PART I

BASIC ELECTROMAGNETIC THEORY

INTRODUCTION TO PART I

Michael Faraday, James Clerk Maxwell, and others[1] postulated that the electromagnetic forces generated by electric charge and current densities (ρ and $\mathbf{J}$) are transmitted by electric and magnetic vector fields ($\mathbf{E}$ and $\mathbf{H}$) that exist in the surrounding free-space. This insight led to rapid progress in understanding the interconnections between time-varying sources and these fields and culminated in Maxwell's Equations as set forth in 1864. Their properties led to the prediction of electromagnetic waves traveling at the speed of light. A more complete understanding of the power and energy associated with these fields followed twenty years later.

[1]Originally, electricity and magnetism were separate fields of inquiry, each with its own nomenclature and units. That changed in 1819 when the Danish physicist Hans Christian Oersted (1777–1851) discovered the interaction of electric currents on a compass needle and coined the word "electromagnetism." Following up on this revelation, the French physicist Andre Ampere (1775–1836) studied the forces between electric currents.

The interaction of fields and macroscopic matter is complicated by the fact that (fortunately) materials have very diverse forms that include insulators, metals, and semiconductors. Almost all of these have dielectric properties; a few are strongly magnetic—sometimes even when no external fields are applied. The principal responses of electromagnetic character that are generated in materials are: conduction currents due to the motion of mobile unpaired charges,[2] electric dipoles created by bound-charge pairs of opposite polarity, and magnetic dipoles generated by either tiny current loops or the coupling of the intrinsic magnetic moments of individual electrons (sometimes protons). Three vector fields are found to characterize a wide range of materials: They are the conduction-current density, $\mathbf{J}_u$, the electric-dipole density, $\mathbf{P}$, and the magnetic-dipole density, $\mathbf{M}$. The constitutive laws that relate the material responses to $\mathbf{E}$ and $\mathbf{H}$ are usually nonlinear (especially in the case of ferromagnets), but are often approximated as linear—at least for weak fields.

Part I is devoted to a fairly conventional, but compact, presentation of both the integral and differential forms of Maxwell's Equations in free-space containing electric charges and currents. Rather than follow the historical development, we simply postulate the validity of conservation of charge, the Lorentz-force density, Faraday's Law, and Ampere's Law—as amended by Maxwell. All sources are assumed to be averaged over microscopic distances and times; therefore, the fields that they generate must also be considered macroscopic in nature. An equivalent representation of these fields, in terms of the magnetic vector potential and the electric scalar potential, is also given for both Coulomb and Lorenz gauges.

Next, inhomogeneous wave equations governing the electric and magnetic fields are derived—as are those governing the vector and scalar potentials in both the time and frequency domains. Solutions are found by direct integration of the currents and charges (assumed known). As an important example, the potentials associated with a line current of incremental length are derived; they provide the basis for understanding electromagnetic radiation. Materials with polarization and magnetization that may be electrically conducting are then considered as are the boundary conditions at material and source interfaces. When wave-propagation effects are negligible, fields that are mainly electric or magnetic are classified as either electroquasistatic (*EQS*) or magnetoquasistatic (*MQS*). The equations governing quasistatic fields are derived, and analysis by approximate methods is outlined; the approach taken follows that of earlier M.I.T. textbooks by Fano, Chu, and Adler [5] and Haus and Melcher [6].

As is customary, electromagnetic power, energy, stress, momentum, and angular momentum are defined within the Maxwell–Poynting representation that is based upon the Poynting vector and Maxwell stress tensor. However, the Alternate representation by Morgenthaler [8] is also presented that embraces localized circuit-theory concepts and merges with quasistatic representations, yet is *exact*; the unification of power, energy, stress, and momentum in these, and other representations, is delayed until Part II (where the appropriate mathematical tools are developed). Power-energy theorems for both representations are developed in the time domain. These are used to calculate Poynting power and energy of a uniform plane wave and a radiating electric dipole and compare them with the Alternate counterparts.

[2]In most metals, diffusion currents are negligible because conduction and charge neutrality combine to remove concentration gradients that drive diffusion, In semiconductors, diffusion can be made to dominate; the bipolar transistor depends upon it.

Next, homogeneous wave equations and free-space waves propagating in one, two, and three dimensions are studied in both the time and frequency domains. Waves of high symmetry are given special attention because their superposition can be used to represent inhomogeneous waves generated by arbitrary, but known, sources. Homogeneous waves in linear dielectric and magnetic materials (that may also be conductive) are considered; sinusoidal steady-state (harmonic) waves are given particular attention. In the limit of high conductivity, simple wave propagation is radically altered and replaced by the interrelated concepts of charge relaxation, magnetic diffusion, and skin depth. For loss-free dielectrics, power, energy, stress, and momentum associated with packets of uniform harmonic plane waves are considered from both wave and particle perspectives; the latter is shown to follow a Hamiltonian formulation.

The final chapter of Part I is devoted to important theorems and principles. Frequency-domain theorems involving electromagnetic power, stress, energy, and momentum are derived. These are followed by duality of fields and sources, the uniqueness theorem, the equivalence principle, the induction theorem, Babinet's Principle for complementary structures, and the reciprocity theorem.

When the Part III examples and explanatory texts are included, the overall coverage of the standard wave-propagation issues is similar to that of other texts but is built upon both the Maxwell–Poynting and Alternate representations. The most recent effort [9] (designed for the M.I.T. EECS undergraduate curriculum) is authored by D. H. Staelin, A. W. Morgenthaler, and J. A. Kong; older standards include Adler et al. [10] and Ramo et al. [11].

The text is designed to serve a variety of curriculums. Because much of Part III depends only upon Part I, an undergraduate course could omit the four-dimensional electrodynamics of Part II and include only basic examples; the focus of a particular graduate course might be restricted to advanced electrodynamics, antennas and diffraction, or transmission lines and microwave circuits.

CHAPTER 1

MAXWELL'S EQUATIONS

1.1 MATHEMATICAL NOTATION

- Scalar quantities, such as Φ, are printed in normal type.
- Vectors, such as $\mathbf{E}$ and $\mathbf{H}$, are printed in **bold** type; their components are printed in normal type with subscripts that indicate coordinate directions.
- Unit vectors are printed in **bold** type, but with $\widehat{}$ over the symbol. Thus, $\mathbf{E} = \widehat{\mathbf{x}}E_x + \widehat{\mathbf{y}}E_y + \widehat{\mathbf{z}}E_z$ represents the electric field expressed in Cartesian coordinates as the sum of three orthogonal vectors. Numerical subscripts $1, 2, 3$ often substitute for x, y, z; for example, E_i can be any one of the components.
- The scalar or dot product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is indicated generally by $\mathbf{A} \cdot \mathbf{B}$ or explicitly by either $A_x B_x + A_y B_y + A_z B_z$ or $\sum_{i=1}^{3} A_i B_i$. When (as here) indices are repeated, the summation symbol is often understood to exist and omitted in order to simplify the expression.
- The vector cross product of two vectors $\mathbf{A}$ and $\mathbf{B}$ is indicated by $\mathbf{A} \times \mathbf{B}$.

The Power and Beauty of Electromagnetic Fields, First Edition. F. R. Morgenthaler.

- The del operator is defined by $\nabla = \widehat{\mathbf{x}}\frac{\partial}{\partial x} + \widehat{\mathbf{y}}\frac{\partial}{\partial y} + \widehat{\mathbf{z}}\frac{\partial}{\partial z}$ or in index notation by $\frac{\partial}{\partial x_i}$. It is used to express the gradient of a scalar and the divergence and curl of a vector. These operations are defined in Appendix B, Section B.2.
- Tensor quantities, such as $\overline{\mathbf{T}}$ are printed in **bold** type, but with a bar over the symbol; the components may be expressed in dyadic notation as two adjacent vectors $\mathbf{A}\,\mathbf{B}$ or in component form using normal type. Examples are: $\overline{\mathbf{T}} = \mathbf{A}\,\mathbf{B} = (\widehat{\mathbf{x}}A_x + \widehat{\mathbf{y}}A_y + \widehat{\mathbf{z}}A_z)(\widehat{\mathbf{x}}B_x + \widehat{\mathbf{y}}B_y + \widehat{\mathbf{z}}B_z)$ or $T_{ij} = A_i B_j$. Notice that, in general, $T_{ij} \neq T_{ji}$.
- Four-dimensional vectors, such as $\mathcal{E}$ and $\mathcal{H}$, and dyads, like $\mathcal{A}\ \mathcal{B}$, are printed in calligraphic type.

1.2 FREE-SPACE FIELDS AND FORCES

In free-space regions that contain electric and magnetic field vectors, **E** and **H**, the Lorentz force on a charge, q_i, moving with velocity, $\mathbf{v}_i$ (the subscript enumerates the charge and is not the index of a Cartesian component) is

$$\mathbf{F}_i = q_i(\mathbf{E} + \mathbf{v}_i \times \mu_o \mathbf{H}) \tag{1.1}$$

We employ SI units[1] here and throughout the text.

