

1 Antiderivative(s) [or Indefinite Integral(s)]

1.1 INTRODUCTION

In mathematics, we are familiar with *many pairs of inverse operations*: addition and subtraction, multiplication and division, raising to powers and extracting roots, taking logarithms and finding antilogarithms, and so on. In this chapter, we discuss the inverse operation of *differentiation*, which we call *antidifferentiation*.

Definition (1): A function $\phi(x)$ is called an *antiderivative* of the given function $f(x)$ on the interval $[a, b]$, if at all points of the interval $[a, b]$,

$$\phi'(x) = f(x)^{(1)}$$

Of course, it is logical to use the terms differentiation and antidifferentiation to mean the operations, which must be inverse of each other. However, *the term integration* is frequently used to stand for the process of antidifferentiation, and the term *an integral* (or an indefinite integral) is generally used to mean *an antiderivative* of a function.

The reason behind using the terminology “an integral” (or an indefinite integral) will be clear only after we have studied the concept of “the definite integral” in Chapter 5. The relation between “the definite integral” and “an antiderivative” or an indefinite integral of a function is established through *first and second fundamental theorems of Calculus*, discussed in Chapter 6a.

For the time being, we agree to use these terms freely, with an understanding that the terms: “an antiderivative” and “an indefinite integral” have the same meaning for all practical purposes and that the logic behind using these terms will be clear later on. If a function f is differentiable in an interval I , [i.e., if its derivative f' exists at each point in I] then a natural question arises: *Given $f'(x)$ which exists at each point of I , can we determine the function $f(x)$?* In this chapter, we shall consider this reverse problem, and study some *methods of finding $f(x)$ from $f'(x)$* .

Note: We know that the derivative of a function $f(x)$, if it exists, is a *unique function*. Let $f'(x) = g(x)$ and that $f(x)$ and $g(x)$ [where $g(x) = f'(x)$] both exist for each $x \in I$, then we say that *an antiderivative (or an integral) of the function $g(x)$ is $f(x)$* .⁽²⁾

1-Anti-differentiation (or integration) as the inverse process of differentiation.

⁽¹⁾ Note that if x is an end point of the interval $[a, b]$, then $\phi'(x)$ will stand for the one-sided derivative at x .

⁽²⁾ Shortly, it will be shown that an integral of the function $g(x)$ [= $f'(x)$] can be expressed in the form $f(x) + c$, where c is any constant. Thus, any two integrals of $g(x)$ can differ only by some constant. We say that an integral (or an antiderivative) of a function is “unique up to a constant.”

To understand the concept of an antiderivative (or an indefinite integral) more clearly, consider the following example.

Example: Find an antiderivative of the function $f(x) = x^3$.

Solution: From the definition of the derivative of a function, and its relation with the given function, it is natural to guess that an integral of x^3 must have the term x^4 . Therefore, we consider the derivative of x^4 . Thus, we have

$$\frac{d}{dx} x^4 = 4x^3.$$

Now, from the definition of antiderivative (or indefinite integral) we can write that antiderivative of $4x^3$ is x^4 . Therefore, antiderivative of x^3 must be $x^4/4$. In other words, the function $\phi(x) = x^4/4$ is an antiderivative of x^3 .

1.1.1 The Constant of Integration

When a function $\phi(x)$ containing a constant term is differentiated, the constant term does not appear in the derivative, since its derivative is zero. For instance, we have,

$$\begin{aligned}\frac{d}{dx} (x^4 + 6) &= 4x^3 + 0 = 4x^3; \\ \frac{d}{dx} x^4 &= 4x^3; \text{ and} \\ \frac{d}{dx} (x^4 - 5) &= 4x^3 - 0 = 4x^3.\end{aligned}$$

Thus, by the definition of antiderivative, we can say that the functions $x^4 + 6$, x^4 , $x^4 - 5$, and in general, $x^4 + c$ (where $c \in \mathbf{R}$), all are antiderivatives of $4x^3$.

Remark: From the above examples, it follows that a given function $f(x)$ can have *infinite number of antiderivatives*. Suppose the antiderivative of $f(x)$ is $\phi(x)$, then not only $\phi(x)$ but also functions like $\phi(x) + 3$, $\phi(x) - 2$, and so on all are called antiderivatives of $f(x)$. Since, the constant term involved with an antiderivative can be any real number, an antiderivative is called an indefinite integral, the indefiniteness being due to the constant term.

In the process of antidifferentiation, we cannot determine the constant term, associated with the (original) function $\phi(x)$. Hence, from this point of view, an antiderivative $\phi(x)$ of the given function $f(x)$ will always be *incomplete up to a constant*. Therefore, to get a complete antiderivative of a function, an *arbitrary constant* (which may be denoted by “ c ” or “ k ” or any other symbol) must be added to the result. This *arbitrary constant* represents the *undetermined constant term* of the function, and is called the *constant of integration*.

