Chapter 1

An Overview of Geometry

In This Chapter

- Surveying the geometric landscape: Shapes and proofs
- Understanding points, lines, rays, segments, angles, and planes
- Cutting segments and angles in two or three congruent pieces

Studying geometry is sort of a Dr. Jekyll-and-Mr. Hyde thing. You have the ordinary geometry of shapes (the Dr. Jekyll part) and the strange world of geometry proofs (the Mr. Hyde part).

Every day, you see various shapes all around you (triangles, rectangles, boxes, circles, balls, and so on), and you're probably already familiar with some of their properties: area, perimeter, and volume, for example. In this book, you discover much more about these basic properties and then explore more advanced geometric ideas about shapes.

Geometry proofs are an entirely different sort of animal. They involve shapes, but instead of doing something straightforward like calculating the area of a shape, you have to come up with a mathematical argument that proves something about a shape. This process requires not only mathematical skills but verbal skills and logical deduction skills as well, and for this reason, proofs trip up many, many students. If you're one of these people and have already started singing the geometry-proof blues, you might even describe proofs — like Mr. Hyde — as monstrous. But I'm confident that, with the help of this book, you'll have no trouble taming them.

The Geometry of Shapes

Have you ever reflected on the fact that you're literally surrounded by shapes? Look around. The rays of the sun are — what else? — rays. The book in your hands has a shape, every table and chair has a shape, every wall has an area, and every container has a shape and a volume; most picture frames are rectangles, DVDs are circles, soup cans are cylinders, and so on.

One-dimensional shapes

There aren't many shapes you can make if you're limited to one dimension. You've got your lines, your segments, and your rays. That's about it. On to something more interesting.

Two-dimensional shapes

As you probably know, two-dimensional shapes are flat things like triangles, circles, squares, rectangles, and pentagons. The two most common characteristics you study about 2-D shapes are their area and perimeter. I devote many chapters in this book to triangles and *quadrilaterals* (shapes with four sides); I give less space to shapes that have more sides, like pentagons and hexagons. Then there are the shapes with curved sides: The only curved 2-D shape I discuss is the circle.

Three-dimensional shapes

In this book, you work with prisms (a box is one example), cylinders, pyramids, cones, and spheres. The two major characteristics of these 3-D shapes are their *surface area* and *volume.* These two concepts come up frequently in the real world; examples include the amount of wrapping paper you need to wrap a gift box (a surface area problem) and the volume of water in a backyard pool (a volume problem).

Geometry Proofs

A *geometry proof* — like any mathematical proof — is an argument that begins with known facts, proceeds from there through a series of logical deductions, and ends with the thing you're trying to prove. Here's a very simple example using the line segments in Figure 1-1.

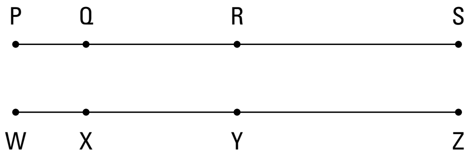

Figure 1-1: *PS* and *WZ*, each made up of three pieces.

For this proof, you're told that segment $\overline{PS}$ is *congruent to* (the same length as) segment $\overline{WZ}$, that $\overline{PQ}$ is congruent to $\overline{WX}$, and that $\overline{QR}$ is congruent to $\overline{XY}$. You have to prove that $\overline{RS}$ is congruent to $\overline{YZ}$.

Now, you may be thinking, "That's obvious — if $\overline{PS}$ is the same length as $\overline{WZ}$ and both segments contain these equal short pieces and the equal medium pieces, then the longer third pieces have to be equal as well." And you'd be right. But that's not how the proof game is played. You have to spell out every little step in your thinking. Here's the whole chain of logical deductions:

1. $\overline{PS} \cong \overline{WZ}$ (this is given).
2. $\overline{PQ} \cong \overline{WX}$ and $\overline{QR} \cong \overline{XY}$ (these facts are also given).
3. Therefore, $\overline{PR} \cong \overline{WY}$ (because if you add equal things to equal things, you get equal totals).
4. Therefore, $\overline{RS} \cong \overline{YZ}$ (because if you start with equal segments, the whole segments $\overline{PS}$ and $\overline{WZ}$, and take away equal parts of them, $\overline{PR}$ and $\overline{WY}$, the parts that are left must be equal).

