1 From Arithmetic to Algebra

1.1 INTRODUCTION

Numbers are symbols used for counting and measuring. Hindu–Arabic numerals 0, 1, 2, 3, ..., 9 are grouped systematically in units, tens, hundreds, and so on, to solve problems containing numerical information. This is the subject of *Arithmetic*. It also involves an understanding of the structure of the number system and the facility to change numbers from one form to another; for example, the changing of *fractions to decimals* and vice versa. A detailed discussion about the *Real Number System* is given in Chapter 3. However, it would be instructive to *recall some important subsets of real numbers*, known to us.

Numbers, which are used in counting, are called *natural numbers* or *positive integers*. The set of *natural numbers* is denoted by

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

1.2 THE SET OF WHOLE NUMBERS

The set of *natural numbers* along with the number "0" makes the set of whole numbers, denoted by W. Thus,

$$W = \{0, 1, 2, 3, 4, \ldots\}$$

Note: "0" is a whole number but it is not a natural number.

1.3 THE SET OF INTEGERS

All *natural numbers*, their *negatives* and *zero* when considered together, form *the set of integers* denoted by Z. Thus,

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

1.4 THE SET OF RATIONAL NUMBERS

The numbers of the form p/q where p and q are *integers*, and the denominator $q \neq 0$, form the set of *rational numbers*, denoted by Q.

Examples: $\frac{3}{5}$, $\frac{-7}{9}$, $\frac{8}{-15}$, $\frac{0}{15}$, $\frac{9}{1}$, $\frac{-121}{-12}$, $\frac{16}{2}$ and so on, are all rational numbers.

What must you know to learn calculus? 1-(The Language of Algebra)

Introduction to Differential Calculus: Systematic Studies with Engineering Applications for Beginners, First Edition.
Ulrich L. Rohde, G. C. Jain, Ajay K. Poddar, and Ajoy K. Ghosh.

Remarks:

(a) *Zero* is a *rational number*, but *division by zero* is not defined. Thus, $5/0$ and $0/0$ are meaningless expressions.
(b) *All integers are rational numbers*, but the converse is not true.
(c) *Positive rational numbers* are called *fractions.*

Let us discuss more about fractions.

Generally, "*fractions*" are used to represent the parts of a given quantity, under consideration. Thus, 3/7 tells us that a given quantity or an object is divided into seven equal parts and three parts are under consideration. A fraction is also used to express a ratio. Thus, 2:5 is also written as 2/5 and similarly 12:5 is written as 12/5. Since the ratio of two natural numbers can be greater than 1, *all positive rational numbers* are called *fractions*. This definition suggests that fractions could be classified more meaningfully as follows:

- When both numerator and denominator are positive integers, the fraction is known as a *simple, common*, or *vulgar fraction* (Examples: 1/2, 3/5, 9/7).
- A *complex fraction* is one in which either the numerator or the denominator or both are fractions (Examples: 3/(7/5), (5/9)/2, (7/3)/(11/4)).
- If the numerator is less than the denominator, the fraction is called a *proper fraction* (Examples: 4/7, 3/5, 1/4).
- If the numerator is greater than the denominator, the fraction is called an *improper fraction* (Examples: 7/4, 5/3, 9/2).
- A *unit fraction* is a special proper fraction, whose numerator is 1 (Examples: 1/7, 1/100).

Note (1): A fraction is said to be in *lowest terms*, if the only common factor of the numerator and denominator is 1. Thus, 3/4 is in lowest terms, but 6/8 is *not* in lowest terms since 6 and 8 have a common factor 2, other than 1. We say that a/b, $2a/2b$, $3a/3b$, ... all belong to the *same family of fractions*, described by a/b.

In fact, we use the fraction in lowest terms to describe the family of fractions. We define the set of all fractions by $F = \{a/b | a, b \in N\}$

1.5 THE SET OF IRRATIONAL NUMBERS

There are numbers that cannot be expressed in the form p/q, where p and q are integers. They are called *irrational numbers*, and the set is denoted by Q' or Q^c. (More details are given in Chapter 3.)

Examples:

$\sqrt{2}$, $\sqrt{5}$, $6\sqrt{3}$, $7\sqrt{11}$, e, π, 1.101001 ..., 5.71071007100071, ... and so on.

1.6 THE SET OF REAL NUMBERS

The set of *rational numbers* together with the set of *irrational numbers*, form the set of *real numbers*, denoted by R.[(1)]

[(1)] The square roots of negative numbers (i.e., $\sqrt{-1}$ or $\sqrt{-7}$, etc.) do not represent real numbers, hence we shall not discuss about such numbers at this stage.

1.6.1 Arithmetic and Algebra

In arithmetic, there are four fundamental operations, namely, addition, subtraction, multiplication, and division, which are performed on the set of natural numbers to make new numbers, namely, the number zero, negative integers, and rational numbers. For the formation of *irrational numbers,* we have to go beyond the four fundamental arithmetic operations given above.

The subject of algebra involves the study of equations and a number of other problems that developed out of the theory of equations. *It is in connection with the solution of algebraic equations that negative numbers, fractions, and rational numbers were developed. The number "0" could enter the family of numbers only after negative numbers were developed.*

In arithmetic, we deal with numbers that have one (single) definite value. On the other hand, in algebra we deal with symbols such as x, y, z, ..., and so on, which represent variable quantities and those like a, b, c, ..., and so on, which may have any value we chose to assign to them. These symbols represent *variable quantities and are hence called variables*. We may operate with all these symbols as numbers without assigning to them any particular numerical value. Note that, both *numbers and letters are symbols*, which were developed to solve various problems.