1.1.2 The Symbol for Integration (or Antidifferentiation)

The symbol chosen for expressing the operation of integration is “ $\int$ ”; it is the old fashioned elongated “S”, and it is selected as being the first letter of the word “Sum”, which is another aspect of integration, as will be seen later.⁽³⁾

⁽³⁾ The symbol $\int$ is also looked upon as a modification of the summation sign $\sum$.

Thus, if an integral of a function $f(x)$ is $\phi(x)$, we write

$$\int f(x)dx = \phi(x) + c, \text{ where } c \text{ is the constant of integration.}$$

Remark: The differential “ dx ” [written by the side of the function $f(x)$ to be integrated] *separately* does not have a meaning. However, “ dx ” indicates *the independent variable “ x ”, with respect to which the original differentiation was made*. It also suggests that the reverse process of integration has to be performed with respect to x .

Note: The concept of differentials “ dy ” and “ dx ” is discussed at length, in Chapter 16. There, we have discussed how the derivative of a function $y = f(x)$ can be looked upon as the ratio dy/dx of differentials. Besides, it is also explained that the equation $dy/dx = f'(x)$ can be expressed in the form

$$dy = f'(x)dx,$$

which defines the differential of the dependent variable [i.e., the differential of the function $y = f(x)$].

Accordingly, $\int f(x)dx$ stands to mean that $f(x)$ is to be integrated with respect to x . In other words, we have to find (or identify) a function $\phi(x)$ such that $\phi'(x) = f(x)$. Once this is done, we can write

$$\int f(x)dx = \phi(x) + c, \quad (c \in \mathbf{R}).$$

Now, we are in a position to clarify *the distinction between an antiderivative and an indefinite integral*.

Definition: If the function $\phi(x)$ is an antiderivative of $f(x)$, then the expression $\phi(x) + c$ is called *the indefinite integral* of $f(x)$ and it is denoted by the symbol $\int f(x)dx$.

Thus, by definition,

$$\int f(x)dx = \phi(x) + c, \quad (c \in \mathbf{R}), \text{ provided } \phi'(x) = f(x).$$

Remark: Note that the function in the form $\phi(x) + c$ exhausts all the antiderivatives of the function $f(x)$. On the other hand, the function $\phi(x)$ with a constant [for instance, $\phi(x) + 3$, or $\phi(x) - 7$, or $\phi(x) + 0$, etc.] is called an antiderivative or an indefinite integral (or simply, an integral) of $f(x)$.

1.1.3 Geometrical Interpretation of the Indefinite Integral

From the geometrical point of view, the indefinite integral of a function is a collection (or family) of curves, each of which is obtained by translating any one curve [representing $\phi(x) + c$] parallel to itself, upwards or downwards along the y -axis. A natural question arises: *Do antiderivatives exist for every function $f(x)$?* The answer is *NO*.

Let us note, however, *without proof*, that if a function $f(x)$ is continuous on an interval $[a, b]$, then the function has an antiderivative.

Now, let us integrate the function $y = f(x) = 2x$. We have,

$$\int f(x)dx = \int 2x \, dx = x^2 + c \tag{1}$$

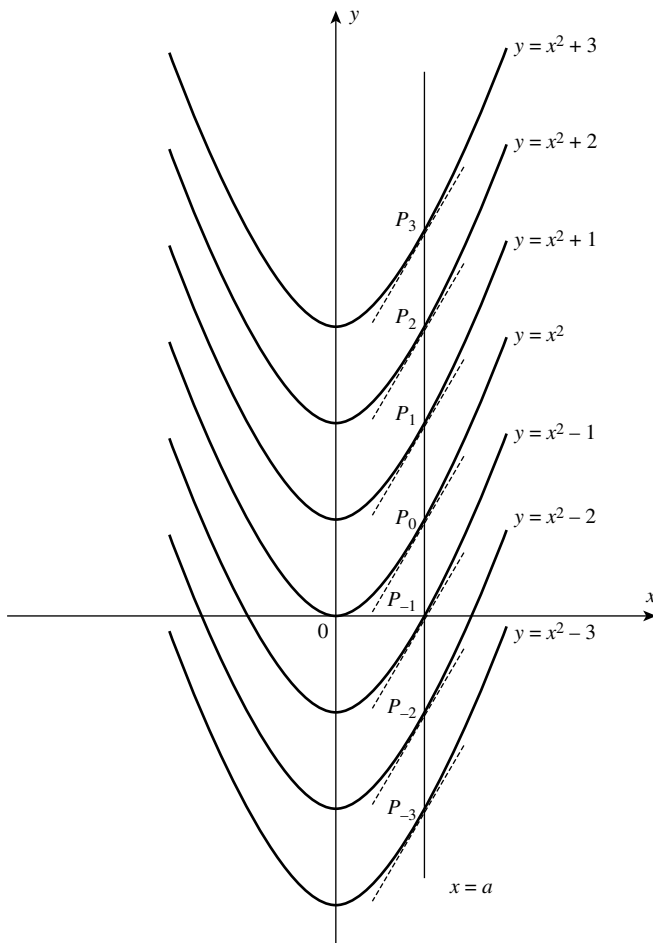


FIGURE 1.1 Shows some curves of $f(x) = 2x$ family.