If there are many charges within a small finite volume, ΔV, the total force within it is

$$\textstyle\sum_i \mathbf{F}_i = (\sum_i q_i)\mathbf{E} + (\sum_i q_i \mathbf{v}_i) \times \mu_o \mathbf{H}$$

When divided by ΔV, Eq. (1.1) reduces to the Lorentz force density,

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mu_o \mathbf{H} \tag{1.2}$$

where $\rho = \lim_{\Delta V \to 0} \frac{\sum_i q_i}{\Delta V}$ is the macroscopic electric charge density and $\mathbf{J} = \lim_{\Delta V \to 0} \frac{\sum_i q_i \mathbf{v}_i}{\Delta V}$ is the macroscopic electric-current density. In these averages, the limiting ΔV must remain large compared to microscopic dimensions. In the event that all of the velocities are equal to **v**, the current, $\mathbf{J} = \rho\mathbf{v}$, is said to be convective. The **E** and **H** fields arise from all of the charges and currents and are themselves macroscopic averages. The equations that follow describe the interactions between these macroscopic quantities.

Integral form of Maxwell's Equations

For a region of free-space containing ρ and **J**, Maxwell's Equations, when expressed in integral form, are

$$\oint_{C_o} \mathbf{H} \cdot d\mathbf{s} - \frac{d}{dt}\int_S \varepsilon_o \mathbf{E} \cdot d\mathbf{a} = \int_S \mathbf{J} \cdot d\mathbf{a} \tag{1.3a}$$

$$\oint_{C_o} \mathbf{E} \cdot d\mathbf{s} + \frac{d}{dt}\int_S \mu_o \mathbf{H} \cdot d\mathbf{a} = 0 \tag{1.3b}$$

[1] In this system of units, the abbreviations used include:
m (meter), s (second), kg (kilogram), C (coulomb), V (volt), A (ampere), F (farad), H (henry), S (siemens), $S^{-1} = \Omega$ (ohms).

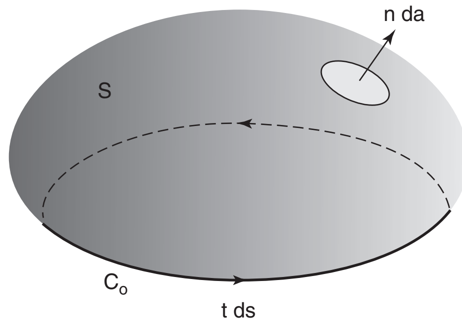

Figure 1.1 Surface spanning closed contour.

where S is an arbitrary surface that spans the closed contour C_o that is depicted in Figure 1.1.

When S is a minimal surface (that of a soap bubble film anchored by the contour), its positive side is defined as that seen from above as we circle the contour in a counterclockwise direction. The unit vector, $\widehat{\mathbf{n}}$, normal to that surface (or any other that can be formed by simple deformation), points outward from its positive side. The vector differential area of S is defined by $d\mathbf{a} = \widehat{\mathbf{n}}da$; the vector differential distance along the contour, C_o, by $d\mathbf{s} = \widehat{\mathbf{t}}ds$, with $\widehat{\mathbf{t}}$ the unit vector tangential to the contour (and of positive polarity as we circle it). The first of these equations is Ampere's Law as amended by Maxwell to include the time-varying term (renowned as the Maxwell displacement current density); the second is Faraday's Law.

Because the electric charge is conserved, the total current flowing out of the closed-surface, S_o, that encloses an arbitrary volume, V, must equal the negative time-rate of change of the charge contained within it. Therefore,

$$\oint_{S_o} \mathbf{J} \cdot d\mathbf{a} + \frac{d}{dt}\int_V \rho \; dV = 0 \tag{1.4}$$

is the integral form of the Law of Conservation of Charge. If the contour C_o is allowed to shrink to zero as $S \rightarrow S_o$, Eq. (1.3a) becomes

$$-\frac{d}{dt}\oint_{S_o} \varepsilon_o \mathbf{E} \cdot d\mathbf{a} = \oint_{S_o} \mathbf{J} \cdot d\mathbf{a}$$

provided that the magnetic-field tangential to C_o is finite and so cannot contribute to the limiting contour integral. When combined with Eq. (1.4) the result,

$$\frac{d}{dt}\left(\oint_{S_o} \varepsilon_o \mathbf{E} \cdot d\mathbf{a} - \int_V \rho \; dV\right) = 0$$

can be integrated with respect to time to yield

$$\oint_{S_o} \varepsilon_o \mathbf{E} \cdot d\mathbf{a} - \int_V \rho \; dV = \text{constant}$$

Consider a very small volume, V, centered upon some chosen point. Because ρ and $\mathbf{E}$ can be forced to vanish in that region for at least one instant of time, the constant must be zero for that and (by extension) *every* region.

It follows that

$$\oint_{S_o} \varepsilon_o \mathbf{E} \cdot d\mathbf{a} = \int_V \rho \, dV \tag{1.5a}$$

applies to an arbitrary volume. Equation (1.5a) is known as Gauss' Law. Similar arguments applied to Eq. (1.3b) lead to

$$\oint_{S_o} \mu_o \mathbf{H} \cdot d\mathbf{a} = 0 \tag{1.5b}$$

These equations are often included in the set of Maxwell's Equations, but are not independent. From Eqs. (1.3a) and (1.3b) and *either* Conservation of Charge or Gauss' Law the other equation follows.

Units and fundamental constants

The units of the field vectors are $[E] = Vm^{-1}$and $[H] = Am^{-1}$, those of the electric charge and current densities are $[\rho] = Cm^{-3}$, and $[J] = Am^{-2}$, and that of the Lorentz force density is $[f] = N\ m^{-3}$. The constants μ_o and ε_o are, respectively, the permeability and permittivity of free-space. The former is legislated to have the value

$$\mu_o = 4\pi \times 10^{-7}\, Hm^{-1}$$

whereas the latter is a measured quantity,

$$\varepsilon_o = 8.854187817 \times 10^{-12}\, Fm^{-1}$$

Because

$$\mathrm{c} = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 2.99792458 \times 10^8\, ms^{-1} \tag{1.6}$$

(the velocity of light in vacuum) is so close to 3×10^8 m s^{-1}, it is often convenient to approximate the permittivity by the exact value,

$$\varepsilon_o = \frac{1}{36\pi} \times 10^{-9}\, Fm^{-1}$$

We shall adhere to that practice. Another combination of these constants that is extremely useful is $\mathrm{c}\mu_o$ defined as the characteristic impedance of free-space. Given the symbol, η_o, it has the value

$$\eta_o = \mathrm{c}\mu_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\,\Omega\ (S^{-1})$$

Linearity and superposition

Because Maxwell's Equations are linear with respect to the fields and the charges and currents, superposition may be applied after dividing the sources into components with respect to either space or time (or some combination). In particular, if

$$\rho = \sum_i \rho_{(i)} \tag{1.7a}$$

$$\mathbf{J} = \sum_i \mathbf{J}_{(i)} \tag{1.7b}$$

and each ith set of sources (not to be confused with a Cartesian index) is chosen so as to satisfy conservation of charge, then

$$\mathbf{E} = \sum_i \mathbf{E}_{(i)} \tag{1.7c}$$

$$\mathbf{H} = \sum_i \mathbf{H}_{(i)} \tag{1.7d}$$

where $\mathbf{E}_{(i)}$ and $\mathbf{H}_{(i)}$ are the Maxwellian fields generated when $\rho_{(i)}$ and $\mathbf{J}_{(i)}$ act alone. The components may be either discrete or differential; in the latter case the summations are replaced by integrals. When it is possible to decompose the sources into elements each providing a high degree of spatial symmetry, finding the field solutions is greatly facilitated. Fields generated by stationary spatially symmetric charge or current sources with either spherical, cylindrical, or planar symmetry are considered in Appendix C, Section C.1. Time-dependent sources are often considered to be the superposition of sinusoidal functions. This gives rise to representations employing Fourier series and Fourier integrals in the respective cases of periodic or aperiodic functions. As we shall learn in later sections of this chapter, representation in terms of complex exponential functions is also a standard technique—applicable to the fields as well as the sources.

Symmetric field-transient example

When a highly symmetric source is suddenly switched on at $t = 0$ and remains constant thereafter, it is sometimes possible to deduce the field solutions using only the integral form of the Maxwell Equations together with symmetry considerations. This method depends solely upon the material that has been presented up to this point and is employed in Part III, Chapter 15, Section 15.1. A more traditional solution of the same problem is presented in Section 15.2; it is based upon the symmetries that are made evident by the differential form of Maxwell's Equations. These are developed and built upon in the remainder of this chapter. Before continuing, the reader is encouraged to jump ahead to the first example in order to gain a preliminary understanding of both propagating electromagnetic transients and the quasistatic fields that follow in their wake. Both the speed of light and the ratio of electric to magnetic radiation field strengths (the characteristic impedance of free-space) emerge from that example.

