CHAPTER 1

CUBIC EQUATIONS

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The quadratic formula states that the solutions of a quadratic equation

$$ax^2+bx+c=0, \quad a,b,c \in \mathbb{C},\ a \neq 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \tag{1.1}$$

In this chapter we will consider a cubic equation

$$ax^3+bx^2+cx+d=0, \quad a,b,c,d \in \mathbb{C},\ a \neq 0,$$

and we will show that the solutions of this equation are given by a similar though somewhat more complicated formula. Finding the formula will not be difficult, but understanding where it comes from and what it means will lead to some interesting questions.

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1.1 CARDAN'S FORMULAS

Given a cubic equation $ax^3+bx^2+cx+d=0$ with $a \neq 0$, we first divide by a to rewrite the equation as

$$x^3+bx^2+cx+d=0, \quad b,c,d \in \mathbb{C},$$

where b/a, c/a, and d/a have been replaced with b, c, and d, respectively. Observe that x^3+bx^2+cx+d is a monic polynomial and that reducing to the monic case has no effect on the roots.

The next step is to remove the coefficient of x^2 by the substitution

$$x = y - \frac{b}{3}.$$

The binomial theorem implies that

$$\begin{aligned} x^2 &= y^2 - 2y\frac{b}{3} + \left(\frac{b}{3}\right)^2 = y^2 - \frac{2b}{3}y + \frac{b^2}{9}, \\ x^3 &= y^3 - 3y^2\frac{b}{3} + 3y\left(\frac{b}{3}\right)^2 - \left(\frac{b}{3}\right)^3 = y^3 - by^2 + \frac{b^2}{3}y - \frac{b^3}{27}, \end{aligned}$$

so that

$$\begin{aligned} 0 &= x^3+bx^2+cx+d \\ &= \left(y^3 - by^2 + \frac{b^2}{3}y - \frac{b^3}{27}\right) + b\left(y^2 - \frac{2b}{3}y + \frac{b^2}{9}\right) + c\left(y - \frac{b}{3}\right) + d. \end{aligned}$$

If we collect terms, then we can write the resulting equation in y as

$$y^3+py+q=0,$$

where

$$\begin{aligned} p &= -\frac{b^2}{3} + c, \\ q &= \frac{2b^3}{27} - \frac{bc}{3} + d. \end{aligned} \tag{1.2}$$

You will verify the details of this calculation in Exercise 1.

We call a cubic of the form $y^3+py+q=0$ a *reduced cubic*. If we can find the roots y_1, y_2, y_3 of the reduced cubic, then we get the roots of the original cubic $x^3+bx^2+cx+d=0$ by adding $-b/3$ to each y_i.

To solve $y^3+py+q=0$, we use the substitution

$$y = z - \frac{p}{3z}. \tag{1.3}$$

This change of variable has a dramatic effect on the equation. Using the binomial theorem again, we obtain

$$y^3 = z^3 - 3z^2\frac{p}{3z} + 3z\left(\frac{p}{3z}\right)^2 - \left(\frac{p}{3z}\right)^3 = z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3}.$$

Combining this with (1.3) gives

$$y^3 + py + q = \left(z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3}\right) + p\left(z - \frac{p}{3z}\right) + q = z^3 - \frac{p^3}{27z^3} + q.$$

Multiplying by z^3, we conclude that $y^3 + py + q = 0$ is equivalent to the equation

$$z^6 + qz^3 - \frac{p^3}{27} = 0. \tag{1.4}$$

This equation is the *cubic resolvent* of the reduced cubic $y^3 + py + q = 0$.

At first glance, (1.4) might not seem useful, since we have replaced a cubic equation with one of degree 6. However, upon closer inspection, we see that the cubic resolvent can be written as

$$(z^3)^2 + qz^3 - \frac{p^3}{27} = 0.$$

By the quadratic formula (1.1), we obtain

$$z^3 = \frac{1}{2}\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right),$$

so that

$$z = \sqrt[3]{\frac{1}{2}\left(-q \pm \sqrt{q^2 + \frac{4p^3}{27}}\right)}. \tag{1.5}$$

Substituting this into (1.3) gives a root of the reduced cubic $y^3 + py + q$, and then $x = y - b/3$ is a root of the cubic $x^3 + bx^2 + cx + d$.

However, before we can claim to have solved the cubic, there are several questions that need to be answered:

- By setting $y^3 + py + q = 0$, we essentially assumed that a solution exists. What justifies this assumption?
- A cubic equation has three roots, yet the cubic resolvent has degree 6. Why?
- The substitution (1.3) assumes that $z \neq 0$. What happens when $z = 0$?
- $y^3 + py + q$ has coefficients in $\mathbb{C}$, since $b, c, d \in \mathbb{C}$. Thus (1.5) involves square roots and cube roots of complex numbers. How are these described?

The first bullet will be answered in Chapter 3 when we discuss the existence of roots. The second bullet will be considered in Section 1.2, though the ultimate answer will involve Galois theory. For the rest of this section, we will concentrate on the last two bullets. Our strategy will be to study the formula (1.5) in more detail.

First assume that $p \neq 0$ in the reduced cubic $y^3 + py + q$. By Section A.2, every nonzero complex number has n distinct nth roots when $n \in \mathbb{Z}$ is positive. In (1.5), the $\pm$ in the formula indicates that a nonzero complex number has two square roots. Similarly, the cube root symbol denotes any of the *three* cube roots of the complex number under the radical. To understand these cube roots, we use the cube roots of unity $1, \zeta_3, \zeta_3^2$ from Section A.2. We will write ζ_3 as ω. Recall that

$$\omega = \zeta_3 = e^{2\pi i/3} = \frac{-1 + i\sqrt{3}}{2}$$

and that given one cube root of a nonzero complex number, we get the other two cube roots by multiplying by ω and ω^2.

We can now make sense of (1.5). Let

$$\sqrt{q^2 + \frac{4p^3}{27}}$$

denote a fixed square root of $q^2 + 4p^3/27 \in \mathbb{C}$. With this choice of square root, let

$$z_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)}$$

denote a fixed cube root of $\frac{1}{2}\left(-q + \sqrt{q^2 + 4p^3/27}\right)$. Then we get the other two cube roots by multiplying by ω and ω^2. Note also that $p \neq 0$ implies that $z_1 \neq 0$ and that z_1 is a root of the cubic resolvent (1.4). It follows easily that if we set

$$z_2 = -\frac{p}{3z_1},$$

then

$$y_1 = z_1 + z_2 = z_1 - \frac{p}{3z_1} \tag{1.6}$$

is a root of the reduced cubic $y^3 + py + q$.