Am I Ever Going to Use This?

You'll likely have plenty of opportunities to use your knowledge about the geometry of shapes. What about geometry proofs? Not so much.

When you'll use your knowledge of shapes

Shapes are everywhere, so every educated person should have a working knowledge of shapes and their properties. If you have to buy fertilizer or grass seed for your lawn, you should know something about area. You might want to understand the volume measurements in cooking recipes, or you may want to help a child with an art or science project that involves geometry. You certainly need to understand something about geometry to build some shelves or a backyard deck. And after finishing your work, you might be hungry — a grasp of how area works can come in handy when you're ordering pizza: a 20-inch pizza is four, not two, times as big as a 10-incher. There's no end to the list of geometry problems that come up in everyday life.

When you'll use your knowledge of proofs

Will you ever use your knowledge of geometry proofs? I'll give you a politically correct answer and a politically incorrect one. Take your pick.

First, the politically correct answer (which is also *actually* correct). Granted, it's extremely unlikely that you'll ever have occasion to do a single geometry proof outside of a high school math course. However, doing geometry proofs teaches you important lessons that you can apply to nonmathematical arguments. Proofs teach you . . .

- Not to assume things are true just because they seem true
- To carefully explain each step in an argument even if you think it should be obvious to everyone
- To search for holes in your arguments
- Not to jump to conclusions

In general, proofs teach you to be disciplined and rigorous in your thinking and in communicating your thoughts.

If you don't buy that PC stuff, I'm sure you'll get this politically incorrect answer: Okay, so you're never going to use geometry proofs, but you want to get a decent grade in geometry, right? So you might as well pay attention in class (what else is there to do, anyway?), do your homework, and use the hints, tips, and strategies I give you in this book. They'll make your life much easier. Promise.

Getting Down with Definitions

The study of geometry begins with the definitions of the five simplest geometric objects: point, line, segment, ray, and angle. And I throw in two extra definitions for you (plane and 3-D space) for no extra charge.

- **Point:** A point is like a dot except that it has no size at all. A point is zero-dimensional, with no height, length, or width, but you draw it as a dot, anyway. You name a point with a single uppercase letter, as with points *A, D,* and *T* in Figure 1-2.

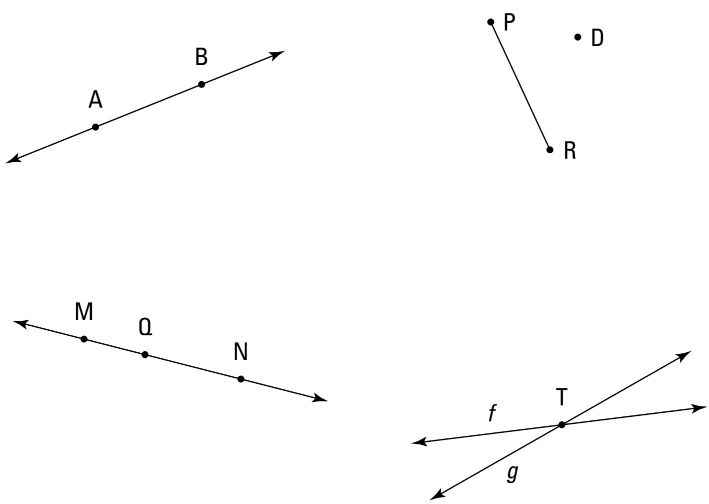

Figure 1-2: Some points, lines, and segments.

- **Line:** A line is like a thin, straight wire (although really it's infinitely thin — or better yet, it has no width at all). Lines have length, so they're one-dimensional. Remember that a line goes on forever in both directions, which is why you use the little double-headed arrow as in $\overleftrightarrow{AB}$ (read as *line AB*).

