CHAPTER 1

INTRODUCTION

We assume that the reader has a basic knowledge of graph theory. Concepts and notation not defined in this book will be used as in standard textbooks on graph theory.

1.1 GRAPHS

By a **graph** G we mean a finite undirected graph without loops, and possibly with multiple edges. The **vertex set** and the **edge set** of G are denoted by $V(G)$ and $E(G)$, respectively. Every edge of G is **incident** with two distinct vertices and the edge is then said to **join** these two vertices. For a vertex $x \in V(G)$, denote by $E_G(x)$ the set of all edges of G that are incident with x. Two distinct edges of G incident to the same vertex will be called **adjacent edges**. Furthermore, for $X, Y \subseteq V(G)$, let $E_G(X, Y)$ denote the set of all edges of G joining a vertex of X with a vertex of Y. When $Y = V(G) \setminus X$, then $E_G(X, Y)$ is called the **coboundary** of X in G and is denoted by $\partial_G(X)$. We write $E_G(x, y)$ instead of $E_G(\{x\}, \{y\})$. Two distinct vertices x, y of G with $E_G(x, y) \neq \emptyset$ are called **adjacent vertices** and **neighbors**. The set of all neighbors of x in G is denoted by $N_G(x)$, i.e., $N_G(x) = \{y \in V(G) \mid E_G(x, y) \neq \emptyset\}$.

Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture, *First Edition*. By M. Stiebitz, D. Scheide, B. Toft, and L. M. Favrholdt

The **degree** of the vertex $x \in V(G)$ is $d_G(x) = |E_G(x)|$, and the **multiplicity** of two distinct vertices $x, y \in V(G)$ is $\mu_G(x,y) = |E_G(x,y)|$. Let $\delta(G)$, $\Delta(G)$ and $\mu(G)$ denote the **minimum degree**, the **maximum degree**, and the **maximum multiplicity** of G, respectively. A graph G is called **simple** if $\mu(G) \leq 1$. A graph G is called **regular** and **r-regular** if $\delta(G) = \Delta(G) = r$, where $r \geq 0$ is an integer.

For a **subgraph** H of G, we write briefly $H \subseteq G$. For a graph G and a set $X \subseteq V(G)$, let $G[X]$ denote the subgraph of G **induced** by X, that is, $V(G[X]) = X$ and $E(G[X]) = E_G(X,X)$. Furthermore, let $G - X = G[V(G) \setminus X]$. We write $G - x$ instead of $G - \{x\}$. For $F \subseteq E(G)$, let $G - F$ denote the subgraph H of G with $V(H) = V(G)$ and $E(H) = E(G) \setminus F$. If $F = \{e\}$ is a singleton, we write $G - e$ rather than $G - \{e\}$.

If S is a sequence consisting of edges and vertices of a given graph G, then we denote by $V(S)$, respectively $E(S)$, the set of all elements of $V(G)$, respectively $E(G)$, that belong to the sequence S. Let G be a graph and let $S = (v_0, e_1, v_1, \dots, v_{p-1}, e_p, v_p)$ be a sequence such that $v_0, \dots, v_p$ are distinct vertices of G and $e_1, \dots, e_p$ are edges of G, where we do not assume anything about incidences of the elements in S. For a vertex $v_i \in V(S)$, we define $Sv_i = (v_0, e_1, \dots, e_i, v_i)$ and $v_iS = (v_i, e_{i+1}, \dots, v_p)$.

By a path, a cycle, or a tree we usually mean a graph or subgraph rather than a sequence consisting of edges and vertices. There are only two exceptions to this: The Kierstead path (Sect. 3.1) and the Tashkinov tree (Sect. 5.1) are considered as sequences. If P is a **path** of length $p \geq 0$ with $V(P) = \{v_0, \dots, v_p\}$ and $E(P) = \{e_1, \dots e_p\}$ such that $e_i \in E_P(v_{i-1}, v_i)$ for $1 \leq i \leq p$, then we also write $P = \mathrm{Path}(v_0, e_1, v_1, \dots, e_p, v_p)$. Note that $\mathrm{Path}(v_0, e_1, \dots, e_p, v_p) = \mathrm{Path}(v_p, e_p, \dots, e_1, v_0)$; but the corresponding sequences are distinct, provided that $p \geq 1$. The vertices $v_0, \dots, v_p$ of the path P are distinct and we say that v_0, v_p are the **endvertices** of the path P and that P is a path **joining** the vertices v_0 and v_p.

The **complete graph** on n vertices is denoted K_n, while the **cycle** on n vertices is denoted C_n. A K_3 (isomorphic to C_3) is often called a **triangle**. A cycle C_n is **odd** or **even**, depending on whether its order n is odd or even. As usual, the number of vertices of a graph is its **order**.

If G and H are two graphs with the same vertex set such that every pair (x, y) of distinct vertices satisfies $\mu_H(x,y) = 0$ if $\mu_G(x,y) = 0$ and $\mu_H(x,y) \geq \mu_G(x,y)$ otherwise, then H is called an **inflation graph** of G. If H is an inflation graph of G such that $\mu_H(x,y) = t\,\mu_G(x,y)$ for every $x, y \in V(G)$, where $t \geq 1$ is an integer, then we simply write $H = tG$ and call H a **multiple** of G. An inflation graph of a cycle C_n with $n \geq 3$ is also called a **ring graph**.

As usual, we shall write $\lfloor x \rfloor$ for the **lower integer part** of the real number x, and $\lceil x \rceil$ for the **upper integer part** of x.

1.2 COLORING PRELIMINARIES

A **k-edge-coloring** of a graph G is a map $\varphi : E(G) \to \{1, \dots, k\}$ that assigns to every edge e of G a color $\varphi(e) \in \{1, \dots, k\}$ such that no two adjacent edges of G

receive the same color. Denote by $\mathcal{C}^k(G)$ the set of all k-edge-colorings of G. The **chromatic index or edge chromatic number** $\chi'(G)$ is the least integer $k \geq 0$ such that $\mathcal{C}^k(G) \neq \emptyset$.

Let φ be a k-edge-coloring of G. For a color $\alpha \in \{1, \ldots, k\}$, the edge set $E_{\varphi,\alpha} = \{e \in E(G) \,|\, \varphi(e) = \alpha\}$ is called a **color class**. Then every vertex x of G is incident with at most one edge of $E_{\varphi,\alpha}$, i.e., $E_{\varphi,\alpha}$ is a **matching** of G (possibly empty). So, there is a one-to-one correspondence between k-edge-colorings of G and partitions $(E_1, \ldots, E_k)$ of $E(G)$ into k matchings (color classes); and the chromatic index of G is the minimum number of matchings into which the edge set of G can be partitioned.

A simple, but very useful recoloring technique for the edge color problem was developed by König [174], Shannon [284], and Vizing [297, 298]. Suppose that G is a graph and φ is a k-edge-coloring of G. To obtain a new coloring, choose two distinct colors α, β and consider the subgraph H with $V(H) = V(G)$ and $E(H) = E_{\varphi,\alpha} \cup E_{\varphi,\beta}$. Then every component of H is either a path or an even cycle and we refer to such a component as an **(α, β)-chain** of G with respect to φ. Now choose an arbitrary (α, β)-chain C of G with respect to φ. If we interchange the colors α and β on C, then we obtain a new k-edge-coloring φ' of G satisfying

$$\varphi'(e) = \begin{cases} \varphi(e) & \text{if } e \in E(G) \setminus E(C), \\ \beta & \text{if } e \in E(C) \text{ and } \varphi(e) = \alpha, \\ \alpha & \text{if } e \in E(C) \text{ and } \varphi(e) = \beta. \end{cases}$$

In what follows, we briefly say that the coloring φ' is obtained from φ by **recoloring** C, and we write $\varphi' = \varphi/C$. This operation is called a **Kempe change**. Furthermore, we say that an (α, β)-chain C has **endvertices** x, y if C is a path joining x and y.