For different values of c , we get different antiderivatives of $f(x)$. But, these antiderivatives (or indefinite integrals) are very similar geometrically. By assigning different values to c , we get different members of the family. All these members considered together constitute the indefinite integral of $f(x) = 2x$. In this case, each antiderivative represents a parabola with its axis along the y -axis.⁽⁴⁾

Note that for each positive value of c , there is a parabola of the family which has its vertex on the positive side of the y -axis, and for each negative value of c , there is a parabola which has its vertex on the negative side of the y -axis.

Let us consider the intersection of all these parabolas by a line $x = a$. In Figure 1.1, we have taken $a > 0$ (the same is true for $a < 0$). If the line $x = a$ intersects the parabolas $y = x^2$,

⁽⁴⁾ For $c = 0$, we obtain $y = x^2$, a parabola with its vertex on the origin. The curve $y = x^2 + 1$ for $c = 1$, is obtained by shifting the parabola $y = x^2$ one unit along y -axis in positive direction. Similarly, for $c = -1$, the curve $y = x^2 - 1$ is obtained by shifting the parabola $y = x^2$ one unit along y -axis in the negative direction. Similarly, all other curves can be obtained.

$y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$, at $P_0, P_1, P_2, P_{-1}, P_{-2}$, and so on, respectively, then dy/dx (i.e., the slope of each curve) at $x = a$ is $2a$. This indicates that *the tangents to the curves $\phi(x) = x^2 + c$ at $x = a$ are parallel. This is the geometrical interpretation of the indefinite integral.*

Now, suppose we want to find the curve that passes through the point $(3, 6)$. These values of x and y can be substituted in the equation of the curve. Thus, on substitution in the equation $y = x^2 + c$,

$$\text{We get, } 6 = 3^2 + c$$

$$\therefore c = -3$$

Thus, $y = x^2 - 3$ is the equation of the particular curve which passes through the point $(3, 6)$. Similarly, we can find the equation of any curve which passes through any given point (a, b) . In the relation,

$$\int f(x)dx = \phi(x) + c, (c \in \mathbf{R}).$$

- The function $f(x)$ is called *the integrand*.
- The expression under the integral sign, that is, “ $f(x)dx$ ” is called the element of integration.

Remark: By the definition of an integral, we have,

$$f(x) = [\phi(x) + c]' = \phi'(x).$$

Thus, we can write,

$$\begin{aligned} \int f(x)dx &= \int \phi'(x)dx \\ &= \int d[\phi(x)] \end{aligned}$$

Observe that the last expression $\int d[\phi(x)]$ does not have “ dx ” attached to it (Why?). Recall that $d[\phi(x)]$ stands for *the differential of the function $\phi(x)$* , which is denoted by $\phi'(x)dx$, as discussed in Chapter 16 of Part I. Thus, we write,

$$\int f(x)dx = \int \phi'(x)dx = \int d[\phi(x)] = \phi(x) + c. \quad (2)$$

Equation(2) tells us that when we *integrate $f(x)$ [or antidifferentiate the differential of a function $\phi(x)$]* we obtain the function “ $\phi(x) + c$ ”, where “ c ” is an arbitrary constant. *Thus, on the differential level, we have a useful interpretation of antiderivative of “ f ”.*

Since we have $\int f(x)dx = \phi(x)$, we can say that *an antiderivative of “ f ” is a function “ ϕ ”, whose differential $\phi'(x)dx$ equals $f(x)dx$* . Thus, we can say that in the symbol $\int f(x)dx$, *the expression “ $f(x)dx$ ” is the differential of some function $\phi(x)$.*

Remark: Equation (2) suggests that differentiation and antidifferentiation (or integration) are inverse processes of each other. (We shall come back to this discussion again in Chapter 6a).