  Check out Figure 1-2 again. Lines are usually named using any two points on the line, with the letters in any order. So $\overleftrightarrow{MQ}$ is the same line as $\overleftrightarrow{QM}$, $\overleftrightarrow{MN}$, $\overleftrightarrow{NM}$, $\overleftrightarrow{QN}$, and $\overleftrightarrow{NQ}$. Occasionally, lines are named with a single, italicized, lowercase letter, such as lines *f* and *g*.

- **Line segment (or just segment):** A segment is a section of a line that has two endpoints. See Figure 1-2 yet again. If a segment goes from *P* to *R*, you call it *segment PR* and write it as $\overline{PR}$. You can also switch the order of the letters and call it $\overline{RP}$. Segments can also appear within lines, as in $\overline{MN}$.

- **Ray:** A ray is a section of a line (kind of like half a line) that has one endpoint and goes on forever in the other direction. If its endpoint is point *K* and it goes through point *S* and then past *S* forever, you call it *ray KS* and write $\overrightarrow{KS}$. See Figure 1-3. Note that a ray like $\overrightarrow{AB}$ can also be called $\overrightarrow{AC}$ because either way, you start at *A* and go forever past *B* and *C*. $\overrightarrow{BC}$, however, is a different ray.

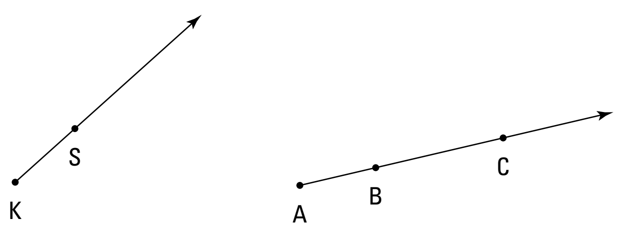

Figure 1-3: Catching a few rays.

- **Angle:** Two rays with the same endpoint form an angle. Each ray is a *side* of the angle, and the common endpoint is the angle's *vertex*. You can name an angle using its vertex alone or three points (first, a point on one ray, then the vertex, and then a point on the other ray).

Check out Figure 1-4. Rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ form the sides of an angle, with point P as the vertex. You can call the angle $\angle P$, $\angle RPQ$, or $\angle QPR$. Angles can also be named with numbers, such as the angle on the right in the figure, which you can call $\angle 4$. The number is just another way of naming the angle, and it has nothing to do with the size of the angle. The angle on the right also illustrates the *interior* and *exterior* of an angle.

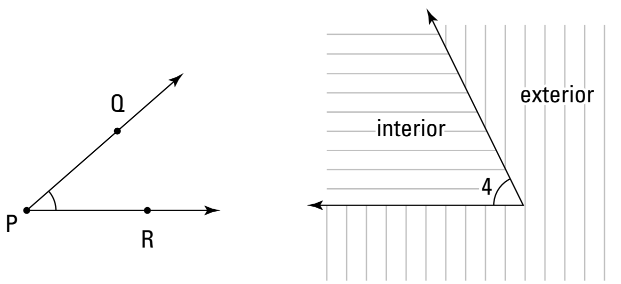

Figure 1-4: Some angles and their parts.

- **Plane:** A plane is like a perfectly flat sheet of paper except that it has no thickness whatsoever and it goes on forever in all directions. You might say it's infinitely thin and has an infinite length and an infinite width. Because it has length and width but no height, it's two-dimensional. Planes are named with a single, italicized, lowercase letter or sometimes with the name of a figure (a rectangle, for example) that lies in the plane.
- **3-D (three-dimensional) space:** 3-D space is everywhere — all of space in every direction. First, picture an infinitely big map that goes forever to the north, south, east, and west. That's a two-dimensional plane. Then, to get 3-D space from this map, add the third dimension by going up and down forever.

A Few Points on Points

There isn't much that can be said about points. They have no features, and each one is the same as every other. Various *groups* of points, however, do merit an explanation:

- **Collinear points:** See the word *line* in *collinear?* Collinear points are points that lie on a line. Any two points are always collinear because you can always connect them with a straight line. Three or more points can be collinear, but they don't have to be.
- **Non-collinear points:** Non-collinear points are three or more points that don't all lie on the same line.
- **Coplanar points:** A group of points that lie in the same plane are coplanar. Any two or three points are always coplanar. Four or more points might or might not be coplanar.

  Look at Figure 1-5, which shows coplanar points *A, B, C,* and *D.* In the box on the right, there are many sets of coplanar points. Points *P, Q, X,* and *W,* for example, are coplanar; the plane that contains them is the left side of the box. Note that points *Q, X, S,* and *Z* are also coplanar even though the plane that contains them isn't shown; it slices the box in half diagonally.

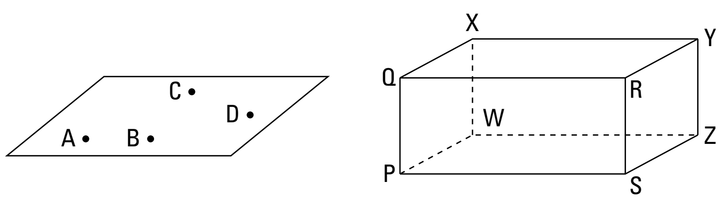

Figure 1-5: Coplanar and non-coplanar points.

- **Non-coplanar points:** A group of points that don't all lie in the same plane are non-coplanar. In Figure 1-5, points *P, Q, X,* and *Y* are non-coplanar. The top of the box contains *Q, X,* and *Y,* and the left side contains *P, Q,* and *X,* but no plane contains all four points.

Lines, Segments, and Rays

In this section, I describe different types of lines (or segments or rays) or pairs of lines (or segments or rays) based on the direction they're pointing or how they relate to each other.

Horizontal and vertical lines

Defining *horizontal* and *vertical* may seem a bit pointless because you probably already know what the terms mean. But, hey, this is a math book, and math books are supposed to define terms. Who am I to question this tradition?

- **Horizontal lines, segments, or rays:** Horizontal lines, segments, and rays go straight across, left and right, not up or down at all — you know, like the horizon.
- **Vertical lines, segments, or rays:** Lines or parts of a line that go straight up and down are vertical. (Shocking!)

Doubling up with pairs of lines

In this section, I give you five terms that describe pairs of lines. The first four are about coplanar lines — you use these a lot. The fifth term describes non-coplanar lines. This term comes up only in 3-D problems, so you won't need it much.

Coplanar lines

Coplanar lines are lines in the same plane. Here are some ways coplanar lines may interact:

- **Parallel lines, segments, or rays**: Lines that run in the same direction and never cross (like two railroad tracks) are called parallel. Segments and rays are parallel if the lines that contain them are parallel. If $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$, you write $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.
- **Intersecting lines, segments, or rays:** Lines, rays, or segments that cross or touch are intersecting.
    - **Perpendicular lines, segments, or rays**: Lines, segments, or rays that intersect at right angles are perpendicular. If $\overline{PQ}$ is perpendicular to $\overline{RS}$, you write $\overline{PQ} \perp \overline{RS}$. See Figure 1-6. The little boxes in the corners of the angles indicate right angles.
    - **Oblique lines, segments, or rays:** Lines or segments or rays that intersect at any angle other than 90° are called *oblique.* See Figure 1-6.

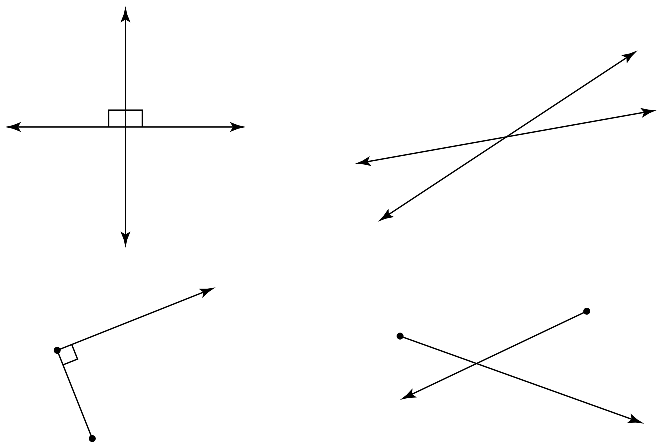

Figure 1-6: Perpendicular and oblique lines, rays, and segments.

Non-coplanar lines

Non-coplanar lines are lines that cannot be contained in a single plane.

Skew lines, segments, or rays: Lines that don't lie in the same plane are called skew lines — *skew* simply means *non-coplanar.* Or you can say that skew lines are lines that are neither parallel nor intersecting.

Investigating the Plane Facts

Here are two terms for a pair of planes (see Figure 1-7):

- **Parallel planes:** Parallel planes are planes that never cross. The ceiling of a room (assuming it's flat) and the floor are parallel planes (though true planes extend forever).
- **Intersecting planes:** Hold onto your hat — intersecting planes are planes that cross, or intersect. When planes intersect, the place where they cross forms a line. The floor and a wall of a room are intersecting planes, and where the floor meets the wall is the line of intersection of the two planes.

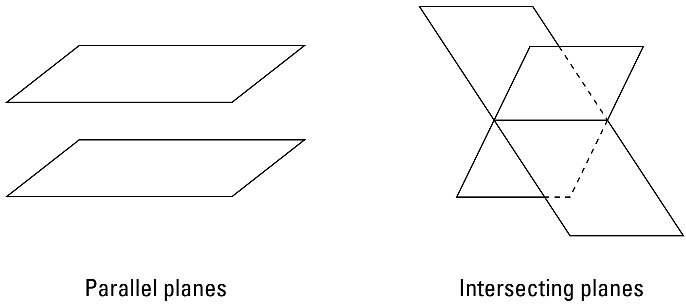

Figure 1-7: Parallel and intersecting planes.

Everybody's Got an Angle

Angles are one of the basic building blocks of triangles and other polygons. Angles appear on virtually every page of every geometry book, so you gotta get up to speed about them — no ifs, ands, or buts.

Five types of angles

Check out the five angle definitions and see Figure 1-8:

- **Acute angle:** An acute angle is less than 90°. Think "a-*cute* little angle."
- **Right angle:** A right angle is a 90° angle. Right angles should be familiar to you from the corners of picture frames, tabletops, boxes, and books, and all kinds of other things that show up in everyday life.
- **Obtuse angle:** An obtuse angle has a measure greater than 90°.
- **Straight angle:** A straight angle has a measure of 180°; it looks just like a line with a point on it.
- **Reflex angle:** A reflex angle has a measure of more than 180°. Basically, a reflex angle is just the other side of an ordinary angle.

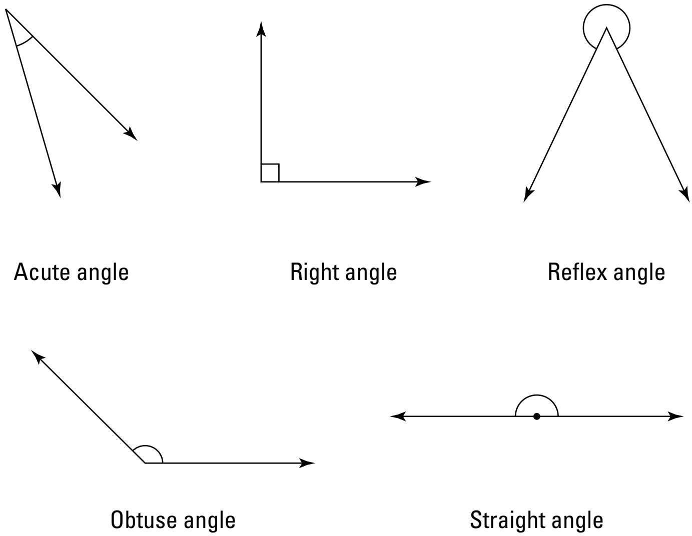

Figure 1-8: Examining all the angles.

Angle pairs

The needy angles in this section have to be in a relationship with another angle for these definitions to mean anything.

- **Adjacent angles:** Adjacent angles are neighboring angles that have the same vertex and that share a side; also, neither angle can be inside the other. I realize that's quite a mouthful. This very simple idea is kind of a pain to define, so just check out Figure 1-9. $\angle BAC$ and $\angle CAD$ are adjacent, as are $\angle 1$ and $\angle 2$. However, neither $\angle 1$ nor $\angle 2$ is adjacent to $\angle XYZ$ because they're both inside $\angle XYZ$. None of the unnamed angles to the right are adjacent because they either don't share a vertex or don't share a side.

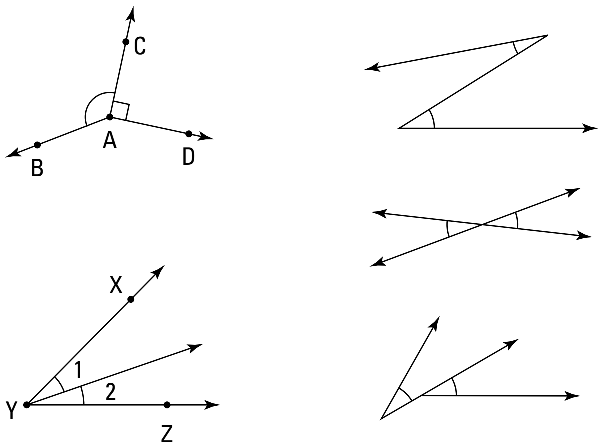

Figure 1-9: Adjacent and non-adjacent angles.

- **Complementary angles:** Two angles that add up to 90° are complementary. They can be adjacent angles but don't have to be. In Figure 1-10, adjacent angles ∠1 and ∠2 are complementary because they make a right angle; ∠*P* and ∠*Q* are complementary because they add up to 90°.

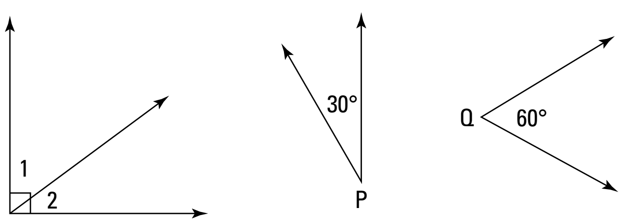

Figure 1-10: Complementary angles can join forces to form a right angle.

- **Supplementary angles:** Two angles that add up to 180° are supplementary. They may or may not be adjacent angles. In Figure 1-11, ∠1 and ∠2, or the two right angles, are supplementary because they form a straight angle. Such angle pairs are called a *linear pair.* Angles *A* and *Z* are supplementary because they add up to 180°.

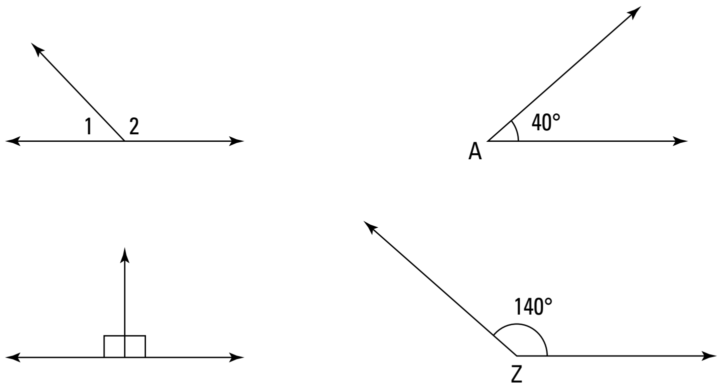

Figure 1-11: Together, supplementary angles can form a straight line.

- **Vertical angles:** When two intersecting lines form an X, the angles on the opposite sides of the X are called vertical angles. Two vertical angles are always congruent. By the way, the *vertical* in *vertical angles* has nothing to do with the up-and-down meaning of the word vertical.

Bisection and Trisection

For all you fans of bicycles and tricycles and bifocals and trifocals — not to mention the biathlon and the triathlon, bifurcation and trifurcation, and bipartition and tripartition — you're really going to love this section on bisection and trisection: cutting something into two or three equal parts.

Segments

REMEMBER

Segment *bisection,* the related term *midpoint,* and segment *trisection* are pretty simple ideas.

- **Segment bisection:** A point, segment, ray, or line that divides a segment into two congruent segments *bisects* the segment.
- **Midpoint:** The point where a segment is bisected is called the *midpoint*; the midpoint cuts the segment into two congruent parts.

- **Segment trisection:** Two things (points, segments, rays, or lines) that divide a segment into three congruent segments *trisect* the segment.

Students often make the mistake of thinking that *divide* means to *bisect,* or cut exactly in half. This error is understandable because when you do ordinary division with numbers, you are, in a sense, dividing the larger number into equal parts ($24 \div 2 = 12$ because $12 + 12 = 24$). But in geometry, to *divide* something just means to cut it into parts of any size, equal or unequal. *Bisect* and *trisect,* of course, *do* mean to cut into exactly equal parts.

Angles

Brace yourself for a shocker: The terms *bisecting* and *trisecting* mean the same thing for angles as they do for segments!

- **Angle bisection:** A ray that cuts an angle into two congruent angles *bisects* the angle. The ray is called the *angle bisector.*
- **Angle trisection:** Two rays that divide an angle into three congruent angles *trisect* the angle. These rays are called *angle trisectors.*

Take a stab at this problem: In Figure 1-12, $\overrightarrow{TP}$ bisects $\angle STL$, which equals $(12x - 24)^\circ$; $\overrightarrow{TL}$ bisects $\angle PTI$, which equals $(8x)^\circ$. Is $\angle STI$ trisected, and what is its measure?

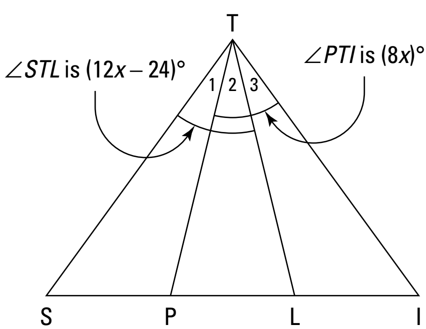

Figure 1-12: A three-way SPLIT.

Nothing to it. First, yes, $\angle STI$ is trisected. You know this because $\angle STL$ is bisected, so $\angle 1$ must equal $\angle 2$. And because $\angle PTI$ is bisected, $\angle 2$ equals $\angle 3$. Thus, all three angles must be equal, and that means $\angle STI$ is trisected.

Now find the measure of $\angle STI$. Because $\angle STL$ — which measures $(12x - 24)°$ — is bisected, $\angle 2$ must be half its size, or $(6x - 12)°$. And because $\angle PTI$ is bisected, $\angle 2$ must also be half the size of $\angle PTI$ — that's half of $(8x)°$, or $(4x)°$. Because $\angle 2$ equals both $(6x - 12)°$ and $(4x)°$, you set those expressions equal to each other and solve for x:

$$\begin{aligned} 6x - 12 &= 4x \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

Then just plug $x = 6$ into, say, $(4x)°$, which gives you $4 \cdot 6$, or $24°$ for $\angle 2$. Angle STI is three times that, or $72°$. That does it.

WARNING!

When rays trisect an angle of a triangle, the opposite side of the triangle is *never* trisected by these rays. (It might be close, but it's never exactly trisected.) In Figure 1-12, for instance, because $\angle STI$ is trisected, $\overline{SI}$ is definitely *not* trisected by points P and L. This warning also works in reverse. Assuming for the sake of argument that P and L *did* trisect $\overline{SI}$, you would know that $\angle STI$ is definitely *not* trisected.