Let G be a graph, let $F \subseteq E(G)$ be an edge set, and let $\varphi \in \mathcal{C}^k(G - F)$ be a coloring for some integer $k \geq 0$. For a vertex $v \in V(G)$, define the two color sets

$$\varphi(v) = \{\varphi(e) |\, e \in E_G(v) \setminus F\} \text{ and } \overline{\varphi}(v) = \{1, \ldots, k\} \setminus \varphi(v).$$

We call $\varphi(v)$ the set of **colors present** at v and $\overline{\varphi}(v)$ the set of **colors missing** at v. Evidently, we have

$$|\overline{\varphi}(v)| = k - d_G(v) + |E_G(v) \cap F|. \tag{1.1}$$

For a color $\alpha \in \{1, \ldots, k\}$, let $m_{\varphi,\alpha}$ denote the number of vertices $v \in V(G)$ such that $\alpha \in \overline{\varphi}(v)$. Since the color class $E_{\varphi,\alpha}$ is a matching of G, we have $m_{\varphi,\alpha} = |V(G)| - 2|E_{\varphi,\alpha}|$. Consequently, we obtain

$$m_{\varphi,\alpha} \equiv |V(G)| \bmod 2 \tag{1.2}$$

for all colors $\alpha \in \{1, \ldots, k\}$ and, moreover, from (1.1)

$$\sum_{v \in V(G)} (k - d_G(v)) + 2|F| = \sum_{v \in V(G)} |\overline{\varphi}(v)| = \sum_{\alpha=1}^{k} m_{\varphi,\alpha}. \tag{1.3}$$

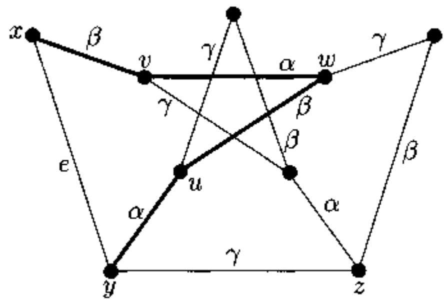

Figure 1.1 A graph G with a chain $P_x(\alpha, \beta, \varphi)$ (bold edges).

For a vertex set $X \subseteq V(G)$, we define

$$\overline{\varphi}(X) = \bigcup_{v \in X} \overline{\varphi}(v).$$

If $X = \{v_1, \dots v_p\}$, then we also write $\overline{\varphi}(v_1, \dots, v_p)$ instead of $\overline{\varphi}(X)$. The set X is called **elementary** with respect to φ if $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. The set X is called **closed** with respect to φ if for every colored edge $f \in \partial_G(X)$ the color $\varphi(f)$ is present at every vertex of X, i.e., $\varphi(f) \in \varphi(v)$ for every $v \in X$. Finally, the set X is called **strongly closed** with respect to φ if X is closed with respect to φ and $\varphi(f) \neq \varphi(f')$ for every two distinct colored edges $f, f' \in \partial_G(X)$.

Let $\alpha, \beta \in \{1, \dots, k\}$ be two distinct colors. For a vertex v of G, we denote by $P_v(\alpha, \beta, \varphi)$ the unique (α, β)-chain of $G - F$ with respect to φ that contains the vertex v. If it is clear that we refer to the coloring φ, then we just write $P_v(\alpha, \beta)$ rather than $P_v(\alpha, \beta, \varphi)$. If exactly one of the two colors α or β belongs to $\overline{\varphi}(v)$, then $P_v(\alpha, \beta)$ is a path, where one endvertex is v and the other endvertex is some vertex $u \neq v$ such that $\overline{\varphi}(u)$ contains either α or β. For two vertices $v, w \in V(G)$, the two chains $P_v(\alpha, \beta)$ and $P_w(\alpha, \beta)$ are either equal or vertex disjoint. For the coloring $\varphi' = \varphi / P_v(\alpha, \beta, \varphi)$, we have $\varphi' \in \mathcal{C}^k(G - F)$, since $\varphi \in \mathcal{C}^k(G - F)$. Furthermore, if x is not an endvertex of $P_v(\alpha, \beta, \varphi)$, then $\overline{\varphi}'(x) = \overline{\varphi}(x)$, else $\overline{\varphi}'(x)$ is obtained from $\overline{\varphi}(x)$ by interchanging α and β. We shall use these simple facts quite often without explicit mention.

Figure 1.1 shows the graph G obtained from the Petersen graph (see Fig. 1.2) by deleting one vertex as well as a 3-edge-coloring φ of $G - e$, where the three colors are α, β, γ. The graph G itself has chromatic index 4. Furthermore, $\overline{\varphi}(x) = \{\alpha, \gamma\}, \overline{\varphi}(y) = \{\beta\}, \overline{\varphi}(u) = \overline{\varphi}(v) = \overline{\varphi}(w) = \overline{\varphi}(z) = \emptyset$, and $P_x(\alpha, \beta)$ is a path of length 4 with vertex set $X = \{x, v, w, u, y\}$. The set X is elementary with respect to φ, but not closed.

If the condition $\varphi(e) \neq \varphi(e')$ for any two adjacent edges $e, e' \in E(G)$ is dropped from the definition of edge coloring, then φ is called an **improper edge coloring** of G. Accordingly, the term **proper edge coloring** is used in the graph theory literature

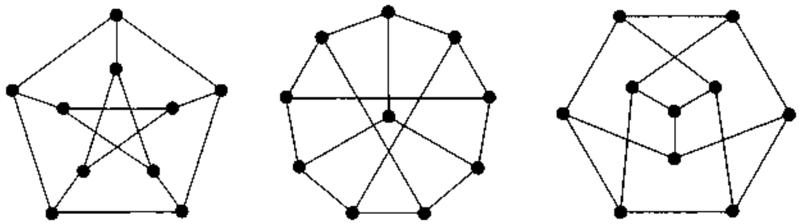

Figure 1.2 Three drawings of the Petersen graph.

in order to emphasize that the condition holds. In Chap. 8 we shall discuss some results about improper edge colorings.

A k-**vertex-coloring** of a graph G is an assignment of k colors to its vertices in such a way that adjacent vertices receive different colors. The minimum k for which a graph G has a k-vertex-coloring is called its **chromatic number**, denoted $\chi(G)$. A trivial lower bound for the chromatic number of a graph G is its **clique number** $\omega(G)$, that is, the maximum p for which G contains a complete graph on p vertices as a subgraph. On the other hand, every graph G satisfies $\chi(G) \leq \Delta(G) + 1$.

A graph with chromatic number at most k is also called a k-**partite graph**, where a 2-partite graph is also called a **bipartite graph**. The **complete bipartite graph** on two sets of n and m vertices is denoted $K_{n,m}$.

Clearly, a graph has chromatic number 0 if it has no vertices and chromatic number 1 if it has vertices, but no edges. A well-known result of König [174] states that a graph G is bipartite if and only if G contains no odd cycle.

