1

RELIABILITY THEORY

A solid foundation in theoretical knowledge surrounding system reliability is fundamental to the analysis of telecommunications systems. All modern system reliability analysis relies heavily on the application of probability and statistics mathematics. This chapter presents a discussion of the theories, mathematics, and concepts required to analyze telecommunications systems. It begins by presenting the system metrics that are most important to telecommunications engineers, managers, and executives. These metrics are the typical desired output of an analysis, design, or concept. They form the basis of contract language, system specifications, and network design. Without a target metric for design or evaluation, a system can be constructed that fails to meet the end customer's expectations. System metrics are calculated by making assumptions or assignments of statistical distributions. These statistical distributions form the basis for an analysis and are crucial to the accuracy of the system model. A fundamental understanding of the statistical models used in reliability is important. The statistical distributions commonly used in telecommunications reliability analysis are presented from a quantitative mathematical perspective. Review of the basic concepts of probability and statistics that are relevant to reliability analysis are also presented.

Having developed a clear, concise understanding of the required probability and statistics theory, this chapter focuses on techniques of reliability analysis. Assumptions

Telecommunications System Reliability Engineering, Theory, and Practice, Mark L. Ayers.

adopted for failure and repair of individual components or systems are incorporated into larger systems made up of many components or systems. Several techniques exist for performing system analysis, each with its own drawbacks and advantages. These analysis techniques include reliability block diagrams (RBDs), Markov analysis, and numerical Monte Carlo simulation. The advantages and disadvantages of each of the presented approaches are discussed along with the technical methodology for conducting each type of analysis.

System sparing considerations are presented in the final section of this chapter. Component sparing levels for large systems is a common consideration in telecommunications systems. Methods for calculating sparing levels based on the RMA repair period, failure rate, and redundancy level are presented in this section.

Chapter 1 makes considerable reference to the well-established and foundational work published in "System Reliability Theory: Models, Statistical Methods and Applications" by M. Rausand and A. Høyland. References to this text are made in Chapter 1 using a superscript[1] indicator.

1.1 SYSTEM METRICS

System metrics are arguably the most important topic presented in this book. The definitions and concepts of reliability, availability, maintainability, and failure rate are fundamental to both defining and analyzing telecommunications systems. During the analysis phase of a system design, metrics such as availability and failure rate may be calculated as predictive values. These calculated values can be used to develop contracts and guide customer expectations in contract negotiations.

This section discusses the metrics of importance in telecommunications from both a detailed technical perspective and a practical operational perspective. The predictive and empirical calculation of each metric is presented along with caveats associated with each approach.

1.1.1 Reliability

MIL-STD-721C (MILSTD,1981) defines reliability with two different complementary definitions.

1. The duration or probability of failure-free performance under stated conditions.
2. The probability that an item can perform its intended function for a specified interval under stated conditions. (For nonredundant items, this is equivalent to definition 1. For redundant items this is equivalent to the definition of mission reliability.)

Both MIL-STD-721C definitions of reliability focus on the same performance measure. The probability of failure-free performance or mission success refers to the likelihood that the system being examined works for a stated period of time. In order to quantify and thus calculate reliability as a system metric, the terms "stated period" and "stated conditions" must be clearly defined for any system or mission.

The stated period defines the duration over which the system analysis is valid. Without definition of the stated period, the term reliability has no meaning. Reliability is a time-dependent function. Defining reliability as a statistical probability becomes a problem of distribution selection and metric calculation.

The stated conditions define the operating parameters under which the reliability function is valid. These conditions are crucial to both defining and limiting the scope under which a reliability analysis or function is valid. Both designers and consumers of telecommunications systems must pay particular attention to the "stated conditions" in order to ensure that the decisions and judgments derived are correct and appropriate.

Reliability taken from a qualitative perspective often invokes personal experience and perceptions. Qualitative analysis of reliability should be done as a broad-brush or high-level analysis based in a quantitative technical understanding of the term. In many cases, qualitative reliability is defined as a sense or "gut feeling" of how well a system can or will perform. The true definition of reliability as defined in MIL-STD-721C is both statistical and technical and thus any discussion of reliability must be based in those terms.