Differential form of Maxwell's Equations

If the contour C_o is chosen to lie in the $y - z$ plane and it and the surface S spanning the contour are allowed to shrink to vanishingly small dimensions, the contour integral $\oint_{C_o} \mathbf{A} \cdot d\mathbf{s}$ (on a per unit area basis) equals the x component of the curl $\mathbf{A}$. Suitable orientation of the contour (in the $z - x$ and $x - y$ planes) will produce the y and z components. Likewise, when the closed surface shrinks toward a point, the integral of the normal flux of a vector over that surface is (on a per unit volume basis) the divergence of that vector.

The set of integral equations is therefore equivalent to the differential forms

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \tag{1.8a}$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} \tag{1.8b}$$

$$\nabla \cdot \ \varepsilon_o \mathbf{E} = \rho \tag{1.8c}$$

$$\nabla \cdot \ \mu_o \mathbf{H} = 0 \tag{1.8d}$$

where the curl ($\nabla\times$) and divergence ($\nabla\cdot$) operators are written in terms of the gradient operator, ∇ (del). Alternatively, the Divergence Theorem and Stoke's Theorem (Appendix B, Eqs. (B.7) and (B.8)) can be applied to produce the same results. We prove in Chapter 6 that both the curl and divergence of a field vector are determining factors in specifying that vector, and so it is not surprising that Maxwell's Equations constrain these quantities for both **E** and **H**.

Because $\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}(\nabla \cdot \varepsilon_o \mathbf{E}) = 0$, the first and third equations imply conservation of electric charge,

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \tag{1.8e}$$

This result also follows directly from Eq. (1.4).

1.3 VECTOR AND SCALAR POTENTIALS

From renewed use of the vector identity, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, it follows that Eq. (1.8d) is automatically satisfied by

$$\mu_o \mathbf{H} = \nabla \times \mathbf{A} \tag{1.9}$$

When substituted into Eq. (1.8b), the result is

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

A second vector identity, $\nabla \times (\nabla \Phi) = 0$, then permits one to express the electric field as

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi \tag{1.10}$$

The vector (**A**) and scalar (Φ) potentials (defined with the conventional polarities) are not unique because the set

$$\mathbf{A}' = \mathbf{A} + \nabla \Psi \tag{1.11a}$$

$$\Phi' = \Phi - \frac{\partial \Psi}{\partial t} \tag{1.11b}$$

produces the same values of both **E** and $\mu_o \mathbf{H}$. The primed and unprimed potentials are said to be related by a gauge transformation (set by the scalar function, Ψ). Evidently, the value of $\nabla \cdot \mathbf{A}$ is not unique and may be specified as desired; two possibilities are of special interest.

Lorenz gauge

The Lorenz gauge[2] is defined by

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0 \tag{1.12}$$

with c given by Eq. (1.6).

If that value is selected and Eqs. (1.9) and (1.10) are substituted into Eqs. (1.8e), the result is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \mathbf{J} \tag{1.13a}$$

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_o} \tag{1.13b}$$

where **A** and **J** are assumed to be expressed in Cartesian components. All four components (A_x, A_y, A_z, Φ) are solutions of inhomogeneous wave equations; they are not independent because conservation of charge must be satisfied. The current and charge densities are those existing in free-space and must satisfy Eq. (1.8e); additional sources that model dielectric and magnetic materials are described in Section 1.7 and, more thoroughly, in Chapter 9.

Coulomb gauge

The Coulomb gauge (sometimes called the radiation gauge) is defined by

$$\nabla \cdot \mathbf{A} = 0 \tag{1.14}$$

If that value is selected and Eqs. (1.9) and (1.10) are substituted into Eqs. (1.8e), the result is

$$\nabla^2 \Phi = -\rho/\varepsilon_o \tag{1.15a}$$

$$\nabla^2 \mathbf{A} = -\mu_o \left(\mathbf{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \right) \tag{1.15b}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o \left(\mathbf{J} - \varepsilon_o \nabla \frac{\partial \Phi}{\partial t} \right) \tag{1.15c}$$

Equation (1.15a) is Poisson's Equation, which does not contain time derivatives; consequently, the scalar potential at any position must respond *instantaneously and simultaneously* to all changes in the electric charge density, $\rho(\mathbf{r}, t)$. Consideration of Eq. (1.15b) reveals that the same is true of the vector potential, with respect to $\mathbf{J}(\mathbf{r},t) + \varepsilon_o \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}$ – the current density that includes the Maxwell displacement current. However, when **E** is expressed in terms of both the scalar and vector potentials, the wave equation,

[2]The Lorenz gauge formulated by the Danish mathematician and physicist, Ludwig V. Lorenz (1829–1891), is often incorrectly attributed to Hendrick Lorentz; the latter is noted for the Lorentz transformation, the Lorentz contraction, and his theory of the electron.

Eq. (1.15c), emerges. Of course, the values of $\mathbf{E}$ and $\mathbf{H}$ calculated from the Coulomb gauge must (and do) agree with those calculated from the Lorenz (or any other) gauge.

Finally, when $\Phi = 0$ (because, either there is no electric charge anywhere or else Eqs. (1.11) have been used), the Lorenz and Coulomb gauges can be identical, but only in charge-free regions of space. The reader should verify that the requisite Ψ satisfies the homogeneous wave equation, except where $\partial(\nabla \cdot \mathbf{A})/\partial t = -\rho/\varepsilon_o$ causes the Coulomb gauge to fail.

1.4 INHOMOGENEOUS WAVE EQUATIONS FOR E AND H

The electric and magnetic fields, interwoven by Maxwell's Equations, can be decoupled by use of the vector identities,

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}$$

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

and substitution of the curls and divergences of $\mathbf{H}$ and $\mathbf{E}$. The result (after reordering partial derivatives) is a pair of inhomogeneous wave equations (expressed in terms of Cartesian components),

$$\nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla \times \mathbf{J} \tag{1.16a}$$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\varepsilon_o}\nabla\rho + \mu_o\frac{\partial \mathbf{J}}{\partial t} \tag{1.16b}$$

The solutions are not independent because, as noted previously, the current and charge densities must satisfy conservation of charge. The velocity of light, c, is given by Eq. (1.6).

An alternate, though less direct, derivation of Eqs. (1.16a) and (1.16b) follows (in the Lorenz gauge) from

$$\nabla \times \left(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \mu_o \mathbf{J}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \mu_o \mathbf{J}\right) + \nabla\left(\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \frac{\rho}{\varepsilon_o}\right) = 0$$

and (after again interchanging the order of the derivatives) substitution of Eqs. (1.9) and (1.10). Naturally, the final result is independent of the choice of gauge.

1.5 STATIC FIELDS

When there is no time variation in the charges and/or currents, the steady-state fields are also static and related by

$$\nabla^2 \mathbf{E} = \frac{1}{\varepsilon_o}\nabla\rho \tag{1.17a}$$

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon_o} \tag{1.17b}$$

$$\nabla^2\mathbf{A} = -\mu_o\mathbf{J} \tag{1.17c}$$

Static electric charge generates electrostatic fields Φ and $\mathbf{E}$; static electric current generates magnetostatic fields, $\mathbf{A}$ and $\mathbf{H}$. When the vectors are expressed in Cartesian coordinates, each of the scalar equations takes the form of Poisson's Equation:

$$\nabla^2\Psi = -s \tag{1.18}$$

When $s = 0$, the homogeneous equation is termed Laplace's Equation. Solutions in various coordinate systems are discussed in Appendix C, Section C.4; they are of great importance in solving both static and quasistatic boundary-value problems. For now, we assume that s is a known function of position and attempt to integrate Poisson's Equation directly.

Integration of Poisson's Equation

Because Eq. (1.18) is a linear partial differential equation, both the source density, s, and the response function, Ψ, can be expressed as the superposition of differential components that individually satisfy this equation.

When s is a three-dimensional unit impulse (delta function) located at the origin of spherical coordinates, the spherically symmetric solution satisfies

$$\nabla^2\Psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) = -\delta(r) \tag{1.19}$$

with $\delta(r) = \frac{1}{4\pi r^2}u_0(r)$. Here $u_0(r)$ is the derivative of the unit step function, $u_{-1}(r)$ (both are defined in Appendix B, Section B.1, but with origins shifted to $r = 0^+$ so that $\int_0^\infty u_0(r)dr = 1$). It follows that $\Psi = 1/(4\pi r)$ and (with a shift of origin) $d\Psi_p = s_q dV_q/(4\pi r_{qp})$ where, as indicated in Figure 1.2,

$\mathbf{r}_{qp} = \mathbf{r}_p - \mathbf{r}_q$ is the vector between the differential source located at $\mathbf{r}_q$ and the response evaluated at $\mathbf{r}_p$ and $r_{qp} = |\mathbf{r}_p - \mathbf{r}_q|$. After integration, the general solution of Eq. (1.18) is found to be

$$\Psi_p(\mathbf{r}_p) = \int_V \frac{s_q(\mathbf{r}_q)dV_q}{4\pi r_{qp}} \tag{1.20}$$

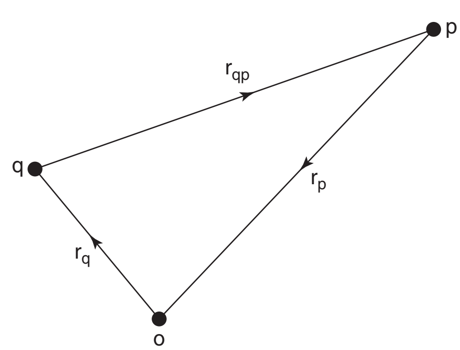

Figure 1.2 Origin, source point, q, and field point, p.