In fact, *traditional algebra* is a generalization of *arithmetic*. Hence, the symbols used in *arithmetic* have the *same meaning in algebra*. Thus, we use $+$ (plus for addition), $-$ (minus for subtraction), $\times$ and $\cdot$ (cross and dot for multiplication), / (slash for division), $=$ (equals for equality), $>$ (for greater than), $<$ (for less than) and so on, in algebra also.

Before we enter the true realm of algebra, it is useful to recall *some more subsets of real numbers*, which will be needed in various discussions.

1.7 EVEN AND ODD NUMBERS

Every integer that is *exactly divisible by 2*, is called an *even number*, otherwise it is *odd*. Thus, an even number is of the form $2n$, where n is an integer.

An odd number is of the form $(2n \pm 1)$. If number "a" is even, then $(a \pm 1)$ is odd and vice versa. *It follows that 0 is an even integer.*

1.8 FACTORS

Natural numbers that exactly divide a given integer are called the factors of that number. For example, the factors of 12 are 1, 2, 3, 4, 6, and 12. We also say that 12 is a *multiple* of 1, 2, 3, 4, 6, and 12. Similarly, the factors of 6 are 1, 2, 3, and 6, and the factors of zero are all the natural numbers.

Remark: The number "0" is not a factor of any number.[(2)]

1.9 PRIME AND COMPOSITE NUMBERS

A *natural* number that has *exactly two unique factors* (namely the number itself and 1) is called a *prime number*. A *natural* number that has three or more factors is called a *composite number.*

[(2)] Factors are considered from *natural numbers* only. Besides, note that *division by zero is not permitted in mathematics*. This is explained at the end of this chapter.

Some examples of prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, ..., and so on.

- Each prime number, *except* 2, is *odd.*
- The number 1 is *neither* prime *nor* composite. Six is a composite number since it has *four factors*, namely 1, 2, 3, and 6.

A given *natural* number can be uniquely expressed as a product of primes.

1.10 COPRIME NUMBERS

Two natural numbers are said to be *coprime (or relatively prime) to each other* if they have no common factor except 1. For example, 8 and 25 are coprime to one another. Obviously, all prime numbers are coprime to each other.

Remark: Coprime numbers need not be prime numbers.[(3)]

1.11 HIGHEST COMMON FACTOR (H.C.F.)

The highest common factor (H.C.F.) of two or more *(natural)* numbers is the greatest number which divides each of them exactly. It is also known as the greatest common divisor (G.C.D.). [The H.C.F. of any two prime numbers (or coprime numbers) is always 1.]

1.12 LEAST COMMON MULTIPLE (L.C.M.)

The least common multiple (L.C.M.) of two or more *(natural)* numbers is the *smallest number* which is exactly divisible by each of them. To find the L.C.M. of two (or more) natural numbers, we find prime factors. If two (or more) numbers have a factor in common, we select it once. This is done for each such common factor and the remaining factors from each number are taken as they are. The product of all these factors taken together, gives the L.C.M. of the given numbers.

$$(\textit{Product of two numbers} = \textit{their H.C.F.} \times \textit{their L.C.M.})$$

1.12.1 Continuous Variables and Arbitrary Constants

A changing quantity, usually denoted by a letter (i.e., x, y, z, etc.), which takes on any one of the possible values, in an interval, is called a *variable*. On the other hand, the set of letters a, b, c, d, and so on are used to denote *arbitrary constants*.

In the case of arbitrary constants, though there is no restriction to the numerical values a letter may represent, it is understood that in the same piece of work, it keeps the same value throughout. For example, in the expression, $f(x) = ax^2 + bx + c$, $(0 \leq x \leq 5)$, x is a continuous variable in the interval [0,5] and a, b, c are arbitrary constants. (The concept of an interval is discussed in Chapter 3.)

(3) There is one more term used in connection with prime numbers. A pair of prime numbers which differ by 2, are called *twin-primes* (Examples: 3 and 5, 5 and 7, 11 and 13, 17 and 19, and so on).
Remark: It is proved that the number of primes is infinite, but it is not yet proved whether the number of twin-primes is finite or infinite. This is because of the fact that, so far there is no formula that can generate all primes.

1.13 THE LANGUAGE OF ALGEBRA

Let us now recall the terminology used in algebra:

- an algebraic expression;
- factors, coefficients, index/exponent (or power) of a quantity;
- positive and negative terms;
- like and unlike terms;
- processes involving addition, subtraction, multiplication, and division among algebraic expressions;
- removal and insertion of brackets;
- simplification of an algebraic expression;
- polynomials and related concepts.

It is assumed that all these terms and processes are known to the reader. However, *it is proposed to extend the terminology and concepts related to polynomials,* since the same will be useful to us, in our discussions to follow.