Quantitative reliability analysis requires a technical understanding of mathematics, statistics, and engineering analysis. The following discussion presents the mathematical derivation of reliability and the conditions under which its application are valid with specific discussions of telecommunications systems applications.

Telecommunications systems reliability analysis has limited application as a useful performance metric. Telecommunications applications for which reliability is a useful metric include nonrepairable systems (such as satellites) or semirepairable systems (such as submarine cables). The reliability metric forms the foundation upon which availability and maintainability are built and thus must be fully understood.

1.1.1.1 The Reliability Function. The reliability function is a mathematical expression analytically relating the probability of success to time. In order to completely describe the reliability function, the concepts of the state variable and time to failure (TTF) must be presented.

The operational state of any item at a time t can be defined in terms of a state variable $X(t)$. The state variable $X(t)$ describes the operational state of a system, item, or mission at any time t. For the purposes of the analysis presented in this section, the state variable $X(t)$ will take on one of two values.[1]

$$X(t) = \begin{cases} 1 & \text{if the item state is functional or successful} \\ 0 & \text{if the item state is failed or unsuccessful} \end{cases} \tag{1.1}$$

The state variable is the fundamental unit of reliability analysis. All of the future analyses will be based on one of two system states at any given time, functional or failed ($X(t) = 1$ or $X(t) = 0$). Although this discussion is limited to the "functional" and "failed" states, the analysis can be expanded to allow $X(t)$ to assume any number of different states. It is not common for telecommunications systems to be analyzed for partial failure conditions, and thus these analyses are not presented in this treatment.

We can describe the operational functionality of an item in terms of how its operational state at time t translates to a TTF. The discrete TTF is a measure of the amount of time elapsed before an item, system, or mission fails. It should be clear that the discrete, single-valued TTF can be easily extended to a statistical model. In telecommunications reliability analysis, the TTF is almost always a function of elapsed time. The TTF can be either a discrete or continuous valued function.

Let the time to failure be given by a random variable T. We can thus write that probability that the time to failure T is greater than $t = 0$ and less than a time t (this is also known as the CDF $F(t)$ on the interval $[0,t)$) as[1]

$$F(t) = Pr(T \leq t) \quad \text{for } [0, t) \tag{1.2}$$

Recall from probability and statistics that the CDF can be derived from the probability density function (PDF) by evaluating the relationship

$$F(t) = \int_0^t f(u)\, du \quad \text{for all } t \geq 0 \tag{1.3}$$

where $f(u)$ is the PDF of the time to failure. Conceptually, the PDF represents a histogram function of time for which $f(t)$ represents the relative frequency of occurrence of TTF events.

The reliability of an item is the probability that an item does not fail for an interval $(0, t]$. For this reason, the reliability function $R(t)$ is also referred to as the survivor function since the item "survives" for a time t. Mathematically, we can write the survivor function $R(t)$ as[1]

$$R(t) = 1 - F(t) \quad \text{for } t > 0 \tag{1.4}$$

Recall that $F(t)$ represents the probability that an item fails on the interval $(0, t]$ so logically that the reliability is simply one minus that probability. Figure 1.1 shows the familiar Gaussian CDF and the associated reliability function $R(t)$.

1.1.2 Availability

In the telecommunications environment, the metric most often used in contracts, designs, and discussion is availability. The dictionary defines available as "present or ready for immediate use." This definition has direct applicability in the world of telecommunications. When an item or a system is referred to as being "available," it is inherently implied that the system is working. When the item or system is referred to as "unavailable," it is implied that the system has failed. Thus, when applied to a telecommunications item or system, the term availability implies how ready a system is for use. The technical definition of availability (according to MIL-STD-721C) is:

> "A measure of the degree to which an item or system is in an operable and committable state at the start of a mission when the mission is called for an unknown (random) time."