When the field point is distant from all of the sources, $1/r_{qp}$ can be expanded in a series of powers of $1/r_p$. Together these constitute the multipole expansion that is developed in Appendix C, Section C.2; the $1/r_p$ term is the monopole, $1/r_p^2$ the dipole, $1/r_p^3$ the quadrupole, and so on.

Electrostatics

The equations of electrostatics in integral form are

$$\Phi_p(\mathbf{r}_p) = \int_V \frac{\rho_q(\mathbf{r}_q)\ dV_q}{4\pi\varepsilon_o r_{qp}} \tag{1.21a}$$

$$\mathbf{E}_p(\mathbf{r}_p) = \int_V \widehat{\mathbf{r}}_{qp} \frac{\rho_q(\mathbf{r}_q)\ dV_q}{4\pi\varepsilon_o r_{qp}^2} \tag{1.21b}$$

and together form Coulomb's Law.

Sphere of uniform charge

As a simple example, consider a sphere of radius R that contains a uniform volume charge density, ρ_o. Without loss of generality we may define spherical coordinates so that the point p lies on the negative z axis. Then Eq. (1.21a) becomes

$$\Phi(r_p) = \int_0^R \int_0^\pi \frac{\rho_o 2\pi r_q^2\ dr_q \sin\theta_q\ d\theta_q}{4\pi\varepsilon_o\sqrt{r_p^2 + r_q^2 + 2r_p r_q \cos\theta_q}}$$

where r_p may be smaller or larger than R. The result of the two integrations (after dropping the subscript p) is

$$\Phi(r) = \frac{\rho_o}{\varepsilon_o}\begin{cases} \frac{1}{2}R^2 - \frac{1}{6}r^2, & r \le R \\ \dfrac{R^3}{3r}, & r \ge R \end{cases}$$

From the gradient operator, or from Eq. (1.21b), the electrostatic electric field is

$$\mathbf{E}(r) = \widehat{\mathbf{r}}\frac{\rho_o}{3\varepsilon_o}\begin{cases} r, & r \le R \\ \dfrac{R^3}{r^2}, & r \ge R \end{cases}$$

This solution was carried out only to illustrate the procedure; the preferred method of solution is to use symmetry and Gauss' Law to immediately solve for the electric field.

Magnetostatics

The equations of magnetostatics in integral form are (with $\nabla \cdot \mathbf{J} = 0$):

$$\mathbf{A}_p(\mathbf{r}_p) = \int_V \frac{\mu_o \mathbf{J}_q(\mathbf{r}_q)\ dV_q}{4\pi r_{qp}} \tag{1.22a}$$

$$\mu_o \mathbf{H}_p(\mathbf{r}_p) = \nabla_p \times \mathbf{A}_p = \frac{\mu_o}{4\pi}\int_V \nabla_p \times \left[\frac{\mathbf{J}_q(\mathbf{r}_q)}{r_{qp}}\right] dV_q$$

$$= \frac{\mu_o}{4\pi}\int_V \nabla_p\left(\frac{1}{r_{qp}}\right) \times \mathbf{J}_q(\mathbf{r}_q) dV_q \tag{1.22b}$$

Because $\nabla_p\left(\frac{1}{r_{qp}}\right) = -\nabla_q\left(\frac{1}{r_{qp}}\right) = -\frac{\widehat{\mathbf{r}}_{qp}}{r_{qp}^2}$,

$$\mathbf{H}_p(\mathbf{r}_p) = \frac{1}{4\pi}\int_V \frac{\mathbf{J}_q \times \mathbf{r}_{qp}}{r_{qp}^3}\, dV_q \tag{1.22c}$$

Equation (1.22c) is known as the Biot–Savart Law.

Circular current loop

As an important example, we calculate the magnetic field on the axis of a planar circular loop of wire carrying current I_o; the wire radius is assumed to be very small compared to the loop radius, R. Integrating over the area of the wire results in

$$\mathbf{H}_p(\mathbf{r}_p) = \frac{I_o}{4\pi}\oint \frac{\mathbf{ds}_q \times \mathbf{r}_{qp}}{r_{qp}^3} \tag{1.23a}$$

where $\mathbf{ds}_q$ is the differential vector length of the current element. If the current loop lies in the plane $z = 0$ with its center at the origin, the on-axis field (which by symmetry is z-directed) is easily evaluated and found to be

$$\mathbf{H}(r = 0, z) = \widehat{\mathbf{z}}\frac{I_o R^2}{2(R^2 + z^2)^{3/2}} \tag{1.23b}$$

The current direction and $\mathbf{H}$ obey the right-hand rule (with fingers curled in the direction of the current, the thumb points in the direction of the field).

1.6 INTEGRATION OF THE INHOMOGENEOUS WAVE EQUATION

The inhomogeneous scalar wave equation is defined by

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -s(\mathbf{r}, t) \tag{1.24}$$

If the spatial derivatives dominate over those with respect to time, the inhomogeneous wave equation reduces to Poisson's Equation, and an approximation to the solution is

$$\Psi_p(\mathbf{r}_p, t) \simeq \int_V \frac{s_q(\mathbf{r}_q, t)\, dV_q}{4\pi r_{qp}} \tag{1.25}$$

Such quasistatic solutions are often useful approximations to the exact solution.

Because there is a time delay of r_{qp}/c before Ψ_p can respond to changes in the source located at $\mathbf{r}_q$, the simplest correction to Eq. (1.25) that might be expected to improve the approximation is to replace t with the retarded time, $t - r_{qp}/c$, where

$$r_{qp} = \sqrt{r_q^2 + r_p^2 - 2\mathbf{r}_q \cdot \mathbf{r}_p} \tag{1.26}$$

Remarkably,

$$\Psi_p(\mathbf{r}_p, t) = \int \frac{s_q(\mathbf{r}_q, t - r_{qp}/c)\, dV_q}{4\pi r_{qp}} \tag{1.27}$$

is an *exact* solution of the wave equation. Although this result can be verified by direct substitution, we postpone proofs until Chapter 4, Section 4.5 and Chapter 7, Section 7.7.

In cases where the source term, $s_q(\mathbf{r}_q, t)$ has a sinusoidal time dependence at frequency ω, it is advantageous to introduce complex functions $\underline{s}_q(\mathbf{r}_q, j\omega)$ and $\underline{\Psi}_p(\mathbf{r}_p, j\omega)$ such that

$$s_q(\mathbf{r}_q, t) = Re\{\underline{s}_q(\mathbf{r}_q, j\omega)\exp(j\omega t)\}$$

$$\Psi_p(\mathbf{r}_p, t) = Re\{\underline{\Psi}_p(\mathbf{r}_p, j\omega)\exp(j\omega t)\}$$

where Re { } is the operator that extracts the real part of the expression contained within the curly brackets.

Substitution into Eq. (1.24) produces the complex form of the inhomogeneous wave equation,

$$\nabla^2\underline{\Psi} + k^2\underline{\Psi} = -\underline{s}_q(\mathbf{r}_q) \tag{1.28}$$

where the wavenumber k is defined by

$$k = \omega\sqrt{\mu_o\varepsilon_o} \tag{1.29}$$

Because it is customary to define $\omega = 2\pi f$ and $k = 2\pi/\lambda_o$, the frequency f and free-space wavelength λ_o are related by

$$f\lambda_o = c \tag{1.30}$$

From Eq. (1.27), the integral form of the complex solution of Eq. (1.28) is

$$\underline{\Psi}_p(\mathbf{r}_p) = \int \frac{\underline{s}_q(\mathbf{r}_q)\exp(-jkr_{qp})\ dV_q}{4\pi r_{qp}} \tag{1.31}$$

The complex exponential accounts for time retardation through the phase factor kr_{qp}. Integrals of this form are commonly encountered in diffraction and antenna theory; they can often be approximated with simpler forms when the observation point, $\mathbf{r}_p$, is very distant from all source locations, $\mathbf{r}_q$. In such cases, Eq. (1.26) is well-approximated by

$$r_{qp} \simeq r_p - \frac{\mathbf{r}_q \cdot \mathbf{r}_p}{r_p}$$

and the so-called "far-field" solution is

$$\underline{\Psi}_p(\mathbf{r}_p) \simeq \frac{\exp(-jkr_p)}{4\pi r_p}\int_V \underline{s}_q(\mathbf{r}_q)\exp\left(jk\frac{\mathbf{r}_q \cdot \mathbf{r}_p}{r_p}\right)\ dV_q \tag{1.32}$$

where the amplitude has been safely approximated using, $r_{qp} \simeq r_p$.

We note that in either the time or frequency domain, Ψ can be replaced by the scalar Φ or any Cartesian component of $\mathbf{A}$, $\mathbf{E}$, or $\mathbf{H}$ with the corresponding s evaluated from ρ and $\mathbf{J}$. Equation (1.32) will be made use of repeatedly in the examples analyzed in Part III, Chapter 22.

