CHAPTER 1

Introduction

A man may imagine things that are false, but he can only understand things that are true.
Isaac Newton

Almost all mathematical problems connected with the electromagnetic phenomena require solutions of the Maxwell differential equations

$$\mathrm{curl}\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{J}, \qquad \mathrm{curl}\mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} \tag{1.1a,b}$$

$$\mathrm{div}\mathbf{D} = \rho, \qquad \mathrm{div}\mathbf{B} = 0 \tag{1.1c,d}$$

under certain supplementary restrictions stipulated on certain surfaces. In (1.1a–d) **E, D, H**, and **B** stand for the *electric field, electric displacement, magnetic field*, and *magnetic induction*, respectively, while ρ and **J** are the volume densities of the charges and currents. As to the parameter t that appears in (1.1a,b), it is, as usual, the time. The surfaces that bear the above-mentioned supplementary restrictions are either the interfaces between bodies of different constitutive parameters or surfaces that support surface charges and currents[†] or *material sheets* that model very thin layers. The supplementary restrictions in

[*]Michael White, *Isaac Newton: The Last Sorcerer*, Basic Books, New York, 1977, p. 5. See also *Sir Isaac Newton's Theological Manuscripts*, H. McLachlan (Ed.), Liverpool University Press, Liverpool, 1950, p. 17.
[†]Line and point charges are also carried by certain appropriately defined surfaces.

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question are the relations that state the *physically admissible discontinuities* that may occur on these surfaces. They consist, in general, of the so-called *boundary conditions* that give the jump discontinuities on the surfaces in questions. If the discontinuity surfaces also involve sharp edges and/or sharp tips, then some components of the field become infinitely large at some points. In this case, in addition to the boundary conditions, one also has to know the physically admissible asymptotic behaviors of the field near those points because, as was shown more than 60 years ago by Bouwkamp [1], one can construct many solutions to the Maxwell equations under the given boundary conditions. Of course some of these solutions are not acceptable from physics point of view. Depending on the nature of the singular point, the relations that state the asymptotic behaviors in question are called the *edge conditions* or the *tip conditions*.

It is worthwhile to remark here that any relation written on a surface cannot be treated as a boundary condition for the electromagnetic field. In order to be so, it must also be compatible with the Maxwell equations. The spectrum of the electromagnetic waves used in the telecommunication is enlarging every day more and more toward very short waves. Hence many types of roughness, which had been assumed to be negligible in earlier investigations in order to reduce mathematical difficulties, became today unavoidable. In studies to be made in forthcoming days, one will have to consider the effects of these types of roughness on the propagation of waves. Therefore rigorous and detailed investigations of the boundary, edge, and tip conditions for various geometrical and physical structures are of crucial importance from both pure scientific and technological applications points of view. The aim of the present monograph is to study the discontinuities (i.e., singularities) in questions in their most general framework and discuss the validity of some particular relations that are in use in current literature. Our fundamental basis will be the so-called *distributions* (or *generalized functions*). In order to clarify the crucial role of this concept in the present study, it will be useful to reexamine the derivation of the *classical* boundary conditions in *almost* all textbooks.

On those days when the Electromagnetic Theory had been established, one had a huge heap of scientific knowledge in theoretical physics, especially in fluid mechanics. With this potential, the scientists of that time (i.e., mathematicians, physicists, and engineers) had formulated and solved many mathematical problems that could have interesting and important interpretations in terms of the electromagnetic notions. First problems were those that needed to find the explicit expressions of the fields created by various sources distributed in the vacuum. They were very easy. Soon later, one had considered the problems connected with bodies having simple geometrical shapes such as infinite planes, infinitely long cylinders, whole spheres, whole ellipsoids, and so on. They were also rather easy and tractable with known techniques provided that the boundary conditions to be satisfied on the surfaces of the bodies were known beforehand. It was at this stage that the struggles to reveal the discontinuities of the electromagnetic field were started. The methods that seemed most propitious were what are based on the applications of the Gauss–Ostrogradski and Stokes theorems (see Smirnow [2, pp. 177, 197]). Although these applications are repeated in almost all textbooks, we want to recapitulate them here in order to clarify the philosophy of the method adopted in the present study. To this end, consider, for example, (1.1c) and integrate it inside a volume ϑ bounded by planar boundaries of very small areas (see Fig. 1.1). Assume that $\mathbf{D}$ as well as its partial derivatives of the first orders are all *continuous inside* ϑ *except on a regular surface* ΔS. To avoid useless complexities, assume also that the field depends only on two space coordinates. Then a *careless* application (without regarding the validity conditions) of the Gauss–Ostrogradski theorem to