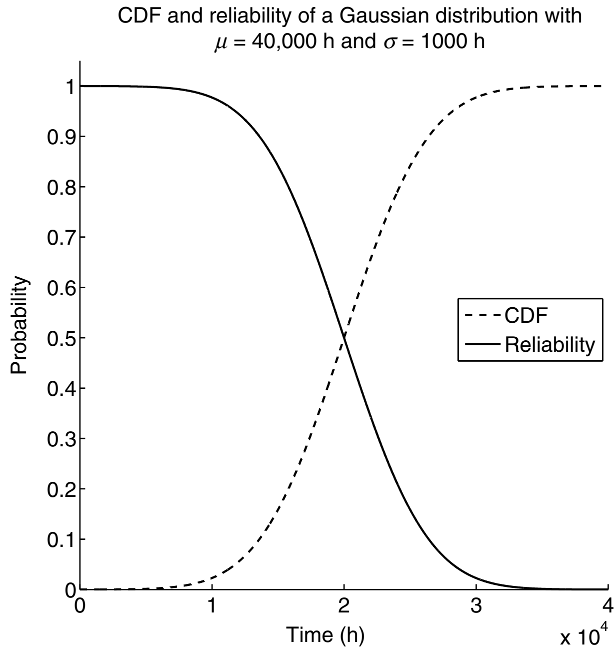

Figure 1.1. Gaussian CDF and associated reliability function $R(t)$.

Both the technical definition and the qualitative dictionary definition have the same fundamental meaning. This meaning can be captured by asking the question: "Is the system ready for use at any particular instant in time?" The answer to this question can clearly be formulated in terms of a statistical probability of readiness.

1.1.2.1 Availability Calculations. When examined as a statistical quantity, the availability of an item or a system can take on two different quantitative definitions. The average availability of an item or a system is the statistical probability of that item or system working over a defined period of time. For example, if an item's or a system's life cycle is considered to be 5 years and the availability of that item or system is of interest, then the availability of that item or system can be calculated as

$$A = \frac{\text{item or system uptime}}{\text{item or system operational time}} \tag{1.5}$$

In this case, the availability of the item or system is defined in terms of the percentage of time the item or system is working with respect to the amount of time the item or system has been in operation. (Note that the term "item" is used as shorthand to denote any item, system, or subsystem being analyzed.) This form of calculation of availability provides an average or mean availability over a specific period of time (defined by the operational time). One interesting item of note in this calculation is that the average availability as presented above provides very little insight with regard to the duration and/or frequency of outages that might occur, particularly in cases of long operational periods. When specifying average availability as a metric or design criteria, it is important to also specify maximum outage duration and failure frequency. Availability lifecycle or evaluation period must be carefully considered, particularly when availability is used as a metric for contract language. Availability targets that are achievable on an annual basis may be very difficult or impossible to achieve on monthly or even quarterly intervals. The time to repair and total system downtime have a great impact on availability over short intervals.

In order to visualize this concept, consider two different system designs, both of which achieve the same life-cycle availability. First, consider a system design with a replacement life cycle of 20 years. The system is designed to provide an average life-cycle availability of 99.9%. That is, the probability that the system is available at any particular instant in time is 0.999. The first system consists of a design with many redundant components. These individual components have a relatively poor reliability and need replacement on a regular basis. As a result, there are relatively frequent, short-duration outages that result from the dual failure of redundant components. This system is brought back online quickly, but has frequent outages. In the second system design, the components in use are extremely reliable but due to design constraints repair is difficult and therefore time consuming. This results in infrequent, long outages. Both systems achieve the same life-cycle availability but they do so in very different manners. The customer that uses the system in question would be well advised to understand both the mean repair time for a system failure as well as the most common expected failure modes in order to ensure that their expectations are met. Figure 1.2 provides a graphical