For a graph G, the **line graph** of G, denoted $L(G)$, is the simple graph whose vertex set corresponds to the edge set of G and in which two vertices are adjacent if the corresponding edges of G have a common endvertex. Evidently, every edge coloring of G is a vertex coloring of $L(G)$, and vice versa; in particular, $\chi'(G) = \chi(L(G))$.

1.3 CRITICAL GRAPHS

By a **graph parameter** we mean a function ρ that assigns to each graph G a real number $\rho(G)$ such that $\rho(G) = \rho(H)$ whenever G and H are isomorphic graphs. A graph parameter ρ is called **monotone** if $\rho(H) \leq \rho(G)$ whenever H is a subgraph of G. Clearly, the set of all graph parameters form a real vector space with respect to the addition of functions and the multiplication of a function by a real number. Let ρ and ρ' be two graph parameters. If $\rho'(G) = c$ for every graph G, then instead of $\rho + \rho'$ we also write $\rho + c$. If $\rho(G) \leq \rho'(G)$ holds for every graph G, then we say that ρ' is an **upper bound** for ρ and ρ is a **lower bound** for ρ'.

Criticality is a general concept in graph theory and can be defined with respect to various graph parameters. The importance of the notion of criticality is that problems for graphs in general may often be reduced to problems for critical graphs whose structure is more restricted. Critical graphs (w. r. t. the chromatic number) were first defined and used by Dirac in 1951, in his Ph.D. thesis [71] and in Dirac [72].

Let ρ be a monotone graph parameter. A graph G is called **ρ-critical** if $\rho(H) < \rho(G)$ for every proper subgraph H of G. We say that $e \in E(G)$ is a **ρ-critical edge** if $\rho(G - e) < \rho(G)$. Evidently, in a ρ-critical graph every edge is ρ-critical.

Proposition 1.1 *Let ρ and ρ' be two monotone graph parameters. Then the following statements hold:*

(a) *Every graph G contains a ρ-critical subgraph H with $\rho(H) = \rho(G)$.*

(b) *If every ρ-critical graph H satisfies $\rho(H) \leq \rho'(H)$, then $\rho(G) \leq \rho'(G)$ for all graphs G.*

Proof: Since ρ is monotone, every graph G contains a minimal subgraph H with $\rho(H) = \rho(G)$. Obviously, H is ρ-critical. This proves (a). For the proof of (b), let G be an arbitrary graph. By (a), G contains a ρ-critical subgraph H with $\rho(H) = \rho(G)$. Then $\rho(H) \leq \rho'(H)$; and since ρ' is monotone, we obtain $\rho(G) = \rho(H) \leq \rho'(H) \leq \rho'(G)$. ∎

For convenience, we allow a graph G to **be empty**[1], i.e., $V(G) = E(G) = \emptyset$. In this case we also write $G = \emptyset$. For the empty graph G, define $\chi'(G) = \delta(G) = \Delta(G) = \mu(G) = 0$. So the empty graph is r-regular only for $r = 0$. If ρ is a monotone graph parameter, then the empty graph is ρ-critical; it is the only ρ-critical graph H with $\rho(H) = \rho(\emptyset)$.

By a **critical graph** we always mean a χ'-critical graph, and by a **critical edge** we always mean a χ'-critical edge. Clearly, an edge e of G is critical if and only if $\chi'(G - e) = \chi'(G) - 1$. For $k = 1, 2$, a graph G with $\chi'(G) = k$ is critical if and only if G is connected and has exactly k edges. It is also easy to show that a graph G with $\chi'(G) = 3$ is critical if and only if G is an odd cycle or G is a connected graph consisting of three edges that are all incident to the same vertex x of G. Furthermore, if a graph G satisfies $\chi'(G) \geq \Delta(G) + 1$, then $|V(G)| \geq 3$ and $\Delta(G) \geq 2$.

1.4 LOWER BOUNDS AND ELEMENTARY GRAPHS

Since, in an edge coloring, no two edges incident to the same vertex can have the same color, every graph G satisfies $\chi'(G) \geq \Delta(G)$. For a **fat triangle** of multiplicity μ, that is, a graph $T = \mu K_3$ consisting of three vertices pairwise joined by μ parallel edges, we obtain $\chi'(T) = 3\mu$ and $\Delta(T) = 2\mu$. This shows that the gap between the chromatic index and the maximum degree can be arbitrarily large.

Apart from the maximum degree there is another trivial lower bound for the chromatic index, sometimes called the **density** of G, written as $\mathcal{W}(G)$. Consider a k-edge-coloring φ of G and a subgraph H of G with $|V(H)| \geq 2$. For every color α, the restricted color class $E_\alpha = E_{\varphi,\alpha} \cap E(H)$ is a matching of H. Consequently, $|E_\alpha| \leq \lfloor |V(H)|/2 \rfloor$ for every color α and, therefore, $|E(H)| \leq k\lfloor |V(H)|/2 \rfloor$.

[1] The empty graph is also called the null-graph.

This observation leads to the following definition of a parameter for graphs G with $|V(G)| \geq 2$, namely,

$$w(G) = \max_{H \subseteq G, |V(H)| \geq 2} \left\lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\rceil. \tag{1.4}$$

For a graph G with $|V(G)| \leq 1$, define $w(G) = 0$. Then, clearly, w is monotone and every graph G satisfies

$$\chi'(G) \geq w(G). \tag{1.5}$$

As Scheinerman and Ullman [276] proved, the maximum in (1.4) can always be achieved for an induced subgraph H of G having odd order, provided that $|V(G)| \geq 3$. To see this, suppose that the maximum in (1.4) is achieved for a graph $H \subseteq G$ having even order. If $|V(H)| = 2$ this gives $w(G) = |E(H)|$, and hence $w(G) = \lceil 2|E(H')|/(|V(H')| - 1) \rceil$ for any subgraph H' of G with three vertices such that $H \subseteq H'$. Otherwise, $|V(H)| \geq 4$ and we argue as follows. Let v be a vertex of minimum degree in H and let $H' = H - v$. Then $|V(H')|$ is odd and $d_H(v) \leq 2|E(H)|/|V(H)| \leq \lceil 2|E(H)|/|V(H)| \rceil = w(G)$ and, therefore,

$$w(G) \geq \left\lceil \frac{2|E(H')|}{|V(H')| - 1} \right\rceil = \left\lceil \frac{2(|E(H)| - d_H(v))}{|V(H)| - 2} \right\rceil \geq \left\lceil \frac{2|E(H)|}{|V(H)|} \right\rceil = w(G),$$

which proves the claim. Hence, for a graph G with $|V(G)| \geq 3$, we have

$$w(G) = \max_{X \subseteq V(G), |X| \geq 3 \text{ odd}} \left\lceil \frac{2|E(G[X])|}{|X| - 1} \right\rceil. \tag{1.6}$$

Clearly, any graph G with $|V(G)| \leq 2$ satisfies $w(G) = \Delta(G) = \chi'(G)$. To see that the gap between $w(G)$ and $\chi'(G)$ can be arbitrarily large, consider the simple graph $G = K_{1,n}$ consisting of $n + 1$ vertices and n edges all incident to the same vertex. Then, for $n \geq 2$, we have $w(K_{1,n}) = 2$ and $\chi'(K_{1,n}) = \Delta(K_{1,n}) = n$. For the Petersen graph P, we have $\chi'(P) = 4$ and $\Delta(P) = w(P) = 3$. However, the situation seems to be different for graphs with $\chi'(G) > \Delta(G) + 1$.