Leibniz introduced the convention of writing *the differential of a function after the integral symbol* “ $\int$ ”. The advantage of using the differential in this manner will be apparent to the reader later when we compute antiderivatives by the method of substitution—to be studied later in Chapters 3a and 3b. Whenever we are asked to evaluate the integral $\int f(x)dx$, we are required to find a function $\phi(x)$, satisfying the condition $\phi'(x)=f(x)$. *But how can we find the function $\phi(x)$?*

Because of certain practical difficulties, it is not possible to formulate a set of rules by which any function may be integrated. However, certain methods have been devised for integrating certain types of functions.

- The knowledge of these methods,
- good grasp of differentiation formulas, and
- necessary practice, should help the students to integrate most of the commonly occurring functions.

The methods of integration, in general, consist of certain mathematical operations applied to the integrand so that it assumes some known form(s) of which the integrals are known. Whenever it is possible to express the integrand in any of the known forms (which we call standard forms), the final solution becomes a matter of recognition and inspection.

Remark: It is important to remember that in the integral $\int f(x)dx$, the variable in the integrand “ $f(x)$ ” and in the differential “ dx ” must be same (Here it is “ x ” in both). Thus, $\int \cos y \, dx$ *cannot be evaluated as it stands*. It would be necessary, if possible, to express $\cos y$ as a function of x . Any other letter may be used to represent the independent variable besides x . Thus, $\int t^2 dt$ indicates that t^2 is to be integrated (wherein t is the independent variable), and we need to integrate it with respect to t (which appears in dt).

Note: Integration has one advantage that the result can always be checked by differentiation. If the function obtained by integration is differentiated, we should get back the original function.

1.2 USEFUL SYMBOLS, TERMS, AND PHRASES FREQUENTLY NEEDED

TABLE 1.1 Useful Symbols, Terms, and Phrases Frequently Needed

Symbols/Terms/Phrases	Meaning
$f(x)$ in $\int f(x)dx$	Integrand
The expression $f(x)dx$ in $\int f(x)dx$	The element of integration
$\int f(x)dx$	Integral of $f(x)$ with respect to x . Here, x in “ dx ” is the variable of integration
Integrate	Find the indefinite integral (i.e., find an antiderivative and add an arbitrary constant to it) ^a
An integral of $f(x)$	A function $\phi(x)$, such that $\phi'(x)=f(x)$
Integration	The process of finding the integral
Constant of integration	An arbitrary real number denoted by “ c ” (or any other symbol) and considered as a constant.

^aThe term integration also stands for the process of computing the definite integral of $f(x)$, to be studied in Chapter 5.

1.3 TABLE(S) OF DERIVATIVES AND THEIR CORRESPONDING INTEGRALS

TABLE 1.2a Table of Derivatives and Corresponding Integrals

S. No.	Differentiation Formulas Already Known to us $\frac{d}{dx}[f(x)] = f'(x)$	Corresponding Formulas for Integrals $\int f'(x)dx = f(x) + c$ (Antiderivative with Arbitrary Constants)
1.	$\frac{d}{dx}(x^n) = nx^{n-1}, n \in \mathbf{R}$	$\int nx^{n-1}dx = x^n + c, n \in \mathbf{R}$
2.	$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = x^n, n \neq -1$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1, n \in \mathbf{R}.$ This form is more useful
3.	$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
4.	$\frac{d}{dx}(a^x) = a^x \cdot \log_e a (a > 0)$ or $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x (a > 0)$	$\int a^x \cdot \log_e a dx = a^x + c (a > 0)$ $\therefore \int a^x dx = \frac{a^x}{\log_e a} + c$
5.	$\frac{d}{dx}(\log_e x) = \frac{1}{x} (x > 0)$	$\int \frac{1}{x} dx = \log_e x + c, x \neq 0^*$

*This formula is discussed at length in Remark (2), which follows.

From the formulas of derivatives of functions, we can write down directly the corresponding formulas for integrals. The formulas for integrals of the important functions given on the right-hand side of the Table 1.2a are referred to as standard formulas which will be used to find integrals of other (similar) functions.

Remark (1): We make *two comments about formula (2) mentioned in Table 1.2a.*

- (i) It is meant to include the case when $n = 0$, that is,

$$\int x^0 dx = \int 1 dx = x + c$$

- (ii) Since no interval is specified, *the conclusion is understood to be valid for any interval on which x^n is defined. In particular, if $n < 0$, we must exclude any interval containing the origin.* (Thus, $\int x^{-3} dx = (x^{-2} / -2) = -(1/2x^2)$, which is valid in any interval not containing zero.)

Remark (2): Refer to formula (5) mentioned in Table 1.2a. *We have to be careful when considering functions whose domain is not the whole real line.* For instance, when we say $d/dx(\log_e x) = 1/x$, it is obvious that in this equality $x \neq 0$. However, it is important to remember that, $\log_e x$ is defined only for positive x .⁽⁵⁾

⁽⁵⁾ Recall that $y = e^x \Leftrightarrow \log_e y = x$. Note that $e^x (=y)$ is always a positive number. It follows that $\log_e y$ is defined only for positive numbers. In fact, in any equality involving the function $\log_e x$ (to any base), it is assumed that $\log x$ is defined only for positive values of x .

