

# Problem 1

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## Cardano and Games of Chance (1564)

**Problem.** How many throws of a fair die do we need in order to have an even chance of at least one six?

**Solution.** Let  $A$  be the event “a six shows in one throw of a die” and  $p_A$  its probability. Then  $p_A = 1/6$ . The probability that a six does not show in one throw is  $q_A = 1 - 1/6 = 5/6$ . Let the number of throws be  $n$ . Therefore, assuming independence between the throws,

$$\begin{aligned} \Pr\{\text{a six shows at least once in } n \text{ throws}\} &= 1 - \Pr\{\text{a six does not show} \\ &\quad \text{at all in } n \text{ throws}\} \\ &= 1 - \underbrace{q_A \times \cdots \times q_A}_{n \text{ terms}} \\ &= 1 - (5/6)^n \end{aligned}$$

We now solve  $1 - (5/6)^n \geq 1/2$  obtaining  $n \geq \ln(1/2)/\ln(5/6) = 3.8$ , so the number of throws is 4.

### 1.1 Discussion

In the history of probability, the physician and mathematician Gerolamo Cardano (1501–1575) (Fig. 1.1) was among the first to attempt a systematic study of the calculus of probabilities. Like those of his contemporaries, Cardano’s studies were primarily driven by games of chance. Concerning his gambling for 25 years, he famously said in his autobiography (Cardano, 1935, p. 146)

...and I do not mean to say only from time to time during those years, but I am ashamed to say it, everyday.

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**Figure 1.1** Gerolamo Cardano (1501–1575).

Cardano’s works on probability were published posthumously in 1663, in the famous 15-page *Liber de ludo aleae*<sup>†</sup> (Fig. 1.2) consisting of 32 small chapters (Cardano, 1564).

Cardano was undoubtedly a great mathematician of his time but stumbled on the question in **Problem 1**, and several others too. In this case, he thought the number of throws should be three. In Chapter 9 of his book, Cardano states regarding a die:

One-half of the total number of faces always represents equality<sup>‡</sup>; thus the chances are equal that a given point will turn up in three throws. . .

Cardano’s mistake stems from a prevalent general confusion between the concepts of probability and expectation. Let’s now dig deeper into Cardano’s

<sup>†</sup>The Book on Games of Chance. An English translation of the book and a thorough analysis of Cardano’s connections with games of chance can be found in Ore’s *Cardano: The Gambling Scholar* (Ore, 1953). More bibliographic details can be found in Gliozzi (1980, pp. 64–67) and Scardovi (2004, pp. 754–758).

<sup>‡</sup>Cardano frequently uses the term “equality” in the *Liber* to denote half of the total number of sample points in the sample space. See Ore (1953, p. 149).

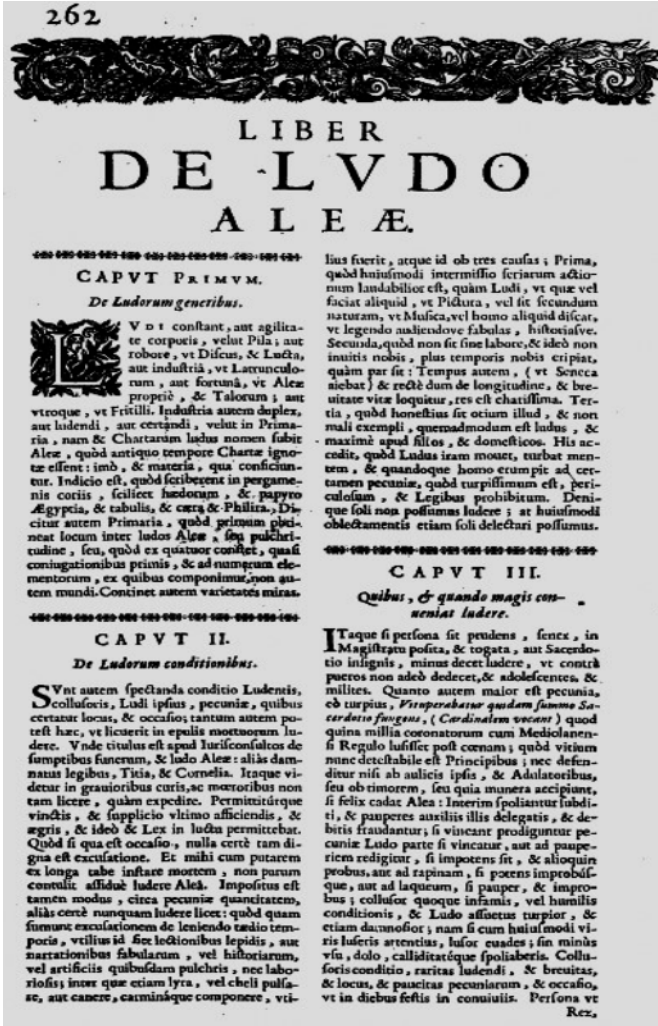


Figure 1.2 First page of the *Liber de ludo aleae*, taken from the *Opera Omnia* (Vol. I) (Cardano, 1564).