Current element (Hertzian electric dipole)

As an important example, we calculate the vector potential of a line current, of length d and magnitude $I_o \sin\omega t$, that is parallel to the z axis and is "electrically short," that is $d \ll \lambda_o (kd \ll 1)$. Because we are assuming a steady-state current, it is convenient to employ complex vectors and scalars. With $\underline{s}_q(\mathbf{r}_q) = \underline{J}_z(x_q, y_q, z_q)$ and $\underline{\Psi}_p(\mathbf{r}_p) = \underline{A}_z(\mathbf{r}_p)$, it follows that for all $r_p \gg d$, Eq. (1.32),

$$\underline{A}_z(\mathbf{r}_p) = \mu_o\frac{\exp(-jkr_p)}{4\pi r_p}\iiint \underline{J}_z(x_q, y_q, z_q)\exp(jk\frac{\mathbf{r}_q \cdot \mathbf{r}_p}{r_p})\, dx_q dy_q dz_q$$

can be used. After integrating over the cross section of the current and realizing that $\exp(jk\frac{\mathbf{r}_q\cdot\mathbf{r}_p}{r_p}) \simeq 1$ (because the argument of the exponential never exceeds kd), the result (after dropping the subscript p) is

$$\underline{A}_z(\mathbf{r}) = \mu_o \frac{\exp(-jkr)}{4\pi r}\int_0^{\mathrm{d}} \underline{\mathrm{I}}(z_q)\ dz_q$$

If the current is uniform, the integral is simply $\underline{\mathrm{I}}_o d$; if not, it can be written as $\underline{\mathrm{I}}_o d_{\mathrm{eff}}$ and

$$\underline{A}_z(\mathbf{r}) = \mu_o \frac{\exp(-jkr)}{4\pi r}\underline{\mathrm{I}}_o d_{\mathrm{eff}} \tag{1.33}$$

Because of charge conservation, the current cannot go to zero at the ends of the line element without electric charges being created at the ends and/or along the length according to

$$\frac{\partial \underline{\mathrm{I}}(z)}{\partial z} + j\omega Q'(z) = 0$$

If $\frac{\partial \underline{\mathrm{I}}(z)}{\partial z} = 0$ (except at the ends), point charges of equal magnitude, $\underline{\mathrm{Q}}_o = \underline{\mathrm{I}}_o/\omega$, but opposite polarity, will exist at the ends and together create an electric dipole of moment, $\underline{\mathrm{Q}}_o d$; otherwise a line-charge density, $Q'(z)$, will exist along the current and form a distributed dipole. In all cases, bipolar charges are the source of a complex scalar potential that satisfies

$$\underline{\Phi}(\mathbf{r}_p) = \frac{\exp(-jkr_p)}{-j\omega 4\pi\varepsilon_o r_p}\int_0^{\mathrm{d}} \frac{\partial \underline{\mathrm{I}}(z_q)}{\partial z_q}\exp\left(jk\frac{\mathbf{r}_q\cdot\mathbf{r}_p}{r_p}\right)\ dz_q \tag{1.34}$$

This integral is a little tricky to evaluate correctly, so we choose the alternate approach of simply invoking the Lorenz gauge, $\nabla\cdot\underline{\mathbf{A}} + j\omega\mu_o\varepsilon_o\underline{\Phi} = 0$ (which by itself imposes conservation of charge). The result is

$$\begin{aligned}\underline{\Phi} &= \frac{-1}{j\omega\mu_o\varepsilon_o}\frac{\partial \underline{A}_z(\mathbf{r})}{\partial z} = \frac{-\underline{\mathrm{I}}_o d_{\mathrm{eff}}}{j\omega 4\pi\varepsilon_o}\frac{\partial}{\partial z}\left[\frac{\exp(-jkr)}{r}\right]\\ &= \frac{\underline{\mathrm{I}}_o d_{\mathrm{eff}}}{j\omega 4\pi\varepsilon_o}\left(jk + \frac{1}{r}\right)\frac{\exp(-jkr)}{r}\cos\theta\end{aligned}$$

The reader should verify that the same result is obtained from Eq. (1.34).

For the $\sin\omega t$ dependence specified at the outset, $\underline{\mathrm{I}}_o = -j\mathrm{I}_o$; therefore the time-dependent versions of the potentials (evaluated by multiplying by $\exp(j\omega t)$ and taking the real part) are

$$A_z(\mathbf{r},t) = \mu_o I_o d_{\mathrm{eff}}\frac{\sin(\omega t - kr)}{4\pi r} \tag{1.35a}$$

$$\Phi(\mathbf{r},t) = \sqrt{\frac{\mu_o}{\varepsilon_o}} I_o d_{\mathrm{eff}}\frac{\sin(\omega t - kr) - \frac{1}{kr}\cos(\omega t - kr)}{4\pi r}\cos\theta \tag{1.35b}$$

The Hertzian electric dipole is revisited in Chapter 3, Section 3.5 and Part III, Chapter 13, Section 13.1.

Current loop (Hertzian magnetic dipole)

As a second example, consider a small circular loop of radius R that lies in the $x-y$ plane, centered at $z = 0$. A uniform current, $\underline{\mathrm{I}}(t) = \underline{\mathrm{I}}_o\cos\omega t$, circulates around the loop.

We continue to define $k = \omega/c$ and employ spherical coordinates with $\widehat{\phi}$ the unit vector in the azimuthal direction. For $kR \ll 1$, the complex vector and scalar retarded potentials are

$$\underline{\mathbf{A}}(\mathbf{r}_p) = \mu_o \frac{\exp(-jkr_p)}{4\pi r_p} \oint_{\text{loop}} \underline{\mathrm{I}}_o \widehat{\phi}_q \exp\left(jk\frac{\mathbf{r}_q \cdot \mathbf{r}_p}{r_p}\right) ds_q$$
$$\underline{\Phi} = 0$$

The reader is cautioned to remember that although the current is $\mathrm{I}_o\widehat{\phi}$, the unit vector *cannot* simply be moved from inside to outside the integral. Instead, it must be converted to its Cartesian components before the integrations that determine **A** are attempted. Afterwards, the components can be converted to spherical coordinates; of course, only the ϕ component will survive. The electric potential is zero because the time-varying current is uniform and does not produce electric charges. The complex exponential when expanded in a Taylor series becomes

$$\exp\left(jk\frac{\mathbf{r}_q \cdot \mathbf{r}_p}{r_p}\right) \simeq 1 + jk\frac{\mathbf{r}_q \cdot \mathbf{r}_p}{r_p} + \cdots$$

Because the leading term vanishes when integrated around the closed loop, the first-order term must be retained (unlike the case of the electric dipole). The far-field complex vector potential evaluates to

$$\underline{\mathbf{A}}(\mathbf{r}) = jk\left(1 + \frac{1}{jkr}\right) R \frac{\mu_o \underline{\mathrm{I}}_o R}{4} \frac{\exp(-jkr)}{r} \sin\theta \ \widehat{\phi}$$

The time-dependent form is

$$\mathbf{A}(\mathbf{r}) = \widehat{\phi}\frac{\mu_o}{4\pi}\frac{m_o[-kr\sin(\omega t - kr) + \cos(\omega t - kr)]}{r^2}\sin\theta \tag{1.36}$$

where $m_o = \underline{\mathrm{I}}_o \pi R^2$ is defined as the magnetic dipole moment.

The Hertzian magnetic dipole is revisited in Part III, Chapter 13, Section 13.2. We postpone further discussion of complex fields or the introduction of the complex Maxwell's Equations until after the time-domain version has been generalized to include dielectric and magnetic materials.

1.7 POLARIZABLE, MAGNETIZABLE, AND CONDUCTING MEDIA

In the free-space formulation considered so far, only free charges and their associated electric currents are present. When dielectric and magnetic materials are considered, the response of electric and/or magnetic dipoles must be considered. Commonly, these are considered to be paired electric charges (bound to each other), in the case of dielectrics, and small electric-current loops (or their equivalent), in the case of magnetics. Other models (permitted by the equivalence principle) are possible; these include using fictitious magnetic charges of opposing polarities to model magnetic dipoles and/or small current loops carrying fictitious magnetic current to model electric dipoles. Moreover, various superpositions of all four types can be employed. In all models, the individual dipoles are of microscopic or mesoscopic dimensions; this permits one to characterize the material in terms of its polarization (**P**) and magnetization (**M**) vectors. Each is defined as the appropriate macroscopically averaged dipole-moment density. There may be spontaneous values of **M** or **P**; more generally, they arise in response to applied electric or magnetic fields. The response of dielectric materials can often be approximated as linear with respect to the excitation. That of magnetic materials is generally nonlinear and hysteretic, yet linear operation with respect to some operating point is sometimes possible. When the material is deforming and/or accelerating, additional complexity and subtlety is involved.