In view of the above, the derivative of $\log_e x$ must also be *considered only for positive values of x* . Further, when we write $\int 1/x(dx) = \log_e x$, one must remember that in this equality the function $1/x$ is to be considered only for positive values of x .

Note: Observe that, though the integrand $1/x$ is defined for negative values of x , it will be wrong to say that, since $1/x$ is defined for all nonzero values of x , the integral of $1/x$ (which is $\log_e x$) may be defined for negative values of x . To overcome this situation, we write

$$\int \frac{1}{x} dx = \log_e |x|, \quad x \neq 0. \text{ Let us prove this.}$$

For $x > 0$, we have,

$$\frac{d}{dx}(\log_e x) = \frac{1}{x},$$

and, for $x < 0$,

$$\frac{d}{dx}[\log_e(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}$$

[Note that for $x < 0$, $(-x) > 0$]. Combining these two results, we get,

$$\frac{d}{dx}(\log_e |x|) = \frac{1}{x}, \quad x \neq 0 \quad \therefore \int \frac{1}{x} dx = \log_e |x| + c, \quad x \neq 0.$$

From this point of view, it is *not appropriate* to write

$$\int \frac{1}{x} dx = \log_e x, \quad x \neq 0. \quad (\text{Why?})$$

The correct statement is:

$$\int \frac{1}{x} dx = \log_e x, \quad x > 0. \quad (\text{A})$$

$$\text{or } \int \frac{1}{x} dx = \log_e |x|, \quad x \neq 0 \quad (\text{B})$$

Note that both the equalities at (A) and (B) above clearly indicate that $\log_e x$ is defined only for positive values of x .

In solving problems involving log functions, generally the base “ e ” is assumed. It is convenient and saves time and effort, both (To avoid confusion, one may like to indicate the base of logarithm, if necessary). Some important formulas for integrals that are directly obtained from the derivatives of certain functions, are listed in Tables 1.2b and 1.2c.

Besides, there are *certain results (formulas) for integration*, which are *not obtained directly* from the formulas for derivatives but obtained indirectly by applying other methods of integration. (These methods will be discussed and developed in subsequent chapters).

Many important formulas for *integration* (whether obtained directly or indirectly) are treated as *standard formulas for integration*, which means that we can use these results to write the integrals of (other) similar looking functions.

TABLE 1.2b Table of Derivatives and Corresponding Integrals

Differentiation Formulas		Corresponding Formulas for Indefinite Integrals $\int f'(x)dx = f(x) + c$
S. No.	Already Known to us $\frac{d}{dx}[f(x)] = f'(x)$	
6.	$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + c$
7. ^a	$\frac{d}{dx}(\cos x) = -\sin x$	$\int (-\sin x)dx = \cos x + c$ $\therefore \int \sin x \, dx = -\cos x + c$
8.	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + c$
9. ^a	$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\int (-\operatorname{cosec}^2 x)dx = \cot x + c$ $\therefore \int \operatorname{cosec}^2 x \, dx = -\cot x + c$
10.	$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \cdot \tan x \, dx = \sec x + c$
11. ^a	$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$	$\int (-\operatorname{cosec} x \cdot \cot x)dx = \operatorname{cosec} x + c$ $\therefore \int \operatorname{cosec} x \cdot \cot x \, dx = -\operatorname{cosec} x + c$

^aObserve that derivatives of trigonometric functions starting with “co,” (i.e., $\cos x$, $\cot x$, and $\operatorname{cosec} x$) are with *negative* sign. Accordingly, the corresponding integrals are also with *negative* sign.

Important Note: *The main problem in evaluating an integral lies in expressing the integrand in the standard form. For this purpose, we may have to use algebraic operations and/or trigonometric identities. For certain integrals, we may have to change the variable of integration by using the method of substitution, to be studied later, in Chapters 3a and 3b. In such cases, the element of integration is changed to a new element of integration, in which the integrand (in a new variable) may be in the standard form. Once the integrand is expressed in the standard*

TABLE 1.2c Derivatives of Inverse Trigonometric Functions and Corresponding Formulas for Indefinite Integrals

Differentiation Formulas		Corresponding Formulas for
S. No.	Already Known to us $\frac{d}{dx}[f(x)] = f'(x)$	Indefinite Integrals $\int f'(x)dx = f(x) + c$
12.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \sin^{-1} x + c \\ \text{or} \\ -\cos^{-1} x + c \end{cases}$
13.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \begin{cases} \tan^{-1} x + c \\ \text{or} \\ -\cot^{-1} x + c \end{cases}$
14.	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x \cdot \sqrt{x^2-1}}$ $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x \cdot \sqrt{x^2-1}}$	$\int \frac{dx}{x \cdot \sqrt{x^2-1}} = \begin{cases} \sec^{-1} x + c \\ \text{or} \\ -\operatorname{cosec}^{-1} x + c \end{cases}$

form, evaluating the integral depends only on recognizing the form and remembering the table of integrals.