1.13.1 Polynomials

A polynomial in x is an expression of the form

$$p(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are *real numbers* called the *coefficients of* $p(x)$ and n in x^n is a *non-negative integer*.[4]

Usually, we write a polynomial in either *descending powers of x or ascending powers of x*. The form of a ploynomial written in this way is called the standard form. From the definition of a polynomial, it is clear that polynomials are *special types of algebraic expressions* involving only *finite number of terms* and one variable.[5]

1.13.2 Degree of a Polynomial

The exponent, in the highest degree term of a nonzero polynomial is called the *degree of the polynomial*. Thus, if $a_n \neq 0$, then n (in x^n) is *the degree of the polynomial*. In particular, the degree of $3x^5 + 2x^3 - x + 7$ is 5 and the degree of $(3/2)y^3 - \sqrt{2}y - 1$ is 3.[6]

A polynomial having only one term is called "monomial".

[4] *By definition, the power of x in each term* of a polynomial must be *a whole number*. If the power of any term is a negative integer or a fraction, then such an expression is not called a polynomial. Note that the power of x in $p(x)$ can be zero. Such a polynomial is called a *constant polynomial*. Another way for getting a constant polynomial could be to make all the coefficients (except a_0) equal to zero, so that we get $p(x) = a_0$, $a_0 \neq 0$. If each of the coefficients $a_0, a_1, a_2, \ldots, a_n$ in $p(x)$ is zero, then such a polynomial is called the *zero polynomial*.

Remark: The zero polynomial is included in the definition of a polynomial.

[5] A polynomial may have more than one variable but our interest lies in the polynomials involving only one variable.

[6] If $n = 1$, it is a linear expression [Example: $f(x) = 2x + 5$].
If $n = 2$, it is a quadratic expression [Example: $f(x) = x^2 + 3x + 1$].
If $n = 3$, it is a cubic expression [Example: $f(x) = x^3 + 3x^2 + 2x + 1$].
If $n = 4$, it is a quartic or biquadratic expression. If $n = 5$, it is a quintic expression.

1.13.3 The Zero Polynomial

We know that a polynomial having all coefficients as zero is called "*the zero polynomial*". Zero polynomial is *unique* and it is denoted by the symbol "0".[(7)]

The degree of "zero polynomial" is not defined. (Note that, $0 = 0{\cdot}x = 0{\cdot}x^5 \ldots = 0{\cdot}x^{107}$, and so on. These are all zero polynomials and obviously, their degree cannot be defined.) In what follows, *a polynomial will mean a nonzero polynomial (in a single variable) with real coefficients.*

1.13.4 Polynomials Behave Like Integers

Many properties possessed by integers are also possessed by the polynomials. Therefore, *we extend the terminology, used in the algebra of numbers, to the algebra of polynomials*. Thus, if $p(x)$ and $q(x)$ are two polynomials, then the expression $p(x)/q(x)$, where $q(x)$ is a nonzero-polynomial, is called a *rational expression.*[(8)]

A rational expression must be expressed in its *lowest terms*, by canceling the common factors in the numerator and denominator. For this purpose, one has to *learn the process of factorization of a polynomial.*

1.13.4.1 Factors of a Polynomial A polynomial $g(x)$ is called a factor of polynomial $p(x)$, if $g(x)$ divides $p(x)$ exactly; that is, on dividing $p(x)$ by $g(x)$ we get zero as the remainder.

1.13.4.2 Division Algorithm (or Procedure) for Polynomials On dividing a polynomial $p(x)$ by a polynomial $g(x)$, let the quotient be $q(x)$ and the remainder be $r(x)$, then we have $p(x) = g(x) \cdot q(x) + r(x)$, where either $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$.

Remark: When a polynomial $p(x)$ is divided by a linear polynomial $(x - \alpha)$ then the remainder is a constant, which may be zero or nonzero. The value of the remainder can be obtained by applying the *remainder theorem.*

1.13.4.3 Remainder Theorem If a polynomial $p(x)$ is divided by a linear polynomial $(x - \alpha)$, then the remainder is $p(\alpha)$. (This theorem can be easily proved using the division algorithm.)

Remark: If $p(x)$ is divided by $(x + \alpha)$, then the remainder $= p(-\alpha)$. Similarly, when $p(x)$ is divided by $(ax + b)$ then the remainder $= p(-b/a)$.

It is sometimes possible to express a polynomial as a product of other polynomials, each of degree ≥ 1. For example, $x^3 - x^2 + 9x - 9 = (x - 1){\cdot}(x^2 + 9)$ and $3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$.

1.13.5 Value of a Polynomial and Zeros of a Polynomial

We know that for every real value of x, a polynomial has a real value. For example, let $p(x) = 3x^4 - 2x^3 + x + 5$. Then, for $x = 1$, we have $p(1) = 7$ and for $x = 0$, $p(0) = 5$.

(7) The role of zero polynomial can be compared with that of number "0", in arithmetic. The symbol "0", in polynomial algebra represents the zero polynomial whereas in arithmetic it represents the real number "0".

(8) Every polynomial may be regarded as a rational expression but the converse is not true. Note that $(x + 3)/(x - \sqrt{x})$ is *not* a rational expression. It is *an irrational algebraic expression.*

An important aspect of the study of a polynomial is *to determine those values of x for which $p(x) = 0$*. Such values of x are called *zeros of the polynomial $p(x)$*. Consider the quadratic polynomial $q(x) = x^2 - x - 6$. It may be seen that $q(3) = 0$ and $q(-2) = 0$. If $x = a$ is a zero of the polynomial $p(x)$ then *$(x - a)$ is a factor of $p(x)$*. This is known as the *factor theorem* of algebra.