Because, at present, we wish to continue using a vector-potential formulation, it is convenient to employ a model for materials that is based solely on electric charge and current. The previous formulation can then be generalized by adding additional electric currents to those generated by the free (unpaired) charges. The form of Maxwell's Equations remains unchanged. Other ways of incorporating magnetization into electrodynamic theories are reviewed in Part II; the Chu formulation is of special interest because electric and magnetic dipoles are modelled symmetrically.

Polarization and Amperian electric currents

The dielectric polarization can be modeled as identical charge distributions of opposite polarity, $\pm\rho_o$, displaced from one another by a small vector distance **d**. This is depicted in Figure 1.3 with the separation greatly exaggerated.

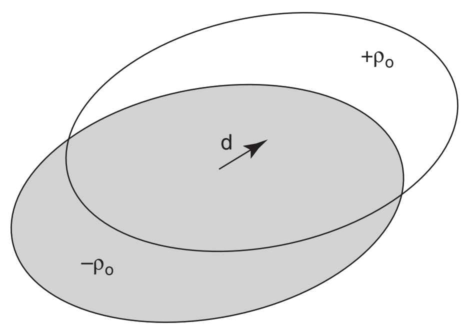

Figure 1.3 Separated bipolar charge densities.

The electric-dipole moment of an incremental volume of the dielectric is the increment $\Delta\mathbf{p} = \rho_o\mathbf{d}\ \Delta V$ [refer to Appendix C, Eq. (C.9)]; therefore

$$\mathbf{p} = \int \mathbf{P}^o dV \tag{1.37}$$

where the polarization vector is defined as $\mathbf{P}^o = \rho_o\mathbf{d}$.[3]

The time rate of change of the polarization produces a current density,

$$\mathbf{J}_{\text{polarization}} = \rho_o\frac{\partial\mathbf{d}}{\partial t} = \frac{\partial\mathbf{P}^o}{\partial t} \tag{1.38}$$

and an associated charge density, $\rho_{\text{polarization}}$. Because the latter is conserved independently of the free charge, we obtain

$$\nabla\cdot\frac{\partial\mathbf{P}^o}{\partial t} = \frac{\partial}{\partial t}(\nabla\cdot\mathbf{P}^o) = -\frac{\partial\rho_{\text{polarization}}}{\partial t}$$

[3]The superscript o is added because we reserve unscripted variables for field quantities in the Chu formulation of electrodynamics. When the material is stationary and nondeformable, there is no difference between corresponding quantities with and without a superscript.

and

$$\rho_{\text{polarization}} = -\nabla \cdot \mathbf{P}^{\text{o}} \tag{1.39}$$

Notice that this charge density is *not* ρ_{o}. Indeed, if $\mathbf{P}^{\text{o}}$ is spatially uniform, $\rho_{\text{polarization}} = 0$, except on the boundaries of the dielectric. There, surface polarization charges reside wherever $\mathbf{P}^{\text{o}} \cdot \widehat{\mathbf{n}}$ is nonzero ($\widehat{\mathbf{n}}$ is the normal to the surface).

For magnetic materials, the magnetization is assumed to arise from tiny Amperian-current loops (shown schematically in Figure 1.4) that produce magnetic-dipole moments.

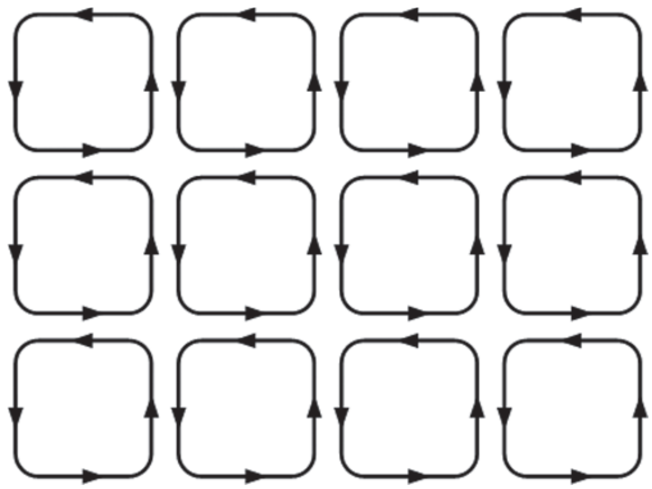

Figure 1.4 Amperian current loops.

If the current circulating around each loop is uniform, no electric charge will be present. Because conservation of Amperian electric charge requires that

$$\nabla \cdot \mathbf{J}_{\text{amperian}} = -\frac{\partial \rho_{\text{amperian}}}{\partial t} = 0$$

it follows that

$$\mathbf{J}_{\text{amperian}} = \nabla \times \mathbf{M}^{\text{o}} \tag{1.40a}$$

$$\rho_{\text{amperian}} = 0 \tag{1.40b}$$

where $\mathbf{M}^{\text{o}}$ is, as yet, an undefined vector.

However, the net magnetic moment, $\mathbf{m}$, of an electric-current distribution is defined as $\int \frac{1}{2}\mathbf{r}' \times \mathbf{J}\, dV$ [refer to Appendix C, Eq. (C.14)], which in this case is

$$\mathbf{m} = \int \frac{1}{2}\mathbf{r}' \times (\nabla \times \mathbf{M}^{\text{o}})\; dV$$

and where the choice of the origin of $\mathbf{r}'$ is arbitrary. The material volume is the sum of increments ΔV; each contributes an incremental magnetic moment $\Delta \mathbf{m}$. As we consider each increment in turn, the origin of $\mathbf{r}' = \mathbf{r}$ can be redefined so as to be centered upon it. Each incremental moment is therefore of the form $\Delta \mathbf{m} = [\lim_{\mathbf{r}\to 0} \frac{1}{2}\mathbf{r} \times (\nabla \times \mathbf{M}^{\text{o}})]\Delta V$.

But

$$\mathbf{r} \times (\nabla \times \mathbf{M}^{\text{o}}) = (\nabla \cdot \mathbf{r} - 1)\mathbf{M}^{\text{o}} + \nabla(\mathbf{M}^{\text{o}} \cdot \mathbf{r}) - \nabla \cdot (\mathbf{M}^{\text{o}}\, \mathbf{r})$$

Therefore, because $\nabla \cdot \mathbf{r} = 3$ and the other terms vanish in the limit, we obtain $\Delta\mathbf{m} = \mathbf{M}^{\text{o}}\Delta V$. Consequently, the total magnetic moment of the material volume can also be written as

$$\mathbf{m} = \int \mathbf{M}^{\text{o}}\, dV \tag{1.41}$$

with $\mathbf{M}^{\text{o}}$ now identified as the magnetization vector (the magnetic-dipole moment per unit volume). If $\mathbf{M}^{\text{o}}$ is spatially uniform, $\mathbf{J}_{\text{amperian}} = 0$, except on the boundaries of the magnetic material. There, surface magnetization currents reside wherever $\mathbf{M}^{\text{o}} \times \widehat{\mathbf{n}}$ is nonzero ($\widehat{\mathbf{n}}$ is the normal to the surface).

When polarization currents and Amperian currents are added to $\mathbf{J}_{\text{u}}$, the current associated with the free (unpaired) charge, it follows that

$$\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{u}} + \mathbf{J}_{\text{polarization}} + \mathbf{J}_{\text{amperian}} = \mathbf{J}_{\text{u}} + \frac{\partial \mathbf{P}^{\text{o}}}{\partial t} + \nabla \times \mathbf{M}^{\text{o}} \tag{1.42a}$$

$$\rho_{\text{total}} = \rho_{\text{u}} + \rho_{\text{polarization}} + \rho_{\text{amperian}} = \rho_{\text{u}} - \nabla \cdot \mathbf{P}^{\text{o}} + 0 \tag{1.42b}$$

With these field sources, it is customary to express Maxwell's Equations in terms of the field vector $\mathbf{B}/\mu_{\text{o}}$ rather than $\mathbf{H}$. With $\mathbf{E}^{\text{o}}$ substituted for $\mathbf{E}$ to distinguish the electric field from that used in the Chu formulation, the result is

$$\frac{1}{\mu_{\text{o}}}\nabla \times \mathbf{B} - \varepsilon_{\text{o}}\frac{\partial \mathbf{E}^{\text{o}}}{\partial t} = \mathbf{J}_{\text{total}} = \mathbf{J}_{\text{u}} + \frac{\partial \mathbf{P}^{\text{o}}}{\partial t} + \nabla \times \mathbf{M}^{\text{o}} \tag{1.43a}$$

$$\varepsilon_{\text{o}}\nabla \cdot \mathbf{E}^{\text{o}} = \rho_{\text{total}} = \rho_{\text{u}} - \nabla \cdot \mathbf{P}^{\text{o}} \tag{1.43b}$$

$$\nabla \times \mathbf{E}^{\text{o}} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{1.43c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.43d}$$

where

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1.44a}$$

$$\mathbf{E}^{\text{o}} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \tag{1.44b}$$

In this Amperian formulation, the Lorentz force density is

$$\mathbf{f} = \rho_{\text{total}}\mathbf{E}^{\text{o}} + \mathbf{J}_{\text{total}} \times \mathbf{B} \tag{1.45}$$