Remark: Thus, integration as such is not at all difficult. The real difficulty lies in applying the necessary algebraic operations and using trigonometric identities needed for converting the integrand to standard form(s).

1.3.1 Table of Integrals of $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$

Now consider Table 1.2b.

Note: Table 1.2b does not include the integrals of $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$. The integrals of these functions will be established by using the method of substitution (to be studied later in Chapter 3a). However, we list below these results for convenience.

1.3.2 Results for the Integrals of $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$

- (i) $\int \tan x \, dx = \log_e |\sec x| + c = \log(\sec x) + c$
- (ii) $\int \cot x \, dx = \log_e |\sin x| + c = \log(\sin x) + c$
- (iii) $\int \sec x \, dx = \log(\sec x + \tan x) + c = \log\left(\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right) + c$
- (iv) $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x) + c = \log\left(\tan\left(\frac{x}{2}\right)\right) + c$

These four integrals are also treated as *standard integrals*.

Now, we consider Table 1.2c.

Remark: Derivatives of inverse circular functions are certain algebraic functions. In fact, *there are only three types of algebraic functions whose integrals are inverse circular functions*.

1.4 INTEGRATION OF CERTAIN COMBINATIONS OF FUNCTIONS

There are some *theorems of differentiation that have their counterparts in integration*. These theorems state the properties of “indefinite integrals” and can be easily proved using the definition of antiderivative. Almost every theorem is proved with the help of differentiation, *thus stressing the concept of antidifferentiation*. To integrate a given function, we shall need these theorems of integration, in addition to the above standard formulas. We give below these results without proof.

$$(a) \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

In words, “*an integral of the sum of two functions, is equal to the sum of integrals of these two functions*”. The above rule can be extended to the sum of a *finite* number of functions. The result also holds good, if the *sum* is replaced by the *difference*. Hence, integration can be extended to the sum or difference of a finite number of functions.

$$(b) \int c \cdot f(x)dx = c \cdot \int f(x)dx, \text{ where } c \text{ is a real number.}$$

Note that result (b) follows from result (a).

Thus, a constant can be taken out of the integral sign. The theorem can also be extended as follows:

Corollary:

$$\int [k_1 f(x) + k_2 g(x)] dx = k_1 \int f(x) dx + k_2 \int g(x) dx, \text{ where } k_1 \text{ and } k_2 \text{ are real numbers}$$

$$(c) \text{ If } \int f(x) dx = F(x) + c$$

$$\text{then } \int f(x+b) dx = F(x+b) + c.$$

Example:

$$\int \cos(x+3) dx = \sin(x+3) + c$$

$$(d) \text{ If } \int f(x) dx = F(x) + c,$$

$$\text{then } \int f(ax+b) dx = \frac{1}{a} F(ax+b) + c.$$

This result is easily proved, by differentiating both the sides.

Proof: It is given that $\int f(x) dx = F(x) + c$

$$\therefore F'(x) = f(x) \quad (\text{By definition})$$

To prove the desired result, we will show that the derivatives of both the sides give the same function.

Now consider,

$$\text{LHS: } \frac{d}{dx} \left[\int f(ax+b) dx \right] = f(ax+b)^{(6)}$$

$$\begin{aligned} \text{RHS: } \frac{d}{dx} \left[\frac{1}{a} F(ax+b) + c \right] &= \frac{1}{a} F'(ax+b) \cdot a \\ &= F'(ax+b) \\ &= f(ax+b) \end{aligned}$$

$$\therefore \text{ L.H.S.} = \text{R.H.S.}$$

Note: This result is very useful since it offers a new set of “standard forms of integrals”, wherein “ x ” is replaced by a linear function $(ax+b)$. Later on, we will show that this result is more conveniently proved by the method of substitution, to be studied in Chapter 3a.

Let us now evaluate the integrals of some functions using the above theorems, and the standard formulas given in Tables 1.2a–1.2c.

⁽⁶⁾ We know that the process of differentiation is the inverse of integration (and vice versa). Hence, differentiation nullifies the integration, and we get the integrand as the result. (Detailed explanation on this is given in Chapter 6a).