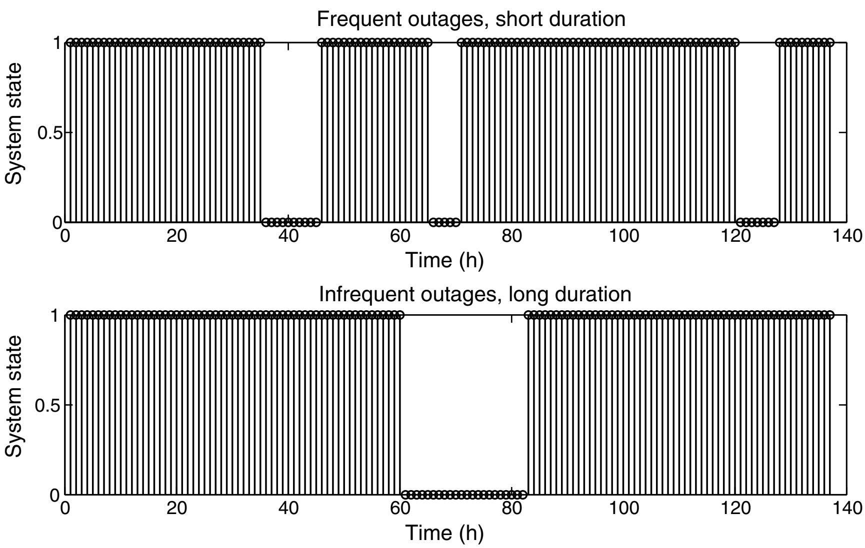

Figure 1.2. Average availability for system 1 (short duration, frequent outages) and system 2 (long duration, infrequent outages).

sketch of the scenario described above (note that the time scale has been exaggerated for emphasis).

The technical definition of availability need not be limited to the average or mean value. Availability can also be defined in terms of a time-dependent function $A(t)$ given by[1]

$$A(t) = Pr(X(t) = 1) \quad \text{for all } t \geq 0 \tag{1.6}$$

The term $A(t)$ specifies availability for a moment in time and is thus referred to as the instantaneous availability. The introduction of time dependence to the calculation of availability implies that the availability of an item can change with time. This could be due to a number of factors including early or late item failures, maintenance/repair practice changes, or sparing considerations. In most telecommunications system analyses, the steady-state availability is commonly used for system design or for contract language definitions. This assumption may not be appropriate for systems that require burn in or significant troubleshooting during system turn-up. Likewise, the system may become more unavailable as the system ages and vendors discontinue the manufacture of components or items begin to see late failures. The instantaneous availability $A(t)$ is related to the average availability A by the expression[1]

$$A_{\text{Average}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} A(t)dt \tag{1.7}$$

The most familiar form that availability takes in telecommunications system analysis is in relation to the mean time between failures (MTBF) and the mean time to repair (MTTR). These terms refer to the average (mean) amount of time that an item or a system is functioning (MTBF) between failure events and the average (mean) amount of time that it takes to place the item or system back into service. The average availability of a system can thus be determined by calculating[1]

$$A_{\text{Average}} = \frac{\text{MTBF}}{(\text{MTBF} + \text{MTTR})} \tag{1.8}$$

Availability is the average time between failures (operational time) divided by the average downtime plus the operational time (total time).

Unavailability is defined as the probability that the system is not functional at any particular instant in time or over a defined period of time. The expression for instantaneous unavailability is

$$U(t) = Pr(X(t) = 0) \quad \text{for all } t \geq 0 \tag{1.9}$$

where $U(t)$ represents time-dependent unavailability. The average value of unavailability is given by

$$U_{\text{Average}} = 1 - A_{\text{Average}} = \frac{\text{MTTR}}{(\text{MTBF} + \text{MTTR})} \tag{1.10}$$

It should be clear to the reader that calculations performed using the average expressions above are broad brush averages and do not give much insight into the variability of repair or failure events in the item or system. Calculation of availability using the expression above assumes that the sample set is very large and that the system achieves the average behavior. The applicability of this average availability value varies from system to system. In cases of relatively small-deployed component counts, this number may significantly misrepresent the actual achieved results of a system. For example, if the availability of a particular device (only one device is installed in the network of interest) is calculated using the average value based on a vendor provided MTBF and an assumed MTTR, one might be led to believe that the device availability is within the specifications desired. Consider a case where the MTTR has a high variability (statistical variance). Also consider that the device MTBF is very large, such that it might only be expected to fail once or twice in its lifetime. The achieved availability and the average availability could have very different values in this case since the variability of the repair period is high and the sample set is very small. The availability analyst must make careful consideration of not only the average system behavior but also the boundary behavior of the system being analyzed.

Bounding the achievable availability of an item or a system places bounds on the risk. Risk can be financial, technical, political, and so on, but risk is always present in a system design. Developing a clear understanding of system failure modes, expected system performance (both average and boundary value), and system cost reduces risk significantly and allows all parties involved to make the best, most informed decisions regarding construction and operations of a telecommunications system.