To understand z_2, observe that

$$z_1^3 z_2^3 = z_1^3\left(-\frac{p}{3z_1}\right)^3 = -\frac{p^3}{27}.$$

An easy calculation shows that

$$z_1^3 \cdot \frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right) = \frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right) \cdot \frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right) = -\frac{p^3}{27}.$$

Since $z_1 \neq 0$, these formulas imply that

$$z_2^3 = \frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right).$$

Hence $z_2 = -p/3z_1$ is a cube root of $\frac{1}{2}\left(-q - \sqrt{q^2+4p^3/27}\right)$, so that

$$(1.7) \qquad z_1 = \sqrt[3]{\frac{1}{2}\left(-q+\sqrt{q^2+\frac{4p^3}{27}}\right)} \text{ and } z_2 = \sqrt[3]{\frac{1}{2}\left(-q-\sqrt{q^2+\frac{4p^3}{27}}\right)}$$

are cube roots with the property that their product is $-p/3$.

From (1.6), we see that $y_1 = z_1 + z_2$ is a root of $y^3 + py + q$ when z_1 and z_2 are the above cube roots. To get the other roots, note that (1.6) gives a root of the cubic whenever the cube roots are chosen so that their product is $-p/3$ (be sure you understand this). For example, if we use the cube root ωz_1, then

$$\omega z_1 \cdot \omega^2 z_2 = z_1 z_2 = -\frac{p}{3}$$

shows that $y_2 = \omega z_1 + \omega^2 z_2$ is also a root. Similarly, using the cube root $\omega^2 z_1$ shows that $y_3 = \omega^2 z_1 + \omega z_2$ is a third root of the reduced cubic.

By (1.7), it follows that the three roots of $y^3 + py + q = 0$ are given by

$$\begin{aligned}
y_1 &= \sqrt[3]{\frac{1}{2}\left(-q+\sqrt{q^2+\frac{4p^3}{27}}\right)} + \sqrt[3]{\frac{1}{2}\left(-q-\sqrt{q^2+\frac{4p^3}{27}}\right)},\\
y_2 &= \omega\sqrt[3]{\frac{1}{2}\left(-q+\sqrt{q^2+\frac{4p^3}{27}}\right)} + \omega^2\sqrt[3]{\frac{1}{2}\left(-q-\sqrt{q^2+\frac{4p^3}{27}}\right)},\\
y_3 &= \omega^2\sqrt[3]{\frac{1}{2}\left(-q+\sqrt{q^2+\frac{4p^3}{27}}\right)} + \omega\sqrt[3]{\frac{1}{2}\left(-q-\sqrt{q^2+\frac{4p^3}{27}}\right)},
\end{aligned}$$

provided the cube roots in (1.7) are chosen so that their product is $-p/3$. These are *Cardan's formulas* for the roots of the reduced cubic $y^3 + py + q$.

Example 1.1.1 For the reduced cubic $y^3 + 3y + 1$, consider the real cube roots

$$\sqrt[3]{\tfrac{1}{2}(-1+\sqrt{5})} \quad \text{and} \quad \sqrt[3]{\tfrac{1}{2}(-1-\sqrt{5})}.$$

Their product is $-1 = -p/3$, so by Cardan's formulas, the roots of $y^3 + 3y + 1$ are

$$\begin{aligned}
y_1 &= \sqrt[3]{\tfrac{1}{2}(-1+\sqrt{5})} + \sqrt[3]{\tfrac{1}{2}(-1-\sqrt{5})},\\
y_2 &= \omega\sqrt[3]{\tfrac{1}{2}(-1+\sqrt{5})} + \omega^2\sqrt[3]{\tfrac{1}{2}(-1-\sqrt{5})},\\
y_3 &= \omega^2\sqrt[3]{\tfrac{1}{2}(-1+\sqrt{5})} + \omega\sqrt[3]{\tfrac{1}{2}(-1-\sqrt{5})}.
\end{aligned}$$

Note that y_1 is real. In Exercise 2 you will show that y_2 and y_3 are complex conjugates of each other. ◁▷

Although Cardan's formulas only apply to a reduced cubic, we get formulas for the roots of an arbitrary monic cubic polynomial $x^3 + bx^2 + cx + d \in \mathbb{C}[x]$ as follows.

The substitution $x = y - b/3$ gives the reduced cubic $y^3 + py + q = 0$, where p and q are as in (1.2). If z_1 and z_2 are the cube roots in Cardan's formulas for $y^3 + py + q = 0$, then the roots of $x^3 + bx^2 + cx + d = 0$ are given by

$$\begin{aligned} x_1 &= -\frac{b}{3} + z_1 + z_2, \\ x_2 &= -\frac{b}{3} + \omega z_1 + \omega^2 z_2, \\ x_3 &= -\frac{b}{3} + \omega^2 z_1 + \omega z_2, \end{aligned}$$

where z_1 and z_2 from (1.7) satisfy $z_1 z_2 = -p/3$. Our derivation assumed $p \neq 0$, but these formulas give the correct roots even when $p = 0$ (see Exercise 3).

We will eventually see that Cardan's formulas make perfect sense from the point of view of Galois theory. For example, the quantity under the square root in (1.5) is

$$q^2 + \frac{4p^3}{27}.$$

Up to a constant factor, this is the *discriminant* of the polynomial $y^3 + py + q$. We will give a careful definition of discriminant in Section 1.2, and Section 1.3 will show that the discriminant gives useful information about the roots of a real cubic.

Here is an example of a puzzle that arises when using Cardan's formula.

Example 1.1.2 The cubic equation $y^3 - 3y = 0$ has roots $y = 0, \pm\sqrt{3}$, all of which are real. When we apply Cardan's formulas, we begin with

$$z_1 = \sqrt[3]{\tfrac{1}{2}\left(-0 + \sqrt{0^2 + \tfrac{4(-3)^3}{27}}\right)} = \sqrt[3]{i}.$$

To pick a specific value for z_1, notice that $(-i)^3 = i$, so that we can take $z_1 = -i$. Thus $z_2 = -p/3z_1 = i$, since $p = -3$. Then Cardan's formulas give the roots

$$\begin{aligned} y_1 &= -i + i = 0, \\ y_2 &= \omega(-i) + \omega^2(i) = \sqrt{3}, \\ y_3 &= \omega^2(-i) + \omega(i) = -\sqrt{3}. \end{aligned}$$

(You will verify the last two formulas in Exercise 4.) ◁▷

The surprise is that Cardan's formulas express the real roots of $y^3 - 3y$ in terms of complex numbers. In Section 1.3, we will prove that for *any* cubic with distinct real roots, Cardan's formulas *always* involve complex numbers.

Historical Notes

The quadratic formula is very old, dating back to the Babylonians, circa 1700 B.C. Cubic equations were first studied systematically by Islamic mathematicians such as

Omar Khayyam, and by the Middle Ages cubic equations had become a popular topic. For example, when Leonardo of Pisa (also known as Fibonacci) was introduced to Emperor Frederick II in 1225, Fibonacci was asked to solve two problems, the second of which was the cubic equation

$$x^3 + 2x^2 + 10x = 20.$$

Fibonacci's solution was

$$x = 1 + \frac{22}{60} + \frac{7}{60^2} + \frac{42}{60^3} + \frac{33}{60^4} + \frac{4}{60^5} + \frac{40}{60^6}.$$

In decimal notation, this gives $x = 1.368808107853\ldots$, which is correct to 10 decimal places. Not bad for 787 years ago!

Challenges and contests involving cubic equations were not uncommon during the Middle Ages, and one such contest played a crucial role in the development of Cardan's formula. Early in the sixteenth century, Scipio del Ferro found a solution for cubics of the form $x^3 + bx = c$, where b and c are positive. His student Florido knew this solution, and in 1535, Florido challenged Niccolò Fontana (also known as Tartaglia) to a contest involving 30 cubic equations. Working feverishly in preparation for the contest, Tartaglia worked out the solution of this and other cases, and went on to defeat Florido. In 1539, Tartaglia told his solution to Girolamo Cardan (or Cardano), who published it in 1545 in his book *Ars Magna* (see [2]).

Rather than present one solution to the cubic, as we have done here, Cardan's treatment in *Ars Magna* requires 13 cases. For example, Chapter XIV considers $x^3 + 64 = 18x^2$, and Chapter XV does $x^3 + 6x^2 = 40$. The reason is that Cardan prefers positive coefficients. However, he makes systematic use of the substitution $x = y - b/3$ to get rid of the coefficient of x^2, and Cardan was also aware that complex numbers can arise in solutions of quadratic equations.

Numerous other people worked to simplify and understand Cardan's solution. In 1550, Rafael Bombelli considered more carefully the role of complex solutions (see Section 1.3), and in two papers published posthumously in 1615, François Viète (or Vieta, in Latin) introduced the substitution (1.3) used in our derivation of Cardan's formulas and gave the trigonometric solution to be discussed in Section 1.3 .

In addition to the cubic, *Ars Magna* also contained a solution for the quartic equation due to Lodovico (or Luigi) Ferrari, a student of Cardan's. We will discuss the solution of the quartic in Chapter 12.

Exercises for Section 1.1

Exercise 1. Complete the demonstration (begun in the text) that the substitution $x = y - b/3$ transforms $x^3 + bx^2 + cx + d$ into $y^3 + py + q$, where p and q are given by (1.2).

Exercise 2. In Example 1.1.1, show that y_2 and y_3 are complex conjugates of each other.

Exercise 3. Show that Cardan's formulas give the roots of $y^3 + py + q$ when $p = 0$.

Exercise 4. Verify the formulas for y_2 and y_3 in Example 1.1.2.

Exercise 5. The substitution $x = y - b/3$ can be adapted to other equations as follows.

(a) Show that $x = y - b/2$ gets rid of the coefficient of x in the quadratic equation $x^2 + bx + c = 0$. Then use this to derive the quadratic formula.

(b) For the quartic equation $x^4 + bx^3 + cx^2 + dx + e = 0$, what substitution should you use to get rid of the coefficient of x^3?

(c) Explain how part (b) generalizes to a monic equation of degree n.

Exercise 6. Consider the equation $x^3 + x - 2 = 0$. Note that $x = 1$ is a root.

(a) Use Cardan's formulas (carefully) to derive the surprising formula

$$1 = \sqrt[3]{1 + \frac{2}{3}\sqrt{\frac{7}{3}}} + \sqrt[3]{1 - \frac{2}{3}\sqrt{\frac{7}{3}}}.$$

(b) Show that $1 + \frac{2}{3}\sqrt{\frac{7}{3}} = \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{7}{3}}\right)^3$, and use this to explain the result of part (a).

Exercise 7. Cardan's formulas, as stated in the text, express the roots as sums of two cube roots. Each cube root has three values, so there are nine different possible values for the sum of the cube roots. Show that these nine values are the roots of the equations $y^3 + py + q = 0$, $y^3 + \omega py + q = 0$, and $y^3 + \omega^2 py + q = 0$, where as usual $\omega = \frac{1}{2}(-1 + i\sqrt{3})$.

Exercise 8. Use Cardan's formulas to solve $y^3 + 3\omega y + 1 = 0$.

1.2 PERMUTATIONS OF THE ROOTS

In Section 1.1 we learned that the roots of $x^3 + bx^2 + cx + d = 0$ are given by

$$\begin{aligned} x_1 &= -\frac{b}{3} + z_1 + z_2, \\ x_2 &= -\frac{b}{3} + \omega z_1 + \omega^2 z_2, \\ x_3 &= -\frac{b}{3} + \omega^2 z_1 + \omega z_2, \end{aligned} \tag{1.8}$$

where z_1 and z_2 are the cube roots (1.7) chosen so that $z_1 z_2 = -p/3$. We also know that z_1 is a root of the cubic resolvent

$$z^6 + qz^3 - \frac{p^3}{27} = 0, \tag{1.9}$$

and in Exercise 1 you will show that z_2 is also a root of (1.9). The goal of this section is to understand more clearly the relation between x_1, x_2, x_3 and z_1, z_2. We will learn that *permutations*, the *discriminant*, and *symmetric polynomials* play an important role in these formulas.

A. Permutations. We begin by observing that we can use (1.8) to express z_1, z_2 in terms of x_1, x_2, x_3. We do this by multiplying the second equation by ω^2 and the third by ω. When we add the three resulting equations, we obtain

$$x_1 + \omega^2 x_2 + \omega x_3 = -(1 + \omega^2 + \omega)\frac{b}{3} + 3z_1 + (1 + \omega + \omega^2)z_2.$$

However, ω is a root of $x^3 - 1 = (x-1)(x^2+x+1)$, which implies $1+\omega+\omega^2 = 0$. Thus the above equation simplifies to

$$x_1 + \omega^2 x_2 + \omega x_3 = 3z_1,$$

so that

$$z_1 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3).$$

Similarly, multiplying the second equation of (1.8) by ω and the third by ω^2 leads to the formula

$$z_2 = \frac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3).$$

This shows that the roots z_1 and z_2 of the cubic resolvent can be expressed in terms of the roots of the original cubic. However, z_1 and z_2 are only two of the six roots of (1.9). What about the other four? In Exercise 1 you will show that the roots of the cubic resolvent (1.9) are

$$z_1,\ z_2,\ \omega z_1,\ \omega z_2,\ \omega^2 z_1,\ \omega^2 z_2,$$

and that these roots are given in terms of x_1, x_2, x_3 by

$$\begin{aligned}
z_1 &= \tfrac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3),\\
z_2 &= \tfrac{1}{3}(x_1 + \omega^2 x_3 + \omega x_2),\\
\omega z_1 &= \tfrac{1}{3}(x_2 + \omega^2 x_3 + \omega x_1),\\
\omega z_2 &= \tfrac{1}{3}(x_3 + \omega^2 x_2 + \omega x_1),\\
\omega^2 z_1 &= \tfrac{1}{3}(x_3 + \omega^2 x_1 + \omega x_2),\\
\omega^2 z_2 &= \tfrac{1}{3}(x_2 + \omega^2 x_1 + \omega x_3).
\end{aligned} \tag{1.10}$$

These expressions for the roots of the resolvent all look similar. What lies behind this similarity is the following crucial fact: *The six roots of the cubic resolvent are obtained from z_1 by permuting x_1, x_2, x_3.* Hence the symmetric group S_3 now enters the picture.

From an intuitive point of view, this is reasonable, since labeling the roots x_1, x_2, x_3 simply lists them in one particular order. If we list the roots in a different order, then we should still get a root of the resolvent. This also explains why the cubic resolvent has degree 6, since $|S_3| = 6$.

B. The Discriminant. We can also use (1.10) to get a better understanding of the square root that appears in Cardan's formulas. If we set

$$D = q^2 + \frac{4p^3}{27}, \tag{1.11}$$

then we can write z_1 and z_2 as

$$\begin{aligned}
z_1 &= \sqrt[3]{\tfrac{1}{2}(-q + \sqrt{D})},\\
z_2 &= \sqrt[3]{\tfrac{1}{2}(-q - \sqrt{D})}.
\end{aligned} \tag{1.12}$$

We claim that D can be expressed in terms of the roots x_1, x_2, x_3. To see why, note that the above formulas imply that

$$z_1^3 - z_2^3 = \tfrac{1}{2}(-q+\sqrt{D}) - \tfrac{1}{2}(-q-\sqrt{D}) = \sqrt{D}.$$

However, (A.15) gives the factorization

$$z_1^3 - z_2^3 = (z_1 - z_2)(z_1 - \omega z_2)(z_1 - \omega^2 z_2). \tag{1.13}$$

Using (1.10), we obtain

$$\begin{aligned} z_1 - z_2 &= \tfrac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3) - \tfrac{1}{3}(x_1 + \omega x_2 + \omega^2 x_3) \\ &= \tfrac{1}{3}(\omega^2 - \omega)(x_2 - x_3) \\ &= \tfrac{-i}{\sqrt{3}}(x_2 - x_3), \end{aligned}$$

where the last line uses $\omega^2 - \omega = -i\sqrt{3}$. Similarly, one can show that

$$\begin{aligned} z_1 - \omega z_2 &= \tfrac{i\omega^2}{\sqrt{3}}(x_1 - x_3), \\ z_1 - \omega^2 z_2 &= \tfrac{-i\omega}{\sqrt{3}}(x_1 - x_2) \end{aligned} \tag{1.14}$$

(see Exercise 2). Combining these formulas with $z_1^3 - z_2^3 = \sqrt{D}$ and (1.13) easily implies that

$$\sqrt{D} = -\frac{i}{3\sqrt{3}}(x_1 - x_2)(x_1 - x_3)(x_2 - x_3). \tag{1.15}$$

If we square this formula for $\sqrt{D}$ and combine it with (1.11), we obtain

$$q^2 + \frac{4p^3}{27} = -\frac{1}{27}(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2. \tag{1.16}$$

It is customary to define the *discriminant* of $x^3 + bx^2 + cx + d$ to be

$$\Delta = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2.$$

Thus Δ is the product of the squares of the differences of the roots. In this notation we can write (1.16) as

$$q^2 + \frac{4p^3}{27} = -\frac{1}{27}\Delta. \tag{1.17}$$

Then (1.12) becomes

$$z_1 = \sqrt[3]{\tfrac{1}{2}\Big(-q + \sqrt{\tfrac{-\Delta}{27}}\Big)} \quad \text{and} \quad z_2 = \sqrt[3]{\tfrac{1}{2}\Big(-q - \sqrt{\tfrac{-\Delta}{27}}\Big)}. \tag{1.18}$$

Substituting this into (1.8), we get a version of Cardan's formulas which uses the square root of the discriminant.

The discriminant is also important in the quadratic case. By the quadratic formula, the roots of x^2+bx+c are

$$x_1 = \frac{-b+\sqrt{\Delta}}{2} \quad \text{and} \quad x_2 = \frac{-b-\sqrt{\Delta}}{2},$$

where $\Delta = b^2-4c$ is the discriminant. This makes it easy to see that

$$\sqrt{\Delta} = x_1 - x_2 \quad \text{and} \quad \Delta = (x_1-x_2)^2.$$

Thus the discriminant is the square of the difference of the roots. In Chapter 2 we will study the discriminant of a polynomial of degree n.

C. Symmetric Polynomials. We begin with two interesting properties of

$$\Delta = (x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2.$$

First suppose that we permute x_1, x_2, x_3 in this formula. The observation is that no matter how we do this, we will still have the product of the squares of the differences of the roots. This shows that Δ is unchanged by permutations of the roots. In the language of Chapter 2 we say that Δ is *symmetric* in the roots x_1, x_2, x_3.

Second, we can also express Δ in terms of the coefficients of x^3+bx^2+cx+d. By (1.17), we know that $\Delta = -4p^3-27q^2$. However, we also have

$$(1.19) \qquad \begin{aligned} p &= -\frac{b^2}{3}+c, \\ q &= \frac{2b^3}{27} - \frac{bc}{3} + d \end{aligned}$$

by Exercise 1 of Section 1.1. If we substitute these into (1.17), then a straightforward calculation shows that

$$(1.20) \qquad \Delta = b^2c^2 + 18bcd - 4c^3 - 4b^3d - 27d^2$$

(see Exercise 3). When $b=0$, it follows that x^3+cx+d has discriminant

$$\Delta = -4c^3 - 27d^2.$$

This will be useful in Section 1.3.

The above formula expresses the discriminant in terms of the coefficients of the original equation, just as the discriminant of $x^2+bx+c=0$ is $\Delta = b^2-4c$. The *Fundamental Theorem of Symmetric Polynomials*, to be proved in Chapter 2, will imply that *any* symmetric polynomial in x_1, x_2, x_3 can be expressed in terms of the coefficients b, c, d. In order to see why b, c, d are so important, note that if x_1, x_2, x_3 are the roots of x^3+bx^2+cx+d, then

$$x^3+bx^2+cx+d = (x-x_1)(x-x_2)(x-x_3).$$

Multiplying out the right-hand side and comparing coefficients leads to the following formulas for b, c, d:

$$
\begin{aligned}
b &= -(x_1 + x_2 + x_3), \\
c &= x_1x_2 + x_1x_3 + x_2x_3, \qquad (1.21) \\
d &= -x_1x_2x_3.
\end{aligned}
$$

These formulas show that the coefficients of a cubic can be expressed as symmetric functions of its roots. The polynomials b, c, d are (up to sign) the *elementary symmetric polynomials* of x_1, x_2, x_3. These polynomials (and their generalization to an arbitrary number of variables) will play a crucial role in Chapter 2.

Mathematical Notes

One aspect of the text needs further discussion.

▪ ***Algebra* versus *Abstract Algebra*.** High school algebra is very different from a course on groups, rings, and fields, yet both are called "algebra." The evolution of algebra can be seen in the difference between Section 1.1, where we used high school algebra, and this section, where questions about the underlying structure (*why does the cubic resolvent have degree* 6*?*) led us to realize the importance of permutations. Many concepts in abstract algebra came from high school algebra in this way.

Historical Notes

In 1770 and 1771, Lagrange's magnificent treatise *Réflexions sur la résolution algébrique des équations* appeared in the Nouvelles Mémoires de l'Academie royale des Sciences et Belles-Lettres de Berlin. This long paper covers pages 205–421 in Volume 3 of Lagrange's collected works [Lagrange]. It is a leisurely account of the known methods for solving equations of degree 3 and 4, together with an analysis of these methods from the point of view of permutations. Lagrange wanted to determine whether these methods could be adapted to equations of degree ≥ 5.

One of Lagrange's powerful ideas is that one should study the roots of a polynomial *without regard to their possible numerical value*. When dealing with functions of the roots, such as

$$z_1 = \frac{1}{3}(x_1 + \omega^2 x_2 + \omega x_3)$$

from (1.10), Lagrange says that he is concerned "only with the form" of such expressions and not "with their numerical quantity" [Lagrange, Vol. 3, p. 385]. In modern terms, Lagrange is saying that we should regard the roots as variables. We will learn more about this idea when we discuss the *universal polynomial* in Chapter 2.

We will see in Chapter 12 that many basic ideas from group theory and Galois theory are implicit in Lagrange's work. However, Lagrange's approach fails when the roots take on specific numerical values. This is part of why Galois's work is so important: he was able to treat the case when the roots were arbitrary. The ideas of

Galois, of course, are the foundation of what we now call Galois theory. This will be the main topic of Chapters 4–7.

Exercises for Section 1.2

Exercise 1. Let z_1, z_2 be the roots of (1.9) chosen at the beginning of the section.
(a) Show that z_1, z_2, ωz_1, ωz_2, $\omega^2 z_1$, $\omega^2 z_2$ are the six roots of the cubic resolvent.
(b) Prove (1.10).

Exercise 2. Prove (1.14) and (1.15).

Exercise 3. Prove (1.20).

Exercise 4. We say that a cubic $x^3 + bx^2 + cx + d$ has a *multiple root* if it can be written as $(x - r_1)^2(x - r_2)$. Prove that $x^3 + bx^2 + cx + d$ has a multiple root if and only if its discriminant is zero.

Exercise 5. Since $\Delta = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$, we can define the square root of Δ to be $\sqrt{\Delta} = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$. Prove that an even permutation of the roots takes $\sqrt{\Delta}$ to $\sqrt{\Delta}$ while an odd permutation takes $\sqrt{\Delta}$ to $-\sqrt{\Delta}$. In Section 2.4 we will see that this generalizes nicely to the case of degree n.

1.3 CUBIC EQUATIONS OVER THE REAL NUMBERS

The final topic of this chapter concerns cubic equations with coefficients in the field $\mathbb{R}$ of real numbers. As in Section 1.1, we can reduce to equations of the form $y^3 + py + q = 0$, where $p, q \in \mathbb{R}$. Then Cardan's formulas show that the roots y_1, y_2, y_3 lie in the field $\mathbb{C}$ of complex numbers. We will show that the *sign* of the discriminant of $y^3 + py + q = 0$ tells us how many of the roots are real. We will also give an unexpected application of trigonometry when the roots are all real.

A. The Number of Real Roots. The discriminant of $y^3 + py + q$ is

$$\Delta = (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2.$$

As we noted in the discussion following (1.20), Δ can be expressed as

$$\Delta = -4p^3 - 27q^2. \tag{1.22}$$

You will give a different proof of this in Exercise 1.

For the rest of the section we will assume that the cubic $y^3 + py + q$ has *distinct* roots y_1, y_2, y_3. It follows that the discriminant Δ is a nonzero real number. We next show that the *sign* of Δ gives interesting information about the roots.

Theorem 1.3.1 *Suppose that the polynomial $y^3 + py + q \in \mathbb{R}[y]$ has distinct roots and discriminant $\Delta \neq 0$. Then:*
(a) *$\Delta > 0$ if and only if the roots of $y^3 + py + q = 0$ are all real.*
(b) *$\Delta < 0$ if and only if $y^3 + py + q = 0$ has only one real root and the other two roots are complex conjugates of each other.*

Proof: First recall from Section A.2 that complex conjugation $z \mapsto \overline{z}$ satisfies $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$. It follows that if y_1 is a root of $y^3 + py + q = 0$, then

$$0 = \overline{0} = \overline{y_1^3 + py_1 + q} = \overline{y_1}^3 + p\overline{y_1} + q,$$

so that $\overline{y_1}$ is also a root. This proves the standard fact that the roots of a polynomial with real coefficients either are real (if $\overline{y_1} = y_1$) or come in complex conjugate pairs (if $\overline{y_1} \neq y_1$).

If y_1, y_2, y_3 are all real and distinct, then $\Delta = (y_1 - y_2)^2(y_1 - y_3)^2(y_2 - y_3)^2$ shows that $\Delta > 0$. If the roots are not all real, then the above discussion shows that we must have one real root, say y_1, and a complex conjugate pair, say y_2 and $\overline{y_2}$. Write $y_2 = u + iv$, where $u, v \in \mathbb{R}$ and $v \neq 0$. Then $\overline{y_2} = u - iv$ and

$$\begin{aligned}
\Delta &= \big(y_1 - (u+iv)\big)^2\big(y_1 - (u-iv)\big)^2\big((u+iv) - (u-iv)\big)^2\\
&= \big((y_1 - u) - iv\big)^2\big((y_1 - u) + iv\big)^2(2iv)^2\\
&= -4v^2\big((y_1 - u)^2 + v^2\big)^2.
\end{aligned}$$

It follows that $\Delta < 0$ when there is only one real root. This completes the proof. ∎

In Exercises 2–5, we will sketch a different proof of Theorem 1.3.1 which uses curve graphing techniques from calculus.

We next apply the theory developed so far to Cardan's formulas

$$\begin{aligned}
y_1 &= z_1 + z_2,\\
y_2 &= \omega z_1 + \omega^2 z_2,\\
y_3 &= \omega^2 z_1 + \omega z_2,
\end{aligned}$$

where the cube roots

$$z_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} \text{ and } z_2 = \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)} \tag{1.23}$$

are chosen so that $z_1 z_2 = -p/3$.

First, suppose that $\Delta < 0$. Then Theorem 1.3.1 implies that $y^3 + py + q = 0$ has precisely *one* real root. Furthermore, by (1.22), we have

$$\Delta = -4p^3 - 27q^2 < 0.$$

Hence the square root $\sqrt{q^2 + 4p^3/27}$ is real, which means that we can take z_1 to be the unique real cube root. Then $z_1 z_2 = -p/3$ implies that z_2 is also the real cube root. It follows that

$$y_1 = \sqrt[3]{\frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right)} + \sqrt[3]{\frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)}$$

expresses the *real* root of $y^3 + py + q = 0$ in terms of *real* radicals. Furthermore, in the above formulas for y_2 and y_3, we see that $y_3 = \overline{y_2}$, since the cube roots are real and

$\omega^2 = \overline{\omega}$. Thus we have a complete understanding of how Cardan's formulas work when the discriminant is negative.

However, the case when $\Delta > 0$ is very different. Here, $y^3 + py + q = 0$ has three real roots by Theorem 1.3.1. Since

$$\Delta = -4p^3 - 27q^2 > 0,$$

one value of the square root $\sqrt{q^2 + 4p^3/27}$ is

$$\sqrt{q^2 + \frac{4p^3}{27}} = \sqrt{\frac{-\Delta}{27}} = i\sqrt{\frac{\Delta}{27}}.$$

Using this and (1.23), we can write z_1 and z_2 as the cube roots

$$z_1 = \sqrt[3]{\frac{1}{2}\left(-q + i\sqrt{\frac{\Delta}{27}}\right)} \quad \text{and} \quad z_2 = \sqrt[3]{\frac{1}{2}\left(-q - i\sqrt{\frac{\Delta}{27}}\right)}.$$

This shows that z_1 and z_2 are both nonreal complex numbers when $\Delta > 0$. You will prove in Exercise 6 that

$$z_1 z_2 = -\frac{p}{3} \Longrightarrow z_2 = \overline{z_1}. \tag{1.24}$$

Combining (1.24) with Cardan's formulas, we see that when $\Delta > 0$, the roots of $y^3 + py + q$ can be written

$$\begin{aligned} y_1 &= z_1 + \overline{z_1}, \\ y_2 &= \omega z_1 + \omega^2 \overline{z_1}, \\ y_3 &= \omega^2 z_1 + \omega \overline{z_1}. \end{aligned}$$

The root y_1 is real, since it is expressed as the sum of a complex number and its conjugate. Furthermore, using $\omega^2 = \overline{\omega}$, one easily sees that

$$\overline{\omega z_1} = \omega^2 \overline{z_1} \quad \text{and} \quad \overline{\omega^2 z_1} = \omega \overline{z_1},$$

so that y_2 and y_3 are also real, since they too are the sum of a complex number and its conjugate.

Notice that, unlike the case when $\Delta < 0$, we no longer have a canonical choice of z_1—it is just one cube root of the complex number $\frac{1}{2}\left(-q + i\sqrt{\Delta/27}\right)$. Furthermore, we get y_1, y_2, y_3 by taking the three cube roots of this number and adding each to its conjugate. This explains how Cardan's formulas work when $\Delta > 0$.

The puzzle, of course, is that we are using complex numbers to express the real roots of a real polynomial. Historically, this is referred to as the *casus irreducibilis*. We will have more to say about this below.

Example 1.3.2 In 1550, Rafael Bombelli applied Cardan's formulas to the cubic $y^3 - 15y - 4 = 0$. This polynomial has discriminant $\Delta = -4(-15)^3 - 27(-4)^2 =$

$13068 > 0$, so that all three roots are real. Bombelli noted that one root is $y = 4$ and used Cardan's formulas to show that

$$4 = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$$

for appropriate choices of cube roots. To understand this formula, Bombelli noted that $(2+i)^3 = 2+11i$ and $(2-i)^3 = 2-11i$. Hence the cube roots in the above formula are $2+i$ and $2-i$, and their sum is clearly 4.

In Exercise 7 below, you will find the other two roots of the equation and explain how Cardan's formulas give these two roots. ◁▷

From the point of view of Cardan's formulas, complex numbers are unavoidable when $\Delta > 0$. But is it possible that there are other ways of expressing the roots which only involve real radicals? In Chapter 8 we will prove that when an irreducible cubic has real roots, the answer to this question is no—using Galois theory, we will see that complex numbers are in fact unavoidable when trying to express the roots of an irreducible cubic with positive discriminant in terms of radicals.

B. Trigonometric Solution of the Cubic. Although complex numbers are unavoidable when applying Cardan's formulas to a cubic with positive discriminant, there is a purely "real" solution provided we use trigonometric functions rather than radicals. This is the *trigonometric solution of the cubic*, due to Viète.

Our starting point is the trigonometric identity

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta,$$

which you will prove in Exercise 8. If we write this as $4\cos^3\theta - 3\cos\theta - \cos(3\theta) = 0$, then $t_1 = \cos\theta$ is a root of the cubic equation $t^3 - 3t - \cos(3\theta) = 0$. However, replacing θ with $\theta + \frac{2\pi}{3}$ gives the same cubic polynomial, since $\cos(3(\theta + \frac{2\pi}{3})) = \cos(3\theta)$. It follows that $t_2 = \cos(\theta + \frac{2\pi}{3})$ is another root of $4t^3 - 3t - \cos(3\theta) = 0$, and similarly, $t_3 = \cos(\theta + \frac{4\pi}{3})$ is also a root.

In Exercise 9 you will show that the discriminant of $4t^3 - 3t - \cos(3\theta)$ is given by $\frac{27}{16}\sin^2(3\theta)$. This is zero if and only if $\sin(3\theta) = 0$, which in turn is equivalent to $\cos(3\theta) = \pm 1$. Thus $\cos(3\theta) \neq \pm 1$ implies that $4t^3 - 3t - \cos(3\theta)$ has roots

$$t_1 = \cos\theta, \quad t_2 = \cos\left(\theta + \tfrac{2\pi}{3}\right), \quad t_3 = \cos\left(\theta + \tfrac{4\pi}{3}\right). \tag{1.25}$$

Hence $4t^3 - 3t - \cos(3\theta) = 0$ is a cubic equation with known roots. Viète's insight was that by a simple change of variable, we can use this to solve *any* cubic equation with positive discriminant. Here is his result.

Theorem 1.3.3 *Let $y^3 + py + q = 0$ be a cubic equation with real coefficients and positive discriminant. Then $p < 0$, and the roots of the equation are*

$$y_1 = 2\sqrt{\frac{-p}{3}}\cos\theta,\ y_2 = 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \tfrac{2\pi}{3}\right),\ \textit{and}\ y_3 = 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \tfrac{4\pi}{3}\right),$$

where θ is the real number defined by

$$\theta = \frac{1}{3}\cos^{-1}\left(\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right).$$

Proof: You will prove this in Exercise 10. ∎

In Exercise 11 you will explore how this relates to Cardan's formulas.

Historical Notes

When Cardan wrote *Ars Magna* in 1545, he and his contemporaries wanted to find real roots of cubic equations. In fact, they worked almost exclusively with positive roots, although they were aware of the existence of negative roots, which Cardan called "false" or "fictitious." However, Cardan does use complex numbers in Chapter XXXVII when he considers the problem of dividing 10 into two parts so that their product is 40. In modern notation this gives the equations $x+y=10$ and $xy=40$. Eliminating y, we get the quadratic equation

$$x^2 - 10x + 40 = 0$$

with roots $5 \pm i\sqrt{15}$. After deriving this solution, Cardan says "Putting aside the mental tortures involved, multiply $5+\sqrt{-15}$ by $5-\sqrt{-15}$, making $25-(-15)\ldots$ Hence this product is 40." Cardan's conclusion is that "This truly is sophisticated" [2, pp. 219–220].

Cardan was also aware of Theorem 1.3.1, though he stated it in very different terms. As an example of a cubic with three real roots, he considers $x^3+9=12x$, for which he gives the "true" (i.e., positive) solutions 3 and $\sqrt{5\frac{1}{4}} - 1\frac{1}{2}$ and the "false" (i.e., negative) solution $-\sqrt{5\frac{1}{4}} - 1\frac{1}{2}$.

However, Cardan never applies his formulas to cubics like $x^3+9=12x$. He only considers cases where there is one real root, which can be expressed in terms of real radicals. Yet Cardan must have known that complex numbers appear in the radicals when the discriminant is positive. This is the *casus irreducibilis* ("irreducible case") mentioned above. According to [1], Tartaglia was also aware of the *casus irreducibilis*, and in fact delayed publication of his results because he was so troubled by it. This is part of the reason why Cardan's work appeared first.

One of the first people to comment directly on the *casus irreducibilis* was Rafael Bombelli. In his book *L'algebra*, written around 1550 but not published until 1572, he treats this case in detail, including the formula

$$4 = \sqrt[3]{2+11i} + \sqrt[3]{2-11i} \tag{1.26}$$

from Example 1.3.2. There we saw how Bombelli explained this formula by showing that $2+11i=(2+i)^3$, so that (1.26) reduces to $4=(2+i)+(2-i)$. Bombelli was pleased with this calculation and commented that

> At first, the thing [equation (1.26)] seemed to me to be based more on sophism than on truth, but I searched until I found a proof.

In working out this solution, Bombelli was the first to give systematic rules for adding and multiplying complex numbers. Exercise 12 will discuss another example of complex cube roots taken from Bombelli's work.

The moral is that cubic equations forced mathematicians to confront complex numbers. For quadratic equations, one could pretend that complex solutions don't exist. But for a cubic with real roots, we've seen that Cardan's formula must involve complex numbers. So it is impossible to ignore complex numbers in this case. See the books [1] and [3] for more background and discussion on the discovery of complex numbers.

We should also say a few words about Viète's trigonometric solution of the cubic. Once we realize that $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ gives a cubic equation with $\cos\theta$ as a root, proving Theorem 1.3.3 is not that difficult. Viète was well aware of such identities. For example, in 1593, Adrianus Romanus (also called Adriaen van Roomen) posed the problem of finding a root of the equation

$$
\begin{aligned}
A = {} & x^{45}-45x^{43}+945x^{41}-12300x^{39}+111150x^{37}-740259x^{35}\\
& +3764565x^{33}-14945040x^{31}+46955700x^{29}-117679100x^{27}\\
& +236030652x^{25}-37865800x^{23}+483841800x^{21}-488494125x^{19}\\
& +384942237x^{17}-232676280x^{15}+105306075x^{13}-3451207x^{11}\\
& +7811375x^{9}-1138500x^{7}+95634x^{5}-3795x^{3}+45x,
\end{aligned}
\tag{1.27}
$$

where

$$
A = \sqrt{\frac{7}{4}-\sqrt{\frac{5}{16}}-\sqrt{\frac{15}{8}-\sqrt{\frac{45}{64}}}}. \tag{1.28}
$$

Viète solved this equation by noting that $2\sin(45\alpha)$ can be expressed as a polynomial of degree 45 in $2\sin\alpha$ whose coefficients match the right-hand side of (1.27). It follows that if $A = 2\sin(45\alpha)$, then $x = 2\sin\alpha$ is a root.

Viète also realized that (1.28) can be written

$$A = 2\sin(\pi/15) = 2\sin(45\cdot\pi/675),$$

which easily implies that one root of (1.27) is $x = 2\sin(\pi/675)$. Using the trick of (1.25), we get the 44 additional solutions

$$x = 2\sin\left(\frac{\pi}{675} + j\frac{2\pi}{45}\right), \quad j = 1,\ldots,44.$$

Viète listed only 23 roots, since he (like Cardan) wanted positive solutions. Nevertheless, Viète's insight is impressive, and his solution of (1.27) makes it clear how he was able to find the trigonometric solution of the cubic.

Exercises for Section 1.3

Exercise 1. Let $f(y) = y^3 + py + q = (y - y_1)(y - y_2)(y - y_3)$, and set

$$\Delta = (y_1 - y_2)^2 (y_1 - y_3)^2 (y_2 - y_3)^2.$$

The goal of this exercise is to give a different proof of (1.22).

(a) Use the product rule to show that $f'(y_1) = (y_1 - y_2)(y_1 - y_3)$, where f' denotes the derivative of f. Also derive similar formulas for $f'(y_2)$ and $f'(y_3)$.

(b) Conclude that $\Delta = -f'(y_1)f'(y_2)f'(y_3)$. Be sure to explain where the minus sign comes from.

(c) The quadratic $f'(y) = 3y^2 + p$ factors as $f'(y) = 3(y - \alpha)(y - \beta)$, where $\alpha = \sqrt{-p/3}$ and $\beta = -\sqrt{-p/3}$ (when $p > 0$, we let $\sqrt{-p/3} = i\sqrt{p/3}$). Prove that $\Delta = -27 f(\alpha) f(\beta)$.

(d) Use $f(y) = y^3 + py + q$ and $\alpha = \sqrt{-p/3}$ to show that

$$f(\alpha) = (\sqrt{-p/3})^3 + p\sqrt{-p/3} + q = (2/3)p\sqrt{-p/3} + q.$$

Similarly, show that $f(\beta) = -(2/3)p\sqrt{-p/3} + q$.

(e) By combining parts (c) and (d), conclude that $\Delta = -4p^3 - 27q^2$.

Exercise 2. Let $f(y) = y^3 + py + q$. The purpose of Exercises 2–5 is to prove Theorem 1.3.1 geometrically using curve graphing techniques. The proof breaks up into three cases corresponding to $p > 0$, $p = 0$, and $p < 0$. This exercise will consider the case $p > 0$.

(a) Explain why $\Delta < 0$.

(b) Analyze the sign of $f'(y)$, and show that $f(y)$ is always increasing.

(c) Explain why $f(y)$ has only one real root.

Exercise 3. Next, consider the case $p = 0$.

(a) Explain why $\Delta < 0$.

(b) Explain why $f(y)$ has only one real root.

Exercise 4. Finally, consider the case $p < 0$. In this case, $f'(y) = 3y^2 + p$ has roots $\alpha = \sqrt{-p/3}$ and $\beta = -\sqrt{-p/3}$, which are real and distinct.

(a) Show that the graph of $f(y)$ has a local minimum at α and a local maximum at β. Thus $f(\alpha)$ is a local minimum value and $f(\beta)$ is a local maximum value. Also show that $f(\alpha) < f(\beta)$.

(b) Explain why $f(y)$ has three real roots if $f(\alpha)$ and $f(\beta)$ have opposite signs and has one real root if they have the same sign. Illustrate your answer with a drawing of the three cases that can occur.

(c) Conclude that $f(y)$ has three real roots if and only if $f(\alpha)f(\beta) < 0$.

(d) Finally, use part (c) of Exercise 1 to show that the roots are all real if and only if $\Delta > 0$.

Exercise 5. Explain how Theorem 1.3.1 follows from Exercises 2, 3, and 4. Notice that the quantity $f(\alpha)f(\beta)$, which appeared earlier in part (c) of Exercise 1, arises naturally in Exercise 4.

Exercise 6. Prove (1.24).

Exercise 7. Example 1.3.2 expressed the root $y = 4$ of $y^3 - 15y - 4$ in terms of Cardan's formulas. Find the other two roots, and explain how Cardan's formulas give these roots.

Exercise 8. Derive the trigonometric identity $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ using $\cos(x + y) = \cos x \cos y - \sin x \sin y$ and $\cos^2\theta + \sin^2\theta = 1$.

Exercise 9. When divided by 4, $4t^3 - 3t - \cos(3\theta)$ gives $t^3 - \frac{3}{4}t - \frac{1}{4}\cos(3\theta)$, which is monic. Show that the discriminant of this polynomial is $\frac{27}{16}\sin^2(3\theta)$.

Exercise 10. The goal of this exercise is to prove Theorem 1.3.3. Let $y^3 + py + q = 0$ be a cubic equation with positive discriminant. Consider the substitution $y = \lambda t$, which transforms the given equation into $\lambda^3 t^3 + \lambda pt + q = 0$.

(a) Show that Exercises 2 and 3 imply that $p < 0$.

(b) The equation $\lambda^3 t^3 + \lambda pt + q = 0$ can be written as

$$4t^3 - \left(\frac{-4p}{\lambda^2}\right)t - \left(\frac{-4q}{\lambda^3}\right) = 0.$$

Show that this coincides with $4t^3 - 3t - \cos(3\theta) = 0$ if and only if

$$\lambda = 2\sqrt{\frac{-p}{3}} \quad \text{and} \quad \cos(3\theta) = \frac{3\sqrt{3}q}{2p\sqrt{-p}}.$$

Note that $\sqrt{-p}$ is real and nonzero by part (a).

(c) Use $\Delta = -(4p^3 + 27q^2) > 0$ to prove that

$$\left|\frac{3\sqrt{3}q}{2p\sqrt{-p}}\right| < 1.$$

(d) Explain how part (c) implies that the second equation of part (b) can be solved for θ. Also show that $\Delta > 0$ implies that $\cos(3\theta) \neq \pm 1$.

(e) By (1.25), $t_1 = \cos\theta$, $t_2 = \cos\left(\theta + \frac{2\pi}{3}\right)$, and $t_3 = \cos\left(\theta + \frac{4\pi}{3}\right)$ are the three roots of $\lambda^3 t^3 + \lambda pt + q = 0$. Then show that the theorem follows by transforming this back to $y = \lambda t$ via part (b).

Exercise 11. Consider the equation $4t^3 - 3t - \cos(3\theta) = 0$, where $\cos(3\theta) \neq \pm 1$. In (1.25), we expressed the roots in terms of trigonometric functions. In this exercise, you will study what happens when we use Cardan's formulas.

(a) Show that Cardan's formulas give the root

$$t_1 = \frac{1}{2}\sqrt[3]{\cos(3\theta) + i\sin(3\theta)} + \frac{1}{2}\sqrt[3]{\cos(3\theta) - i\sin(3\theta)}.$$

(b) Explain why $\frac{1}{2}e^{i\theta} = \frac{1}{2}(\cos\theta + i\sin\theta)$ is a value of $\frac{1}{2}\sqrt[3]{\cos(3\theta) + i\sin(3\theta)}$, and use this to show that t_1 is just $\cos\theta$.

(c) Similarly, show that Cardan's formulas also give the roots t_2 and t_3 as predicted by (1.25).

Exercise 12. Example 1.3.2 discusses Bombelli's discovery that $\sqrt[3]{2 + 11i} = 2 + i$. But not all cube roots can be expressed so simply. This exercise will show that $\sqrt[3]{4 + \sqrt{11}i}$ is not of the form $a + b\sqrt{11}i$ for $a, b \in \mathbb{Z}$.

(a) Suppose that $4 + \sqrt{11}i = (a + b\sqrt{11}i)^3$ for some $a, b \in \mathbb{Z}$. Show that this implies that $4 = a^3 - 33ab^2$ and $1 = 3a^2b - 11b^3$.

(b) Show that the equations of part (a) imply that $b = \pm 1$ and $a|4$. Conclude that the equation $4 + \sqrt{11}i = (a + b\sqrt{11}i)^3$ has no solutions with $a, b \in \mathbb{Z}$.

(c) Find a cubic polynomial of the form $x^3 + px + q$ with $p, q \in \mathbb{Z}$ which has the number $\sqrt[3]{4 + \sqrt{11}i} + \sqrt[3]{4 - \sqrt{11}i}$ as a root.

In contrast to $\sqrt[3]{2 + 11i} = 2 + i$, Bombelli was not certain that $\sqrt[3]{4 + \sqrt{11}i}$ was a complex number. He calls $\sqrt[3]{4 + \sqrt{11}i}$ "another sort of cubic radical." Bombelli never deals with this radical by itself, but rather considers the sum $\sqrt[3]{4+\sqrt{11}i}+\sqrt[3]{4-\sqrt{11}i}$, which is a root of the cubic equation found in part (c).

Exercise 13. Suppose that a quartic polynomial $f = x^4+bx^3+cx^2+dx+e$ in $\mathbb{R}[x]$ has distinct roots $x_1, x_2, x_3, x_4 \in \mathbb{C}$. The *discriminant* of f is defined by the equation

$$\Delta = (x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2(x_2-x_3)^2(x_2-x_4)^2(x_3-x_4)^2.$$

The theory developed in Chapter 2 will imply that $\Delta \in \mathbb{R}$, and $\Delta \neq 0$, since the x_i are distinct. Adapt the proof of Theorem 1.3.1 to show that

$$\Delta < 0 \iff x^4+bx^3+cx^2+dx+e = 0 \text{ has exactly two real roots.}$$

Exercise 14. In Section 1.1, we discussed the equation $x^3+2x^2+10x = 20$ considered by Fibonacci.

(a) Show that this equation has precisely one real root. This is the root Fibonacci approximated so well.

(b) Use Cardan's formulas and a calculator to work out numerically the three roots of this polynomial.

Exercise 15. Use a calculator and Theorem 1.3.3 to compute the roots of the cubic equation $y^3-7y+3=0$ to eight decimal places of accuracy.

REFERENCES

1. I. G. Bashmakova and G. S. Smirnova, *The Beginnings and Evolution of Algebra*, English translation by A. Shenitzer, MAA, Washington, DC, 1999.

2. G. Cardan, *Ars Magna*, Johann Petrieus, Nürnberg, 1545. English translation *The Great Art* by T. R. Witmer, MIT Press, Cambridge, MA, 1968.

3. B. Mazur, *Imagining Numbers (particularly the square root of minus fifteen)*, Farrar Straus Giroux, New York, 2003.