Thus, the factor theorem helps in finding *the linear factors* of a polynomial, provided such factors exist. There are no standard methods available for finding linear factors of polynomials of higher degrees, except in some very special cases.

Every *quadratic polynomial* can have *at most two zeros*, a *cubic polynomial at most three zeros*, and so on. *Some polynomials do not have any real zero. In other words, there may be no real number "x" for which the value of the polynomial becomes zero*. For example, there is no real number "x" for which $x^2 + 3$ will be zero.

Now the following question arises: How do we determine the zeros of a given polynomial $p(x)$?

This leads us to the question: *How to solve the equation $p(x) = 0$?*

1.13.6 Polynomial Equations and Their Solutions (or Roots)

If $p(x)$ is a *quadratic polynomial*, then the equation $p(x) = 0$ is called a quadratic equation. If $p(x)$ is a cubic polynomial, then the corresponding equation $p(x) = 0$ is called a cubic equation, and so on. If the numbers α and β are two zeros of the quadratic polynomial $p(x)$, we say that *α and β are the roots of the corresponding quadratic equation $p(x) = 0$.*[(9)]

Note: The fundamental theorem of algebra states that a nonzero nth degree polynomial equation has at most n roots, in which some roots may be *repeated roots*.

Thus, starting from the concept of an algebraic expression we have revised the concepts of *polynomials, zeros of a polynomial, and the solution of simple polynomial equations.*

1.14 ALGEBRA AS A LANGUAGE FOR THINKING

We know that algebra has a set of rules; but we should not feel satisfied to have learnt algebra merely as a set of rules. It is more important to have some understanding of: *What is algebra all about? How does it grow out of arithmetic? And how is it used to convey concepts of arithmetic?* For instance, the following statements belong to arithmetic:

3^2 is 1 bigger than 2×4

4^2 is 1 bigger than 3×5

5^2 is 1 bigger than 4×6

[(9)] It is easy to solve equations of degree one and two. Thus, we get from a$x + b = 0$, $(a \neq 0)$, $x = -b/a$ and from a $x^2 + bx + c = 0$, $x = \left(-b \pm \sqrt{(b^2 - 4ac)}\right)/2a$. Mathematicians also solved a number of particular equations of degree three but were finding it difficult to express x in terms of general coefficients a, b, c, and d. This problem was finally solved by the Italian mathematician Tartaglia (1499–1557). Later Lodovico Ferari (1522–1565) solved the general fourth degree equation. It seemed almost certain to the mathematicians that the general fifth degree equation and still higher degree equations could also be solved. For 300 years this problem was a classic one. The Frenchman Evariste Galois (1811–1832) showed that the general equation of degree higher than the fourth cannot be solved by algebraic operations including radicals such as square root, cube root, and so on. To establish this result Galois created the Theory of Groups, a subject that is now at the base of modern abstract algebra and that transformed algebra from a series of elementary techniques to a broad, abstract, and basic branch of mathematics. [*Mathematics and the Physical World* by Morris Kline (pp. 71–72).]

These results suggest that "the square of any natural number is 1 bigger than the result of multiplying two numbers of which one is less by one and the other is more by one, than the given number". Thus, we should guess that 87^2 would be 1 bigger than 86×88.

The general result is stated most conveniently in the language of algebra. Let n be any natural number. Then *"the number before n" will be written as (n – 1) and "the number after n" is (n + 1)*. We shall now say, n^2 is 1 bigger than $(n-1)(n+1)$, or, completely in symbols,

$$n^2 = 1 + (n-1)(n+1) \tag{1}$$

Note that, *the above equation holds not only for natural numbers but also for all numbers. It expresses* what we guessed at by looking at particular results in arithmetic. The beauty of algebra lies in its utility. Here, it enables us to prove that our guess is correct. By the usual procedures of algebra, we can simplify the expression on the right-hand side of Equation (1) and see that it equals the left-hand side.

In algebra itself, we often pass from particular results to more general ones. For example, we get from Equation (1)

$$n^2 - 1 = (n-1)(n+1)$$

but we know that $n^2 - 1 = n^2 - 1^2 = (n-1)(n+1)$

In general, we have $a^2 - b^2 = (a-b)(a+b)$

$$\text{or } a^2 = (a-b)(a+b) + b^2 \tag{2}$$

This result is more general than the one expressed by Equation (1).

We can make use of Equation (2) in simple calculations. For example,

$$\begin{aligned} 27^2 &= (27-3)(27+3) + 3^2 \\ &= (24 \times 30) + 9 \\ &= 720 + 9 = 729 \end{aligned}$$

Similarly, $103 \times 97 = (100 + 3)(100 - 3)$

$$\begin{aligned} &= (100)^2 - 3^2 = 10000 - 9 \\ &= 9991 \end{aligned}$$

Now consider the following products:

$$\begin{aligned} (x+3)(x+4) &= x^2 + 7x + 12 \\ &= x^2 + (3+4)x + 3 \cdot 4 \\ (x+5)(x+3) &= x^2 + 8x + 15 \\ &= x^2 + (5+3)x + 5 \cdot 3 \end{aligned}$$

In algebraic symbols, we guess that:

$$(x+a)(x+b) = x^2 + (a+b)x + a \cdot b$$

We can easily prove that our guess is correct. *This type of thinking is very useful in the study of mathematics.*

1.14.1 Algebra is the Best Language for Thinking About Laws

Consider the following table

x:	0	1	2	3	4	5	...
y:	0	2	4	6	8	10	...