1.1.3 Maintainability

Maintainability as a metric is a measure of how quickly and efficiently a system can be repaired in order to ensure performance within the required specifications. MIL-STD-721C defines maintainability as:

> "The measure of the ability of an item to be retained in or restored to specified condition when maintenance is performed by personnel having specified skill levels, using prescribed procedures and resources, at each prescribed level of maintenance and repair."

The most common metric of maintainability used in telecommunications systems is the MTTR. This term refers to the average amount of time that a system is "down" or not operational. This restoral period can apply to either planned or unplanned outage events.

In the telecommunications environment, two types of downtime are typically tracked or observed. There are downtime events due to planned system maintenance such as preventative maintenance (PM), system upgrades, and system reconfiguration or growth. These types of events are typically coordinated with between the system operator and the customer and commonly fall outside of the contractual availability calculations. The second type of downtime event is the outage that occurs due to a failure in the system that results in a service outage. This system downtime is most commonly of primary interest to system designers, operators, and customers.

Scheduled or coordinated maintenance activities typically have predetermined downtime that are carefully controlled to ensure compliance with customer expectations. Such planned maintenance normally has shorter outage durations than unplanned maintenance or repair. Unplanned outages usually require additional time to detect the outage, diagnose its location, mobilize the repair activity, and get to the location of the failure to effect the repair. Unplanned outages that result from system failures result in downtime with varying durations. The duration and variability of the outage durations is dependent on the system's maintainability. A highly maintainable system will have a mean restoral period that is low relative to the system's interfailure period. In addition, the variance of the restoral period will also be small that ensures consistent, predictable outage durations in the case of a system failure event.

MTTR is commonly used interchangeably with the term mean downtime (MDT). MDT represents the sum of the MTTR and the time it takes to identify the failure and to dispatch for repair. Failure identification and dispatch in telecommunications systems can vary from minutes to hours depending on the system type and criticality.

In simple analyses, MDT is modeled assuming an exponential statistical distribution in which a repair rate is specified. Although this simplifying assumption makes the calculations more straightforward, it can result in significant inaccuracies in the resulting conclusions. Telecommunications system repairs more accurately follow normal or lognormal statistical distributions in which the repair of an item or a system has boundaries on the minimum and maximum values observed. The boundaries can be controlled by specifying both the mean and standard deviation of the repair period and by defining the distribution of repair based on those specifications.

MDT can be empirically calculated by collecting real repair data and applying best-fit statistical analysis to determine the distribution model and parameters that best represent the collected dataset.

1.1.4 Mean Time Between Failures, Failure Rates, and FITs

The most fundamental metric used in the analysis, definition, and design of telecommunications components is the MTBF. The MTBF is commonly specified by vendors and system engineers. It is a figure of merit describing the expected performance to be obtained by a component or a system. MTBF is typically provided in hours for telecommunications systems.

The failure rate metric is sometimes encountered in telecommunications systems. The failure rate describes the rate of failures (typically in failures per hour) as a function of time and in the general case is not a constant value. The most common visualization of failure rate is the bathtub curve where the early and late failure rates are much higher than the steady-state failure rate of a component (bottom of the bathtub). Figure 1.3 shows a sketch of the commonly observed "bathtub" curve for electronic systems. Note that although Figure 1.3 shows the failure rates early in system life and late in system life as being identical, in general, both the rate of failure rate change $dz(t)/dt$ and the initial and final values of failure rate are different.

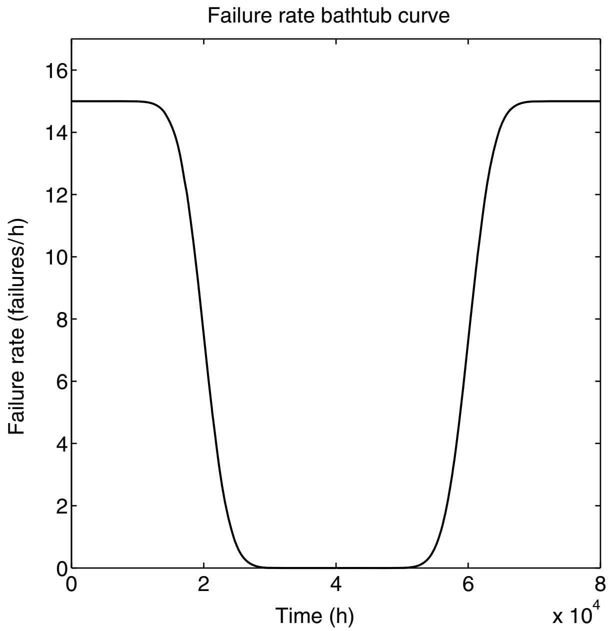

Figure 1.3. Bathtub curve for electronic systems.

A special case of the failure rate metric is the failures in time (FITs) metric. FITs are simply the failure rate of an item per billion hours:

$$\mathrm{FITS} = \frac{z(t)}{10^9} \tag{1.11}$$

where $z(t)$ is the time-dependent failure rate expression. FITS values provided for telecommunications items are almost exclusively constant.

1.1.4.1 MTBF. The mean time to failure defines the average or more specifically the expected value of the TTF of an item, subsystem, or a system. Reliability and availability models rely upon the use of random variables to model component performance. The TTF of an item, subsystem or system is represented by a statistically distributed random variable. The MTTF is the mean value of this variable. In almost all telecommunications models (with the exception of software and firmware), it is assumed that the TTF of a component is exponentially distributed and thus the failure rate is constant (as will be shown in Section 1.2.1). The mean time to failure can be mathematically calculated by applying (Bain and Englehardt, 1992)

$$\mathrm{MTTF} = E[\mathrm{TTF}] = \int_0^\infty t \cdot f(t)dt \tag{1.12}$$

This definition is the familiar first moment or expected value of a random variable. The commonly used MTBF can be approximated by the MTTF when the repair or restoral time (MDT) is small with respect to the MTTF. Furthermore, if the MTTF $< \infty$, then we can write the MTTF as (by applying $f(t) = -R'(t)$)[1]

$$\text{MTTF} = \int_0^\infty R(t)dt \tag{1.13}$$

This expression is particularly useful for calculating the MTTF (or MTBF) in many circumstances.

Telecommunications engineers must be particularly careful when using vendor-provided MTBF values. In many cases, the MTBF and the failure rate are presented as interchangeable inverses of each other. This special case is only true if one assumes that the TTF of a component is exponentially distributed. If the TTF of a component is not assumed to be exponentially distributed, this condition does not hold.

$$z(t) = -\frac{d}{dt}\ln R(t) \tag{1.14}$$

Note that except in the case where the TTF or TTR is exponentially distributed, the resultant failure rate is not constant. It is typically safe to assume that the MTBF and failure rate are inverses of each other if steady-state operation is assumed (see Figure 1.3). In the steady-state operation case, the failure rate is constant and the assumption of exponentially distributed TTFs holds. Early and late failure rates are time dependent and the exponential distribution assumption is invalid. Furthermore, if the system being considered employs redundancy, it does not necessarily hold that the redundant combination of components is exponentially distributed.

1.1.4.2 Failure Rates and FITs. The mathematical definition of failure rate is the probability that an item fails on an infinitesimally small interval (Δt) given that it has not failed at time t[1]

$$Pr(t < T \le t + \Delta t \mid T > t) = \frac{Pr(t < T \le t + \Delta t)}{Pr(T > t)} = \frac{F(t + \Delta t) - F(t)}{R(t)} \tag{1.15}$$

If we take equation 1.15 and divide by an infinitesimally small time Δt (on both the LHS and RHS), then the failure rate $z(t)$ is given by[1]

$$z(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \cdot \frac{1}{R(t)} = \frac{f(t)}{R(t)} \tag{1.16}$$

The failure rate of an item or a component can be empirically determined by examining the histogram statistics of failure events. Empirical determination of the failure rate of a component in telecommunications can provide valuable information. It is therefore important to collect failure data in an organized, searchable format such as a database.

This allows post processors to determine time to failure and failure mode. Empirical failure rate determination is of particular value for systems where the deployed component count is relatively high (generally greater than approximately 25 items for failure rates observed in nonredundant telecommunications systems). In these cases, the system will begin to exhibit observable statistical behavior. Observation of these statistics allows the system operator or system user to identify and address systemic or recurring problems within the system.

The empirical failure rate of a system can be tabulated by separating the time interval of interest into k disjoint intervals of duration Δt. Let $n(k)$ be the number of components that fail on the kth interval and let $m(k)$ be the number of functioning components on the kth interval. The empirical failure rate is the number of failures per interval functioning time. Thus, if each interval duration is Δt[1]

$$z(k) = \frac{n(k)}{m(k) \cdot \Delta t} \tag{1.17}$$

In cases of a large number of deployed components, the calculation of empirical failure rate can allow engineers to validate assumptions about failure distributions and steady-state conditions. Continuous or ongoing calculations of empirical failure rate can allow operators to identify infant mortality conditions or wear out proactively and preemptively deal with these issues before they cause major service-affecting outages.

Typical telecommunications engineers commonly encounter failure rates and FITs values when specifying subsystems or components during the system design process. Failure rates are rarely specified by vendors as time-dependent values and must be carefully examined when used in reliability or availability analyses. The engineer must ask him or herself whether the component failure rate is constant from a practical standpoint. If the constant failure rate assumption is valid, the engineer must then apply any redundancy conditions or requirements to the analysis. As will be seen later in this book, analysis of redundant systems involves several complications and subtleties that must be considered in order to produce meaningful results.

1.2 STATISTICAL DISTRIBUTIONS

System reliability analysis relies heavily on the application of theories developed in the field of mathematical probability and statistics. In order to model the behavior of telecommunications systems, the system analyst must understand the fundamentals of probability and statistics and their implications to reliability theory. Telecommunications system and component models typically use a small subset of the modern statistical distribution library. These distributions form the basis for complex failure and repair models. This section presents the mathematical details of each distribution of interest and discusses the applications for which those models are most relevant. The last section discusses distributions that may be encountered or needed on rare occasions. Each distribution discussion presents the distribution probability density function (PDF) and cumulative distribution function (CDF) as well as the failure rate or repair rate of the distribution.

1.2.1 Exponential Distribution

The exponential distribution is a continuous statistical distribution used extensively in reliability modeling of telecommunications systems. In reliability engineering, the exponential distribution is used because of its memory-less property and its relatively accurate representation of electronic component time to failure.[1] As will be shown in Section 1.3, there are significant simplifications that can be made if a component's time to failure is assumed to exponential.

The PDF of the exponential distribution is given by (Bain and Englehardt, 1992)

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{1.18}$$

Figure 1.4 shows a plot of the exponential PDF for varying values of λ. The values of λ selected for the figure reflect failure rates of one failure every 1, 3, or 5 years. These selections are reasonable expectations for the field of telecommunications and depend upon the equipment type and configuration.

Recalling that the CDF (Figure 1.5) for the exponential distribution can be calculated from the PDF (Bain and Englehardt, 1992)

$$F(x) = \int_0^\infty f(x)dx = \begin{cases} 1 - e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{1.19}$$

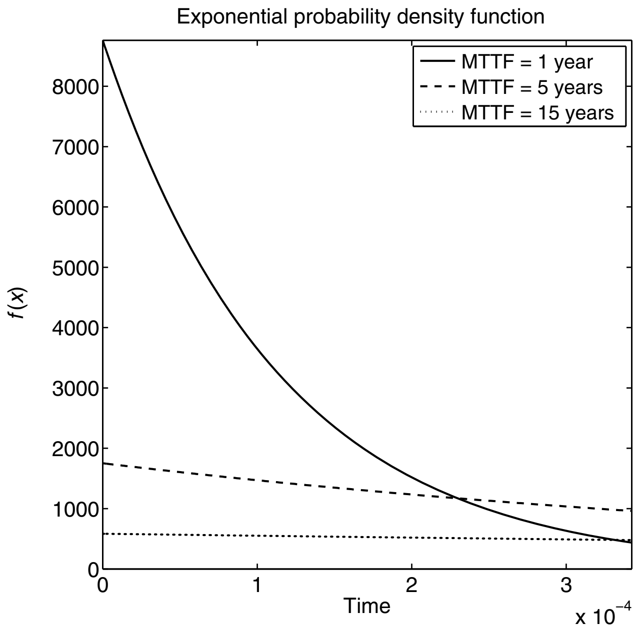

Figure 1.4. