$$\lambda_2' = L_{l2}'i_2' + L_{m1}(i_1 + i_2') \tag{1.2-32}$$

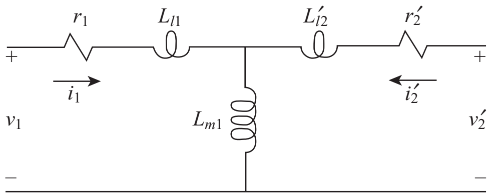

Figure 1.2-2. Equivalent circuit with coil 1 selected as reference coil.

where

$$L'_{l2} = \left(\frac{N_1}{N_2}\right)^2 L_{l2} \tag{1.2-33}$$

The voltage equations become

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt} \tag{1.2-34}$$

$$v'_2 = r'_2 i'_2 + \frac{d\lambda'_2}{dt} \tag{1.2-35}$$

where

$$r'_2 = \left(\frac{N_1}{N_2}\right)^2 r_2 \tag{1.2-36}$$

The above voltage equations suggest the T equivalent circuit shown in Figure 1.2-2. It is apparent that this method may be extended to include any number of coils wound on the same core.

EXAMPLE 1A It is instructive to illustrate the method of deriving an equivalent T circuit from open- and short-circuit measurements. For this purpose, let us assume that when coil 2 of the transformer shown in Figure 1.2-1 is open-circuited, the power input to coil 2 is 12 W when the applied voltage is 110 V (rms) at 60 Hz and the current is 1 A (rms). When coil 2 is short-circuited, the current flowing in coil 1 is 1 A when the applied voltage is 30 V at 60 Hz. The power during this test is 22 W. If we assume $L_{l1} = L'_{l2}$, an approximate equivalent T circuit can be determined from these measurements with coil 1 selected as the reference coil.

The power may be expressed as

$$P_1 = |\tilde{V}_1||\tilde{I}_1|\cos\phi \tag{1A-1}$$

where $\tilde{V}$ and $\tilde{I}$ are phasors, and ϕ is the phase angle between $\tilde{V}_1$ and $\tilde{I}_1$ (power factor angle). Solving for ϕ during the open-circuit test, we have

$$\begin{aligned}\phi &= \cos^{-1}\frac{P_1}{|\tilde{V}_1||\tilde{I}_1|} \\ &= \cos^{-1}\frac{12}{110\times 1} \\ &= 83.7^\circ \end{aligned} \tag{1A-2}$$

With $\tilde{V}_1$ as the reference phasor and assuming an inductive circuit where $\tilde{I}_1$ lags $\tilde{V}_1$,

$$\begin{aligned}Z &= \frac{\tilde{V}_1}{\tilde{I}_1} \\ &= \frac{110\underline{/0^\circ}}{1\underline{/-83.7^\circ}} \\ &= 12 + j109.3\ \Omega \end{aligned} \tag{1A-3}$$

If we neglect hysteresis (core) losses, then r_1 = 12 Ω. We also know from the above calculation that $X_{l1} + X_{m1}$ = 109.3 Ω.

For the short-circuit test, we will assume that $i_1 = -i_2'$, since transformers are designed so that $X_{m1} >> |r_2' + jX_{l2}'|$. Hence, using (1A-1) again

$$\begin{aligned}\phi &= \cos^{-1}\frac{22}{30\times 1} \\ &= 42.8^\circ \end{aligned} \tag{1A-4}$$

In this case, the input impedance is $(r_1 + r_2') + j(X_{l1} + X_{l2}')$. This may be determined as follows:

$$\begin{aligned}Z &= \frac{30\underline{/0^\circ}}{1\underline{/-42.8^\circ}} \\ &= 22 + j20.4\ \Omega \end{aligned} \tag{1A-5}$$

Hence, $r_2' = 10\ \Omega$ and, since it is assumed that $X_{l1} = X_{l2}'$, both are 10.2 Ω. Therefore, X_{m1} = 109.3 − 10.2 = 99.1 Ω. In summary

$$\begin{array}{lll} r_1 = 12\ \Omega & L_{m1} = 262.9\ \text{mH} & r_2' = 10\ \Omega \\ L_{l1} = 27.1\ \text{mH} & & L_{l2}' = 27.1\ \text{mH} \end{array}$$

Nonlinear Magnetic System

Although the analysis of transformers and electric machines is generally performed assuming a linear magnetic system, economics dictate that in the practical design of many of these devices, some saturation occurs and that heating of the magnetic material exists due to hysteresis loss. The magnetization characteristics of transformer or machine materials are given in the form of the magnitude of flux density versus

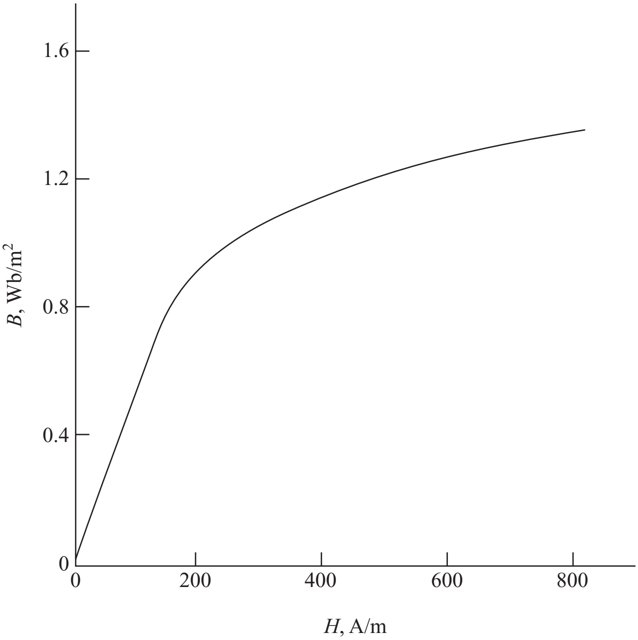

Figure 1.2-3. *B–H* curve for typical silicon steel used in transformers.

magnitude of field strength (*B–H* curve) as shown in Figure 1.2-3. If it is assumed that the magnetic flux is uniform through most of the core, then *B* is proportional to Φ and *H* is proportional to MMF. Hence, a plot of flux versus current is of the same shape as the *B–H* curve. A transformer is generally designed so that some saturation occurs during normal operation. Electric machines are also designed similarly in that a machine generally operates slightly in the saturated region during normal, rated operating conditions. Since saturation causes coefficients of the differential equations describing the behavior of an electromagnetic device to be functions of the coil currents, a transient analysis is difficult without the aid of a computer. Our purpose here is not to set forth methods of analyzing nonlinear magnetic systems. A method of incorporating the effects of saturation into a computer representation is of interest.

Formulating the voltage equations of stationary coupled coils appropriate for computer simulation is straightforward, and yet this technique is fundamental to the computer simulation of ac machines. Therefore, it is to our advantage to consider this method here. For this purpose, let us first write (1.2-31) and (1.2-32) as

$$\lambda_1 = L_{l1} i_1 + \lambda_m \tag{1.2-37}$$

$$\lambda_2' = L_{l2}' i_2' + \lambda_m \tag{1.2-38}$$

where

$$\lambda_m = L_{m1}(i_1 + i_2') \tag{1.2-39}$$

Solving (1.2-37) and (1.2-38) for the currents yields

$$i_1 = \frac{1}{L_{l1}}(\lambda_1 - \lambda_m) \tag{1.2-40}$$

$$i_2' = \frac{1}{L_{l2}'}(\lambda_2' - \lambda_m) \tag{1.2-41}$$

If (1.2-40) and (1.2-41) are substituted into the voltage equations (1.2-34) and (1.2-35), and if we solve the resulting equations for flux linkages, the following equations are obtained:

$$\lambda_1 = \int \left[v_1 + \frac{r_1}{L_{l1}}(\lambda_m - \lambda_1) \right] dt \tag{1.2-42}$$

$$\lambda_2' = \int \left[v_2' + \frac{r_2'}{L_{l2}'}(\lambda_m - \lambda_2') \right] dt \tag{1.2-43}$$

Substituting (1.2-40) and (1.2-41) into (1.2-39) yields

$$\lambda_m = L_a \left(\frac{\lambda_1}{L_{l1}} + \frac{\lambda_2'}{L_{l2}'} \right) \tag{1.2-44}$$

where

$$L_a = \left(\frac{1}{L_{m1}} + \frac{1}{L_{l1}} + \frac{1}{L_{l2}'} \right)^{-1} \tag{1.2-45}$$

We now have the equations expressed with λ_1 and λ_2' as state variables. In the computer simulation, (1.2-42) and (1.2-43) are used to solve for λ_1 and λ_2', and (1.2-44) is used to solve for λ_m. The currents can then be obtained from (1.2-40) and (1.2-41). It is clear that (1.2-44) could be substituted into (1.2-40)–(1.2-43) and λ_m eliminated from the equations, whereupon it would not appear in the computer simulation. However, we will find λ_m (the magnetizing flux linkage) an important variable when we include the effects of saturation.

If the magnetization characteristics (magnetization curve) of the coupled coil are known, the effects of saturation of the mutual flux path may be incorporated into the computer simulation. Generally, the magnetization curve can be adequately determined from a test wherein one of the coils is open-circuited (coil 2, for example) and the input impedance of coil 1 is determined from measurements as the applied voltage is increased in magnitude from 0 to say 150% of the rated value. With information obtained from this type of test, we can plot λ_m versus $(i_1' + i_2')$ as shown in Figure 1.2-4, wherein the slope of the linear portion of the curve is L_{m1}. From Figure 1.2-4, it is clear that in the region of saturation, we have

$$\lambda_m = L_{m1}(i_1 + i_2') - f(\lambda_m) \tag{1.2-46}$$

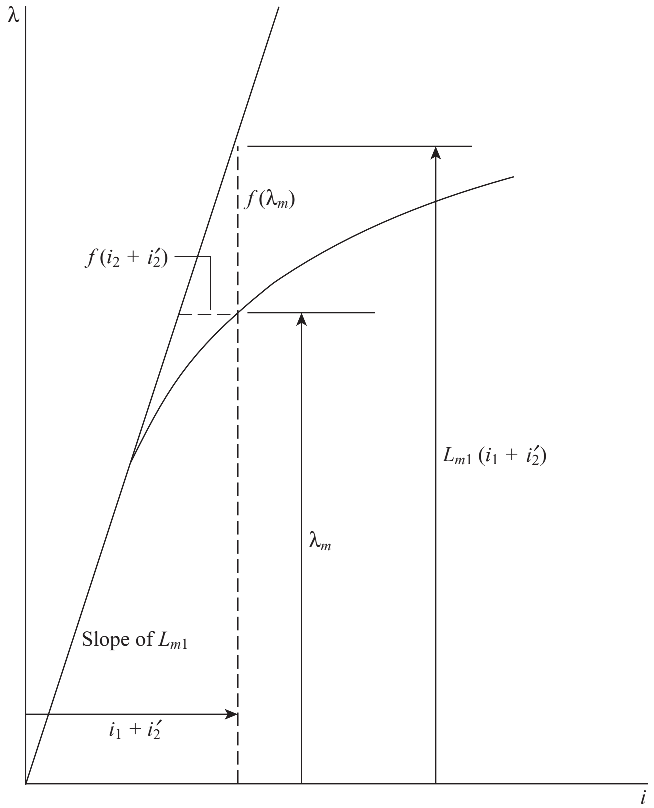

Figure 1.2-4. Magnetization curve.

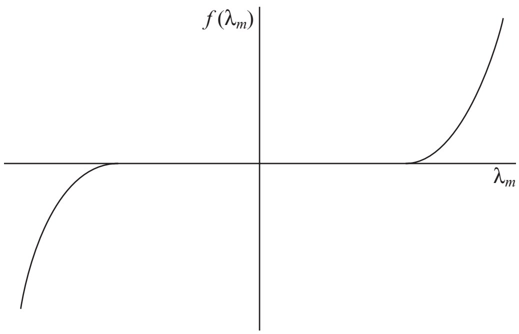

Figure 1.2-5. $f(\lambda_m)$ versus λ_m from Figure 1.2-4.

where $f(\lambda_m)$ may be determined from the magnetization curve for each value of λ_m. In particular, $f(\lambda_m)$ is a function of λ_m as shown in Figure 1.2-5. Therefore, the effects of saturation of the mutual flux path may be taken into account by replacing (1.2-39) with (1.2-46) for λ_m. Substituting (1.2-40) and (1.2-41) for i_1 and i_2', respectively, into (1.2-46) yields the following equation for λ_m

$$\lambda_m = L_a\left(\frac{\lambda_1}{L_{l1}} + \frac{\lambda_2'}{L_{l2}'}\right) - \frac{L_a}{L_{m1}} f(\lambda_m) \tag{1.2-47}$$

Hence, the computer simulation for including saturation involves replacing λ_m given by (1.2-44) with (1.2-47), where $f(\lambda_m)$ is a generated function of λ_m determined from the plot shown in Figure 1.2-5.

1.3. ELECTROMECHANICAL ENERGY CONVERSION

Although electromechanical devices are used in some manner in a wide variety of systems, electric machines are by far the most common. It is desirable, however, to establish methods of analysis that may be applied to all electromechanical devices. Prior to proceeding, it is helpful to clarify that throughout the book, the words "winding" and "coil" are used to describe conductor arrangements. To distinguish, a winding consists of one or more coils connected in series or parallel.

Energy Relationships

Electromechanical systems are comprised of an electrical system, a mechanical system, and a means whereby the electrical and mechanical systems can interact. Interaction can take place through any and all electromagnetic and electrostatic fields that are common to both systems, and energy is transferred from one system to the other as a result of this interaction. Both electrostatic and electromagnetic coupling fields may exist simultaneously and the electromechanical system may have any number of electrical and mechanical systems. However, before considering an involved system, it is helpful to analyze the electromechanical system in a simplified form. An electromechanical system with one electrical system, one mechanical system, and with one coupling field is depicted in Figure 1.3-1. Electromagnetic radiation is neglected, and it is assumed that the electrical system operates at a frequency sufficiently low so that the electrical system may be considered as a lumped parameter system.

Losses occur in all components of the electromechanical system. Heat loss will occur in the mechanical system due to friction and the electrical system will dissipate heat due to the resistance of the current-carrying conductors. Eddy current and hysteresis losses occur in the ferromagnetic material of all magnetic fields while dielectric losses occur in all electric fields. If W_E is the total energy supplied by the electrical source and W_M the total energy supplied by the mechanical source, then the energy distribution could be expressed as

$$W_E = W_e + W_{eL} + W_{eS} \tag{1.3-1}$$

$$W_M = W_m + W_{mL} + W_{mS} \tag{1.3-2}$$

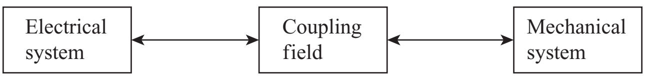

Figure 1.3-1. Block diagram of elementary electromechanical system.

In (1.3-1), W_{eS} is the energy stored in the electric or magnetic fields that are not coupled with the mechanical system. The energy W_{eL} is the heat losses associated with the electrical system. These losses occur due to the resistance of the current-carrying conductors, as well as the energy dissipated from these fields in the form of heat due to hysteresis, eddy currents, and dielectric losses. The energy W_e is the energy transferred to the coupling field by the electrical system. The energies common to the mechanical system may be defined in a similar manner. In (1.3-2), W_{mS} is the energy stored in the moving member and compliances of the mechanical system, W_{mL} is the energy losses of the mechanical system in the form of heat, and W_m is the energy transferred to the coupling field. It is important to note that with the convention adopted, the energy supplied by either source is considered positive. Therefore, $W_E(W_M)$ is negative when energy is supplied to the electrical source (mechanical source).