We can easily guess the law that lies behind this table. Each number in the bottom row is twice the number that lies above it. The law behind the table is $y = 2x$. In the same way, the law behind the following table is $y = x^2$.

x:	0	1	2	3	4	5	...
y:	0	1	4	9	16	25	...

Incidentally, as a rule, there is little point in putting a law into words. *It is far easier to see what the formula* $y = 2x^2 - 5x + 7$ *means (by preparing a table, as given above)* than to understand the same formula expressed in words.

1.15 INDUCTION

In mathematics, it is not always wise to proceed by analogy and draw conclusions. The process of reasoning from some particular results to general one is called "*induction*".

As we know, *induction begins by observation*. We observe particular result(s) and use our intuition to arrive at a tentative conclusion—*tentative, because it is an educated guess or a conjecture*. It may be true or false. If the *general result is proved by systematic deductive reasoning, then it is accepted as true*. On the other hand, the result will be considered false if we are able to show a *counter example* where the conjecture fails.

Remember that, a conjecture remains *a conjecture no matter how many examples we can find to support it*. The great French mathematician Pierre de Fermat (1601–1665) observed that:

$$(2^{2^1} + 1) = (2^2 + 1) = 5 \text{ is a prime number.}$$
$$(2^{2^2} + 1) = (2^4 + 1) = 17 \text{ is a prime number.}$$
$$(2^{2^3} + 1) = (2^8 + 1) = 257 \text{ is a prime number.}$$

Accordingly, *he conjectured that* $(2^{2^n} + 1)$ *is a prime number* for every natural number n and had challenged the mathematicians of his day to prove otherwise. It was several years later that the Swiss mathematician Leonhard Euler (1707–1783) showed that $(2^{2^5} + 1) = 4{,}294{,}967{,}297$ *is not a prime number since it is divisible by 641*. Another interesting example is the following:

We observe that the *absolute values of the coefficients of various terms* in each of the following factorization are equal to 1

$$x^1 - 1 = (x - 1); \qquad x^2 - 1 = (x - 1)(x + 1)$$
$$x^3 - 1 = (x - 1)(x^2 + x + 1); \qquad x^4 - 1 = (x - 1)(x + 1)(x^2 + 1)$$
$$x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$$

Therefore, it was conjectured that when $x^n - 1$ *(n, a natural number) is expressed into factors, with integer coefficients, none of the coefficients is greater than 1, in absolute value.*

All attempts to prove this general statement failed, until 1941, when a Russian mathematician, V. Ivanov came up with a *counter-example*. He found that one of the factors of $x^{105} - 1$ violates the conjecture. This factor is a polynomial of degree 48, as given below.

$$\left.\begin{array}{l} x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-2x^{41}-x^{40}-x^{39}+x^{36}+x^{35}+x^{34}+x^{33}+x^{32} \\ +x^{31}-x^{28}-x^{26}-x^{24}-x^{22}-x^{20}-x^{17}+x^{16}+x^{15}+x^{14}+x^{12}-x^{9}-x^{8} \\ -2x^{7}-x^{6}-x^{5}+x^{2}+x+1. \end{array}\right\} \text{(10)}$$

In mathematics, we have several such conjectures, which have remained conjectures for lack of proof, even though literally thousands of examples have been found in support of them. *Having employed intuition and arrived at a conjecture, the very difficult task of proving the conjecture begins.* If the conjecture is in the form of a statement, say $P(n)$, involving natural numbers, a method of proof is provided by *the principle of mathematical induction.*(11) [For example, let $P(n)$ represent the statements: (i) $n(n + 1)$ is even or (ii) $3^n > n$, or (iii) $n^3 + n$ is divisible by 3, or (iv) $2^{3n} - 1$ is divisible by 7, etc.]

1.16 AN IMPORTANT RESULT: THE NUMBER OF PRIMES IS INFINITE

There is no known formula that relates successive primes to successive integers. Therefore, it is not possible to use the principle of mathematical induction to prove this result. Yet, algebra provides a simple method to prove it. An indirect approach is needed.(12)

1.17 ALGEBRA AS THE SHORTHAND OF MATHEMATICS

Algebra can be compared to writing shorthand in ordinary life. *It can be used either to make statements or to give instructions in a concise form.* Mathematical statements in ordinary language can be translated into algebraic statements and similarly statements in algebra can be translated into ordinary language. For example, consider the following instructions translated into the language of algebra:

(10) *A Textbook of Mathematics for Classes XI–XII* (Book No. 1, p. 100) NCERT Publication, 1978.

(11) To prove that a statement $P(n)$ is true for all natural numbers, we have to go through two steps.

Step (1): We must verify that $P(1)$ is true.

Step (2): Assuming that $P(k)$ is true for some $k \in N$, we must prove that $P(k + 1)$ is true. For this purpose, we obtain an algebraic expression for $P(k + 1)$ and put it in desired form (if possible) to show that $P(k + 1)$ is true. If this is achieved the result is proved to be true for all n.