Examples: We can write,

$$(i) \int \sin (5x + 7) dx = -\frac{1}{5} \cos (5x + 7) + c$$

$$\left[\because \int \sin x \, dx = -\cos x + c \right]$$

Similarly,

$$(ii) \int e^{3x-2} dx = \frac{1}{3} e^{3x-2} + c$$

$$\begin{aligned} (iii) \int \left[\sin (2x + 1) + \frac{x-2}{x} - 4^x \right] dx \\ = \int \sin(2x + 1) dx + \int 1 dx - 2 \int \frac{1}{x} dx - \int 4^x dx \\ = -\frac{1}{2} \cos(2x + 1) + x - 2 \log_e x - \frac{4^x}{\log_e 4} + c \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} (iv) \int [x(x + 3) - 5 \sec^2 x - 3e^{6x-1}] \\ = \int x^2 dx + 3 \int x \, dx - 5 \int \sec^2 x \, dx - 3 \int e^{6x-1} dx \\ = \frac{x^3}{3} + \frac{3x^2}{2} - 5 \tan x - \frac{3e^{6x-1}}{6} + c \\ = \frac{x^3}{3} + \frac{3}{2}x^2 - 5 \tan x - \frac{1}{2}e^{6x-1} + c \quad \text{Ans.} \end{aligned}$$

$$\begin{aligned} (v) \int \frac{x + 2\sqrt{x} + 7}{\sqrt{x}} dx = I \quad (\text{say}) \\ \therefore I = \int (x^{1/2} + 2 + 7 \cdot x^{-1/2}) dx \\ = \frac{x^{3/2}}{3/2} + 2x + 7 \frac{x^{1/2}}{1/2} + c \\ = \frac{2}{3}x^{3/2} + 2x + 14x^{1/2} + c \quad \text{Ans.} \end{aligned}$$

Here, *the integrand is in the form of a ratio*, which can be easily reduced to a sum of functions in the standard form and hence their antiderivatives can be written, using the tables.

$$\begin{aligned} (vi) \int \frac{3x + 1}{x - 3} dx &= \int \frac{3x - 9 + 9 + 1}{x - 3} dx \\ &= \int \frac{3(x - 3) + 10}{x - 3} dx \\ &= 3 \int dx + \int \frac{10}{x - 3} dx \\ &= 3x + 10 \log_e (x - 3) + c \quad \text{Ans.} \end{aligned}$$

Here again, *the integrand is in the form of a ratio, which can be easily reduced to the standard form*. If the degree of numerator and denominator is same, then *creating the same factor as the denominator* (as shown above) is a quicker method than actual division.

$$\begin{aligned}
 \text{(vii)} \quad \int (2x + 3)\sqrt{x - 4} \, dx &= I \quad (\text{say}) \\
 \therefore I &= \int (2x - 8 + 11)\sqrt{x - 4} \, dx \\
 &= \int [2(x - 4) + 11] \cdot \sqrt{x - 4} \, dx \\
 &= 2 \int (x - 4)^{3/2} dx + 11 \int (x - 4)^{1/2} dx \\
 &= 2 \cdot \frac{(x - 4)^{5/2}}{5/2} + 11 \cdot \frac{(x - 4)^{3/2}}{3/2} \\
 &= \frac{4}{5} (x - 4)^{5/2} + \frac{22}{3} (x - 4)^{3/2} + c \quad \text{Ans.}
 \end{aligned}$$

Here, the *integrand is in the form of a product*, which can be easily reduced to the standard forms, as indicated above.

In solving the above problems, it has been possible to evaluate the integrals in the form of quotients and products of functions, simply because the integrands *can be converted to standard forms, by applying certain algebraic operations*. In fact, there are different methods for handling integrals involving quotients and products and so on. For example, consider the following integrals.

$$\text{(a)} \quad \int (3x^2 - 5)^{100} x \, dx$$

$$\text{(b)} \quad \int \sin^3 x \cos x \, dx$$

$$\text{(c)} \quad \int \frac{\sin x}{1 + \sin x} dx$$

$$\text{(d)} \quad \int \frac{1 - \cos 2x}{1 + \cos 2x} dx$$

$$\text{(e)} \quad \int x^2 \sin x \, dx$$

$$\text{(f)} \quad \int \frac{3x - 4}{x^2 - 3x + 2} dx$$

The above integrals are not in the standard form(s), but they *can be reduced to the standard forms, by using algebraic operations, trigonometric identities, and some special methods to be studied later*.

Note: We emphasize that the main problem in evaluating integrals lies in converting the given integrals into standard forms. *Some integrands can be reduced to standard forms by using*

algebraic operations and trigonometric identities. For instance, consider $\int \sin^2 x \, dx$. Here, the integrand $\sin^2 x$ is not in the standard form. But, we know the trigonometric identity $\cos 2x = 1 - 2 \sin^2 x$. $\therefore \sin^2 x = (1 - \cos 2x)/2$.

Thus, $\int \sin^2 x \, dx = \int ((1 - \cos 2x)/2) dx = 1/2 \int dx - (1/2) \int \cos 2x \, dx$, where the integrands are in the standard form and so their (indefinite) integrals can be written easily. Note that, here we could express the integrand in a standard form by using a trigonometric identity. Similarly, we can show that

$$\int \frac{\sin x}{1 + \sin x} dx = \int (\sec x \cdot \tan x - \sec^2 x + 1) dx$$

and
$$\int \frac{1 - \cos 2x}{1 + \cos 2x} dx = \int (\sec^2 x - 1) dx$$

wherein, the integrands on the right-hand side are in the standard form(s). A good number of such integrals, *involving trigonometric functions, are evaluated in Chapter 2, using trigonometric identities and algebraic operations.* Naturally, the variable of integration remains unchanged in these operations.

Now, consider the integral $\int f(x) dx = \int \sin^3 x \cos x \, dx$. Here, again the integrand is not in the standard form. Moreover, *it is not possible to convert it to a standard form by using algebraic operations and/or trigonometric identities. However, it is possible to convert it into a standard form as follows:*

We put $\sin x = t$ and differentiate both sides of this equation with respect to t to obtain $\cos x \, dx = dt$. Now, by using these relations in the expression for the element of integration, we get $\int \sin^3 x \cos x \, dx = \int t^3 dt$, which can be easily evaluated. We have

$$\int t^3 dt = \frac{t^4}{4} + c = \frac{\sin^4 x}{4} + c$$

Note that, in the process of *converting the above integrand into a standard form, we had to change the variable of integration from x to t .* This method is known as the *method of substitution* which is to be studied later.

The method of substitution is a very useful method for integration, *associated with the change of variable of integration.* Besides these there are other methods of integration. In this book, our interest is restricted to study the following methods of integration.

- (a) Integration of certain trigonometric functions by using algebraic operations and/or trigonometric identities.
- (b) *Method of substitution.* This method involves the change of variable.
- (c) *Integration by parts.* This method is applicable for integrating product(s) of two different functions. It is also used for evaluating integrals of powers of trigonometric functions (reduction formula). Finer details of this method will be appreciated only while solving problems in Chapters 4a and 4b
- (d) *Method of integration by partial fractions.* For integrating rational functions like $\int (3x - 4)/(x^2 - 3x + 2) dx$.

The purpose of each method is to reduce the integrand into the standard form.

Before going for discussions about the above methods of integration, it is useful to realize and appreciate the following points related to the processes of *differentiation* and *integration*, in connection with the similarities and differences in these operations.

1.5 COMPARISON BETWEEN THE OPERATIONS OF DIFFERENTIATION AND INTEGRATION

- (1) Both operate on functions.
- (2) Both satisfy the property of linearity, that is,
 - (i) $\frac{d}{dx}[k_1f_1(x) + k_2f_2(x)] = k_1\frac{d}{dx}f_1(x) + k_2\frac{d}{dx}f_2(x)$, where k_1 and k_2 are constants.
 - (ii) $\int [k_1f_1(x) + k_2f_2(x)]dx = k_1\int f_1(x)dx + k_2\int f_2(x)dx$, where k_1 and k_2 are constants.
- (3) We have seen that *all functions are not differentiable*. Similarly, *all functions are not integrable*. We will learn about this later in Chapter 5.
- (4) The derivative of a function (when it exists) is a unique function. The integral of a function is not so. However, integrals are *unique up to an additive constant*, that is, *any two integrals of a function differ by a constant*.
- (5) When a polynomial function P is differentiated, the result is a polynomial whose degree is one less than the degree of P . When a polynomial function P is *integrated*, the result is a polynomial *whose degree is one more than that of P* .
- (6) We can speak of the derivative at a point. We do not speak of an integral at a point. We speak of an integral over an interval on which the integral is defined. (This will be seen in the Chapter 5).
- (7) The derivative of a function has a *geometrical meaning*, namely *the slope of the tangent to the corresponding curve at a point*. Similarly, *the indefinite integral of a function represents geometrically, a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curve* by the family of lines perpendicular to the axis representing the variable of integration. (Definite integral has a geometrical meaning as an area under a curve).
- (8) The derivative is used to find some physical quantities such as the *velocity of a moving particle*, when the distance traversed at any time t is known. *Similarly, the integral is used in calculating the distance traversed, when the velocity at time t is known*.
- (9) Differentiation is the process involving limits. *So is the process of integration*, as will be seen in Chapter 5. Both processes deal with situations where the quantities vary.
- (10) The process of differentiation and integration are inverses of each other as will be clear in Chapter 6a.