$$\int_{\vartheta} \operatorname{div}\mathbf{D}\, d\vartheta = \int_{\vartheta} \rho\, d\vartheta = Q \tag{1.2a}$$

yields

$$\int_{\Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4} \mathbf{D} \cdot \mathbf{n}\, dS = Q. \tag{1.2b}$$

Here Q stands for the total charge existing inside ϑ. If Q consists of the surface charge distributed with density ρ_S on S, then by making

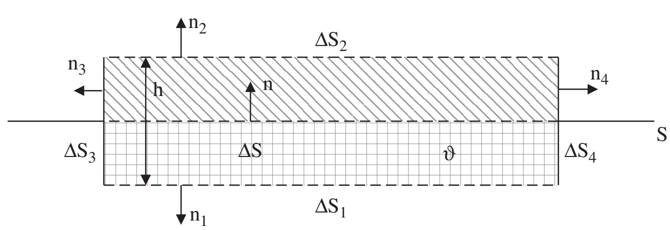

Figure 1.1. A neighborhood of a surface of discontinuity.

$h \to 0$ in (1.2b) one gets

$$\int_{\Delta S} [D_n^{(2)} - D_n^{(1)}]\, dS = \int_{\Delta S} \rho_S\, dS \tag{1.2c}$$

or

$$D_n^{(2)} - D_n^{(1)} = \rho_S. \tag{1.2d}$$

Here $D_n^{(1)}$ and $D_n^{(2)}$ show the values of the normal component of **D** which are observed when one approaches S from lower and upper sides, respectively.

Quite similarly, from (1.1a,b,d) one gets

$$B_n^{(2)} - B_n^{(1)} = 0, \tag{1.2e}$$

$$\mathbf{n} \times \mathbf{E}^{(2)} - \mathbf{n} \times \mathbf{E}^{(1)} = 0, \tag{1.2f}$$

$$\mathbf{n} \times \mathbf{H}^{(2)} - \mathbf{n} \times \mathbf{H}^{(1)} = \mathbf{J}_S. \tag{1.2g}$$

The field $\mathbf{J}_S$ appearing in (1.2g) is the density of the surface current that flows on S.

The relations in (1.2.d–g) are the *classical boundary conditions* that are satisfied on the interface between two regions filled with different materials. They are in use since the first days of the Theory to find solutions of the electromagnetic problems, which are in complete agreements with experiments. But the mathematical analysis made to reveal them is obviously *not legitimate*. Indeed, in order for the Gauss–Ostrogradski and Stokes theorems to be applicable, the fields **E, D, H**, and **B** have to be continuous inside ϑ. But, as the results themselves show, this is not the case. However, as we have already stated, the results are *correct from physics point of view*. A few writers who are familiar with this contrast warn readers by indicating that the results to be obtained by this kind of faulty applications are *assumed* to be acceptable for physics. D. S. Jones [3, p. 46] and S. A. Schelkunoff [4] epitomize this group of meticulous scientists.* It goes without saying that this assumption is nothing but an *additional postulate* to the Maxwell equations. Regarding the swift developments in the contemporary technology, which create and use very different materials and geometries, one can easily grasp that this approach cannot enable one

*Schelkunoff characterizes this kind of an approach to be "a proof which is not a proof but a swindle" (see Schelkunoff [4, Section 5]).

to overcome the difficulties completely. Hence one has to contrive a general and robust method.