Remark: If $P(1)$ is not true, the principle of induction does not apply. [See Example (iii) above.]

(12) We assume that every natural number greater than 1, which is not prime can be represented by a product $P_1, P_2, P_3, P_4, \ldots, P_n$ of prime integers P_i. *This is known as the fundamental theorem of arithmetic.*

Proof: Assume that there is but a finite number of primes and hence a last (largest) prime, P.

Let N be *the product of all primes up to P*: i.e., $N = 2, 3, 5, 7, 11, \ldots, P$. Now consider $N + 1 = (2, 3, 5, 7, 11, \ldots, P) + 1$. Let r be one of the prime numbers 2, 3, 5, …, P. If we divide $(N + 1)$ by r then we will always get the remainder 1. Therefore, $N + 1$ itself must be a prime, which is larger than P. This contradicts the assumption that P is the largest prime. [The largest known prime as of March 2011 is $(2^{43,112,609} - 1)$. It has about 700 digits and a modern computer was used to perform the necessary computation. *Mathematics can be Fun* by Yakov Perelman (p. 288), Mir Publishers, Moscow, 1985.]

Statements in Ordinary Language	Equivalent Statements in the Language of Algebra
(i) Think of a number, add 7 to it and double the result.	$2(x + 7)$
(ii) Choose a number, multiply it by 5, add 2, square this expression, and divide the result by 8.	$(5x + 2)^2/8$

Algebra puts mathematical statements in a small space. *The statement is shorter to write, easier to read, quicker to say, and simpler to understand, than the corresponding sentence in ordinary English.*

Next, though it is easy to say that $2n$ (where n is a natural number) represents an even number, *it is not obvious that the number ($n^2 \pm n$) also represents an even number*. Yet, algebra tells us that $(n^2 \pm n) = n{\cdot}(n \pm 1)$ *must always be even (Why?).*

When we say that algebra is a language, we mean that it has its own words and symbols for expressing what might otherwise be expressed in ordinary language such as French or German. However, we do not look at algebra from this point of view. *For us, algebra is a special kind of language for the following two reasons*:

(a) Algebra is concerned primarily with statement(s) about numbers, items, symbols, or quantities.

(b) The language of algebra uses symbols in place of words.

For example, *to discuss about a class of numbers* (say the class of natural numbers) a mathematician may say: Let "α" be any natural number. Thereafter, in the entire discussion whenever he wishes to refer to *an arbitrary natural number, he will use the letter* α and thus save words and space. Of course, he will have to be careful because any statement(s) he makes about α applies to all natural numbers.

1.18 NOTATIONS IN ALGEBRA

One important difference between the notation of arithmetic and algebra is as follows.

In arithmetic, the product of 3 and 5 is written as 3×5, whereas *in algebra, the product of a and b may be written in any of the forms $a \times b$, $a \cdot b$, or ab*. The form *ab* is the most useful. *In arithmetic, this is not permitted since* 35 means $(3 \times 10) + 5$ and is read as "thirty-five". *Acceptance of such notations in algebra may be treated as a special feature of algebra.*

There are many notations in algebra with which the reader is familiar. For example,

- $a^n = a \cdot \mathrm{a} \cdot \mathrm{a} \cdot \mathrm{a} \cdot \mathrm{a} \ldots$ (n times)
  Example, $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$
  We know that, $a^7/a^4 = a^{7-4} = a^3$
  $\therefore \quad a^n/a^n = a^{n-n} = a^0 = 1$, (provided $a \neq 0$)
  $(a^0 = 1)$, $a \neq 0$, since, 0^0 is not defined.
- Product of first n natural numbers is given by
  $n! = n \cdot (n-1) \cdot (n-2) \cdot (n = 3) \ldots 3 \cdot 2 \cdot 1$
  Example, $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

- Number of permutations (arrangements) of *n different things taken r at a time is given by*

$$^{n}p_r = \frac{n!}{(n-r)!} \quad \text{Example, } ^{5}p_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!}$$

$$= \frac{\text{Product of first 5 natural numbers}}{\text{Product of first 2 natural numbers}}$$

$$= \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1} = 60$$

$$^{n}p_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{\text{Product of first } n \text{ natural numbers}}{\text{Product of first “zero” natural numbers}}$$

$$= n!$$

It follows that $0! = 1$. (This is taken as the definition of 0!)

- Number of combinations of n different items taken r at a time; is given by $^{n}C_r = \frac{n!}{r!(n-r!)}$.

Example: $^{7}C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{(3!)(4!)} = \frac{7.6.5.4.3.2.1}{(3.2.1)(4.3.2.1)} = 35.$

$$^{n}C_r = {}^{n}C_{(n-r)},\ ^{n}C_0 = 1,\ ^{n}C_n = 1$$

Note that in all these notations, *n is a natural number and r is a whole number, with* $n \geq r$.

A beginner may complain about some difficulty in learning the language of algebra. However, *one who has mastered this language of mathematics* and has grasped the ideas and reasoning, *does appreciate the mathematical symbolism.* It is a relatively modern invention and *mathematicians should be complimented for designing “symbols” and “notations”, out of necessity.*

It is important to realize that, while all the languages of the world are quite different from one another, *the language of algebra* is a common one (as is the language of mathematics) and *serves the purpose so well.*

1.19 EXPRESSIONS AND IDENTITIES IN ALGEBRA

The basic function of algebra is to convert expressions into more useful ones. For example, the sum.

$$\sum_{k=1}^{n} k = \sum n = 1 + 2 + 3 + 4 + \ldots\ldots + n$$

was converted by Gauss to the more useful form $(n(n+1)/2)$.

How do you prove this?

The method is not obvious and yet a simple idea does the trick, as follows:

$$\text{Let } S = 1 + 2 + 3 + 4 + \ldots\ldots + (n-1) + n \tag{3}$$

$$\text{Also, } S = n + (n-1) + (n-2) + \ldots\ldots + 2 + 1 \tag{4}$$

Adding corresponding terms in (3) and (4), we get

$$\begin{aligned} 2S &= (n+1)+(n+1)+(n+1)\ldots\ldots(n \text{ times}) \\ &= n(n+1). \text{ Hence, } S = (n(n+1)/2) \end{aligned}$$

The ideas in this proof must arouse some excitement in the reader's mind. *Here, it is important to realize that by simple means we have converted the cumbersome expression to a simpler and readily computable expression.*

Similarly, using algebra, many such useful expressions can be obtained easily. For example,

- $\sum n^2 = 1^2+2^2+3^2+4^2+\ldots+n^2$
  $= \dfrac{n(n+1)(2n+1)}{6}$
- $\sum n^3 = 1^3+2^3+3^3+4^3+\ldots+n^3$
  $= \dfrac{n^2(n+1)^2}{4}$

$\left[\text{Note that } \sum n^3 = \left(\sum n\right)^2\right]$

- $a+ar+ar^2+ar^3+\ldots+ar^{n-1}$

$= \dfrac{a(1-r^n)}{(1-r)}, \ (r<1)$

$= \dfrac{a(r^n-1)}{(r-1)}, \ (r>1)$

It is sometimes possible that a question may have two answers which at first sight appear different, but which are actually the same. This can be checked by simplifying both the algebraic expressions. *An important part of algebra therefore consists in learning how to express any result in the simplest form.* Algebraic identities,[13] and methods available for factorizing polynomials, are helpful in simplifying algebraic expressions.

Some important identities are given below:[14]

- $(x+y)(x-y) = x^2-y^2.$
  Thus, $(a+b)(a-b) = a^2-b^2.$
- $(x+y)^2 = x^2+y^2+2xy.$
  Thus, $a^2+b^2 = (a+b)^2-2ab.$

[13] An algebraic statement expressed in two (or more) forms with a symbol of equality (=) between them is called an algebraic identity. *Obviously, an identity is true for all real value(s) of the variable(s) involved.*

[14] For some purpose, the expression a^2-b^2 is useful as it stands, but for others it may be better to write it in the equivalent form $(a+b)(a-b)$. This statement is also applicable for other expressions to follow.

- $(x-y)^2 = x^2+y^2-2xy.$

  Thus, $a^2+b^2 = (a-b)^2+2ab,$

  $(a+b)^2+(a-b)^2 = 2(a^2+b^2),$

  and $(a+b)^2-(a-b)^2 = 4ab.$

- $(x+y)^3 = x^3+y^3+3xy(x+y).$

  Thus, $a^3+b^3 = (a+b)^3-3ab(a+b),$

  or $a^3+b^3 = (a+b)(a^2-ab+b^2).$

- $(x-y)^3 = x^3-y^3-3xy(x-y).$

  Thus, $a^3-b^3 = (a-b)^3+3ab(a-b),$

  or $a^3-b^3 = (a-b)(a^2+ab+b^2).$

From the expression(s) for $(a\pm b)^3$ and $(a\pm b)^2$ many useful identities can be obtained. For example,

$$\frac{a^3+b^3}{a^2+b^2-ab} = (a+b),\ \frac{a^3-b^3}{a^2+b^2+ab} = (a-b)$$

$$\frac{(a+b)^2+(a-b)^2}{(a^2+b^2)} = \frac{2(a^2+b^2)}{(a^2+b^2)} = 2,$$

$$\frac{(a+b)^2-(a-b)^2}{ab} = \frac{4ab}{ab} = 4.$$

Next, observe that,

- $$\left.\begin{aligned}\left(a+\frac{1}{a}\right)^2 &= a^2+\frac{1}{a^2}+2\\ \left(a-\frac{1}{a}\right)^2 &= a^2+\frac{1}{a^2}-2\end{aligned}\right\}\therefore\left(a+\frac{1}{a}\right)^2-\left(a-\frac{1}{a}\right)^2 = 4$$

- $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$
- $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$
- If $a+b+c = 0$, then $a^3+b^3+c^3 = 3abc.$
- $\dfrac{1}{a\cdot b} = \dfrac{1}{b-a}\left[\dfrac{1}{a}-\dfrac{1}{b}\right]$
- If n is a *natural number*, then the expansion $(x+y)^n = {}^nC_0x^n+{}^nC_1x^{n-1}\cdot y+{}^nC_2x^{n-2}\cdot y^2+\ldots+{}^nC_ny^n$ is *called the binomial expansion*, where x and y can be any *real numbers.*
  - This expansion has $(n+1)$ terms.
  - The general term is of the form ${}^nC_rx^{n-r}y^r$ and *it is the* $(r+1)$th *term* in the expansion.
  - In each term, *the sum of the indices* of x and y, is n.
- If m is *a negative integer* or *a rational number*, then the *binomial expansion* is

$$(b+x)^m = b^m+mb^{m-1}x+\frac{m(m-1)}{2!}b^{m-2}x^2+\ldots + \frac{m(m-1)(m-2)\ldots(m-r+1)}{r!}b^{m-r}x^r+\ldots \tag{5}$$

provided $|x|<b$

Remark (1): Note that the coefficients m, $(m(m-1)/2!)$, and so on, *look like combinatorial coefficients* (i.e., ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, $\ldots$, ${}^nC_r$, and so on). However, recall that *${}^nC_r$ is defined for natural number n and whole number r* (with $n \geq r$), and as such has *no meaning* in other cases.

Remark (2): When m is *a negative integer or a rational number*, there are *infinite number of terms* in the expansion of $(b + x)^m$.

Remark (3): The following results are very useful and can be easily obtained by using the expansion in Equation (5).

- $\dfrac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots; \ |x| < 1$
- $\dfrac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots; \ |x| < 1$
- $\dfrac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots; \ |x| < 1$
- $\dfrac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots; \ |x| < 1$

1.20 OPERATIONS INVOLVING NEGATIVE NUMBERS

A good deal of the machinery of elementary algebra is concerned with *the solution of equations involving unknowns*. However, we should note that this simple machinery can lead directly to useful results in numerous other types of problems.

The most difficult item in algebra is that devoted to operations involving negative numbers. The difficulty is twofold:

(i) Why introduce negative numbers?

(ii) Why does multiplication of two negative numbers (or division of a negative number by another negative number) yield a positive number?

In fact, it is in connection with the solution of equations, that both questions can be answered. For example, note that if we do not accept negative numbers then even a simple equation, like $2x + 5 = 0$ cannot be solved. Next, consider the equation

$$7x - 5 = 10x - 11 \tag{6}$$

To solve this equation, *we can transpose the terms in two ways so that the unknowns are on one side and the knowns are on the other side*. (Of course, we will expect that in both the cases the solution should be same.)

Thus, we get

$$11 - 5 = 10x - 7x$$

$$\text{or} \quad 6 = 3x \quad \text{so} \quad x = 2$$

Also, we get

$$7x - 10x = -11 + 5$$

$$-3x = -6.$$

$$x = \frac{-6}{-3} = \frac{(-1)\cdot 6}{(-1)\cdot 3} = \frac{(-1)}{(-1)}\cdot\frac{6}{3} = \frac{(-1)}{(-1)}\cdot 2 \tag{7}$$

$$\begin{aligned} \text{Also,}\quad \frac{-6}{-3} &= (-1)\cdot 6\cdot\frac{(-1)}{3}\left[\because \frac{1}{-3} = \frac{1}{(-1)3} = (-1)\frac{1}{3} = \frac{(-1)}{3}\right] \\ &= (-1)\cdot(-1)\cdot\frac{6}{3} = (-1)\cdot(-1)\cdot 2 \end{aligned} \tag{8}$$

Now in order that the solution of the Equation (6) should be same, it is necessary that $(-1)/(-1) = 1$ in (7) and $(-1)(-1) = 1$ in (8).

1.21 DIVISION BY ZERO

The question, *"Why is division by zero not permitted in mathematics?"* is answered through algebra.

In arithmetic (or more generally in algebra), the operation of division is defined in terms of the operation of multiplication. Thus according to the existing rule, the division of an arbitrary number "a" by another number "b" means to find a number x such that

$$a\cdot\frac{1}{b} = x \quad \text{where} \quad b \neq 0$$

$$b\cdot x = a$$

or

Let us see what happens if division by zero is permitted. If $b = 0$, then we must consider the following two cases.

(i) when $a \neq 0$, and
(ii) when $a = 0$

Case (i): We try to solve the equation

$$b\cdot x = a, \text{ (where } b = 0, \text{ but } a \neq 0)$$

We get $$0\cdot x = a$$

It follows that $a = 0$, which is against our assumption that $a \neq 0$. This situation arises because there is no number x, which could be multiplied by "0" to get a fixed (nonzero) number "a". It follows that if a nonzero number is divided by zero than we get a meaningless result.

Case (ii): We try to solve the equation

$$b\cdot x = a, \text{ (where } b = 0, \text{ and } a = 0)$$

We get $$0\cdot x = 0$$

Unfortunately, this is true. Here any number x satisfies this equation. Let us see the consequence of this situation.

If division by zero is permitted, then we get from the equation $0\cdot x = 0$, $x = 0/0$. Similarly from $0\cdot y = 0$, we get $y = 0/0$, where $x, y, \ldots$ are all different (nonzero) numbers. From the above, it follows that $0/0 = x = y = z \ldots$, which means that all different numbers are equal.

Thus, if $a = 0$, and $b = 0$, then we have $a/b = 0/0$ and it represents any number whatever we choose. But mathematicians require that the division of "*a*" by "*b*" should yield a unique number as a result. But this is again not achieved.

From the above, we observe that *division by zero leads either to no number or any arbitrary number*. (Note that this is the consequence of permitting division by the number zero.) Thus, *division by zero leads to meaningless results and hence it is not permitted in mathematics.*