If W_F is defined as the total energy transferred to the coupling field, then

$$W_F = W_f + W_{fL} \tag{1.3-3}$$

where W_f is energy stored in the coupling field and W_{fL} is the energy dissipated in the form of heat due to losses within the coupling field (eddy current, hysteresis, or dielectric losses). The electromechanical system must obey the law of conservation of energy, thus

$$W_f + W_{fL} = (W_E - W_{eL} - W_{eS}) + (W_M - W_{mL} - W_{mS}) \tag{1.3-4}$$

which may be written as

$$W_f + W_{fL} = W_e + W_m \tag{1.3-5}$$

This energy relationship is shown schematically in Figure 1.3-2.

The actual process of converting electrical energy to mechanical energy (or vice versa) is independent of (1) the loss of energy in either the electrical or the mechanical systems (W_{eL} and W_{mL}), (2) the energies stored in the electric or magnetic fields that are not common to both systems (W_{eS}), or (3) the energies stored in the mechanical system (W_{mS}). If the losses of the coupling field are neglected, then the field is conservative and (1.3-5) becomes [1]

$$W_f = W_e + W_m \tag{1.3-6}$$

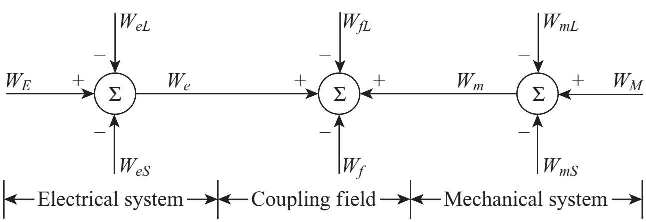

Figure 1.3-2. Energy balance.

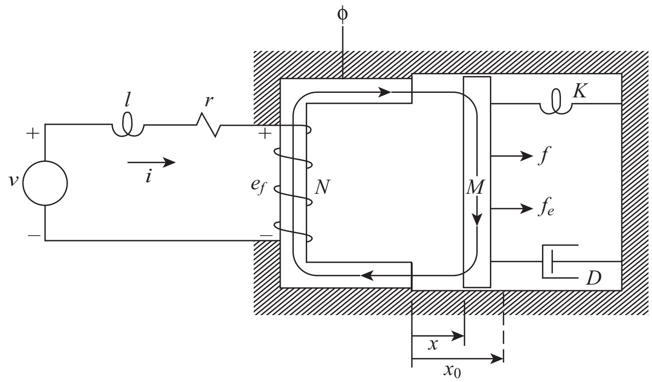

Figure 1.3-3. Electromechanical system with magnetic field.

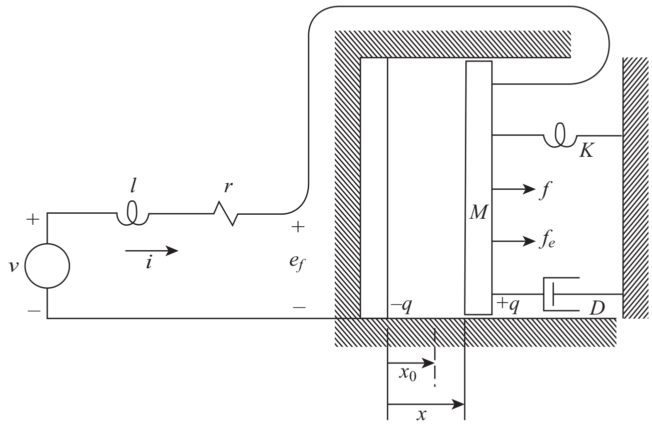

Figure 1.3-4. Electromechanical system with electric field.

Examples of elementary electromechanical systems are shown in Figure 1.3-3 and Figure 1.3-4. The system shown in Figure 1.3-3 has a magnetic coupling field, while the electromechanical system shown in Figure 1.3-4 employs an electric field as a means of transferring energy between the electrical and mechanical systems. In these systems, v is the voltage of the electric source and f is the external mechanical force applied to the mechanical system. The electromagnetic or electrostatic force is denoted by f_e. The resistance of the current-carrying conductors is denoted by r, and l denotes the inductance of a linear (conservative) electromagnetic system that does not couple the mechanical system. In the mechanical system, M is the mass of the movable member, while the linear compliance and damper are represented by a spring constant K and a damping coefficient D, respectively. The displacement x_0 is the zero force or equilibrium position of the mechanical system that is the steady-state position of the mass with f_e and f equal to zero. A series or shunt capacitance may be included in the electrical system wherein energy would also be stored in an electric field external to the electromechanical process.

The voltage equation that describes both electrical systems may be written as

$$v = ri + l\frac{di}{dt} + e_f \tag{1.3-7}$$

where e_f is the voltage drop across the coupling field. The dynamic behavior of the translational mechanical systems may be expressed by employing Newton's law of motion. Thus,

$$f = M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + K(x - x_0) - f_e \tag{1.3-8}$$

The total energy supplied by the electric source is

$$W_E = \int vi dt \tag{1.3-9}$$

The total energy supplied by the mechanical source is

$$W_M = \int f dx \tag{1.3-10}$$

which may also be expressed as

$$W_M = \int f\frac{dx}{dt}dt \tag{1.3-11}$$

Substituting (1.3-7) into (1.3-9) yields

$$W_E = r\int i^2 dt + l\int i di + \int e_f i dt \tag{1.3-12}$$

The first term on the right-hand side of (1.3-12) represents the energy loss due to the resistance of the conductors (W_{eL}). The second term represents the energy stored in the linear electromagnetic field external to the coupling field (W_{eS}). Therefore, the total energy transferred to the coupling field from the electrical system is

$$W_e = \int e_f i dt \tag{1.3-13}$$

Similarly, for the mechanical system, we have

$$W_M = M\int \frac{d^2x}{dt^2}dx + D\int \left(\frac{dx}{dt}\right)^2 dt + K\int (x - x_0)dx - \int f_e dx \tag{1.3-14}$$

Here, the first and third terms on the right-hand side of (1.3-14) represent the energy stored in the mass and spring, respectively (W_{mS}). The second term is the heat loss due to friction (W_{mL}). Thus, the total energy transferred to the coupling field from the mechanical system with one mechanical input is

$$W_m = -\int f_e dx \tag{1.3-15}$$

It is important to note that a positive force, f_e, is assumed to be in the same direction as a positive displacement, x. Substituting (1.3-13) and (1.3-15) into the energy balance relation, (1.3-6), yields

$$W_f = \int e_f i dt - \int f_e dx \tag{1.3-16}$$

The equations set forth may be readily extended to include an electromechanical system with any number of electrical inputs. Thus,

$$W_f = \sum_{j=1}^{J} W_{ej} + W_m \tag{1.3-17}$$

wherein J electrical inputs exist. The J here should not be confused with that used later for the inertia of rotational systems. The total energy supplied to the coupling field from the electrical inputs is

$$\sum_{j=1}^{J} W_{ej} = \int \sum_{j=1}^{J} e_{fj} i_j dt \tag{1.3-18}$$

The total energy supplied to the coupling field from the mechanical input is

$$W_m = -\int f_e dx \tag{1.3-19}$$

The energy balance equation becomes

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j dt - \int f_e dx \tag{1.3-20}$$

In differential form

$$dW_f = \sum_{j=1}^{J} e_{fj} i_j dt - f_e dx \tag{1.3-21}$$

Energy in Coupling Fields

Before using (1.3-21) to obtain an expression for the electromagnetic force f_e, it is necessary to derive an expression for the energy stored in the coupling fields. Once we have an expression for W_f, we can take the total derivative to obtain dW_f that can then be substituted into (1.3-21). When expressing the energy in the coupling fields, it is convenient to neglect all losses associated with the electric and magnetic fields, whereupon the fields are assumed to be conservative and the energy stored therein is a function of the state of the electrical and mechanical variables. Although the effects of the field losses may be functionally taken into account by appropriately introducing a resistance in the electric circuit, this refinement is generally not necessary since the ferromagnetic material is selected and arranged in laminations so as to minimize the hysteresis and eddy current losses. Moreover, nearly all of the energy stored in the coupling fields is stored in the air gaps of the electromechanical device. Since air is a conservative medium, all of the energy stored therein can be returned to the electrical or mechanical systems. Therefore, the assumption of lossless coupling fields is not as restrictive as it might first appear.

The energy stored in a conservative field is a function of the state of the system variables and not the manner in which the variables reached that state. It is convenient to take advantage of this feature when developing a mathematical expression for the field energy. In particular, it is convenient to fix mathematically the position of the mechanical systems associated with the coupling fields and then excite the electrical systems with the displacements of the mechanical systems held fixed. During the excitation of the electrical systems, W_m is zero, since dx is zero, even though electromagnetic or electrostatic forces occur. Therefore, with the displacements held fixed, the energy stored in the coupling fields during the excitation of the electrical systems is equal to the energy supplied to the coupling fields by the electrical systems. Thus, with $W_m = 0$, the energy supplied from the electrical system may be expressed from (1.3-20) as

$$W_f = \int \sum_{j=1}^{J} e_{fj} i_j dt \tag{1.3-22}$$

It is instructive to consider a single-excited electromagnetic system similar to that shown in Figure 1.3-3. In this case, $e_f = d\lambda/dt$ and (1.3-22) becomes

$$W_f = \int i d\lambda \tag{1.3-23}$$

Here $J = 1$, however, the subscript is omitted for the sake of brevity. The area to the left of the λ–i relationship, shown in Figure 1.3-5, for a singly excited electromagnetic device is the area described by (1.3-23). In Figure 1.3-5, this area represents the energy stored in the field at the instant when $\lambda = \lambda_a$ and $i = i_a$. The λ–i relationship need not be linear, it need only be single valued, a property that is characteristic to a conservative or lossless field. Moreover, since the coupling field is conservative, the energy stored in the field with $\lambda = \lambda_a$ and $i = i_a$ is independent of the excursion of the electrical and mechanical variables before reaching this state.

The area to the right of the λ–i curve is called the *coenergy*, and it is defined as

$$W_c = \int \lambda di \tag{1.3-24}$$

which may also be written as

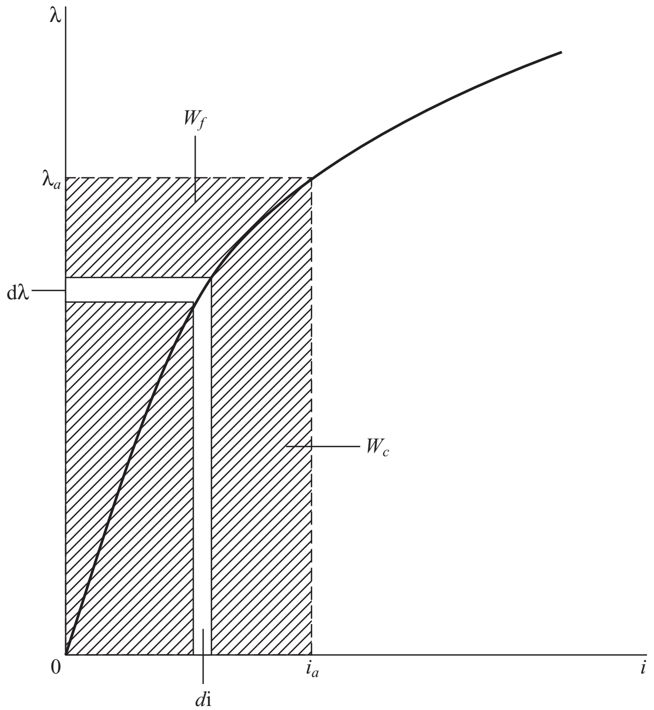

Figure 1.3-5. Stored energy and coenergy in a magnetic field of a singly excited electromagnetic device.

$$W_c = \lambda i - W_f \tag{1.3-25}$$

For multiple electrical inputs, λi in (1.3-25) becomes $\sum_{j=1}^{J} \lambda_j i_j$. Although the coenergy has little or no physical significance, we will find it a convenient quantity for expressing the electromagnetic force. It should be clear that $W_f = W_c$ for a linear magnetic system where the λ–i plots are straight-line relationships.

The displacement x defines completely the influence of the mechanical system upon the coupling field; however, since λ and i are related, only one is needed in addition to x in order to describe the state of the electromechanical system. Therefore, either λ and x or i and x may be selected as independent variables. If i and x are selected as independent variables, it is convenient to express the field energy and the flux linkages as

$$W_f = W_f(i, x) \tag{1.3-26}$$

$$\lambda = \lambda(i, x) \tag{1.3-27}$$

With i and x as independent variables, we must express $d\lambda$ in terms of di before substituting into (1.3-23). Thus, from (1.3-27)

$$d\lambda(i, x) = \frac{\partial \lambda(i, x)}{\partial i} di + \frac{\partial \lambda(i, x)}{\partial x} dx \tag{1.3-28}$$

In the derivation of an expression for the energy stored in the field, dx is set equal to zero. Hence, in the evaluation of field energy, $d\lambda$ is equal to the first term on the right-hand side of (1.3-28). Substituting into (1.3-23) yields

$$W_f(i, x) = \int i \frac{\partial \lambda(i, x)}{\partial i} di = \int_0^i \xi \frac{\partial \lambda(\xi, x)}{\partial \xi} d\xi \tag{1.3-29}$$

where ξ is the dummy variable of integration. Evaluation of (1.3-29) gives the energy stored in the field of a singly excited system. The coenergy in terms of i and x may be evaluated from (1.3-24) as

$$W_c(i, x) = \int \lambda(i, x) di = \int_0^i \lambda(\xi, x) d\xi \tag{1.3-30}$$

With λ and x as independent variables

$$W_f = W_f(\lambda, x) \tag{1.3-31}$$

$$i = i(\lambda, x). \tag{1.3-32}$$

The field energy may be evaluated from (1.3-23) as

$$W_f(\lambda, x) = \int i(\lambda, x) d\lambda = \int_0^\lambda i(\xi, x) d\xi \tag{1.3-33}$$

In order to evaluate the coenergy with λ and x as independent variables, we need to express di in terms of $d\lambda$; thus, from (1.3-32), we obtain

$$di(\lambda, x) = \frac{\partial i(\lambda, x)}{\partial \lambda} d\lambda + \frac{\partial i(\lambda, x)}{\partial x} dx \tag{1.3-34}$$

Since $dx = 0$ in this evaluation, (1.3-24) becomes

$$W_c(\lambda, x) = \int \lambda \frac{\partial i(\lambda, x)}{\partial \lambda} d\lambda = \int_0^\lambda \xi \frac{\partial i(\xi, x)}{\partial \xi} d\xi \tag{1.3-35}$$

For a linear electromagnetic system, the λ–i plots are straight-line relationships; thus, for the singly excited system, we have

$$\lambda(i, x) = L(x) i \tag{1.3-36}$$

or

$$i(\lambda, x) = \frac{\lambda}{L(x)} \tag{1.3-37}$$

Let us evaluate $W_f(i,x)$. From (1.3-28), with $dx = 0$

$$d\lambda(i, x) = L(x)di \tag{1.3-38}$$

Hence, from (1.3-29)

$$W_f(i, x) = \int_0^i \xi L(x)d\xi = \frac{1}{2}L(x)i^2 \tag{1.3-39}$$

It is left to the reader to show that $W_f(\lambda,x)$, $W_c(i,x)$, and $W_c(\lambda,x)$ are equal to (1.3-39) for this magnetically linear system.

The field energy is a state function, and the expression describing the field energy in terms of system variables is valid regardless of the variations in the system variables. For example, (1.3-39) expresses the field energy regardless of the variations in $L(x)$ and i. The fixing of the mechanical system so as to obtain an expression for the field energy is a mathematical convenience and not a restriction upon the result.

In the case of a multiexcited, electromagnetic system, an expression for the field energy may be obtained by evaluating the following relation with $dx = 0$:

$$W_f = \int \sum_{j=1}^{J} i_j d\lambda_j \tag{1.3-40}$$

Because the coupling fields are considered conservative, (1.3-40) may be evaluated independent of the order in which the flux linkages or currents are brought to their final values. To illustrate the evaluation of (1.3-40) for a multiexcited system, we will allow the currents to establish their final states one at a time while all other currents are mathematically fixed either in their final or unexcited state. This procedure may be illustrated by considering a doubly excited electric system. An electromechanical system of this type could be constructed by placing a second coil, supplied from a second electrical system, on either the stationary or movable member of the system shown in Figure 1.3-3. In this evaluation, it is convenient to use currents and displacement as the independent variables. Hence, for a doubly excited electric system

$$W_f(i_1, i_2, x) = \int [i_1 d\lambda_1(i_1, i_2, x) + i_2 d\lambda_2(i_1, i_2, x)] \tag{1.3-41}$$

In this determination of an expression for W_f, the mechanical displacement is held constant ($dx = 0$); thus (1.3-41) becomes

$$\begin{aligned} W_f(i_1, i_2, x) = \int i_1 &\left[\frac{\partial\lambda_1(i_1, i_2, x)}{\partial i_1}di_1 + \frac{\partial\lambda_1(i_1, i_2, x)}{\partial i_2}di_2\right] \\ &+ i_2\left[\frac{\partial\lambda_2(i_1, i_2, x)}{\partial i_1}di_1 + \frac{\partial\lambda_2(i_1, i_2, x)}{\partial i_2}di_2\right] \end{aligned} \tag{1.3-42}$$

We will evaluate the energy stored in the field by employing (1.3-42) twice. First, we will mathematically bring the current i_1 to the desired value while holding i_2 at zero.

Thus, i_1 is the variable of integration and $di_2 = 0$. Energy is supplied to the coupling field from the source connected to coil 1. As the second evaluation of (1.3-42), i_2 is brought to its desired current while holding i_1 at its desired value. Hence, i_2 is the variable of integration and $di_1 = 0$. During this time, energy is supplied from both sources to the coupling field since $i_1 d\lambda_1$ is nonzero. The total energy stored in the coupling field is the sum of the two evaluations. Following this two-step procedure, the evaluation of (1.3-42) for the total field energy becomes

$$W_f(i_1, i_2, x) = \int i_1 \frac{\partial \lambda_1(i_1, i_2, x)}{\partial i_1} di_1 + \int \left[i_1 \frac{\partial \lambda_1(i_1, i_2, x)}{\partial i_2} di_2 + i_2 \frac{\partial \lambda_2(i_1, i_2, x)}{\partial i_2} di_2 \right] \tag{1.3-43}$$

which should be written as

$$W_f(i_1, i_2, x) = \int_0^{i_1} \xi \frac{\partial \lambda_1(\xi, i_2, x)}{\partial \xi} d\xi + \int_0^{i_2} \left[i_1 \frac{\partial \lambda_1(i_1, \xi, x)}{\partial \xi} d\xi + \xi \frac{\partial \lambda_2(i_1, \xi, x)}{\partial \xi} d\xi \right] \tag{1.3-44}$$

The first integral on the right-hand side of (1.3-43) or (1.3-44) results from the first step of the evaluation, with i_1 as the variable of integration and with $i_2 = 0$ and $di_2 = 0$. The second integral comes from the second step of the evaluation with $i_1 = i_1$, $di_1 = 0$, and i_2 as the variable of integration. It is clear that the order of allowing the currents to reach their final state is irrelevant; that is, as our first step, we could have made i_2 the variable of integration while holding i_1 at zero ($di_1 = 0$) and then let i_1 become the variable of integration while holding i_2 at its final value. The result would be the same. It is also clear that for three electrical inputs, the evaluation procedure would require three steps, one for each current to be brought mathematically to its final state.

Let us now evaluate the energy stored in a magnetically linear electromechanical system with two electric inputs. For this, let

$$\lambda_1(i_1, i_2, x) = L_{11}(x)i_1 + L_{12}(x)i_2 \tag{1.3-45}$$

$$\lambda_2(i_1, i_2, x) = L_{21}(x)i_1 + L_{22}(x)i_2 \tag{1.3-46}$$

With that mechanical displacement held constant ($dx = 0$),

$$d\lambda_1(i_1, i_2, x) = L_{11}(x)di_1 + L_{12}(x)di_2 \tag{1.3-47}$$

$$d\lambda_2(i_1, i_2, x) = L_{12}(x)di_1 + L_{22}(x)di_2. \tag{1.3-48}$$

It is clear that the coefficients on the right-hand side of (1.3-47) and (1.3-48) are the partial derivatives. For example, $L_{11}(x)$ is the partial derivative of $\lambda_1(i_1,i_2,x)$ with respect to i_1. Appropriate substitution into (1.3-44) gives

$$W_f(i_1, i_2, x) = \int_0^{i_1} \xi L_{11}(x) d\xi + \int_0^{i_2} [i_1 L_{12}(x) + \xi L_{22}(x)] d\xi \tag{1.3-49}$$

which yields

$$W_f(i_1, i_2, x) = \frac{1}{2}L_{11}(x)i_1^2 + L_{12}(x)i_1 i_2 + \frac{1}{2}L_{22}(x)i_2^2 \tag{1.3-50}$$

The extension to a linear electromagnetic system with J electrical inputs is straightforward, whereupon the following expression for the total field energy is obtained as

$$W_f(i_1, \ldots, i_J, x) = \frac{1}{2}\sum_{p=1}^{J}\sum_{q=1}^{J} L_{pq} i_p i_q \tag{1.3-51}$$

It is left to the reader to show that the equivalent of (1.3-22) for a multiexcited electrostatic system is

$$W_f = \int \sum_{j=1}^{J} e_{fj}\, dq_j \tag{1.3-52}$$

Graphical Interpretation of Energy Conversion

Before proceeding to the derivation of expressions for the electromagnetic force, it is instructive to consider briefly a graphical interpretation of the energy conversion process. For this purpose, let us again refer to the elementary system shown in Figure 1.3-3, and let us assume that as the movable member moves from $x = x_a$ to $x = x_b$, where $x_b < x_a$, the λ–i characteristics are given by Figure 1.3-6. Let us further assume that as the member moves from x_a to x_b, the λ–i trajectory moves from point A to point B. It is clear that the exact trajectory from A to B is determined by the combined dynamics of the electrical and mechanical systems. Now, the area $OACO$ represents the original energy stored in field; area $OBDO$ represents the final energy stored in the field. Therefore, the change in field energy is

$$\Delta W_f = \text{area } OBDO - \text{area } OACO \tag{1.3-53}$$

The change in W_e, denoted as ΔW_e, is

$$\Delta W_e = \int_{\lambda_A}^{\lambda_B} i\, d\lambda = \text{area } CABDC \tag{1.3-54}$$

We know that

$$\Delta W_m = \Delta W_f - \Delta W_e \tag{1.3-55}$$

Hence,

$$\Delta W_m = \text{ area } OBDO - \text{ area } OACO - \text{ area } CABDC = -\text{area } OABO \tag{1.3-56}$$

Here, ΔW_m is negative; energy has been supplied to the mechanical system from the coupling field, part of which came from the energy stored in the field and part from the

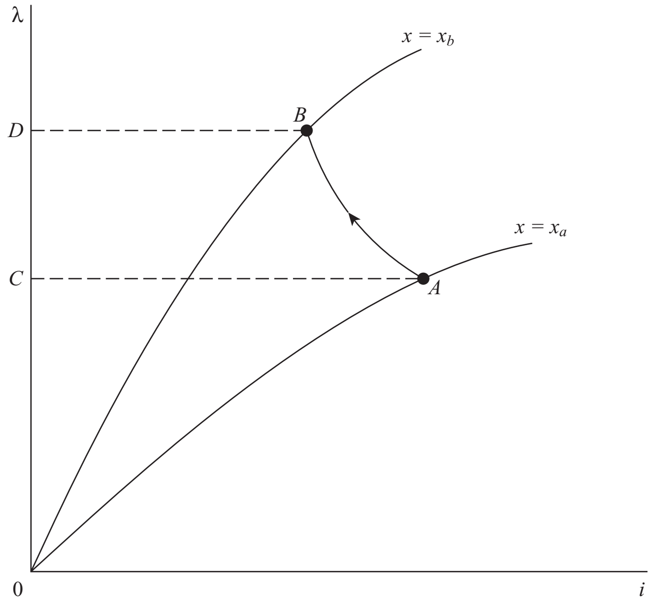

Figure 1.3-6. Graphical representation of electromechanical energy conversion for λ–i path A to B.

electrical system. If the member is now moved back to x_a, the λ–i trajectory may be as shown in Figure 1.3-7. Hence ΔW_m is still area *OABO*, but it is now positive, which means that energy was supplied from the mechanical system to the coupling field, part of which is stored in the field and part of which is transferred to the electrical system. The net ΔW_m for the cycle from A to B back to A is the shaded area shown in Figure 1.3-8. Since ΔW_f is zero for this cycle

$$\Delta W_m = -\Delta W_e \tag{1.3-57}$$

For the cycle shown, the net ΔW_e is negative, thus ΔW_m is positive; we have generator action. If the trajectory had been in the counterclockwise direction, the net ΔW_e would have been positive and the net ΔW_m negative, which would represent motor action.

Electromagnetic and Electrostatic Forces

The energy balance relationships given by (1.3-21) may be arranged as

$$f_e dx = \sum_{j=1}^{J} e_{fj} i_j \, dt - dW_f \tag{1.3-58}$$

In order to obtain an expression for f_e, it is necessary to first express W_f and then take its total derivative. One is tempted to substitute the integrand of (1.3-22) into (1.3-58)

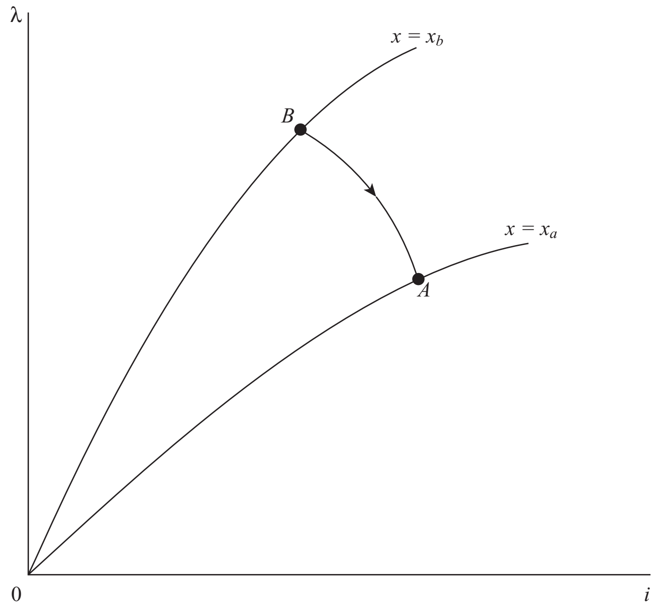

Figure 1.3-7. Graphical representation of electromechanical energy conversion for λ–i path *B* to *A*.

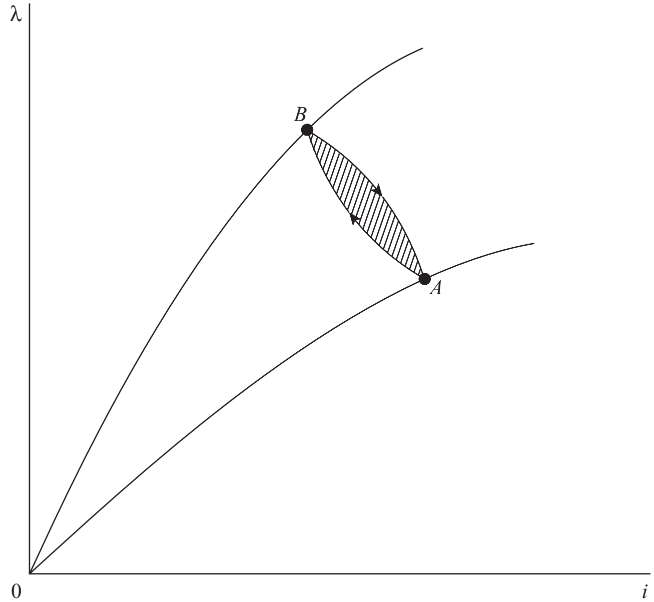

Figure 1.3-8. Graphical representation of electromechanical energy conversion for λ–i path *A* to *B* to *A*.

for the infinitesimal change of field energy. This procedure is, of course, incorrect, since the integrand of (1.3-22) was obtained with the mechanical displacement held fixed ($dx = 0$), where the total differential of the field energy is required in (1.3-58). In the following derivation, we will consider multiple electrical inputs; however, we will consider only one mechanical input, as we noted earlier in (1.3-15). Electromechnical systems with more than one mechanical input are not common; therefore, the additional notation necessary to include multiple mechanical inputs is not warranted. Moreover, the final results of the following derivation may be readily extended to include multiple mechanical inputs.

The force or torque in any electromechanical system may be evaluated by employing (1.3-58). In many respects, one gains a much better understanding of the energy conversion process of a particular system by starting the derivation of the force or torque expression with (1.3-58) rather than selecting a relationship from a table. However, for the sake of completeness, derivation of the force equations will be set forth and tabulated for electromechanical systems with one mechanical input and J electrical inputs.

For an electromagnetic system, (1.3-58) may be written as

$$f_e dx = \sum_{j=1}^{J} i_j d\lambda_j - dW_f \tag{1.3-59}$$

Although we will use (1.3-59), it is helpful to express it in an alternative form. For this purpose, let us first write (1.3-25) for multiple electrical inputs

$$\sum_{j=1}^{J} \lambda_j i_j = W_c + W_f \tag{1.3-60}$$

If we take the total derivative of (1.3-60), we obtain

$$\sum_{j=1}^{J} \lambda_j di_j + \sum_{j=1}^{J} i_j d\lambda_j = dW_c + dW_f \tag{1.3-61}$$

We realize that when we evaluate the force f_e we must select the independent variables; that is, either the flux linkages or the currents and the mechanical displacement x. The flux linkages and the currents cannot simultaneously be considered independent variables when evaluating the f_e. Nevertheless, (1.3-61), wherein both $d\lambda_j$ and di_j appear, is valid in general, before a selection of independent variables is made to evaluate f_e. If we solve (1.3-61) for the total derivative of field energy, dW_f, and substitute the result into (1.3-59), we obtain

$$f_e dx = -\sum_{j=1}^{J} \lambda_j di_j + dW_c \tag{1.3-62}$$

Either (1.3-59) or (1.3-62) can be used to evaluate the electromagnetic force f_e. If flux linkages and x are selected as independent variables, (1.3-59) is the most direct, while (1.3-62) is the most direct if currents and x are selected.

With flux linkages and x as the independent variables, the currents are expressed functionally as

$$i_j(\lambda_1, \ldots, \lambda_j, x) \tag{1.3-63}$$

For the purpose of compactness, we will denote $(\lambda_1, \ldots, \lambda_j, x)$ as $(\boldsymbol{\lambda}, x)$, where $\boldsymbol{\lambda}$ is an abbreviation for the complete set of flux linkages associated with the J windings. Let us write (1.3-59) with flux linkages and x as independent variables

$$f_e(\boldsymbol{\lambda}, x)dx = \sum_{j=1}^{J} i_j(\boldsymbol{\lambda}, x)d\lambda_j - dW_f(\boldsymbol{\lambda}, x) \tag{1.3-64}$$

If we take the total derivative of the field energy with respect to $\boldsymbol{\lambda}$ and x, and substitute that result into (1.3-64), we obtain

$$f_e(\boldsymbol{\lambda}, x)dx = \sum_{j=1}^{J} i_j(\boldsymbol{\lambda}, x)d\lambda_j - \sum_{j=1}^{J} \frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial \lambda_j} d\lambda_j - \frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial x} dx \tag{1.3-65}$$

Equating the coefficients of dx gives

$$f_e(\boldsymbol{\lambda}, x) = -\frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial x} \tag{1.3-66}$$

A second expression for $f_e(\boldsymbol{\lambda},x)$ may be obtained by expressing (1.3-59) with flux linkages and x as independent variables, solving for $W_f(\boldsymbol{\lambda},x)$ and then taking the partial derivative with respect to x. Thus,

$$f_e(\boldsymbol{\lambda}, x) = -\sum_{j=1}^{J} \left[\lambda_j \frac{\partial i_j(\boldsymbol{\lambda}, x)}{\partial x} \right] + \frac{\partial W_c(\boldsymbol{\lambda}, x)}{\partial x} \tag{1.3-67}$$

If we now select $\mathbf{i}$ and x as independent variables, where $\mathbf{i}$ is the abbreviated notation for $(i_1, \ldots, i_J, x)$, then (1.3-62) can be written

$$f_e(\mathbf{i}, x)dx = -\sum_{j=1}^{J} \lambda_j(\mathbf{i}, x)di_j + dW_c(\mathbf{i}, x) \tag{1.3-68}$$

If we take the total derivative of $W_c(\mathbf{i},x)$ and substitute the result into (1.3-68), we obtain

TABLE 1.3-1. Electromagnetic Force at Mechanical Input

$f_e(\mathbf{i}, x) = \sum_{j=1}^{J}\left[i_j \frac{\partial \lambda_j(\mathbf{i}, x)}{\partial x}\right] - \frac{\partial W_f(\mathbf{i}, x)}{\partial x}$
$f_e(\mathbf{i}, x) = \frac{\partial W_c(\mathbf{i}, x)}{\partial x}$
$f_e(\boldsymbol{\lambda}, x) = -\frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial x}$
$f_e(\boldsymbol{\lambda}, x) = -\sum_{j=1}^{J}\left[\lambda_j \frac{\partial i_j(\boldsymbol{\lambda}, x)}{\partial x}\right] + \frac{\partial W_c(\boldsymbol{\lambda}, x)}{\partial x}$

Note: For rotational systems, replace $\boldsymbol{f_e}$ with $\boldsymbol{T_e}$ and $\boldsymbol{x}$ with $\boldsymbol{\theta}$.

$$f_e(\mathbf{i}, x)dx = -\sum_{j=1}^{J}\lambda_j(\mathbf{i}, x)di_j + \sum_{j=1}^{J}\frac{\partial W_c(\mathbf{i}, x)}{\partial i_j}di_j + \frac{\partial W_c(\mathbf{i}, x)}{\partial x}dx \tag{1.3-69}$$

Equating coefficients of dx yields

$$f_e(\mathbf{i}, x) = \frac{\partial W_c(\mathbf{i}, x)}{\partial x} \tag{1.3-70}$$

We will make extensive use of this expression. If we now solve (1.3-60) for $W_c(\mathbf{i},x)$ and then take the partial derivative with respect to x, we can obtain a second expression for $f_e(\mathbf{i},x)$. That is

$$f_e(\mathbf{i}, x) = \sum_{j=1}^{J}\left[i_j \frac{\partial \lambda_j(\mathbf{i}, x)}{\partial x}\right] - \frac{\partial W_f(\mathbf{i}, x)}{\partial x} \tag{1.3-71}$$

We have derived four expressions for the electromagnetic force, which are summarized in Table 1.3-1. Since we will generally use currents and x as independent variables, the two expressions for $f_e(\mathbf{i},x)$ are listed first in Table 1.3-1.

Before proceeding to the next section, it is important to take a moment to look back. In order to obtain $f_e(\boldsymbol{\lambda},x)$, we equated the coefficients of dx in (1.3-65). If, however, we equate the coefficients of $d\lambda_j$ in (1.3-65), we obtain

$$\sum_{j=1}^{J}\frac{\partial W_f(\boldsymbol{\lambda}, x)}{\partial \lambda_j} = \sum_{j=1}^{J} i_j(\boldsymbol{\lambda}, x) \tag{1.3-72}$$

Similarly, if we equate the coefficients of di_j in (1.3-69), we obtain

$$\sum_{j=1}^{J}\frac{\partial W_c(\mathbf{i}, x)}{\partial i_j} = \sum_{j=1}^{J}\lambda_j(\mathbf{i}, x) \tag{1.3-73}$$

TABLE 1.3-2. Electrostatic Force at Mechanical Input

$$f_e(\mathbf{e}_f, x) = \sum_{j=1}^{J}\left[e_{fj}\frac{\partial q_j(\mathbf{e}_f, x)}{\partial x}\right] - \frac{\partial W_f(\mathbf{e}_f, x)}{\partial x}$$

$$f_e(\mathbf{e}_f, x) = \frac{\partial W_c(\mathbf{e}_f, x)}{\partial x}$$

$$f_e(\mathbf{q}, x) = -\frac{\partial W_f(\mathbf{q}, x)}{\partial x}$$

$$f_e(\mathbf{q}, x) = -\sum_{j=1}^{J}\left[q_j\frac{\partial e_{fj}(\mathbf{q}, x)}{\partial x}\right] + \frac{\partial W_c(\mathbf{q}, x)}{\partial x}$$

Note: For rotational systems, replace $\boldsymbol{f_e}$ with $\boldsymbol{T_e}$ and $\boldsymbol{x}$ with $\boldsymbol{\theta}$.

Equations (1.3-72) and (1.3-73) are readily verified by recalling the definitions of W_f and W_c that were obtained by holding x fixed ($dx = 0$).

In Table 1.3-1, the independent variables to be used are designated in each equation by the abbreviated functional notation. Although only translational mechanical systems have been considered, all force relationships developed herein may be modified for the purpose of evaluating the torque in rotational systems. In particular, when considering a rotational system, f_e is replaced with the electromagnetic torque T_e and x with the angular displacement θ. These substitutions are justified since the change of mechanical energy in a rotational system is expressed as

$$dW_m = -T_e d\theta \tag{1.3-74}$$

The force equation for an electromechanical system with electric coupling fields may be derived by following a procedure similar to that used in the case of magnetic coupling fields. These relationships are given in Table 1.3-2 without explanation or proof.

It is instructive to derive the expression for the electromagnetic force of a singly excited electric system as shown in Figure 1.3-3. It is clear that the expressions given in Table 1.3-1 are valid for magnetically linear or nonlinear systems. If we assume the magnetic system is linear, then $\lambda(i,x)$ is expressed by (1.3-36) and $W_f(i,x)$ by (1.3-39), which is also equal to the coenergy. Hence, either the first or second entry of Table 1.3-1 can be used to express f_e. In particular

$$f_e(i, x) = \frac{\partial W_c(i, x)}{\partial x} = \frac{1}{2}i^2\frac{dL(x)}{dx} \tag{1.3-75}$$

With the convention established, a positive electromagnetic force is assumed to act in the direction of increasing x. Thus, with (1.3-15) expressed in differential form as

$$dW_m = -f_e dx \tag{1.3-76}$$

we see that energy is supplied to the coupling field from the mechanical system when f_e and dx are opposite in sign, and energy is supplied to the mechanical system from the coupling field when f_e and dx are the same in sign.

From (1.3-75) it is apparent that when the change of $L(x)$ with respect to x is negative, f_e is negative. In the electromechanical system shown in Figure 1.3-3, the change $L(x)$ with respect to x is always negative, therefore, the electromagnetic force is in the direction so as to pull the movable member to the stationary member. In other words, an electromagnetic force is set up so as to maximize the inductance of the coupling system, or, since reluctance is inversely proportional to the inductance, the force tends to minimize the reluctance. Since f_e is always negative in the system shown in Figure 1.3-3, energy is supplied to the coupling field from the mechanical system (generator action) when dx is positive and from the coupling field to the mechanical system (motor action) when dx is negative.

Steady-State and Dynamic Performance of an Electromechanical System

It is instructive to consider the steady-state and dynamic performance of the elementary electromagnetic system shown in Figure 1.3-3. The differential equations that describe this system are given by (1.3-7) for the electrical system and (1.3-8) for the mechanical system. The electromagnetic force f_e is expressed by (1.3-75). If the applied voltage, v, and the applied mechanical force, f, are constant, all derivatives with respect to time are zero during steady-state operation, and the behavior can be predicted by

$$v = ri \tag{1.3-77}$$

$$f = K(x - x_0) - f_e \tag{1.3-78}$$

Equation (1.3-78) may be written as

$$-f_e = f - K(x - x_0) \tag{1.3-79}$$

The magnetic core of the system in Figure 1.3-3 is generally constructed of ferromagnetic material with a relative permeability in the order of 2000–4000. In this case, the inductance $L(x)$ can be adequately approximated by

$$L(x) = \frac{k}{x} \tag{1.3-80}$$

In the actual system, the inductance will be a large finite value rather than infinity, as predicted by (1.3-80), when $x = 0$. Nevertheless, (1.3-80) is quite sufficient to illustrate the action of the system for $x > 0$. Substituting (1.3-80) into (1.3-75) yields

$$f_e(i, x) = -\frac{ki^2}{2x^2} \tag{1.3-81}$$

A plot of (1.3-79), with f_e replaced by (1.3-81), is shown in Figure 1.3-9 for the following system parameters [1]:

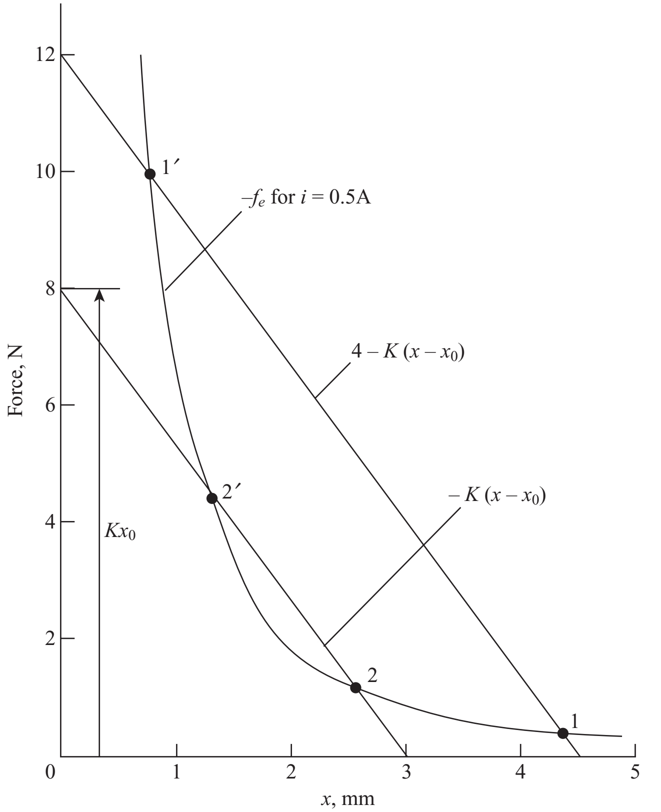

Figure 1.3-9. Steady state operation of electromechanical system shown in Figure 1.3-3.

$$r = 10\ \Omega \qquad x_0 = 3\ \text{mm}$$
$$K = 2667\ \text{N/m} \qquad k = 6.293 \times 10^{-5}\ \text{H}\cdot\text{m}$$

In Figure 1.3-9, the plot of the negative of the electromagnetic force is for an applied voltage of 5 V whereupon the steady-state current is 0.5 A. The straight lines represent the right-hand side of (1.3-79) with $f = 0$ (lower straight line) and $f = 4$ N (upper straight line). Both lines intersect the $-f_e$ curve at two points. In particular, the upper line intersects the $-f_e$ curve at 1 and 1′; the lower line intersects at 2 and 2′. Stable operation occurs at only points 1 and 2. The system will not operate stably at points 1′ and 2′. This can be explained by assuming the system is operating at one of these points (1′ and 2′) and then show that any system disturbance whatsoever will cause the system to move away from these points. If, for example, x increases slightly from its value corresponding to point 1′, the restraining force $f - K(x - x_0)$ is larger in magnitude than $-f_e$, and x will continue to increase until the system reaches operating point 1. If x increases beyond its value corresponding to operating point 1, the restraining force is less than the electromagnetic force. Therefore, the system will establish steady-state operation at 1. If, on the other hand, x decreases from point 1′, the electromagnetic

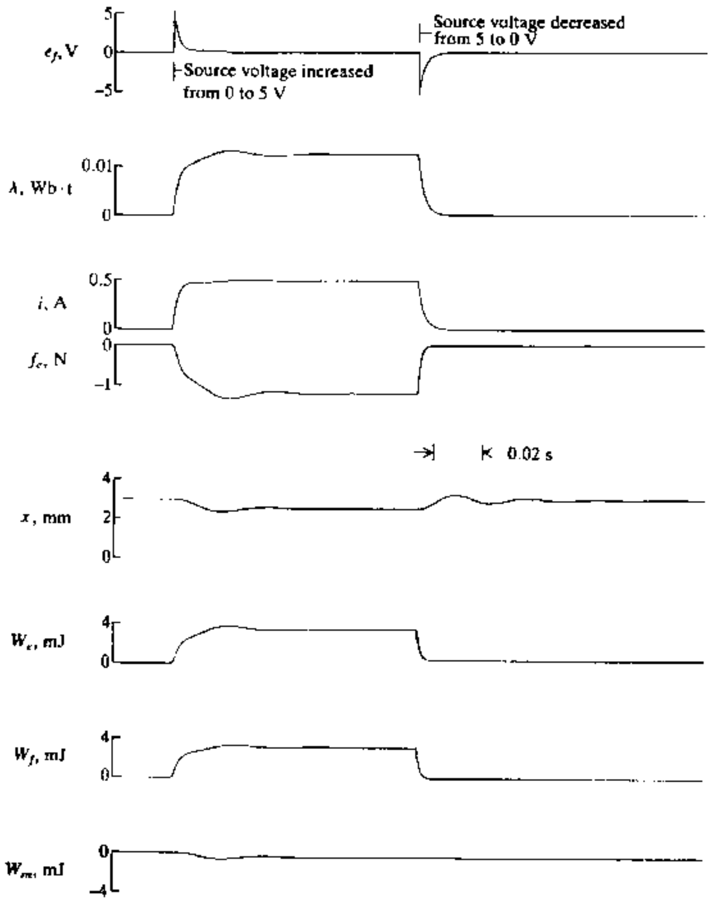

Figure 1.3-10. Dynamic performance of the electromechanical system shown in Figure 1.3-3 during step changes in the source voltage.

force is larger than the restraining force. Therefore, the movable member will move until it comes in contact with the stationary member ($x = 0$). It is clear that the restraining force that yields a straight line below the $-f_e$ curve will not permit stable operation with $x > 0$.

The dynamic behavior of the system during step changes in the source voltage v is shown in Figure 1.3-10, and in Figure 1.3-11 and Figure 1.3-12 for step changes in the applied force f. The following system parameters were used in addition to those given previously:

$$l = 0 \quad M = 0.055 \text{ kg} \quad D = 4 \text{ N}\cdot\text{s/m}$$

The computer traces shown in Figure 1.3-10 depict the dynamic performance of the example system when the applied voltage is stepped from 0 to 5 V and then back to 0 with the applied mechanical force held equal to 0. The following system variables: e_f, λ, i, f_e, x, W_e, W_f, and W_m are shown. The energies are plotted in millijoules (mJ). Initially, the

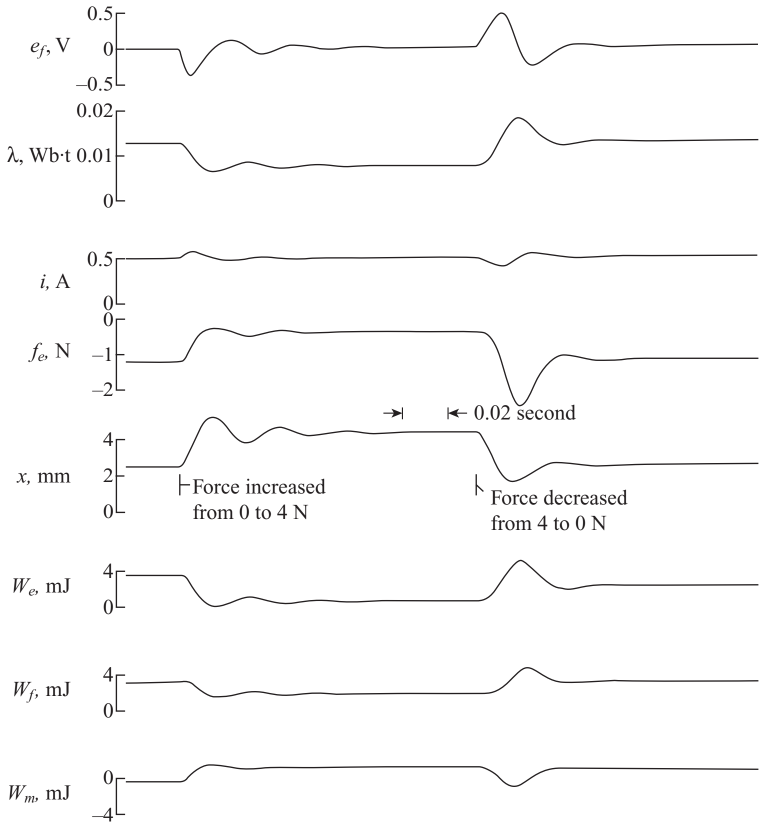

Figure 1.3-11. Dynamic performance of the electromechanical system shown in Figure 1.3-3 during step changes in the applied force.

mechanical system is at rest with $x = x_0$ (3 mm). When the source voltage is applied, x decreases, and when steady-state operation is reestablished, x is approximately 2.5 mm. Energy enters the coupling field via W_e. The bulk of this energy is stored in the field (W_f), with a smaller amount transferred to the mechanical system, some of which is dissipated in the damper during the transient period while the remainder is stored in the spring. When the applied voltage is removed, the electrical and mechanical systems return to their original states. The change in W_m is small, increasing only slightly. Hence, during the transient period, there is an interchange of energy between the spring and mass that is finally dissipated in the damper. The net change in W_f during the application and removal of the applied voltage is zero, hence the net change in W_e is positive and equal to the negative of the net change in W_m. The energy transferred to the mechanical system during this cycle is dissipated in the damper, since f is fixed at zero, and the mechanical system returns to its initial rest position with zero energy stored in the spring.

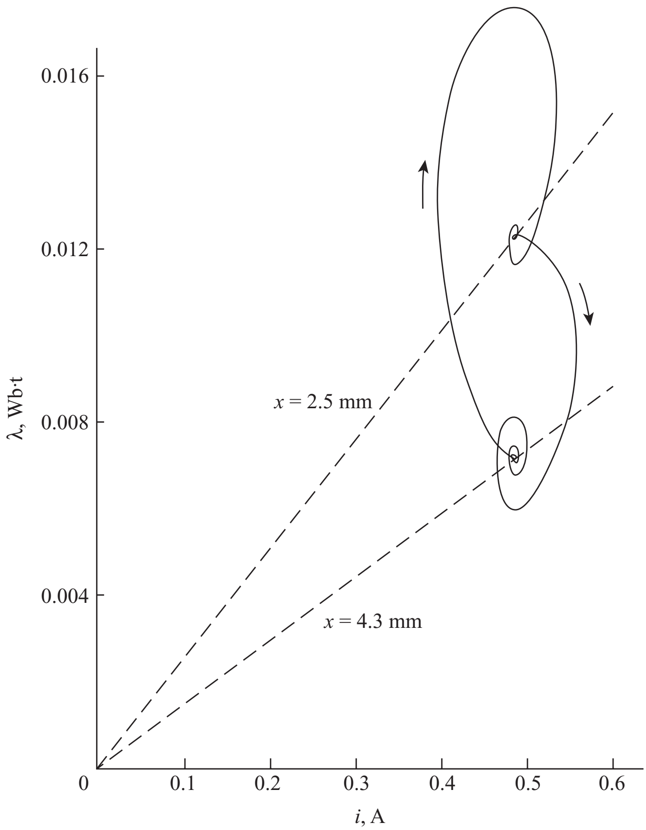

Figure 1.3-12. System response shown in Figure 1.3-3.

In Figure 1.3-11, the initial state is that shown in Figure 1.3-10, with 5 V applied to the electrical system. The mechanical force f is increased from zero to 4 N, whereupon energy enters the coupling field from the mechanical system. Energy is transferred from the coupling field to the electrical system, some coming from the mechanical system and some from the energy originally stored in the magnetic field. Next, the force is stepped back to zero from 4 N. The electrical and mechanical systems return to their original states. During the cycle, a net energy has been transferred from the mechanical system to the electrical system that is dissipated in the resistance. This cycle is depicted on the λ–i plot shown in Figure 1.3-12.

EXAMPLE 1B It is helpful to formulate an expression for the electromagnetic torque of the elementary rotational device shown in Figure 1B-1. This device consists of two conductors. Conductor 1 is placed on the stationary member (stator); conductor 2 is fixed on the rotating member (rotor). The crossed lines inside a circle indicate that the assumed direction of positive current flow is into the paper (we are seeing the tail of the arrow), while a dot inside a circle indicates positive current flow is out of the paper

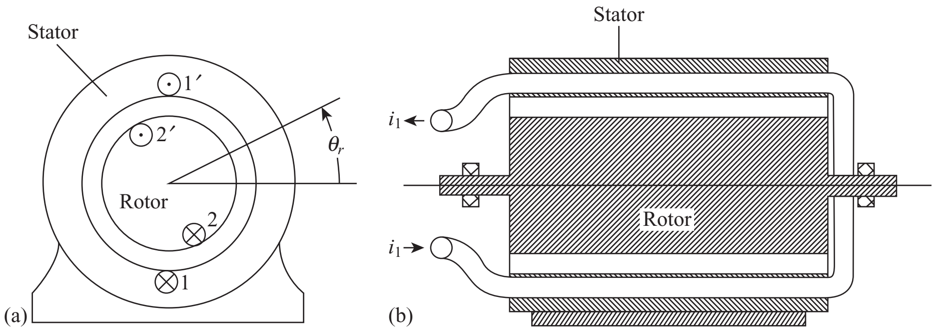

Figure 1B-1. Elementary rotational electromechanical device. (a) End view; (b) cross-sectional view.

(the point of the arrow). The length of the air gap between the stator and rotor is shown exaggerated relative to the inside diameter of the stator.

The voltage equations may be written as

$$v_1 = i_1 r_1 + \frac{d\lambda_1}{dt} \tag{1B-1}$$

$$v_2 = i_2 r_2 + \frac{d\lambda_2}{dt} \tag{1B-2}$$

where r_1 and r_2 are the resistances of conductor 1 and 2, respectively. The magnetic system is assumed linear; therefore the flux linkages may be expressed

$$\lambda_1 = L_{11} i_1 + L_{12} i_2 \tag{1B-3}$$

$$\lambda_2 = L_{21} i_1 + L_{22} i_2 \tag{1B-4}$$

The self-inductances L_{11} and L_{22} are constant. Let us assume that the mutual inductance may be approximated by

$$L_{12} = L_{21} = M \cos\theta_r \tag{1B-5}$$

where θ_r is defined in Figure 1B-1. The reader should be able to justify the form of (1B-5) by considering the mutual coupling between the two conductors as θ_r varies from 0 to 2π rad.

$$T_e(i_1, i_2, \theta_r) = \frac{\partial W_c(i_1, i_2, \theta_r)}{\partial \theta_r} \tag{1B-6}$$

Because the magnetic system is assumed to be linear, we have

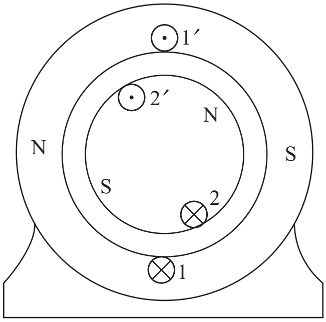

Figure 1B-2. Stator and rotor poles for constant currents.

$$W_c(i_1, i_2, \theta_r) = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1i_2 + \frac{1}{2}L_{22}i_2^2 \qquad \text{(1B-7)}$$

Substituting into (1B-6) yields

$$T_e = -i_1i_2M\sin\theta_r \qquad \text{(1B-8)}$$

Consider for a moment the form of the torque if i_1 and i_2 are both constant. For the positive direction of current shown, the torque is of the form

$$T_e = -K\sin\theta_r, \qquad \text{(1B-9)}$$

where K is a positive constant. We can visualize the production of torque by considering the interaction of the magnetic poles produced by the current flowing in the conductors. If both i_1 and i_2 are positive, the poles produced are as shown in Figure 1B-2. One should recall that by definition flux issues from a north pole. Also, the stator and rotor each must be considered as separate electromagnetic systems. Thus, flux produced by the 1–1′ winding issues from the north pole of the stator into the air gap. Similarly, the flux produced by the 2–2′ winding enters the air gap from the north pole of the rotor. It is left to the reader to justify the fact that the range of θ_r over which stable operation can occur for the expression of electromagnetic torque given by (1B-9) is $-\pi/2 \le \theta_r \le \pi/2$.

1.4. ELEMENTARY AC MACHINES

In Chapter 2, winding distributions, rotating air-gap magnetomotive force (MMF), and winding inductances common to several classes of electric machinery are derived in analytical detail. However, Example 1B has set the stage for us to take a preliminary look at what is coming in Chapter 2 from an elementary standpoint. For this purpose, let us consider a two-phase induction machine by adding two windings to Figure 1B-1; one on the stator and one on the rotor as shown in Figure 1.4-1. This device has two identical windings (same resistance and same number of turns) on the stator and two identical

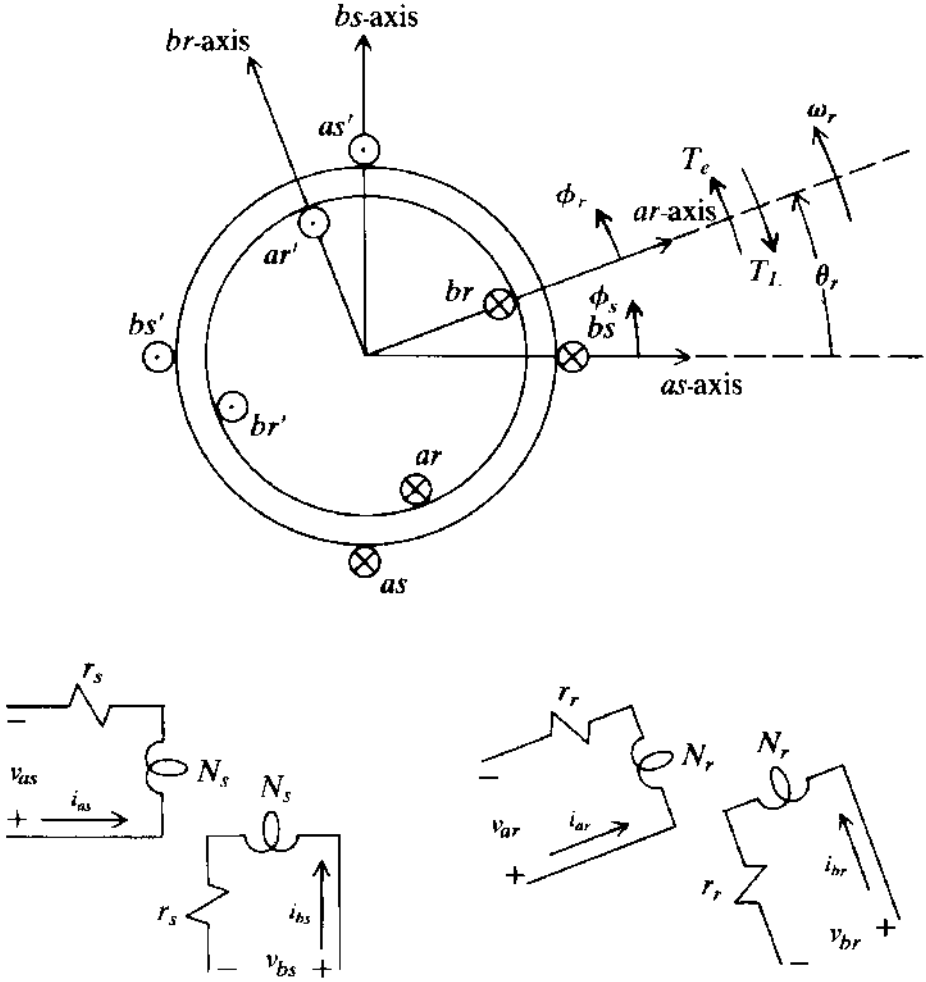

Figure 1.4-1. A two-pole, two-phase induction machine.

windings on the rotor. It is referred to as a symmetrical two-phase induction machine. We can write the flux linkage equations for the *as* and *ar* windings from our work in Example 1B. Following a similar procedure, we can write the flux linkage equations for all of the windings (assuming a linear magnetic system) as

$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{ar} \\ \lambda_{br} \end{bmatrix} = \begin{bmatrix} L_{asas} & 0 & L_{asar} & L_{asbr} \\ 0 & L_{bsbs} & L_{bsar} & L_{bsbr} \\ L_{aras} & L_{arbs} & L_{arar} & 0 \\ L_{bras} & L_{brbs} & 0 & L_{brbr} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{ar} \\ i_{br} \end{bmatrix} \tag{1.4-1}$$

Because the stator (rotor) windings are identical and the air gap is uniform, the self-inductances L_{asas} and L_{bsbs} (L_{arar} and L_{brbr}) are equal. It is clear that $L_{asar} = L_{aras}$, $L_{asbr} = L_{bras}$, and so on. The self-inductances are constant, consisting of a leakage and a magnetizing inductance. The mutual inductances between stator and rotor phases are constant amplitude sinusoidal variations that are rotor position dependent. Thus, (1.4-1) can be written as

$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{ar} \\ \lambda_{br} \end{bmatrix} = \begin{bmatrix} L_{ls}+L_{ms} & 0 & L_{sr}\cos\theta_r & -L_{sr}\sin\theta_r \\ 0 & L_{ls}+L_{ms} & L_{sr}\sin\theta_r & L_{sr}\cos\theta_r \\ L_{sr}\cos\theta_r & L_{sr}\sin\theta_r & L_{lr}+L_{mr} & 0 \\ -L_{sr}\sin\theta_r & L_{sr}\cos\theta_r & 0 & L_{lr}+L_{mr} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{ar} \\ i_{br} \end{bmatrix} \tag{1.4-2}$$

The stator self-inductances, L_{asas} and $L_{asar} = L_{bsbs}$, are generally expressed as L_{ss} or $L_{ls} + L_{ms}$, and the rotor self-inductance, L_{arar} and L_{brbr}, as L_{rr} or $L_{lr} + L_{mr}$. The inductance L_{sr} is the amplitude of the mutual inductances between the stator and rotor windings.

In order to take a preliminary look at a rotating air-gap MMF (rotating poles), let the stator currents be a balanced two-phase set expressed as

$$I_{as} = \sqrt{2}I_s \cos\omega_e t \tag{1.4-3}$$

$$I_{bs} = \sqrt{2}I_s \sin\omega_e t \tag{1.4-4}$$

At time zero, when I_{as} is $\sqrt{2}I_s$ and I_{bs} is zero, the electromagnet system established by these currents is centered about the *as*-axis with the maximum air-gap MMF drop at the positive *as*-axis (to the right), which is a stator south pole, while the stator north pole is at the negative *as*-axis. As time progresses to where I_{as} is zero, I_{bs} is $\sqrt{2}I_s$, the magnetic field (poles) have rotated to where it is now centered about the *bs*-axis. Thus, as the electrical system "moved" or "rotated" ($\pi/2$) rad, the poles or air-gap MMF has moved ($\pi/2$) rad. Currents induced in the rotor windings create a magnetic system as well (rotor poles), and these will also rotate about the air-gap of the machine.

A four-pole, two-phase symmetrical induction machine is shown in Figure 1.4-2. In this case, the flux issuing along the *as*1-axis; one-half returns across the air-gap in the top half of the stator and one-half in the lower one-half. Similarly the flux issuing along the *as*2-axis; one-half returns across the air gap in the top one-half of the stator and one-half on the lower one-half.

It is interesting to note that when balanced two-phase currents flow in the stator windings, the air-gap MMF (poles) created by the stator currents rotate from the *as*1- and *as*2-axes to the *bs*1- and *bs*2-axes or ($\pi/4$) rad, while the electrical system has rotated ($\pi/2$) rad, as in the case of the two-pole system. In other words, the mechanical rotation of the air-gap MMF is determined by the number of poles created by the winding arrangement; however, the electrical system is unaware of the number of poles.

The flux linkage equations of the four-pole machine may be expressed as

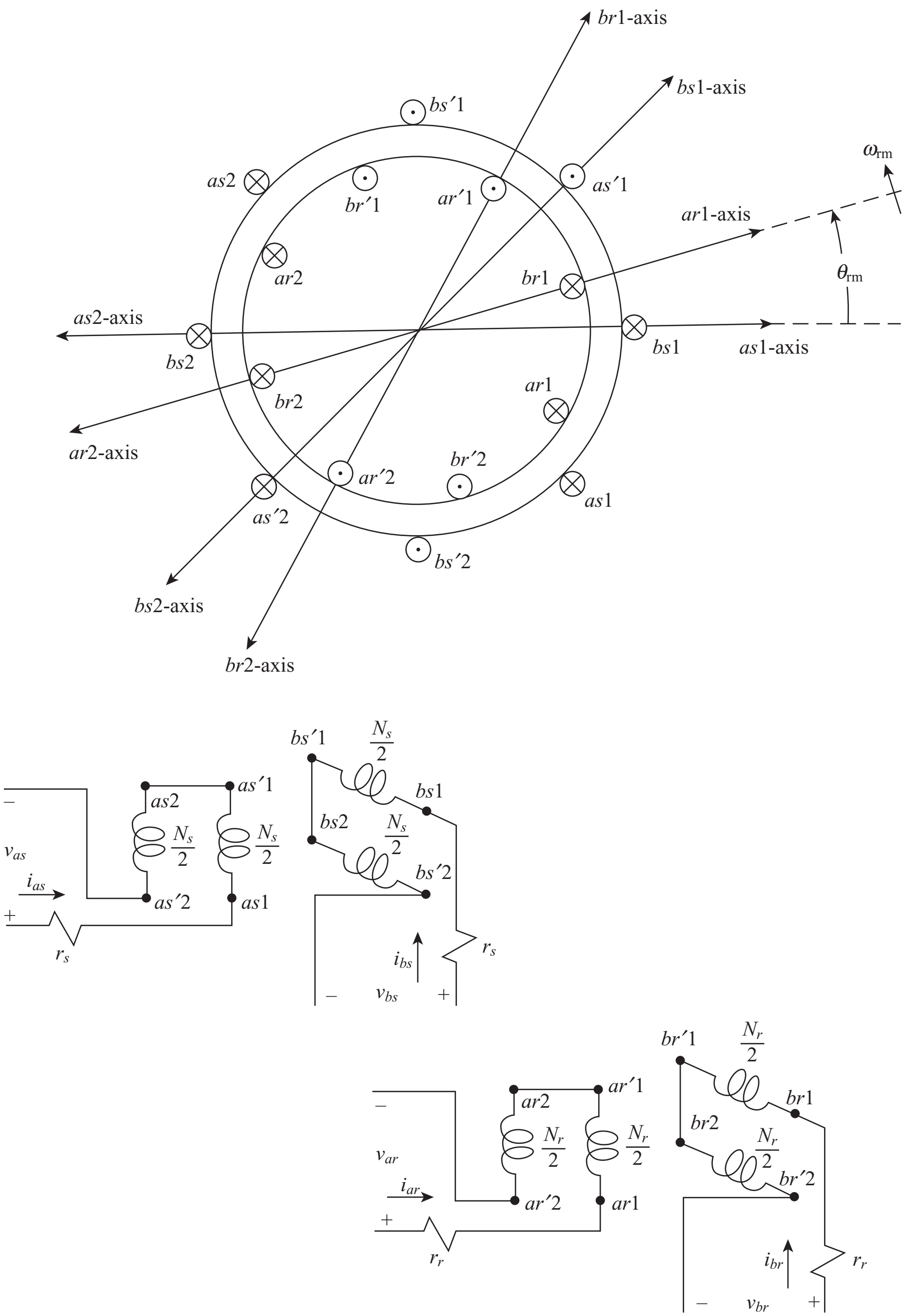

Figure 1.4-2. A four-pole, two-phase induction machine.

$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{ar} \\ \lambda_{br} \end{bmatrix} = \begin{bmatrix} L_{ls}+L_{ms} & 0 & L_{sr}cos2\theta_{rm} & -L_{sr}\sin 2\theta_{rm} \\ 0 & L_{ls}+L_{ms} & L_{sr}\sin 2\theta_{rm} & L_{sr}\cos 2\theta_{rm} \\ L_{sr}\cos 2\theta_{rm} & L_{sr}\sin 2\theta_{rm} & L_{lr}+L_{mr} & 0 \\ -L_{sr}\sin 2\theta_{rm} & L_{sr}\cos 2\theta_{rm} & 0 & L_{lr}+L_{mr} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{ar} \\ i_{br} \end{bmatrix} \tag{1.4-5}$$

where θ_{rm} is the mechanical displacement of the rotor. We will distinguish it from θ_r presently.

It is important to note the difference between (1.4-2), the flux linkage equations for a two-pole machine, and (1.4-5) for a four-pole machine. Clearly, the inductances will generally be different in magnitude; however, the notable difference is that θ_r is the angular displacement for the two-pole while θ_{rm} is the displacement for the four-pole. In particular, we see from comparing (1.4-2) and (1.4-5) that if we substituted θ_r for $2\theta_{rm}$, then the two equations would be identical in form. In general, we can define

$$\theta_r = \frac{P}{2}\theta_{rm} \tag{1.4-6}$$

This relation allows us to assume that all machines are two-pole machines, whereupon (1.4-2) will be the form of the flux linkage equations regardless of the number of poles. This appears reasonable in light of the previous discussion of the rotation of the magnetic poles produced by two-pole and four-pole machines. The displacement θ_r is then referred to as the electrical angular displacement of the rotor. The actual angular rotor displacement can always be determined from (1.4-6). It follows that

$$\omega_r = \frac{P}{2}\omega_{rm} \tag{1.4-7}$$

where ω_r is the electrical angular velocity of the rotor and ω_{rm} is the actual angular velocity. We will find that we can consider all machines as two-pole devices and take the $P/2$ factor into account when evaluating the torque.

An elementary two-pole, three-phase symmetrical induction machine is shown in Figure 1.4-3. Here, the flux linkage equations may be expressed as

$$\begin{bmatrix} \boldsymbol{\lambda}_{abcs} \\ \boldsymbol{\lambda}_{abcr} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & \mathbf{L}_{sr} \\ (\mathbf{L}_{sr})^T & \mathbf{L}_r \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abcs} \\ \mathbf{i}_{abcr} \end{bmatrix} \tag{1.4-8}$$

where

$$(\mathbf{f}_{abcs})^T = [f_{as} \quad f_{bs} \quad f_{cs}] \tag{1.4-9}$$

$$(\mathbf{f}_{abcr})^T = [f_{ar} \quad f_{br} \quad f_{cr}] \tag{1.4-10}$$

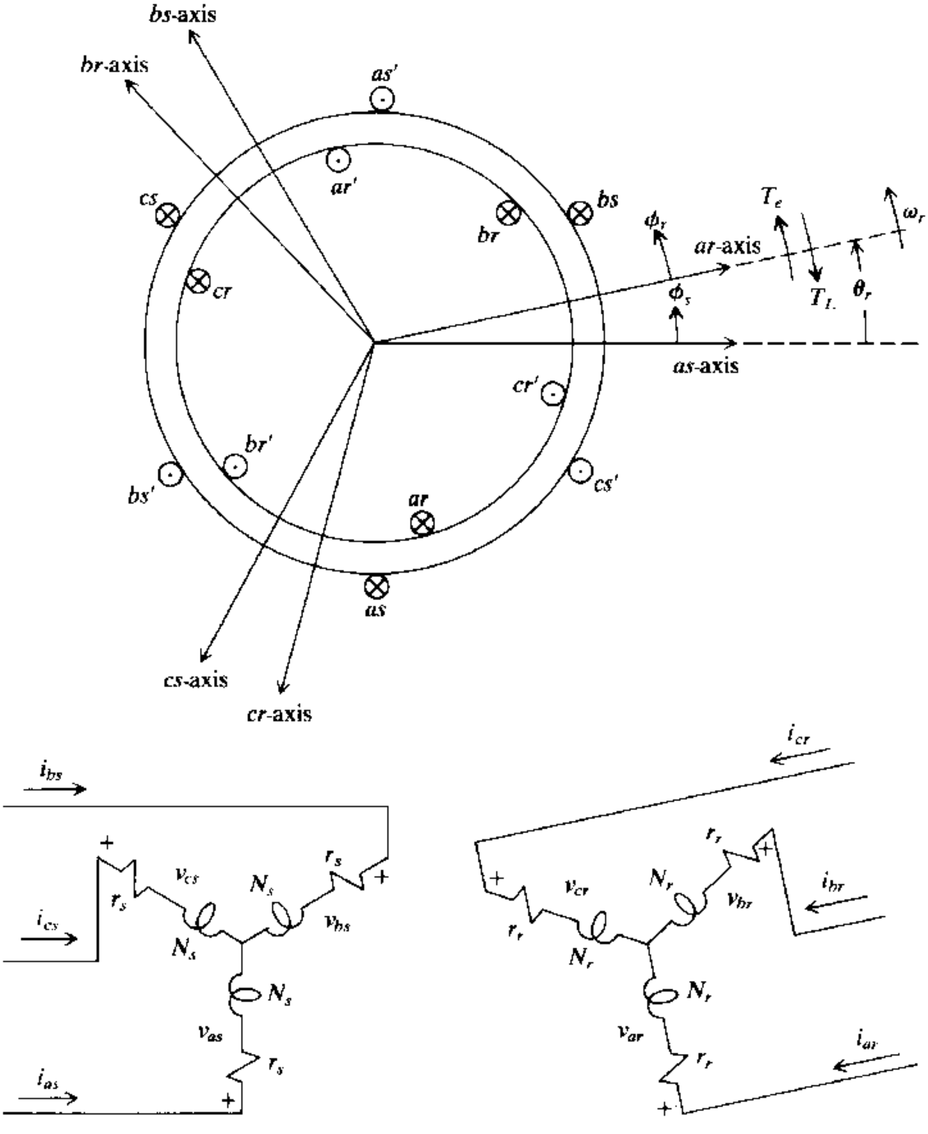

Figure 1.4-3. A two-pole, three-phase induction machine.

As a first approximation (a more detailed representation will be considered in Chapter 2),

$$\mathbf{L}_s = \begin{bmatrix} L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ls} + L_{ms} \end{bmatrix} \tag{1.4-11}$$

$$\mathbf{L}_r = \begin{bmatrix} L_{lr}+L_{mr} & -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & L_{lr}+L_{mr} & -\frac{1}{2}L_{mr} \\ -\frac{1}{2}L_{mr} & -\frac{1}{2}L_{mr} & L_{lr}+L_{mr} \end{bmatrix} \tag{1.4-12}$$

$$\mathbf{L}_{sr} = L_{sr}\begin{bmatrix} \cos\theta_r & \cos\left(\theta_r+\frac{2\pi}{3}\right) & \cos\left(\theta_r-\frac{2\pi}{3}\right) \\ \cos\left(\theta_r-\frac{2\pi}{3}\right) & \cos\theta_r & \cos\left(\theta_r+\frac{2\pi}{3}\right) \\ \cos\left(\theta_r+\frac{2\pi}{3}\right) & \cos\left(\theta_r-\frac{2\pi}{3}\right) & \cos\theta_r \end{bmatrix} \tag{1.4-13}$$

We see that the three-phase stator (rotor) windings are coupled, unlike the two-phase machine (Fig. 1.4-1). The difference in coupling is also true for the four-pole, two-phase machine (Fig. 1.4-2). Why?

Although the above inductance matrices are rigorously derived in Chapter 2, we can get a “first look” from a simplistic consideration. We have previously defined the leakage, magnetizing, and mutual inductances; it is now the off-diagonal terms of $\mathbf{L}_s$ and $\mathbf{L}_r$ that are of concern. To explain these terms, let us first consider the coupling between the *as* and *bs* windings. They are displaced $(2/3)\pi$ from each other. Let us assume that we can take the *bs*-winding and turn (twist) it clockwise through the stator iron for $(2/3)\pi$ rad, whereupon it would be “on top” of the *as*-winding. In this case, the mutual inductance between the *as* and *bs* windings would be L_{ms}, neglecting any coupling of the leakage fluxes. Now let us turn the *bs*-winding counterclockwise back through the stator iron. The mutual inductance would vary as $L_{ms}\cos\alpha$, where α is the angle measured counterclockwise from the *as*-axes; when we reach $\alpha = (\pi/2)$, there is no coupling between the two windings, just as in the case of the two-phase machine. When we have twisted the *bs*-winding back to its original position, the mutual inductance is $L_{ms}\cos(2\pi/3)$ or $-(1/2)L_{ms}$. Following this type of simplistic reasoning, we can justify all of the off-diagonal terms of $\mathbf{L}_s$ and $\mathbf{L}_r$.

In Chapter 2, we will derive the expressions for all inductances as functions of machine dimensions and the type of winding distribution; however, the resulting form of the inductance matrices are nearly the same as given in (1.4-11)–(1.4-13).

An elementary two-pole, two-phase synchronous machine is shown in Figure 1.4-4. The flux linkage equations may be expressed as

$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{fd} \end{bmatrix} = \begin{bmatrix} L_{ls}+L_A-L_B\cos 2\theta_r & -L_B\cos 2\theta_r & L_{sfd}\sin\theta_r \\ -L_B\cos 2\theta_r & L_{ls}+L_A+L_B\cos 2\theta_r & -L_{sfd}\cos\theta_r \\ L_{sfd}\sin\theta_r & -L_{sfd}\cos\theta_r & L_{lfd}+L_{mfd} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{fd} \end{bmatrix} \tag{1.4-14}$$

The stator windings are identical in that they have the same resistance and the same number of turns.

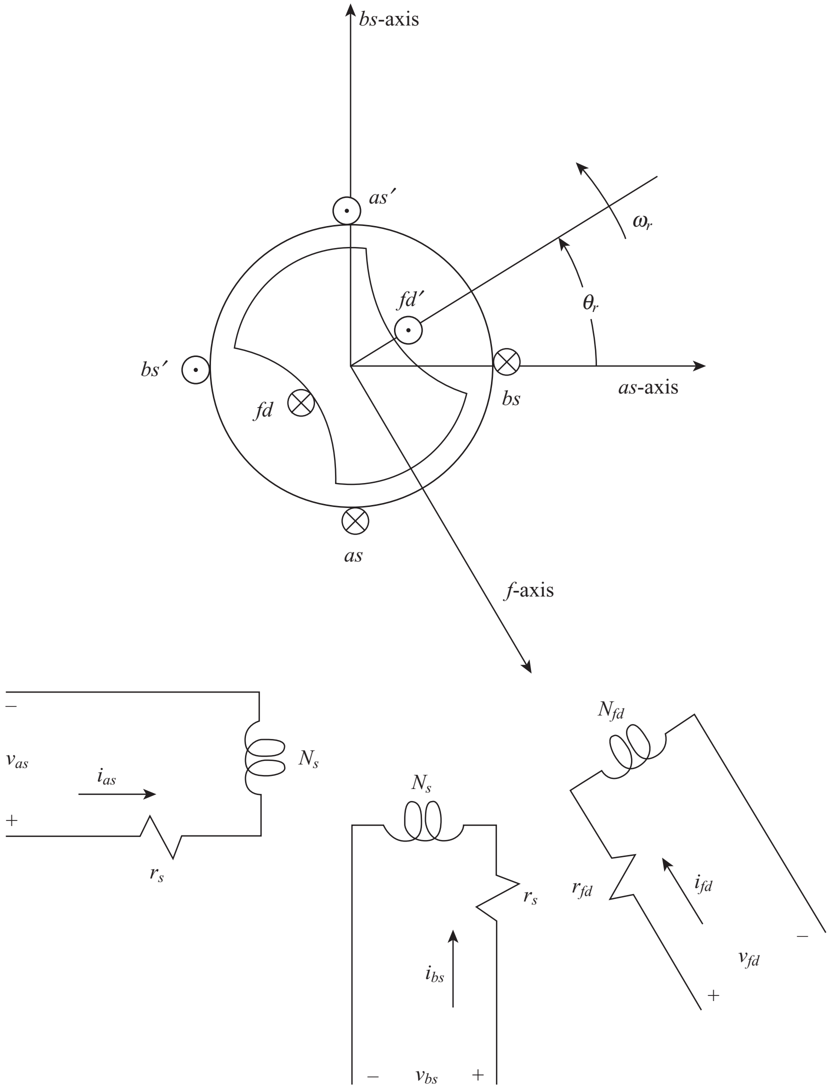

Figure 1.4-4. A two-pole, two-phase, salient-pole synchronous machine.

Due to the saliency of the rotor, the stator windings experience a change in self-inductance as the rotor rotates, which is here approximated as a double-angle variation about an average value. Moreover, the saliency of the rotor also causes a mutual coupling between the orthogonal stator windings. It is interesting that the L_B associated with the self-inductances is also the coefficient of the double-angle mutual inductance between stator phases. This is shown in Chapter 2. It is left to the reader to show that the mutual inductance between the stator phases is a negative with the direction of rotation and the current directions given in Figure 1.4-4.

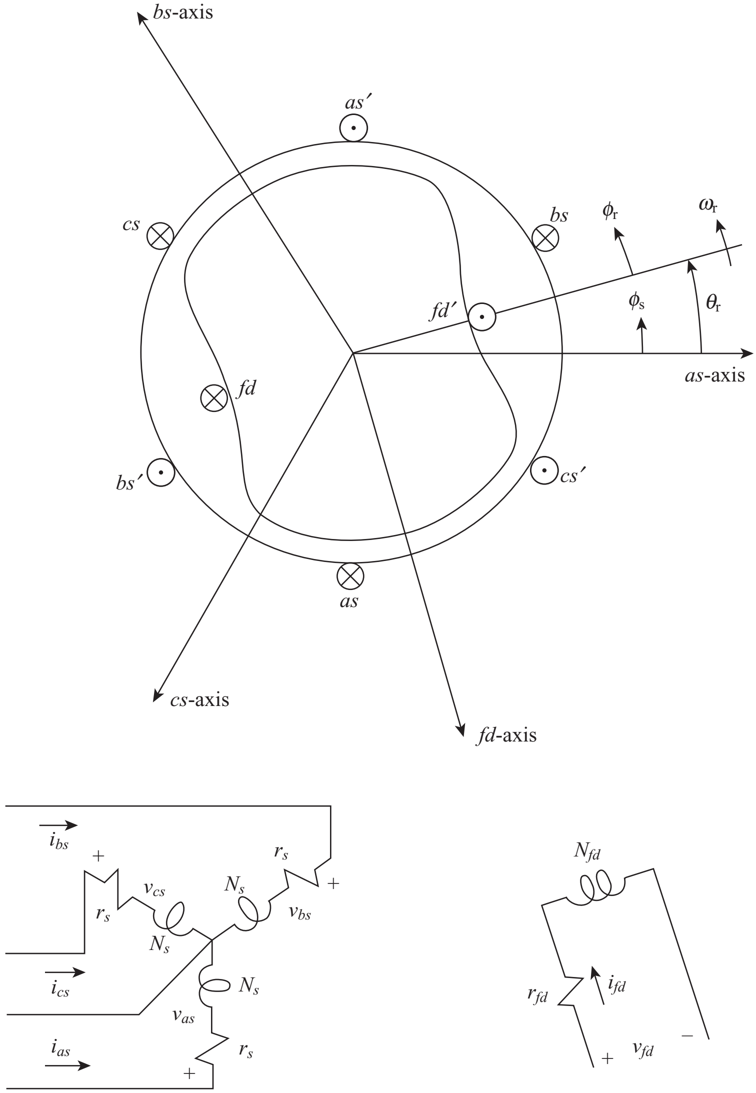

Figure 1.4-5. A two-pole, three-phase, salient-pole synchronous machine.

An elementary two-pole, three-phase synchronous machine is shown in Figure 1.4-5. The flux linkage equations may be written as

$$\begin{bmatrix} \boldsymbol{\lambda}_{abcs} \\ \lambda_{fd} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_s & L_{sfd}\sin\theta_r \\ \mathbf{L}_{sf} & L_{lfd} + L_{mfd} \end{bmatrix} \begin{bmatrix} \mathbf{i}_{abcs} \\ i_{fd} \end{bmatrix} \tag{1.4-15}$$

where

$$\mathbf{L}_s = \begin{bmatrix} L_{ls} + L_A - L_B\cos 2\theta_r & -\frac{1}{2}L_A - L_B\cos 2\left(\theta_r - \frac{\pi}{3}\right) & -\frac{1}{2}L_A - L_B\cos 2\left(\theta_r + \frac{\pi}{3}\right) \\ -\frac{1}{2}L_A - L_B\cos 2\left(\theta_r - \frac{\pi}{3}\right) & L_{ls} + L_A - L_B\cos 2\left(\theta_r - \frac{2\pi}{3}\right) & -\frac{1}{2}L_A - L_B\cos 2(\theta_r + \pi) \\ -\frac{1}{2}L_A - L_B\cos 2\left(\theta_r + \frac{\pi}{3}\right) & -\frac{1}{2}L_A - L_B\cos 2(\theta_r + \pi) & L_{ls} + L_A - L_B\cos 2\left(\theta_r + \frac{2\pi}{3}\right) \end{bmatrix} \tag{1.4-16}$$

where

$$\mathbf{L}_{sf} = \left[L_{sfd}\sin\theta_r \quad L_{sfd}\sin\left(\theta_r - \frac{2\pi}{3}\right) \quad L_{sfd}\sin\left(\theta_r + \frac{2\pi}{3}\right) \right] \tag{1.4-17}$$

It is left to the reader to verify the stator mutual inductances. A practical synchronous machine is equipped with damper windings on the rotor that by induction motor action, damp low-frequency oscillations about a steady-state operating point. The inductances associated with these windings are incorporated in the performance analysis of synchronous machines in Chapter 5.

REFERENCE

[1] D.C. White and H.H. Woodson, Electromechanical Energy Conversion, John Wiley and Sons, New York, 1959.

PROBLEMS

1. A two-winding, iron-core transformer is shown in Figure 1P-1. $N_1 = 50$ turns, $N_2 = 100$ turns, and $\mu_R = 4000$. Calculate L_{m1} and L_{m2}.

2. Repeat Problem 1 if the iron core has an air gap of 0.2 cm in length and is cut through the complete cross section. Assume that fringing (a curvature of the flux lines around the air gap) does not occur; that is, the flux lines follow a straight path through the air gap in which the effective cross-sectional area is 25 cm^2.

3. Two coupled coils have the following parameters:

$$\begin{aligned} L_{11} &= 100 \text{ mH} & r_1 &= 10\ \Omega \\ L_{22} &= 25 \text{ mH} & r_2 &= 2.5\ \Omega \\ N_1 &= 1000 \text{ turns} & N_2 &= 500 \text{ turns} \\ L_{l1} &= 0.1 L_{11} & L_{l2} &= 0.1 L_{22} \end{aligned}$$

Develop an equivalent T circuit with coil 1 as the reference coil. Repeat with coil 2 as the reference coil.

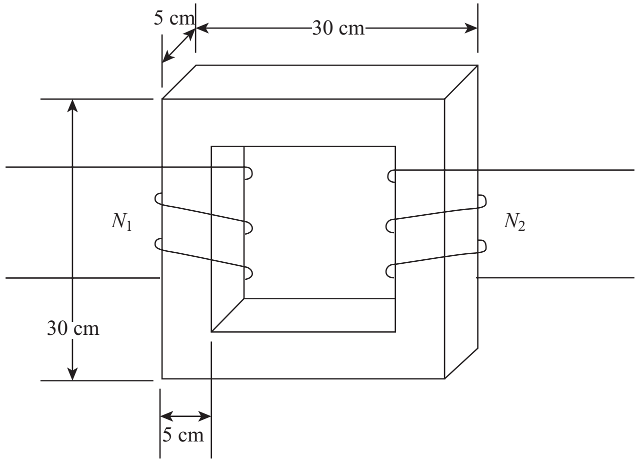

Figure 1P-1. Two-winding, iron-core transformer.

4. A system with two windings has a flux linkage versus current profile of

$$\lambda_1 = \left(0.1 + \frac{0.03}{x}\right) i_1 - \frac{0.01}{x} i_2$$
$$\lambda_2 = \left(0.0111 + \frac{0.03}{9x}\right) i_2 - \frac{0.01}{x} i_1$$

The resistance of the coils is $r_1 = 1\ \Omega$ and $r_2 = 0.3\ \Omega$, respectively. The winding voltage equations can be expressed in a form (where $p = d/dt$):

$$v_1 = r_1 i_1 + p\lambda_1$$
$$v_2 = r_2 i_2 + p\lambda_2$$

(a) Derive the equivalent T circuit model for this system, assuming coil 1 as the reference. Show all component values. Label directions of all currents and voltages.

(b) For this system, describe two conditions where you cannot make the common approximation that $|\tilde{I}_1| \approx |\tilde{I}'_2|$.

5. A constant 10 V is suddenly applied to coil 1 of the coupled circuits given in Problem 3. Coil 2 is short-circuited. Calculate the transient and steady-state current flowing in each coil.

6. Determine the input impedance of the coupled circuits given in Problem 3 if the applied frequency to coil 1 is 60 Hz with coil 2 (a) open-circuited and (b) short-circuited. Repeat (b) with the current flowing in the magnetizing reactance neglected.

7. A third coil is wound on the ferromagnetic core shown in Figure 1.2-1. The resistance is r_3 and the leakage and magnetizing inductances are L_{l3} and L_{m3}, respectively. The coil is wound so that positive current (i_3) produces Φ_{m3} in the same direction

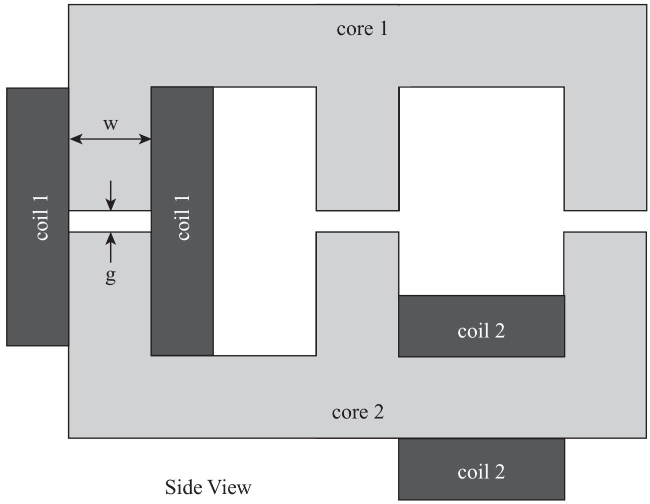

Figure 1P-2. EE iron-core transformer.

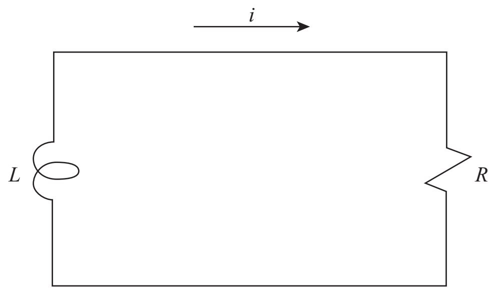

Figure 1P-3. *R–L* circuit.

as Φ_{m1} and Φ_{m2}. Derive the equivalent T circuit for this three-winding transformer. Actually, one should be able to develop the equivalent circuit without derivation.

8. Consider the magnetic device shown in Figure 1P-2, which is commonly referred to as an E-core. The permeability of the cores is infinite, $g = \pi$ mm, $w = 2.5$ cm, and the depth into the page is 10 cm. Coil 1 has $I_1 = 10\cos 100t$ A, 100 turns, and positive current causes the positive flux direction to be upward. Coil 2 has $I_2 = 40\cos 100t$ A, 100 turns, and positive current causes positive flux to travel to the left through the coil. The resistance of both coils, fringing around the gaps, and leakage fluxes are all negligible. Determine the voltage across coil 2 as a function of time.

9. Use Σ and $1/p$ to denote summation and integration, respectively. Draw a time-domain block diagram for two coupled windings with saturation and (a) with leakage inductance and (b) without leakage inductance.

10. A resistor and an inductor are connected as shown in Figure 1P-3 with $R = 15\ \Omega$ and $L = 250$ mH. Determine the energy stored in the inductor W_{eS} and the energy dissipated by the resistor W_{eL} for $i > 0$ if $i(0) = 10$ A.

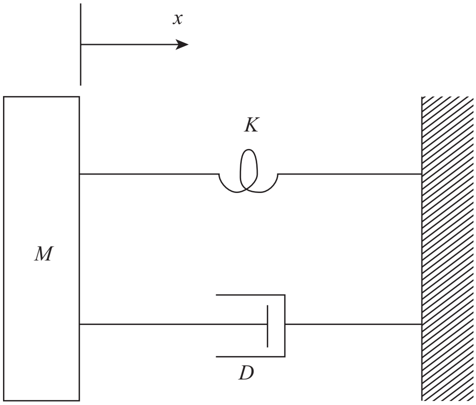

Figure 1P-4. Spring–mass–damper system.

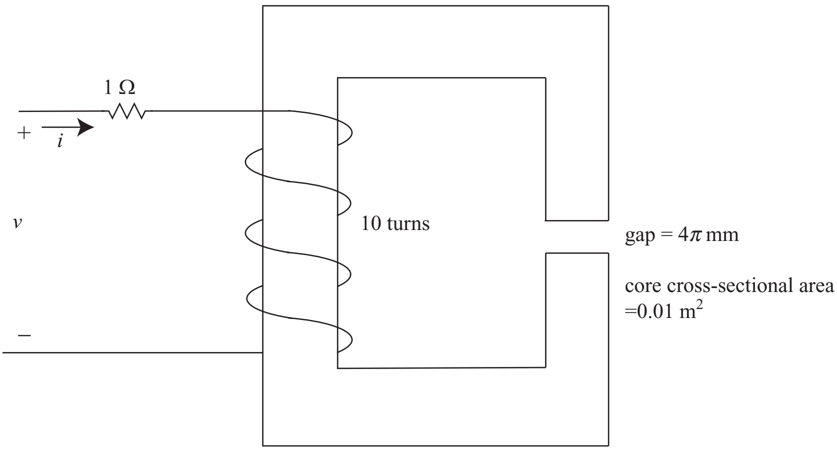

Figure 1P-5. C-core inductor.

11. Consider the spring-mass-damper system shown in Figure 1P-4. At $t = 0$, $x(0) = x_0$ (rest position) and $dx/dt = 1.5$ m/s. $M = 0.8$ kg, $D = 10$ N·s/m and $K = 120$ N·m. For $t > 0$, determine the energy stored in the spring W_{mS1}, the kinetic energy of the mass W_{mS2}, and the energy dissipated by the damper W_{mL}.

12. True/false: Magnetic hysteresis leads to a field that is nonconservative. Explain.

13. For the system shown in Figure 1P-5, which is often referred to as a "C-core," determine the winding inductance if the leakage inductance is 1/10 the magnetizing inductance. If 10 V is applied to the winding at $t = 0$ second, determine W_f and the force of attraction that acts to attempt to reduce the gap at $t = 1$ second. Where is the energy of the coupling field stored in this system?

14. Given the UU-core transformer shown in Figure 1P-6. Assume leakage inductances and the MMF drop across the core is negligible. The cross-sectional area of the

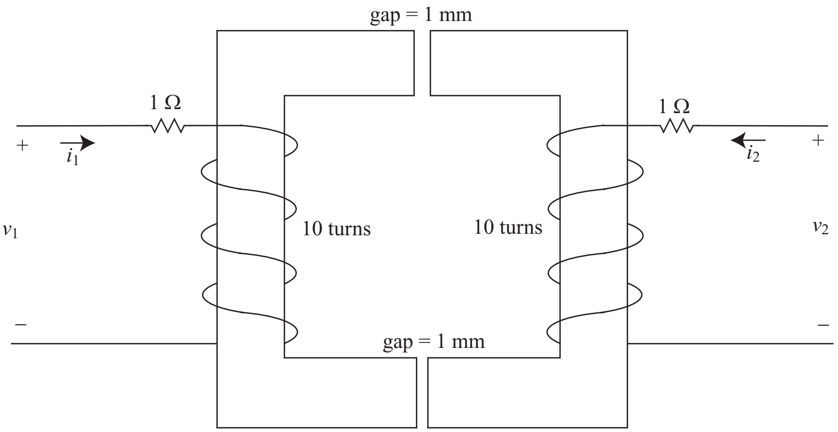

Figure 1P-6. UU-core transformer for Problem 14.

core is 15.9154 m^2. At $t = 0$ second, an input voltage of $v_1 = 10$ V is applied. The secondary is open-circuited. At $t = 1$ second, the primary is open circuited and i_1 goes from 6.32 A to 0 instantaneously. At the same instant, the secondary is short-circuited i_2 goes from 0 to 6.32 A instantaneously. Determine W_E, W_f, W_{eL}, and W_m for $t \geq 0$ second.

15. Express $W_f(i,x)$ and $W_c(i,x)$ for (a) $\lambda(i,x) = i^{2/3}x^2$; (b) $\lambda(i,x) = ki\sin(x/a)\pi - xi$.

16. The energy stored in the coupling field of a magnetically linear system with two electrical inputs may be expressed as

$$W_f(\lambda_1, \lambda_2, x) = \frac{1}{2}B_{11}\lambda_1^2 + B_{12}\lambda_1\lambda_2 + \frac{1}{2}B_{22}\lambda_2^2$$

Express B_{11}, B_{12}, and B_{22} in terms of inductances L_{11}, L_{12}, and L_{22}.

17. An electromechanical system has two electrical inputs. The flux linkages may be expressed as

$$\lambda_1(i_1, i_2, x) = x^2 i_1^2 + x i_2$$
$$\lambda_2(i_1, i_2, x) = x^2 i_2^2 + x i_1$$

Express $W_f(i_1,i_2,x)$ and $W_c(i_1,i_2,x)$.

18. Express $f_e(i,x)$ for the electromechanical systems described by the relations given in Problem 15.

19. Express $f_e(i_1,i_2,x)$ for the electromechanical system given in Problem 17.

20. The flux-linkage equations for a two-phase electromagnetic device are expressed as:

$$\lambda_{as} = 5i_{as} - 3\cos(2\theta_{rm})(i_{as} + i_{bs})^{1/2} + 2\cos(\theta_{rm})$$
$$\lambda_{bs} = 5i_{bs} - 3\cos(2\theta_{rm})(i_{as} + i_{bs})^{1/2} + 2\sin(\theta_{rm})$$

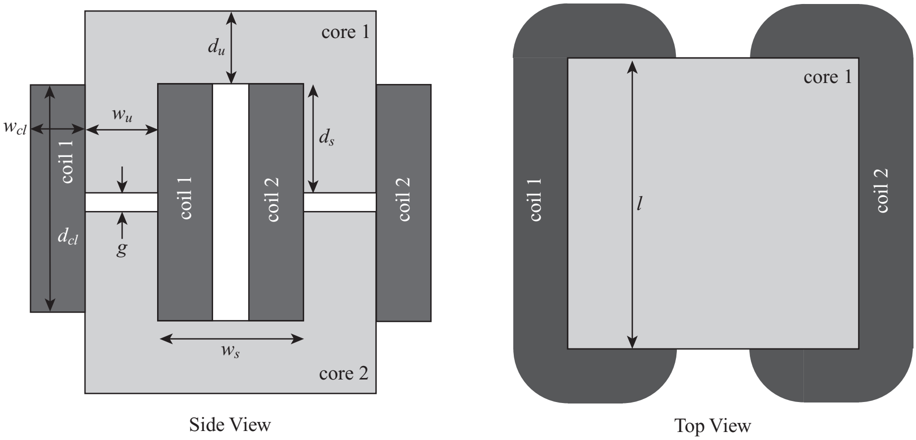

Figure 1P-7. UU-core transformer for Problem 21.

where θ_{rm} is the mechanical rotor position. Determine an expression for torque in terms of i_{as}, i_{bs}, and θ_{rm}. Assume that both currents are greater than or equal to zero.

21. Consider the UU-core transformer shown in Figure 1P-7. Each coil is wound in a direction such that positive current will cause positive flux to flow in a clockwise direction. Neglecting leakage and fringing flux, derive an expression for the electromagnetic force of attraction between the cores in terms of the coil currents i_1 and i_2, turns N_1 and N_2, μ_0, μ_r, and the dimensions given in the figure.

22. Consider an electromechanical system whose flux-linkage equations given by

$$\lambda_1 = 5i_1 + 10\left(1 - \frac{1}{1+i_m}\right)\cos\theta_r$$

$$\lambda_2 = i_2 + 20\left(1 - \frac{1}{1+i_m}\right)\cos\theta_r$$

where $i_1 \geq 0, i_2 \geq 0, i_m = i_1 + 2i_2$ and where θ_r is the mechanical rotor position. Derive expressions for the coenergy and torque in terms of i_1, i_2, and θ_r.

23. Derive an expression for the forces $f_{e1}(\mathbf{i},x_1,x_2)$ and $f_{e2}(\mathbf{i},x_1,x_2)$ in an electromechanical system with two degrees of mechanical motion.

24. For the multicore system shown in Figure 1P-8, assume the components are fixed with $x_1 = x_2 = 1$ mm. The core cross-sectional area is 0.05 m^2, $I_1 = 10 \cos 377t$ A, $I_2 = -20 \cos 377t$ A. Determine the force acting on all components and W_c of the system at $t = 1$ second. Neglect leakage flux.

25. Refer to Figure 1.3-6. As the system moves from x_a to x_b, the λ–i trajectory moves from A to B where both A and B are steady-state operating conditions. Does the voltage v increase or decrease? Does the applied force f increase or decrease? Explain.

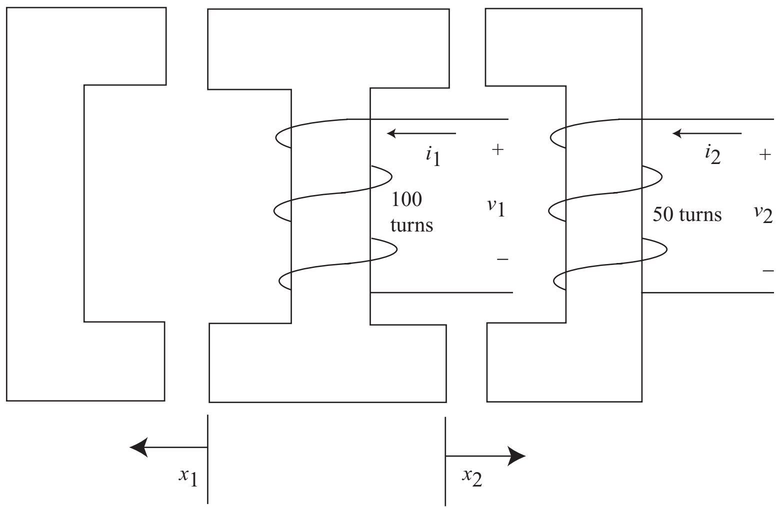

Figure 1P-8. Multicore system.

26. Refer to Figure 1.3-10. Following the system transients due to the application of the source voltage (v = 5 V), the system assumes steady-state operation. For this steady-state operation, calculate W_{eS}, W_f, and W_{mS}.

27. Refer to Figure 1.3-11. Repeat Problem 26 for steady-state operation following the application of $f = 4N$.

28. Refer to Figure 1.3-12. Identify the area corresponding to ΔW_m when (a) x moves from 2.5 mm to 4.3 mm, and when (b) x moves from 4.3 to 2.5 mm.

29. Assume the steady-state currents flowing in the conductors of the device shown in Figure 1B-1 are

$$I_1 = I_{s1}\cos\omega_1 t$$

$$I_2 = I_{s2}\cos(\omega_2 t + \varphi_2)$$

Assume also that during steady-state operation the rotor speed is constant, thus

$$\theta_r = \omega_r t + \theta_r(0)$$

where $\theta_r(0)$ is the rotor displacement at time zero. Determine the rotor speeds at which the device produces a nonzero average torque during steady-state operation if (a) $\omega_1 = \omega_2 = 0$; (b) $\omega_1 = \omega_2 \neq 0$; ($c$) $\omega_2 = 0$.

30. An elementary two-pole, two-phase, salient-pole synchronous machine is shown in Figure 1.4-4. The winding inductances may be expressed as

$$L_{asas} = L_{ls} + L_A - L_B\cos 2\theta_r$$

$$L_{bsbs} = L_{ls} + L_A + L_B\cos 2\theta_r$$

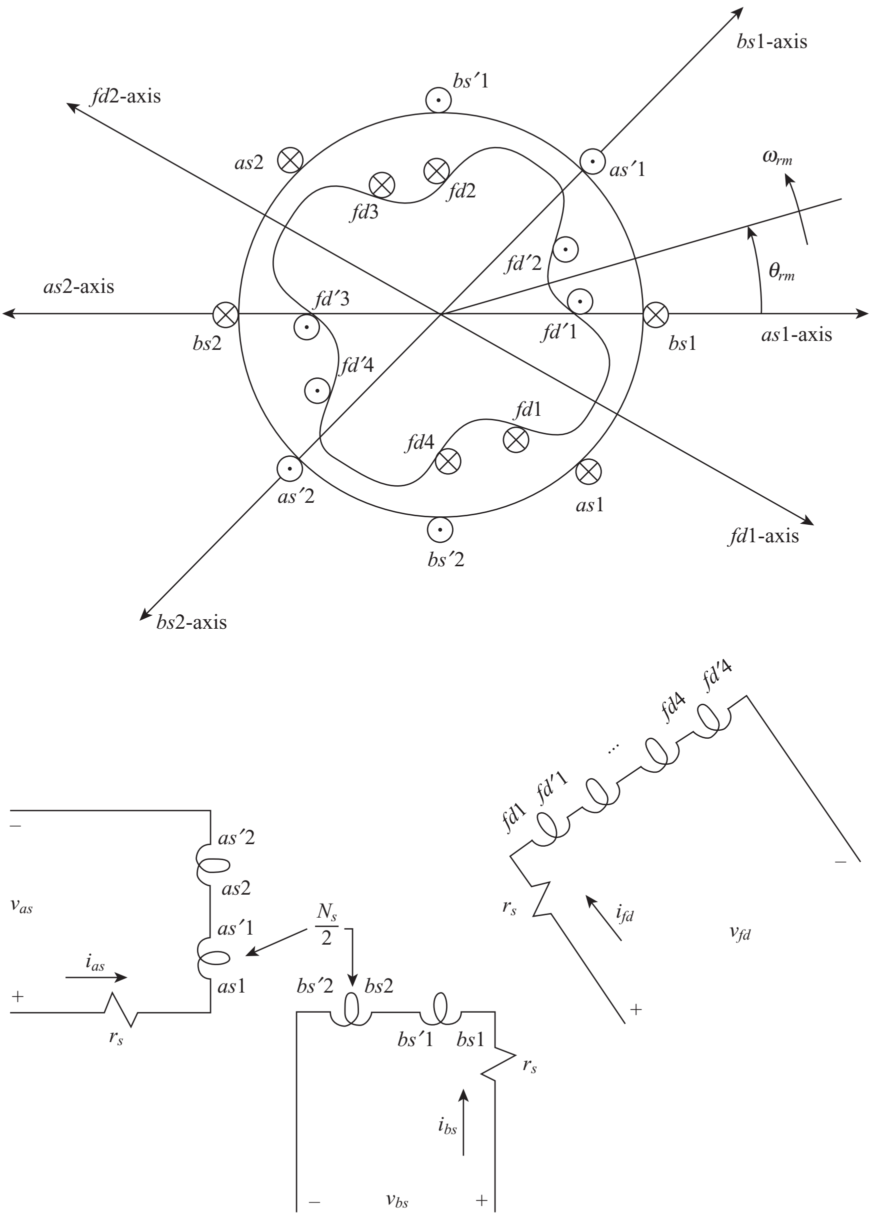

Figure 1P-9. Elementary four-pole, two-phase, salient-pole synchronous machine.

$$L_{asbs} = -L_B \sin 2\theta_r$$

$$L_{fdfd} = L_{lfd} + L_{mfd}$$

$$L_{asfd} = L_{sfd} \sin \theta_r$$

$$L_{bsfd} = -L_{sfd} \cos \theta_r$$

Modify these inductance relationships so that they will describe a two-phase, uniform air-gap synchronous machine.

31. Write the voltage equations for the elementary two-pole, two-phase, salient-pole synchronous machine shown in Figure 1.4-4 and derive the expression for $T_e(i_{as}, i_{bs}, i_{fd}, \theta_r)$.

32. An elementary four-pole, two-phase, salient-pole synchronous machine is shown in Figure 1P-9. Use this machine as a guide to derive expressions for the winding inductances of a P-pole synchronous machine. Show that these inductances are of the same form as those given in Problem 30 if $(P/2)\theta_{rm}$ is replaced by θ_r.

33. Derive an expression for the electromagnetic $T_e(i_{as}, i_{bs}, i_{fd}, \theta_r)$, for a P-pole, two-phase, salient-pole synchronous machine. This expression should be identical in form to that obtained in Problem 31 multiplied by $P/2$.

34. A reluctance machine has no field winding on the rotor. Modify the inductance relationships given in Problem 30 so as to describe the winding inductances of a two-pole, two-phase, reluctance machine. Write the voltage equations and derive an expression for $T_e(i_{as}, i_{bs}, \theta_r)$.

35. An elementary two-pole, two-phase, symmetrical induction machine is shown in Figure 1.4-1. If $L_{asas} = L_{ls} + L_{ms}$, $L_{arar} = L_{lr} + L_{mr}$, and $L_{asar} = L_{sr} \cos\theta_r$, express the remaining self- and mutual inductances of all stator and rotor windings. Following the transformer derivation, refer the rotor quantities to the stator quantities. Express the stator and rotor flux linkages in terms of the referred variables.

36. Write the voltage equations for the induction machine shown in Figure 1.4-1 and derive an expression for the electromagnetic torque $T_e(i_{as}, i_{bs}, i'_{ar}, i'_{br}, \theta_r)$ using the results obtained in Problem 35.

37. An elementary four-pole, two-phase, symmetrical induction machine is shown in Figure 1.4-2. Use this machine as a guide to derive expressions for the winding inductances of a P-pole induction machine. Show that these inductances are of the same form as those given in Problem 35 if $(P/2)\theta_{rm}$ is replaced by θ_r.