A graph G is called an **elementary graph** if $\chi'(G) = w(G)$. The significance of this equation is that the chromatic index is characterized by a min–max equality. The following conjecture seems to have been thought of first by Goldberg [110, 114] around 1970 and, independently, by Seymour [280, 281].

Conjecture 1.2 (Goldberg [110] 1973, Seymour [281] 1979) *Every graph G such that $\chi'(G) \geq \Delta(G) + 2$ is elementary, i.e., $\chi'(G) = w(G)$.*

In this book, we will refer to Conjecture 1.2 briefly as Goldberg's conjecture. For a proof of this conjecture, it is sufficient to consider critical graphs. To see why, let G be an arbitrary graph with $\chi'(G) \geq \Delta(G) + 2$. Clearly, G contains a critical graph H with $\chi'(H) = \chi'(G)$. This implies that $\chi'(H) \geq \Delta(H) + 2$. If the graph H is known to be elementary, then also G is elementary, since in this case we have $w(G) \leq \chi'(G) = \chi'(H) = w(H) \leq w(G)$ and, therefore, $\chi'(G) = w(G)$.

Proposition 1.3 *Let G be a graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G)$. If G is critical and elementary, then the following statements hold:*

(a) $\chi'(G) = \mathcal{W}(G) = \lceil |E(G)| / \lfloor \frac{1}{2}|V(G)| \rfloor \rceil$ *and* $|V(G)|$ *is odd.*

(b) *For every edge $e \in E(G)$ and every coloring $\varphi \in \mathcal{C}^k(G - e)$, we have $m_{\varphi,\alpha} = 1$ for all colors $\alpha \in \{1, \ldots, k\}$; i.e., the color α is missing at exactly one vertex of G.*

(c) $|E(G)| = k \lfloor \frac{1}{2}|V(G)| \rfloor + 1$.

Proof: Since $\chi'(G) \geq \Delta(G) + 1$, we have $|V(G)| \geq 3$ and $E(G) \neq \emptyset$. Since the graph G is both critical and elementary, every proper subgraph H of G satisfies $\mathcal{W}(H) \leq \chi'(H) < \chi'(G) = \mathcal{W}(G)$. Clearly, this implies that

$$\chi'(G) = \mathcal{W}(G) = \left\lceil \frac{|E(G)|}{\lfloor \frac{1}{2}|V(G)| \rfloor} \right\rceil.$$

Consequently, $|V(G)|$ is odd, since otherwise

$$\chi'(G) = \lceil 2|E(G)|/|V(G)| \rceil \leq \Delta(G),$$

a contradiction. This proves (a). For the proof of (b), let $e \in E(G)$ and $\varphi \in \mathcal{C}^k(G-e)$. By (a) and (1.2), we have $m_{\varphi,\alpha} \equiv |V(G)| \equiv 1 \bmod 2$. Hence, if $m_{\varphi,\alpha} \neq 1$ for some color α, then (1.3) implies that $\sum_{v \in V(G)}(k - d_G(v)) + 2 \geq k + 2$. Since $|V(G)|$ is odd, this yields $\lceil |E(G)| / \lfloor |V(G)|/2 \rfloor \rceil \leq k = \chi'(G) - 1$, a contradiction to (a). For the proof of (c), choose an edge $e \in E(G)$. Since G is critical, there is a coloring $\varphi \in \mathcal{C}^k(G-e)$. Then we deduce from (b) and (1.3) that $\sum_{v \in V(G)}(k - d_G(v)) + 2 = k$. Since $|V(G)|$ is odd, this implies that $|E(G)| = k\lfloor |V(G)|/2 \rfloor + 1$. This completes the proof. ■

The following result shows that elementary graphs and elementary sets are closely related to each other. This result is implicitly contained in the papers by Andersen [5] and Goldberg [114].

Theorem 1.4 *Let G be a graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G)$. If G is critical, then the following conditions are equivalent:*

(a) *G is elementary.*

(b) *For every edge $e \in E(G)$ and every coloring $\varphi \in \mathcal{C}^k(G - e)$, the set $V(G)$ is elementary with respect to φ.*

(c) *There is an edge $e \in E(G)$ and a coloring $\varphi \in \mathcal{C}^k(G - e)$ such that $V(G)$ is elementary with respect to φ.*

(d) *There is an edge $e \in E(G)$, a coloring $\varphi \in \mathcal{C}^k(G - e)$ and a set $X \subseteq V(G)$ such that X contains the two endvertices of e and X is elementary as well as strongly closed with respect to φ.*

Proof: That (a) implies (b) follows from Proposition 1.3(b). Evidently, (b) implies (c) and (c) implies (d) with $X = V(G)$. To prove that (d) implies (a), suppose that, for some edge $e \in E(G)$ and some coloring $\varphi \in \mathcal{C}^k(G-e)$, there is a subset X of $V(G)$ such that both endvertices of e are contained in X and X is elementary as well as strongly closed with respect to φ. Let $H = G[X]$ and, for each color $\alpha \in \{1, \dots, k\}$, let $E_\alpha = E_{\varphi,\alpha} \cap E(H)$. Since the edge e is uncolored and both endvertices of e belong to X, the set $\overline{\varphi}(X)$ is nonempty and $|X| \geq 2$.

First, consider an arbitrary color $\alpha \in \overline{\varphi}(X)$. Since X is elementary with respect to φ, color α is missing at exactly one vertex of H. Furthermore, since X is closed, no edge in $\partial_G(X)$ is colored with α. Since E_α is a matching of H, this implies that $|X| = |V(H)|$ is odd and $|E_\alpha| = \lfloor |V(H)|/2 \rfloor$. Now, consider an arbitrary color $\alpha \notin \overline{\varphi}(X)$. Then, clearly, color α is present at every vertex of $X = V(H)$. Since X is strongly closed, at most one edge of $\partial_G(X)$ is colored with α. Since E_α is a matching of H and $|V(H)|$ is odd, this implies that exactly one edge of $\partial_G(X)$ is colored with α and, moreover, $|E_\alpha| = \lfloor |V(H)|/2 \rfloor$, too. This proves that

$$|E(H)| = 1 + k \left\lfloor \frac{1}{2}|V(H)| \right\rfloor.$$

Since H is a subgraph of G with $|V(H)| \geq 2$, we then deduce that

$$w(G) \leq \chi'(G) = k + 1 = \left\lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \right\rceil \leq w(G).$$

Therefore, G is an elementary graph. This shows that (d) implies (a). Hence the proof of Theorem 1.4 is complete. ∎

Combining Proposition 1.3 and Theorem 1.4 together with the equations (1.2) and (1.3), we obtain the following result.

Corollary 1.5 *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)$. If $|V(G)|$ is odd, then $\sum_{v \in V(G)}(k - d_G(v)) + 2 \geq k$, where equality holds if and only if G is elementary. Furthermore, G is elementary if and only if $|V(G)|$ is odd and $\sum_{v \in V(G)}(k - d_G(v)) = k - 2$.*

Since it suffices to verify Goldberg's conjecture for the class of critical graphs, it follows from Corollary 1.5 that Goldberg's conjecture is equivalent to the following conjecture.

Conjecture 1.6 (Critical Multigraph Conjecture) *Every critical graph G with $\chi'(G) \geq \Delta(G) + 2$ is of odd order and satisfies*

$$2|E(G)| = (\chi'(G) - 1)(|V(G)| - 1) + 2.$$

We conclude this section with some basic facts about elementary sets that are useful for our further investigations.

Proposition 1.7 *Let G be a graph with $\Delta(G) = \Delta \geq 2$, let $e \in E_G(x, y)$ be an edge, and let $\varphi \in \mathcal{C}^k(G - e)$ be a coloring for an integer $k \geq \Delta$. If $X \subseteq V(G)$ is an elementary set with respect to φ such that both endvertices of e are contained in X, then the following statements hold:*

(a) $|X| \leq \frac{|\overline{\varphi}(X)|-2}{k-\Delta} \leq \frac{k-2}{k-\Delta}$, *provided that* $k \geq \Delta + 1$.

(b) $\sum_{v \in X} d_G(v) \geq k(|X| - 1) + 2$.

(c) *Suppose that*

$$k + 1 > \frac{m}{m-1}\Delta + \frac{m-3}{m-1}$$

for an integer $m \geq 3$. Then $|X| \leq m - 1$ and, moreover, $|\overline{\varphi}(X)| \geq \Delta + 1$, provided that $|X| = m - 1$.

Proof: Since the set X is elementary with respect to $\varphi \in \mathcal{C}^k(G - e)$, we deduce that

$$\sum_{v \in X} |\overline{\varphi}(v)| = |\overline{\varphi}(X)| \leq k.$$

The edge $e \in E_G(x, y)$ being uncolored, for a vertex $v \in V(G)$, we have

$$|\overline{\varphi}(v)| = \begin{cases} k - d_G(v) + 1 & \text{if } v \in \{x, y\}, \\ k - d_G(v) & \text{otherwise.} \end{cases}$$

Then, since $x, y \in X$, we obtain

$$2 + |X|(k - \Delta) \leq 2 + \sum_{v \in X}(k - d_G(v)) = 2 + k|X| - \sum_{v \in X} d_G(v) = |\overline{\varphi}(X)| \leq k,$$

which implies (a) and (b). To prove (c), we first deduce from the hypothesis that $k - \Delta > (\Delta - 2)/(m - 1)$. Since $\Delta \geq 2$ and $m \geq 3$, this implies that $k \geq \Delta + 1$. By (a), we then obtain

$$|X| \leq \frac{k-2}{k-\Delta} = 1 + \frac{\Delta - 2}{k - \Delta} < 1 + (m-1) = m$$

and, therefore, $|X| \leq m - 1$. Now, assume that $|X| = m - 1$. Then, by (a), we have $|\overline{\varphi}(X)| \geq (k - \Delta)|X| + 2 = (k - \Delta)(m - 1) + 2 > \Delta$ and, therefore, $|\overline{\varphi}(X)| \geq \Delta + 1$. This completes the proof of (c). ■

Let G be a critical graph with maximum degree Δ, and let $m \geq 3$ be an integer. Suppose that G satisfies

$$\chi'(G) > \frac{m}{m-1}\Delta + \frac{m-3}{m-1}.$$

Then $\chi'(G) = k + 1 \geq \Delta + 2 \geq 4$. Consequently, Goldberg's conjecture (that G is elementary) implies that $|V(G)| \leq m-1$ (Theorem 1.4, Proposition 1.7), respectively

$|V(G)| \leq m - 2$ if m is odd (Proposition 1.3). Thus the following conjecture, first posed by Jakobsen [156], may be seen as a weaker form of Goldberg's conjecture.

Conjecture 1.8 (Jakobsen [156] 1975) *Let G be a critical graph, and let*

$$\chi'(G) > \frac{m}{m-1}\,\Delta(G) + \frac{m-3}{m-1}$$

for an odd integer $m \geq 3$. Then $|V(G)| \leq m - 2$.

Thus for fixed $\Delta(G)$, or for fixed $\chi'(G)$, there are only finitely many critical graphs G with $\chi'(G) \geq \Delta(G) + 2$, assuming Goldberg's conjecture is true.

A **fat odd cycle**, i.e., a graph $G = \mu C_m$ for an odd integer $m \geq 3$, has for $\mu \equiv 1 \bmod (m-1)/2$

$$\chi'(G) = w(G) = \frac{m}{m-1}\,\Delta(G) + \frac{m-3}{m-1}$$

and it is critical with m vertices. Thus Conjecture 1.8 is in this sense best possible. To see why G is elementary and critical, note first that $|E(G)| = m\mu$, $\Delta(G) = 2\mu(G) = 2\mu$ and, by (1.5),

$$\chi'(G) \geq w(G) \geq \left\lceil \frac{2|E(G)|}{m-1} \right\rceil = \left\lceil \frac{2m\mu}{m-1} \right\rceil = \left\lceil \frac{m\Delta(G)}{m-1} \right\rceil.$$

By assumption, there are integers $\ell \geq 1$ and $p \geq 0$ such that $m = 2\ell + 1$ and $\mu = 1 + p\ell$. Then $|E(G)| = m\mu = \ell(2p\ell + p + 2) + 1$ and there is an integer $k \geq 2$ such that

$$k + 1 = \left\lceil \frac{m\Delta(G)}{m-1} \right\rceil = \frac{m}{m-1}\,\Delta(G) + \frac{m-3}{m-1} = 2p\ell + p + 3.$$

If e is an arbitrary edge of G, then it is easy to check that the remaining edge set of G can be partitioned into $k = 2p\ell + p + 2$ matchings, each having ℓ edges implying that $\chi'(G - e) \leq k$. Since G is connected and $\chi'(G) \geq w(G) \geq k + 1$, this implies that G is critical and $\chi'(G) = w(G) = k + 1$.

1.5 UPPER BOUNDS AND COLORING ALGORITHMS

The **Edge Color Problem** asks for an optimal edge coloring of a graph G, that is, an edge coloring with $\chi'(G)$ colors. Holyer [150] proved that the determination of the chromatic index is NP-hard, even for 3-regular simple graphs, where the chromatic index is either 3 or 4. Hence it is reasonable to search for upper bounds for the chromatic index, in particular for those bounds that are efficiently realized by a coloring algorithm. A graph parameter ρ is said to be an **efficiently realizable** upper bound for χ' if there exists an algorithm that computes, for every graph $G = (V, E)$, an edge coloring using at most $\rho(G)$ colors, where the algorithm has time complexity

t bounded from above by a polynomial in $|V|$ and $|E|$, that is, $t(G) \leq p(|E|, |V|)$ for some polynomial $p = p(x, y)$ over the real numbers in two variables.

Note that edge coloring algorithms may have an execution time polynomial in $|E|$, but being only pseudopolynomial in the number of bits needed to describe the graph, since edge multiplicities may be encoded as binary numbers, and the size of the input graph therefore may be of order less than the order of E.

A typical algorithm colors the edges of the input graph sequentially. Such an algorithm first fixes an edge order of the input graph, either an arbitrary order or one that satisfies a certain property. The core of the algorithm is a subroutine **Ext** that extends a given partial coloring of the input graph. The input of **Ext** is a tuple (G, e, x, y, k, φ), where G is the graph consisting of all edges that are already colored as well as the next uncolored edge $e \in E_G(x, y)$ with respect to the given edge order, and a coloring $\varphi \in \mathcal{C}^k(G - e)$. The output of **Ext** is a pair (k', φ'), where $k' \in \{k, k+1\}$ and $\varphi' \in \mathcal{C}^{k'}(G)$.

Now, to explain how **Ext** works, a well-defined set $\mathcal{O}(G, e, \varphi)$ of so-called **test objects** will be introduced. A test object $T \in \mathcal{O}(G, e, \varphi)$ is usually a labeled subgraph of G that fulfills a certain property with respect to the uncolored edge e and the coloring $\varphi \in \mathcal{C}^k(G - e)$. In most cases, we start with the test object that only consists of the uncolored edge e. When a test object $T \in \mathcal{O}(G, e, \varphi)$ is investigated, then, using an exhaustive case distinction, three basic outcomes are possible. The first possible outcome is that the vertex set $V(T)$ is not elementary with respect to φ; i.e., a color $\alpha \in \{1, \ldots, k\}$ is missing at two distinct vertices of T with respect to φ. In this case **Ext** returns (k, φ'), where the coloring $\varphi' \in \mathcal{C}^k(G)$ is obtained from φ by Kempe changes, possibly involving more than one pair of colors in a small number of successive Kempe changes. The second possible outcome is that the vertex set $V(T)$ is elementary with respect to φ, but T cannot be enlarged. In that case e is colored with a new color resulting in a coloring $\varphi' \in \mathcal{C}^{k+1}(G)$. Then **Ext** returns $(k+1, \varphi')$. The third possible outcome is that the vertex set $V(T)$ is elementary with respect to φ, but T can be enlarged. Then an exhaustive search for a larger test object is needed. This process eventually terminates, because for sufficiently large test objects $T \in \mathcal{O}(G, e, \varphi)$, one of the first two cases has to be applicable.

To ensure that the subroutine **Ext**, and hence the algorithm, works correctly, we need a statement about the test objects of the following type.

(1) *Let G be a graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G)$, let $e \in E_G(x, y)$ be a critical edge of G, and let $\varphi \in \mathcal{C}^k(G - e)$ be a coloring. Then the vertex set of each test object $T \in \mathcal{O}(G, e, \varphi)$ is elementary with respect to φ.*

This statement is equivalent to the statement that if $\varphi \in \mathcal{C}^k(G - e)$ is a coloring and the vertex set of a test object $T \in \mathcal{O}(G, e, \varphi)$ is not elementary with respect to φ, then $\chi'(G) \leq k$, i.e., there is a coloring $\varphi' \in \mathcal{C}^k(G)$. For the correctness of the algorithm it is, however, important that the proof of (1) is constructive and can be transformed into an efficient procedure for obtaining such a coloring $\varphi' \in \mathcal{C}^k(G)$.

To control the number of colors used by a coloring algorithm of the above type, we need some further information about **maximal test objects**, which means test objects

$T \in \mathcal{O}(G, e, \varphi)$ that cannot be extended to some larger test object $T' \in \mathcal{O}(G, e, \varphi)$. For the proof of Goldberg's conjecture a statement of the following type would be sufficient.

(2) *Let G be a graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$, let $e \in E_G(x, y)$ be a critical edge of G, and let $\varphi \in \mathcal{C}^k(G - e)$ be a coloring. Then the vertex set of each maximal test object $T \in \mathcal{O}(G, e, \varphi)$ is both elementary and strongly closed with respect to φ.*

Suppose our test objects satisfies (1) and (2) and we start our coloring algorithm with $k = \Delta(G) + 1$ colors. If the algorithm never uses a new color, then $\chi'(G) \leq \Delta(G) + 1$. Otherwise, let us consider the last call of **Ext** where we use a new color. The input is a tuple $(G', e, x, y, k, \varphi)$, where G' is a subgraph of G, $e \in E_{G'}(x, y)$, and $\varphi \in \mathcal{C}^k(G' - e)$. Since **Ext** returns a coloring $\varphi' \in \mathcal{C}^{k+1}(G')$, there exist a maximal test object $T \in \mathcal{O}(G', e, \varphi)$ such that $X = V(T)$ is elementary and strongly closed both with respect to φ. Clearly, the coloring algorithm terminates with a $(k + 1)$-edge-coloring of G implying $\chi'(G) \leq k + 1$. Now, let H be the subgraph of G with $V(H) = X$ and $E(H) = E(G[X]) \cap E(G')$. Then $E(H)$ consists of the uncolored edge e and all edges of G that are already colored and have both endvertices in X. Since X is elementary and strongly closed both with respect to $\varphi \in \mathcal{C}^k(G' - e)$, it then follows that $|X| = |V(H)| \geq 3$ is odd and $|E(H)| = 1 + k\lfloor|V(H)|/2\rfloor$ (see the proof of Theorem 1.4, the part where we show that (d) implies (a)). Consequently, we have $w(G) \geq w(H) \geq \lceil|E(H)|/(\lfloor|V(H)|/2\rfloor)\rceil \geq k + 1 \geq \chi'(G) \geq w(G)$ and, therefore, $\chi'(G) = w(G)$. Hence our algorithm colors the edges of G with at most $\max\{\Delta(G) + 1, w(G)\}$ colors.

Classical kinds of test objects are the fans, first used by Shannon [284] and by Vizing [297], the critical chains introduced independently by Andersen [5] and by Goldberg [111, 114], and the Kierstead paths introduced by Kierstead [166]. A more recent kind of test objects, namely Tashkinov trees, were invented by Tashkinov [291]. All these kinds of test objects satisfy (1), but up to now no test objects that fulfill both conditions (1) and (2) are known. A possible way out of this situation is to modify the subroutine Ext and to add further heuristics before using a new color. If the vertex set X of a maximal test object $T \in \mathcal{O}(G, e, \varphi)$ is both elementary and strongly closed with respect to φ, then we just color e with a new color. However, if X is elementary, but not strongly closed with respect to φ, it might be reasonable to use a small number of Kempe changes to obtain a better test object $T' \in \mathcal{O}(G, e', \varphi')$ and to continue with T' instead of T. We shall use this approach to get some partial results related to Goldberg's conjecture.

One obvious way to find an edge coloring of an arbitrary graph G with at least one edge is the following **greedy algorithm**: Starting from a fixed edge order $e_1, \ldots, e_m$ of G, we consider the edges in turn and color each edge e_i with the smallest positive integer not already used to color any adjacent edge of e_i among $e_1, \ldots, e_{i-1}$. Since no edge is adjacent to more than $2(\Delta(G) - 1)$ other edges, this simple greedy algorithm never uses more that $2\Delta(G) - 1$ colors. Hence, every graph G with $E(G) \neq \emptyset$ satisfies $\chi'(G) \leq 2\Delta(G) - 1$. Observe that this greedy strategy is the simplest version of a coloring algorithm that fits into our general approach; there is only one test object in $\mathcal{O}(G, e, \varphi)$, namely the graph consisting of the uncolored edge e and its two endvertices. As an immediate consequence, we obtain that 2Δ is an efficiently realizable upper bound for χ' (including the case $E(G) = \emptyset$). Since Δ is a lower bound for χ', this implies that $2\chi'$ is an efficiently realizable upper bound for χ'. Goldberg's conjecture supports the following suggestion by Hochbaum, Nishizeki, and Shmoys [146].

Conjecture 1.9 $\chi' + 1$ *is an efficiently realizable upper bound for* χ'.

The upper bound $2\Delta - 1$ on the number of colors used by the greedy algorithm is rather generous, and in most graphs there will be scope for an improvement of this bound by choosing a particularly suitable edge order to start with. Let us say that an **edge order** of a graph G is of **depth** p if each edge in this order is preceded by fewer than p of its adjacent edges. Clearly, if we start the greedy algorithm with an edge order of depth p, then the algorithm terminates with a p-edge-coloring. The least number $p \geq 1$ such that G has an edge order of depth p is called the **coloring index** $\mathrm{col}'(G)$ of G. Observe that the coloring index of a graph is nothing else than the so-called coloring number of its line graph. Obviously, every graph G with at least one edge satisfies $\mathrm{col}'(G) \leq 2\Delta(G) - 1$. For an edgeless graph G, we have $\mathrm{col}'(G) = 1$. It is also known (see, e.g., Jensen and Toft [158]), that an edge order $e_1, \dots, e_m$ of depth $\mathrm{col}'(G)$ can be obtained by letting e_i be an edge having a minimum number of adjacent edges in the subgraph $G_i = G - \{e_{i+1}, \dots, e_m\}$ for $i = m, m-1, \dots, 1$, where $G_m = G$. Hence, col' is an efficiently realizable upper bound for χ', obviously the best upper bound that can be realized by the greedy algorithm.

Finally, we discuss some implementation details. The time complexity t of our coloring algorithms has the form $t = t_1 + |E|t_2$, where t_1 is the time complexity for computing the required edge order of the input graph $G = (V, E)$ and t_2 is the (worst case) time complexity for one call of the subroutine Ext.

The running time t_2 depends on the manner in which the partial coloring is stored. As long as we are satisfied with an overall running time t that is polynomial in $|E|$ and $|V|$, one can use the approach by Hochbaum, Nishizeki, and Shmoys [146]. The idea is to combine the standard **incidence lists** for the vertices with the **same-color lists** for the colors. An edge $e \in E_G(u, v)$ receiving color α is stored in the two incidence lists for u and v, and in addition to that also in the same-color list for the color α. The elements in the corresponding three lists are linked to each other by pointers. Furthermore, a list of all uncolored edges is stored.

For the number of colors k, we may assume that $k = O(\Delta)$, where $\Delta = \Delta(G)$. Then, as explained in Hochbaum et al. [146], each set $\overline{\varphi}(x)$ can be found in time $O(\Delta)$ and, therefore, one can decide in time $O(\Delta)$ whether two vertices have a common missing color. Furthermore, it takes time $O(|V|)$ to find an (α, β)-chain $P = P_x(\alpha, \beta, \varphi)$. The colors of P can be interchanged in time $O(|V|)$, and updating the same-color list for the coloring $\varphi' = \varphi/P$ can be carried out in time $O(|V| + \Delta)$.

1.6 NOTES

Edge colorings of graphs were first considered in two short papers by Tait [290] published in the same proceedings between 1878 and 1880. Tait proved a theorem relating face colorings and edge colorings of plane graphs, i.e., graphs embedded in the plane or sphere. Tait's theorem deals with 3-regular graphs, which are also referred to as **cubic** graphs. A **cut-edge** or a **cut-vertex** of a graph is an edge or vertex whose deletion increases the number of components. A graph without cut-edges is also said to be a **bridgeless** graph. Tait's theorem says that if G is a bridgeless cubic plane (simple) graph, then G admits a 3-edge-coloring if and only if the faces of G can be colored with four colors such that adjacent faces receive different colors. Tait's result implies that the following three statements are equivalent.

(A) The faces of any bridgeless plane graph can be colored with four colors such that any two adjacent faces get different colors.

(B) Every bridgeless cubic planar simple graph G satisfies $\chi'(G) = 3$.

(C) Every bridgeless cubic planar graph G satisfies $\chi'(G) = 3$.

Tait did not prove any of the statements in his papers since he did not consider this necessary because of an already existing proof of (A) by Kempe [165].

The Four - Color Problem was first mentioned in writing in a letter from A. De Morgan to W. R. Hamilton, written in 1852 on the same day as De Morgan first heard about the problem from his student Frederic Guthrie, who had the problem from his brother Francis Guthrie. A proposed solution of the problem by Kempe [165] stood for more than a decade until it was refuted by Heawood [133]. Heawood proved, using Kempe's method, the Five - Color Theorem for planar graphs.

Statement (A) is equivalent to the same statement ($\tilde{\text{A}}$) with the words *faces* replaced by *vertices*, as already observed by A. B. Kempe:

($\tilde{\text{A}}$) The vertices of any planar simple graph can be colored with four colors such that any two adjacent vertices get different colors.

It was however first with the famous paper of Brooks [33] that vertex coloring of general graphs became a topic of study. Brooks [33] proved that the complete graphs and the odd cycles are the only connected simple graphs whose chromatic number is larger than their maximum degree.

Even if Kempe's 1879 paper contained a serious flaw, it contained the idea of recoloring a connected component in the subgraph spanned by two colors, a so-called **Kempe chain**, by simply interchanging the two colors on the vertices of the component (we consider here vertex colorings rather than face colorings). This idea has since been a main tool in graph coloring theory, and, as explained in Sect. 1.2, also for edge colorings. Some recent results about Kempe changes and Kempe equivalence of edge colorings can be found in references [11, 219, 225].

The Four - Color Theorem was proved by K. Appel, W. Haken, and J. Koch [9, 10] in 1977, and later by Robertson, Sanders, Seymour, and Thomas [258] with an improved proof, essentially using the same approach as Appel, Haken, and Koch, but the proof is shorter and clearer, avoiding the problematic details of the original proof. In this way the Four - Color Problem has become the Four - Color Theorem.

In the 1890s there was some confusion about Tait's theorem. Some believed that Tait's theorem asserted that every bridgeless cubic simple graph is 3-edge-colorable. This motivated Petersen [242] to present, as a counterexample, the graph that has became famous as the Petersen graph (see Fig. 1.2).

Petersen [241], collaborating with James Joseph Sylvester, was the first mathematician who studied the problem of factorizing graphs in a general context. One of his fundamental results says that every bridgeless cubic graph G has a **perfect matching** M, i.e., $M \subseteq E(G)$ and every vertex x of G is incident with exactly one edge of M. That Petersen's result implies that every cubic graph has chromatic index 3 or 4 was pointed out by Sainte-Laguë [261] in 1926 (without a precise argument). In particular, the Petersen graph has chromatic index 4. We shall apply the elegant argument by Naserasr and Škrekovski [234] to prove the following slightly stronger statement.

(a) *Let P^* be the Petersen graph with one vertex deleted. Then* $\chi'(P^*) = 4$.

Proof of (a) *:* Obviously, $\chi'(P^*) \leq 4$. Now, suppose there is a 3-edge-coloring of

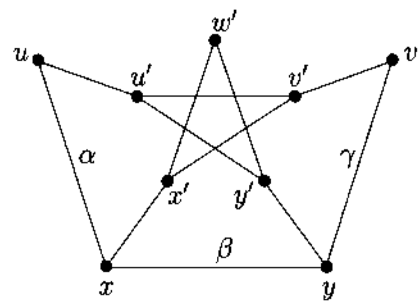

Figure 1.3 The graph P^*.

P^*. Let α, β, γ be the colors of the edges ux, xy, yv, respectively (see Fig. 1.3). Then α and γ may be equal, but $\beta \neq \alpha, \gamma$. Obviously, at each vertex of degree 3 each color must appear. Since β cannot appear on xx' or yy', color β appears on two distinct edges of the inner cycle $C' = (x', v', u', y', w', x')$, one of which must be $x'v'$ or $y'u'$. The same argument works for the colors α and γ. Since C' has only five edges, this implies that $\alpha = \gamma$. But then α has to appear on $u'v'$ and on two more edges of C', a contradiction. ■

If we delete an arbitrary edge of P^*, then it is easy to show that the resulting subgraph has a 3-edge-coloring. Hence P^* is a critical graph.

The basic problem in the theory of graph factorization is decomposing a regular graph into other regular graphs on the same set of vertices. An **r-factor** of an arbitrary graph G is a **spanning subgraph** H of G, i.e., $H \subseteq G$ and $V(H) = V(G)$, such that H is r-regular. Evidently, a graph has a 1-factor if and only if it has a perfect matching. Another result of Petersen's fundamental paper [241] is his even factor theorem, that every $2r$-regular graph has a 2-factor. The statement that every bridgeless cubic graph has a 1-factor should perhaps also be formulated as a 2-factor theorem. As pointed out by Hanson, Loten, and Toft [129], every $(2r + 1)$-regular graph with at most $2r$ bridges has a 2-factor, thus the general Petersen theorem is about 2-factors rather than 1-factors.

As we know, every graph G satisfies $\Delta(G) \leq \chi'(G)$. In 1916 König [174] proved that equality holds for the class of bipartite graphs, and he deduced as a simple corollary that every regular bipartite graph has a perfect matching. König's proof uses induction on the number of edges and a simple recoloring argument. In particular, the proof yields an $O(mn)$ algorithm to find a $\Delta(G)$-edge-coloring for a bipartite graph G with n vertices and m edges. The algorithm is a simplified version of the coloring algorithm described in Sect. 1.5. The only test object in $\mathcal{O}(G, e, \varphi)$ is the graph consisting of the uncolored edge e and the two endvertices x, y of e. Since we have $k = \Delta(G)$ colors, we can chose two colors $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y)$. If $\alpha = \beta$, then we color e with α and continue with the next uncolored edge. Otherwise, we recolor $P = P_x(\alpha, \beta, \varphi)$. Since G contains no odd cycle, y does not belong to P and, therefore, for the coloring $\varphi' = \varphi/P$ we obtain $\alpha \in \overline{\varphi}'(x) \cap \overline{\varphi}'(y)$. Hence we can color e with α and continue with the next uncolored edge.

A simple, self-contained proof of König's theorem that does not use any alternating path argument was given by Rizzi [256].

In 1949 Shannon [284] proved that every graph G satisfies $\chi'(G) \leq \lfloor 3\Delta(G)/2 \rfloor$. The fat triangles are graphs for which Shannon's bound is attained. From Shannon's proof it also follows that $3\Delta/2$ is an efficiently realizable upper bound for χ'.

Shannon's proof uses induction and Kempe changes. He starts by remarking that if $\Delta(G) = 2r$ then G is a subgraph of a $2r$-regular graph. By Petersen's even factor theorem, this graph can be factorized into r 2-factors, each of which has a 3-edge-coloring. This immediately gives the desired result.

For $\Delta(G) = 2r + 1$ Shannon explains that there is a conjecture by Petersen that a $(2r + 1)$-regular bridgeless graph has a 1-factor. If true for graphs with at most one bridge this would imply the result like in the even case. But this conjecture is not true! Shannon did not know, but it is easy to construct a counterexample, namely a $(2r + 1)$-regular bridgeless graph without a 1-factor ($r \geq 2$), using Tutte's 1-factor criterion. For $\Delta(G) = 2r + 1$ one may, however, use a different factorization result. As explained above, it is now known that a $(2r + 1)$-regular graph with at most one bridge has a 2-factor. From this Shannon's theorem follows easily.

Shannon's own proof for the case $\Delta(G) = 2r + 1$ is by induction over the number of vertices. He removes a vertex x of degree $2r+1$ from G, colors $G - x$ by induction using $3r + 1$ colors. Then he colors the edges incident at x one by one, using Kempe changes as the main tool.

It would seem more appropriate to use induction on the number of edges. Let $e \in E_G(x, y)$ be an edge of G and assume that $G - e$ is edge-colored with $3r + 1$ colors by induction. We want to extend this coloring $\varphi \in \mathcal{C}^{3r+1}(G - e)$ by including the remaining edge e.

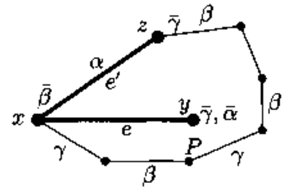

Figure 1.4 Shannon's Kempe change.

Since at most $2r$ edges incident with x are colored, $|\overline{\varphi}(x)| \geq r + 1$. Similarly, $|\overline{\varphi}(y)| \geq r + 1$. If $\overline{\varphi}(x)$ and $\overline{\varphi}(y)$ have a common color, then this color may be given to e and a $(3r + 1)$-edge-coloring of G is obtained. Hence we may assume that $\overline{\varphi}(x)$ and $\overline{\varphi}(y)$ are disjoint. Let α be a color in $\overline{\varphi}(y)$. Then α is present at x, i.e., $\alpha = \varphi(e')$ for an edge $e' \in E_G(x, z)$, where z is different from y. If a color $\beta \in \overline{\varphi}(x)$ is missing at z, then e' may be colored β and e may then be colored α. Hence we may assume that all colors from $\overline{\varphi}(x)$ are present at z. At z there are at most $2r + 1$ colors present, hence at most r colors from $\overline{\varphi}(y)$ belong to $\overline{\varphi}(z)$. This means that there is a color $\gamma \in \overline{\varphi}(y) \cap \overline{\varphi}(z)$. Let β be a color from $\overline{\varphi}(x)$, see Fig. 1.4; note that in Fig. 1.4, and from here on in this book, a bar above a color name means that the color is missing at a particular vertex. Consider the chain $P = P_x(\beta, \gamma)$. If the chain does not end at y, recolor P and color e by γ. If the chain does not end at z, recolor P, recolor e' by γ, and color e by α. One of the two cases must occur.

This proves Shannon's theorem. The proof is essentially Shannon's proof. He formulates it using a $(2n+1) \times (3n+1)$ 0-1-matrix with 1-entries showing the colors possible for the edges from x. He then rearranges columns and rows of the matrix (changes the order of colors and of edges) and makes Kempe changes, corresponding to the arguments above, to see that the matrix may be changed into one with a 1 in all places (i, i) of the matrix, thus showing that it is possible to extend a coloring of $G - x$ to include all edges from x also.

Following Tait, König, and Shannon, the next breakthrough was the theorem of Vizing [297, 298], obtained independently by Gupta [120]. This theorem, from 1964, says that $\chi'(G) \leq \Delta(G) + \mu(G)$ for every graph G.

By Vizing's result, the chromatic index of a simple graph G is either $\Delta(G)$ or $\Delta(G) + 1$. Vizing's proof yields a polynomial-time algorithm that colors the edges of any simple graph G with $\Delta(G) + 1$ colors. On the other hand, Holyer [150] proved that it is NP-complete to decide whether a cubic simple graph has chromatic index 3. These two results answer the edge coloring problem for the class of simple graphs – at least from an algorithmic point of view. Our knowledge about edge coloring of (multi)graphs, however, remains unsatisfactory. Goldberg's conjecture supports the conjecture that there is a polynomial-time algorithm that colors the edges of any graph G with $\chi'(G) + 1$ colors. Furthermore, Goldberg's conjecture implies that the only difficulty in determining the chromatic index of an arbitrary graph G in polynomial time is to distinguish between $\chi'(G) = \Delta(G)$ and $\chi'(G) = \Delta(G) + 1$.