One could also think that the difficulty in deriving (1.2d) resulted from the derivatives existing in (1.1c); if one started from the integral equation (Gauss' law)

$$\oint_S \mathbf{D} \cdot \mathbf{n}\, dS = Q, \tag{1.3a}$$

which is equivalent to (1.1c), one would not have the same difficulty. Indeed, in this case from (1.3a) one writes directly

$$\int_{\Delta S_1+\Delta S_2+\Delta S_3+\Delta S_4} \mathbf{D} \cdot \mathbf{n}\, dS = \int_{\Delta S} \rho_S\, dS, \tag{1.3b}$$

which, after applications of the Gauss–Ostrogradski theorem to the partial regions lying above and below the surface ΔS, yields (1.2d). But this approach too, which at first glance seems to be propitious to overcome the difficulty, has severe defects. For example, in the case when S consists of a double sheet that carries only dipoles, the right-hand side of (1.3b) becomes naught and claims $D_n^{(2)} - D_n^{(1)} = 0$, which is not correct.* Furthermore, in cases of material sheets that are represented, for example, by impedance or resistive or other more general type of boundary conditions one cannot guess the right-hand side of (1.3b).

To avoid the difficulties mentioned above, we propose to add the following assumption to the Maxwell equations [5]:

Maxwell equations are valid in the whole of the four-dimensional space in the sense of distribution.

The results of this assumption (or postulate) will be seen after Chapter 2. Here we confine ourselves to make only the following remarks that are of importance.

i. This assumption not only legitimizes the use of the Dirac delta *functions* to represent surface (or line or point) charges and currents (i.e., ρ and $\mathbf{J}$) but also claims that the field components **E,**

*Correct expression is $\lambda[D_n^{(2)} - D_{1n}] = -\mathrm{div}\{(1/\lambda)\rho_1\mathbf{n}\}$. Here ρ_1 stands for the density of the dipole distribution while λ is a parameter depending on the curvature of S and $\mathbf{n}$ is the unit normal vector to S [see Section 3.2.1, formula (3.19c)].

D, H, and **B** are also distributions. In other words, the field components themselves can contain *singular* terms concentrated on certain surfaces.

ii. This assumption concerns not only the space coordinates and boundary conditions but also the time parameter and initial values. That means that the field components can involve also singular terms concentrated at certain isolated instants (such as flash of lightning).

iii. In 1873, when Maxwell had established his theory, as well as during the following 75 years, the concept of distribution did not exist in the scientific literature. It appeared after 1950 first in mathematics and then in physics and engineering sciences. Hence, the claim that the Maxwell equations are valid in the sense of distribution is in fact a new postulate added to the Maxwell Theory.

The arrangement of the present monograph is as follows: In Chapter 2 the concepts of distribution and derivatives in the sense of distribution are explained. Grad, curl, and div operators as well as distributions concentrated on a surface are discussed in some detail. In Chapter 3 the Maxwell equations are reconsidered and discussed in the framework of these new concepts. The so-called *universal boundary conditions* are derived as a natural result. Chapter 4 is devoted to an extensive analysis of the boundary conditions on a *material sheet* at rest. The so-called impedance and resistive-type particular conditions are discussed and their validity conditions are derived. The case of moving boundaries is considered in Chapter 5. In this chapter the connection with the Special Theory of Relativity is also established whenever the motion is uniform. Chapter 6 is devoted to the edge conditions on a wedge bounded by planar walls. In this chapter, one shows also that the origin of the logarithmic-type singularities is the confluence of two algebraic singularities. In Chapter 7, one considers the tip singularities that occur at the apex of a rotationally curved material cone. For this kind of geometry, also the logarithmic singularities are derived as a result of confluence of two algebraic singularities. In Chapter 8, one considers the temporal discontinuities localized at certain times.