It is customary to define

$$\mathbf{D} = \varepsilon_{\text{o}}\mathbf{E}^{\text{o}} + \mathbf{P}^{\text{o}} \tag{1.46a}$$

$$\mathbf{B} = \mu_{\text{o}}(\mathbf{H}^{\text{o}} + \mathbf{M}^{\text{o}}) \tag{1.46b}$$

because, with their use, Maxwell's Equations simplify to the Minkowski form,

$$\nabla \times \mathbf{H}^{\text{o}} = \mathbf{J}_{\text{u}} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.47a}$$

$$\nabla \cdot \mathbf{D} = \rho_{\text{u}} \tag{1.47b}$$

$$\nabla \times \mathbf{E}^{\text{o}} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.47c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.47d}$$

Chu formulation

An alternate formulation, due to Lan Jen Chu, uses electric charges to model the polarization and magnetic charges to model the magnetization. In this case, the sources are duals of one another (as discussed in Chapter 6, Section 6.5) and Maxwell's Equations become

$$\nabla \times \mathbf{H} - \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}_e = \mathbf{J}_u + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times (\mathbf{P} \times \mathbf{v}) \tag{1.48a}$$

$$\varepsilon_o \nabla \cdot \mathbf{E} = \rho_e = \rho_u - \nabla \cdot \mathbf{P} \tag{1.48b}$$

$$\nabla \times \mathbf{E} + \mu_o \frac{\partial \mathbf{H}}{\partial t} = -\mathbf{J}_m = -\mu_o \left[\frac{\partial \mathbf{M}}{\partial t} + \nabla \times (\mathbf{M} \times \mathbf{v}) \right] \tag{1.48c}$$

$$\mu_o \nabla \cdot \mathbf{H} = \rho_m = -\mu_o \nabla \cdot \mathbf{M} \tag{1.48d}$$

These are consistent with Eqs. (1.47) provided

$$\mathbf{E}^o = \mathbf{E} + \mu_o \mathbf{M} \times \mathbf{v} \tag{1.49a}$$

$$\mathbf{H}^o = \mathbf{H} - \mathbf{P} \times \mathbf{v} \tag{1.49b}$$

$$\mathbf{P}^o = \mathbf{P} - \mu_o \varepsilon_o \mathbf{M} \times \mathbf{v} \tag{1.49c}$$

$$\mathbf{M}^o = \mathbf{M} + \mathbf{P} \times \mathbf{v} \tag{1.49d}$$

Notice that neither **D** nor **B** requires superscript labeling because

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} = \varepsilon_o \mathbf{E}^o + \mathbf{P}^o \tag{1.50a}$$

$$\mathbf{B} = \mu_o (\mathbf{H} + \mathbf{M}) = \mu_o (\mathbf{H}^o + \mathbf{M}^o) \tag{1.50b}$$

In the Chu formulation, there are separate Lorentz-force densities acting upon both the electric and magnetic charges:

$$\mathbf{f}_e = \rho_e \mathbf{E} + \mathbf{J}_e \times \mu_o \mathbf{H} \tag{1.51a}$$

$$\mathbf{f}_m = \rho_m \mathbf{E} - \mathbf{J}_m \times \varepsilon_o \mathbf{E} \tag{1.51b}$$

The total force density is their sum which is different from Eq. (1.45) because magnetic charges have been employed; the Minkowski formulation leads to yet another force density. These are all forces of "electromagnetic origin" that are dependent on how the sources are modelled; they must be augmented by mechanical and/or other forces. When done so properly, the *total* force density is independent of the electromagnetic formulation chosen. Despite the explicit velocity dependence, the Chu formulation is often (but not always) simpler to use. However, because we wish to employ the vector-potential, we continue at present with the Amperian formulation (expressed in terms of the Minkowski electric and magnetic fields).

For regions of free-space it follows that the constitutive laws that relate the four field vectors are (with $\mathbf{E}^o = \mathbf{E}$, $\mathbf{H}^o = \mathbf{H}$):

$$\mathbf{D} = \varepsilon_o \mathbf{E} \tag{1.52a}$$

$$\mathbf{B} = \mu_o \mathbf{H} \tag{1.52b}$$

Electrically conducting materials

For linear isotropic (stationary) conductors that obey Ohm's Law,

$$\mathbf{J}_u = \sigma \mathbf{E} \tag{1.53a}$$

where σ is the electrical-conductivity with units of $S\ m^{-1} = A\ V^{-1} m^{-1}$.

For such materials, conservation of charge becomes

$$\nabla \cdot (\sigma \mathbf{E}) + \frac{\partial \rho_u}{\partial t} = 0 \tag{1.53b}$$

These materials may also have dielectric/magnetic properties.

Perfect conductors

A perfect conductor is defined by $\sigma \to \infty$ and therefore unless $\mathbf{J}_u$ is infinite, the electric field inside it must vanish. It follows that current must flow only in surface layers of thickness $\delta \to 0$ and $\mathbf{J}_u \to \infty$ such that there is a finite surface current density, $\mathbf{K}_u = \lim_{\delta \to 0} \mathbf{J}_u \delta$. Because the time rate of change of any magnetic field inside the conductor would produce electric fields, only static magnetic fields are tenable. Nevertheless, we *define* a perfect conductor as one that is *completely field-free*.

Dielectric and magnetic materials

For linear isotropic (stationary) materials, the constitutive laws that relate the four field vectors are

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{1.54a}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{1.54b}$$

or equivalently

$$\mathbf{P} = (\varepsilon - \varepsilon_o)\mathbf{E} = \chi_e \varepsilon_o \mathbf{E} \tag{1.55a}$$

$$\mathbf{M} = (\mu/\mu_o - 1)\mathbf{H} = \chi_m \mathbf{H} \tag{1.55b}$$

where ε and μ are, respectively, the dielectric permittivity and the magnetic permeability. The ratios, $\varepsilon/\varepsilon_o$and μ/μ_o are defined as the relative permittivity (dielectric constant) and relative permeability. The electric and magnetic susceptibilities are the dimensionless quantities defined by

$$\chi_e = \varepsilon/\varepsilon_o - 1 \tag{1.56a}$$

$$\chi_m = \mu/\mu_o - 1 \tag{1.56b}$$

The reader should again be cautioned that when the material is moving and/or deforming, the values of $\mathbf{E}$ and $\mathbf{H}$ in this formulation are not identical to those in the free-space equations. These complications are taken up in Part II of the text.

1.8 BOUNDARY CONDITIONS

The form of Maxwell's Equations reveals that in regions where the charges and currents are continuous functions of position and time, the divergence, curl, and time-derivative operators must all produce continuous functions. This fact requires the electric and magnetic fields to be continuous functions of space and time. However, there may be locations (such as the surface of a conductor or the interface between different materials) where there is an accumulation of charge and/or current that can be modeled as a finite surface density that within zero thickness has infinite volume density. At such locations $\nabla \cdot \mathbf{E}$ and/or $\nabla \times \mathbf{H}$ are infinite; these singularities lead to discontinuities in the respective field. We wish to determine exactly how large these will be.

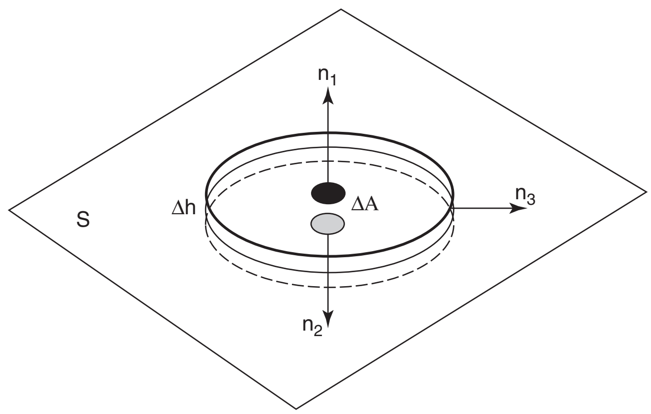

Figure 1.5 Closed surface (normal fields).

Electric surface charges

Assume that an unpaired electric surface charge of density σ_{u}^{s} resides on the surface shown in Figure 1.5. A circular pillbox is constructed with equal top and bottom areas, ΔA, that are located on opposite sides of the surface; the height of the box is Δh. We apply Eqs. (1.5a) and (1.47b) to the closed surface defined by the Gaussian pillbox

$$\oint_{\text{top+bottom+sides}} \mathbf{D} \cdot \widehat{\mathbf{n}}\, da = \int_{\text{area}} \sigma_{\mathrm{u}}^{s}\, da$$

$$[\mathbf{D}^{(1)} \cdot \widehat{\mathbf{n}}_1 + \mathbf{D}^{(2)} \cdot \widehat{\mathbf{n}}_2]\Delta A + \mathbf{D}^{(3)} \cdot \widehat{\mathbf{n}}_3 2\sqrt{\pi\,\Delta A}\,\Delta h = \sigma_{\mathrm{u}}^{s}\,\Delta A$$

If $\Delta h \ll \sqrt{\Delta A} \to 0$ and $\mathbf{E}$ is finite, it follows that because $\widehat{\mathbf{n}}_1 = -\widehat{\mathbf{n}}_2 = \widehat{\mathbf{n}}$, we obtain

$$\widehat{\mathbf{n}} \cdot [\mathbf{D}^{(1)} - \mathbf{D}^{(2)}] = \sigma_{\mathrm{u}}^{s} \tag{1.57}$$

Electric surface currents

Assume that an unpaired electric surface current of density $\mathbf{K}_{\mathrm{u}}^{s}$ resides on the horizontal surface shown in Figure 1.6.

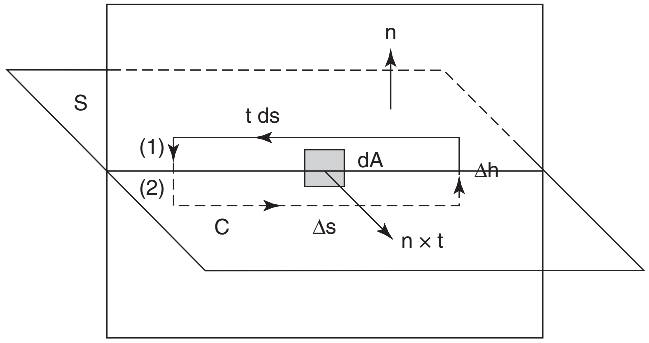

Figure 1.6 Closed contour (tangential fields).

A rectangular area is constructed with equal top and bottom lengths, Δs, that are located on opposite sides of the surface; the height of the rectangle is Δh. We apply Eqs. (1.3a) and (1.47a) to the surface that is the plane of the rectangle:

$$\oint_{\text{perimeter}} \mathbf{H}\cdot\widehat{\mathbf{t}}\,ds = \int_{\text{area}} \mathbf{J}_{\text{u}}\cdot(\widehat{\mathbf{n}}\times\widehat{\mathbf{t}})\,da = \int_{\text{length}} \mathbf{K}_{\text{u}}^{s}\cdot(\widehat{\mathbf{n}}\times\widehat{\mathbf{t}})\,ds$$

$$[\mathbf{H}^{(1)}\cdot\widehat{\mathbf{t}}_1 + \mathbf{H}^{(2)}\cdot\widehat{\mathbf{t}}_2]\Delta s + [\mathbf{H}^{(3)}\cdot\widehat{\mathbf{t}}_3 + \mathbf{H}^{(4)}\cdot\widehat{\mathbf{t}}_4]\Delta h = \mathbf{K}_{\text{u}}^{s}\cdot(\widehat{\mathbf{n}}\times\widehat{\mathbf{t}})\Delta s$$

If $\Delta h \ll \Delta s \to 0$ and $\mathbf{H}$ is finite, it follows that because $\widehat{\mathbf{t}}_1 = -\widehat{\mathbf{t}}_2 = \widehat{\mathbf{t}}$ and $\widehat{\mathbf{n}}\cdot\widehat{\mathbf{t}} = 0$, we have

$$[\mathbf{H}^{(1)} - \mathbf{H}^{(2)}]\cdot\widehat{\mathbf{t}} = \mathbf{K}_{\text{u}}^{s}\cdot(\widehat{\mathbf{n}}\times\widehat{\mathbf{t}}) = (\mathbf{K}_{\text{u}}^{s}\times\widehat{\mathbf{n}})\cdot\widehat{\mathbf{t}} \tag{1.58a}$$

$$\mathbf{H}^{(1)} - \mathbf{H}^{(2)} = \mathbf{K}_{\text{u}}^{s}\times\widehat{\mathbf{n}} \tag{1.58b}$$

$$\widehat{\mathbf{n}}\times[\mathbf{H}^{(1)} - \mathbf{H}^{(2)}] = \mathbf{K}_{\text{u}}^{s} \tag{1.58c}$$

Conservation of charge

If a surface of discontinuity supports σ_{u}^{s} and/or $\mathbf{K}_{\text{u}}^{s}$ and the material on one or both sides supports volume currents, $\mathbf{J}_{\text{u}}$, the boundary conditions must maintain conservation of unpaired electric charge. A Gaussian pillbox is again erected to provide a closed surface over which

$$\nabla\cdot\mathbf{J}_{\text{u}} + \frac{\partial\rho_{\text{u}}}{\partial t} = 0$$

can be integrated. Comparison with Eq. (1.47b) suggests that

$$\mathbf{n}\cdot[\mathbf{J}_{\text{u}}^{(1)} - \mathbf{J}_{\text{u}}^{(2)}] = -\frac{\partial\sigma_{\text{u}}^{s}}{\partial t}$$

However, in this case, $\mathbf{J}_{\text{u}} \to \infty$, and so contributions from the sides cannot be ignored in the limit $\Delta h \to 0$. The correct boundary condition is

$$\mathbf{n}\cdot[\mathbf{J}_{\text{u}}^{(1)} - \mathbf{J}_{\text{u}}^{(2)}] + \nabla_{\Sigma}\cdot\mathbf{K}_{\text{u}}^{s} + \frac{\partial\sigma_{\text{u}}^{s}}{\partial t} = 0 \tag{1.59}$$

where $\nabla_{\Sigma}\cdot\mathbf{K}_{\text{u}}^{s}$ is the two-dimensional surface divergence that accounts for the net outward flow along the surface. It is defined by

$$\nabla_\Sigma \cdot \mathbf{K}_u^s = \lim_{\Delta A \to 0} \frac{\oint_{C_o} \mathbf{K}_u^s \cdot (\mathbf{t} \times \mathbf{n})\, ds}{\Delta A} \tag{1.60}$$

Because polarization charges are separately conserved, this form of boundary condition applies to them as well as to the total current and charge densities. There is no electric charge density associated with the Amperian current density, $\nabla \times \mathbf{M}$; but a surface density,

$$\mathbf{K}_M^s = \widehat{\mathbf{n}} \times [\mathbf{M}^{(1)} - \mathbf{M}^{(2)}]$$

exists wherever there are discontinuities in the tangential magnetization vector.

Because there are no magnetic charges or magnetic currents, the boundary conditions for $\mathbf{E}$ and $\mathbf{B}$ are

$$\mathbf{n} \times [\mathbf{E}^{(1)} - \mathbf{E}^{(2)}] = 0 \tag{1.61a}$$

$$\mathbf{n} \cdot [\mathbf{B}^{(1)} - \mathbf{B}^{(2)}] = 0 \tag{1.61b}$$

It also follows that

$$\mathbf{n} \cdot \varepsilon_o [\mathbf{E}^{(1)} - \mathbf{E}^{(2)}] = \sigma_{\text{total}}^s = \sigma_u^s + \sigma_{\text{polarization}}^s \tag{1.62a}$$

$$\mathbf{n} \cdot [\mathbf{P}^{(1)} - \mathbf{P}^{(2)}] = -\sigma_{\text{polarization}}^s \tag{1.62b}$$

$$\mathbf{n} \cdot [\mathbf{H}^{(1)} - \mathbf{H}^{(2)}] = -\mathbf{n} \cdot [\mathbf{M}^{(1)} - \mathbf{M}^{(2)}] \tag{1.62c}$$

1.9 THE COMPLEX MAXWELL EQUATIONS

In the sinusoidal steady state, time-harmonic vectors and scalars can be conveniently expressed as

$$\mathbf{F}(\mathbf{r},t) = Re\left\{\underline{\mathbf{F}}(\mathbf{r}) \exp(j\omega t)\right\}$$

$$\Psi = Re\left\{\underline{\Psi}(\mathbf{r}) \exp(j\omega t)\right\}$$

where the symbol *Re* stands for the real part of the expression in the curly brackets {}, and the vector or scalar that is underscored is a complex quantity that may be a constant or a function of position. Because spatial and temporal derivatives are operations that commute with taking the real part, one may initially suppress the *Re* until the very last stage of analysis and simply insert the bracketed terms into Maxwell's Equations. When this is done, each time derivative will cause multiplication by the factor $j\omega$, but $\exp(j\omega t)$ will remain a common factor in every term; these are cumbersome and can be ignored if we remember to reinsert it before taking the real part.

The complex forms of Eqs. (1.47a)–(1.47d) (with the time dependence suppressed) are therefore

$$\nabla \times \underline{\mathbf{H}}^o - j\omega \underline{\mathbf{D}} = \underline{\mathbf{J}_u} \tag{1.63a}$$

$$\nabla \cdot \underline{\mathbf{D}} = \underline{\rho_u} \tag{1.63b}$$

$$\nabla \times \underline{\mathbf{E}}^o + j\omega \underline{\mathbf{B}} = 0 \tag{1.63c}$$

$$\nabla \cdot \underline{\mathbf{B}} = 0 \tag{1.63d}$$

The complex forms of Eqs. (1.44a) and (1.44b) are

$$\underline{\mathbf{B}} = \nabla \times \underline{\mathbf{A}} \tag{1.64a}$$

$$\underline{\mathbf{E}}^{\mathrm{o}} = -j\omega\underline{\mathbf{A}} - \nabla\underline{\Phi} \tag{1.64b}$$

In most cases, we assume that the material is stationary and nondeforming; then, the superscripts can be removed from the electric and magnetic fields.