reasoning. In the *Liber*, Cardano frequently makes use of an erroneous principle, which Ore calls a “reasoning on the mean” (ROTM) (Ore, 1953, p. 150),<sup>†</sup> to deal with various probability problems. According to the ROTM, if an event has a probability  $p$  in one trial of an experiment, then in  $n$  independent trials the event will occur  $np$  times on average, which is then wrongly taken to represent the *probability* that the event will occur in  $n$  trials. For the question in **Problem 1**, we have  $p = 1/6$  so that,

<sup>†</sup> See also Williams (2005).

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with  $n = 3$  throws, the event “at least a six” is wrongly taken to occur an average  $np = 3(1/6) = 1/2$  of the time (i.e., with probability  $1/2$ ).

Using modern notation, let us see why the ROTM is wrong. Suppose an event has a probability  $p$  of occurring in a single repetition of an experiment. Then in  $n$  independent and identical repetitions of that experiment, the expected number of the times the event occurs is  $np$ . Thus, for the die example, the expectation for the number of times a six appears in three throws is  $3 \times 1/6 = 1/2$ . However, an expectation of  $1/2$  in three throws is not the same as a probability of  $1/2$  in three throws. These facts can formally be seen by using a binomial model.<sup>†</sup> Let  $X$  be the number of sixes in three throws. Then  $X$  has a binomial distribution with parameters  $n = 3$  and  $p = 1/6$ , that is,  $X \sim B(3, 1/6)$ , and its probability mass function is

$$\Pr\{X = x\} = \binom{n}{x} p^x (1-p)^{n-x} = \binom{3}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{3-x} \quad \text{for } x = 0, 1, 2, 3.$$

From this formula, the probability of one six in three throws is

$$\Pr\{X = 1\} = \binom{3}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 = .347,$$

and the probability of at least one six is

$$\begin{aligned} \Pr\{X \geq 1\} &= 1 - \Pr\{X = 0\} \\ &= 1 - (5/6)^3 \\ &= .421. \end{aligned}$$

Finally, the expected value of  $X$  is

$$\mathcal{E}X = \sum_{x=0}^n x \Pr\{X = x\} = \sum_{x=0}^n x \binom{n}{x} p^x (1-p)^{n-x},$$

which can be simplified to give

$$\mathcal{E}X = np = (3)(1/6) = .5.$$

Thus, we see that although the expected number of sixes in three throws is  $1/2$ , neither the probability of one six or at least one six is  $1/2$ .

Cardano has not got the recognition that he perhaps deserves for his contributions to the field of probability, for in the *Liber de ludo aleae* he touched on many rules and problems that were later to become classics. Let us now outline some of these.

In Chapter 14 of the *Liber*, Cardano gives what some would consider the first definition of classical (or mathematical) probability:

So there is one general rule, namely, that we should consider the whole circuit, and the number of those casts which represents in how many ways the favorable result can occur,

<sup>†</sup> For the origin of the binomial model, see **Problem 8**.

and compare that number to the rest of the circuit, and according to that proportion should the mutual wagers be laid so that one may contend on equal terms.

Cardano thus calls the “circuit” what is known as the sample space today, that is, the set of all possible outcomes when an experiment is performed. If the sample space is made up of  $r$  outcomes that are favorable to an event, and  $s$  outcomes that are unfavorable, and if all outcomes are equally likely, then Cardano correctly defines the odds in favor of the event by  $r : s$ . This corresponds to a probability of  $r/(r + s)$ . Compare Cardano’s definition to

- The definition given by Leibniz (1646–1716) in 1710 (Leibniz, 1969, p. 161):  
If a situation can lead to different advantageous results ruling out each other, the estimation of the expectation will be the sum of the possible advantages for the set of all these results, divided into the total number of results.
- Jacob Bernoulli’s (1654–1705) statement from the *Ars Conjectandi* (Bernoulli, 1713, p. 211)<sup>†</sup>:  
... if the integral and absolute certainty, which we designate by letter  $\alpha$  or by unity 1, will be thought to consist, for example, of five probabilities, as though of five parts, three of which favor the existence or realization of some events, with the other ones, however, being against it, we will say that this event has  $3/5\alpha$ , or  $3/5$ , of certainty.
- De Moivre’s (1667–1754) definition from the *De Mensura Sortis* (de Moivre, 1711; Hald, 1984):  
If  $p$  is the number of chances by which a certain event may happen, and  $q$  is the number of chances by which it may fail, the happenings as much as the failings have their degree of probability; but if all the chances by which the event may happen or fail were equally easy, the probability of happening will be to the probability of failing as  $p$  to  $q$ .
- The definition given in 1774 by Laplace (1749–1827), with whom the formal definition of classical probability is usually associated. In his first probability paper, Laplace (1774b) states:  
The probability of an event is the ratio of the number of cases favorable to it, to the number of possible cases, when there is nothing to make us believe that one case should occur rather than any other, so that these cases are, for us, equally possible.

However, although the first four definitions (starting from Cardano’s) all anteceded Laplace’s, it is with the latter that the classical definition was fully appreciated and began to be formally used. In modern notation, if a sample space consists of  $N$  equally likely outcomes, of which  $n_A$  are favorable to an event  $A$ , then Laplace’s classical definition of the probability of the event  $A$  is

$$\Pr\{A\} \equiv \frac{n_A}{N}.$$

<sup>†</sup> The translation that follows is taken from Oscar Sheynin’s translations of Chapter 4 of the *Ars Conjectandi* (Sheynin, 2005).

One of Cardano's other important contributions to the theory of probability is "Cardano's formula."<sup>†</sup> Suppose an experiment consists of  $t$  equally likely outcomes of which  $r$  are favorable to an event. Then the odds in favor of the event in one trial of the experiment are  $r : (t - r)$ .<sup>‡</sup> Cardano's formula then states that, in  $n$  independent and identical trials of the experiment, the odds in favor of the event occurring  $n$  times are  $r^n : (t^n - r^n)$ .<sup>§</sup> While this is an elementary result nowadays, Cardano had some difficulty establishing it.<sup>\*\*</sup> At first he thought it was the odds that ought to be multiplied. Cardano calculated the odds against obtaining at least one 1 appearing in a toss of three dice as 125 to 91. Cardano then proceeded to obtain the odds against obtaining at least one 1 in two tosses of three dice as  $(125/91)^2 \approx 2:1$ . Thus, on the last paragraph of Chapter 14 of the *Liber*, Cardano writes (Ore, 1953, p. 202)

Thus, if it is necessary for someone that he should throw an ace twice, then you know that the throws favorable for it are 91 in number, and the remainder is 125; so we multiply each of these numbers by itself and get 8281 and 15,625, and the odds are about 2 to 1.<sup>††</sup> Thus, if he should wager double, he will contend under an unfair condition, although in the opinion of some the condition of the one offering double stakes would be better.

However, in the very next chapter entitled *On an Error Which Is Made About This*, Cardano realizes that it is not the odds that must be multiplied. He comes to understand this by considering an event with odds 1:1 in one trial of an experiment. His multiplication rule for the odds would still give an odds of  $(1/1)^3 = 1:1$  for three trials of the experiment, which is clearly wrong. Cardano thus writes (Ore, 1953, pp. 202–203)

But this reasoning seems to be false, even in the case of equality, as, for example, the chance of getting one of any three chosen faces in one cast of one die is equal to the chance of getting one of the other three, but according to this reasoning there would be an even chance of getting a chosen face each time in two casts, and thus in three, and four, which is most absurd. For if a player with two dice can with equal chances throw an even and an odd number, it does not follow that he can with equal fortune throw an even number in each of three successive casts.

Cardano thus correctly calls his initial reasoning "most absurd," and then gives the following correct reasoning (Ore, 1953):

<sup>†</sup> Not to be confused with the other Cardano's formula having to do with the general solution of a "reduced" cubic equation (i.e., a cubic equation with no second-degree term). Cardano also provided methods to convert the general cubic equation to the reduced one. These results appeared in Cardano's opus *Ars Magna* (The Great Art) and had been communicated to him previously by the mathematician Niccolò Tartaglia of Brescia (1499–1557) after swearing that he would never disclose the results. A bitter dispute thereby ensued between Cardano and Tartaglia, and is nicely documented in Hellman's *Great Feuds in Mathematics* (Hellman, 2006, pp. 7–25).

<sup>‡</sup> Thus, the odds *against* the event in one trial are  $(t - r) : r$ .

<sup>§</sup> This is the same as saying that, if an event has probability  $p (=r/t)$  of occurring in one trial of an experiment, then the probability that it will occur in all of  $n$  independent and identical trials of the experiment is  $p^n$ .

<sup>\*\*</sup> See also Katz (1998, p. 450).

<sup>††</sup> The odds calculated by Cardano are the odds *against* the event in question.

Therefore, in comparisons where the probability is one-half, as of even faces with odd, we shall multiply the number of casts by itself and subtract one from the product, and the proportion which the remainder bears to unity will be the proportion of the wagers to be staked. Thus, in 2 successive casts we shall multiply 2 by itself, which will be 4; we shall subtract 1; the remainder is 3; therefore a player will rightly wager 3 against 1; for if he is striving for odd and throws even, that is, if after an even he throws either even or odd, he is beaten, or if after an odd, an even. Thus he loses three times and wins once.

Cardano thus realizes that it is the probability, not the odds, that ought to be multiplied.<sup>†</sup> However, in the very next sentence following his previous correct reasoning, he makes a mistake again when considering three consecutive casts for an event with odds 1:1. Cardano wrongly states that the odds against the event happening in three casts are  $1:(3^2 - 1) = 1:8$ , instead of  $1:(2^3 - 1) = 1:7$ . Nevertheless, further in the book, Cardano does give the correct general rule (Ore, 1953, p. 205):

Thus, in the case of one die, let the ace and the deuce be favorable to us; we shall multiply 6, the number of faces, into itself: the result is 36; and two multiplied into itself will be 4; therefore the odds are 4 to 32, or, when inverted, 8 to 1.

If three throws are necessary, we shall multiply 3 times; thus, 6 multiplied into itself and then again into itself gives 216; and 2 multiplied into itself and again into 2, gives 8; take away 8 from 216: the result will be 208; and so the odds are 208 to 8, or 26 to 1. And if four throws are necessary, the numbers will be found by the same reasoning, as you see in the table; and if one of them be subtracted from the other, the odds are found to be 80 to 1.

In the above, Cardano has considered an event with probability  $1/3$ , and correctly gives the odds against the event happening twice as  $(3^2 - 1):1 = 8:1$ , happening thrice as  $(3^3 - 1):1 = 26:1$ , and so on. Cardano thus finally reaches the following correct rule: if the odds in favor of an event happening in one trial of an experiment are  $r : (t - r)$ , then in  $n$  independent and identical trials of the experiment, the odds against the event happening  $n$  times are  $(t^n - r^n) : r^n$ .

Cardano also anticipated the law of large numbers (see **Problem 8**), although he never explicitly stated it. Ore writes (1953, p. 170)

It is clear. . . that he [Cardano] is aware of the so-called law of large numbers in its most rudimentary form. Cardano's mathematics belongs to the period antedating the expression by means of formulas, so that he is not able to express the law explicitly in this way, but he uses it as follows: when the probability for an event is  $p$  then by a large number  $n$  of repetitions the number of times it will occur does not lie far from the value  $m = np$ .

Moreover, in Chapter 11 of the *De ludo aleae*, Cardano investigated the *Problem of Dice* (see **Problem 3**); in the *Practica arithmetice* (Cardano, 1539), he discussed the *Problem of Points* (see **Problem 4**), the *Gambler's Ruin* (see **Problem 5**) (Coumet, 1965a), and the *St Petersburg Problem* (see **Problem 11**) (Dutka, 1988); finally, in the *Opus novum de proportionibus* (Cardano, 1570), Cardano also made use of what later came to be known as Pascal's Arithmetic Triangle (see **Problem 4**) (Boyer, 1950).

<sup>†</sup> Thus for the 125:91 example, the correct odds against in two trials are  $(216^2 - 91^2)/91^2 \approx 4.63:1$  and for the 1:1 example, the correct odds in favor in three trials are  $1^3:(2^3 - 1^3) = 1:7$ .

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However, in none of these problems did Cardano reach the level of mathematical sophistication and maturity that was later to be evidenced in the hands of his successors.

We make a final comment on Cardano's investigations in probability. From the *Liber de ludo aleae* it is clear that Cardano is unable to disassociate the unscientific concept of luck from the mathematical concept of chance. He identifies luck with some supernatural force that he calls the "authority of the Prince" (Ore, 1953, p. 227). In Chapter 20 entitled *On Luck in Play*, Cardano states (Ore, 1953, pp. 215–216)

In these matters, luck seems to play a very great role, so that some meet with unexpected success while others fail in what they might expect. . .

If anyone should throw with an outcome tending more in one direction than it should and less in another, or else it is always just equal to what it should be, then, in the case of a fair game there will be a reason and a basis for it, and it is not the play of chance; but if there are diverse results at every placing of the wagers, then some other factor is present to a greater or less extent; there is no rational knowledge of luck to be found in this, though it is necessarily luck.

Cardano thus believes that there is some external force that is responsible for the fluctuations of outcomes from their expectations. He fails to recognize such fluctuations are germane to chance and not because of the workings of supernatural forces. Gigerenzer et al. (1989, p. 12) thus write

. . .He [Cardano] thus relinquished his claim to founding the mathematical theory of probability. Classical probability arrived when luck was banished; it required a climate of determinism so thorough as to embrace even variable events as expressions of stable underlying probabilities, at least in the long run.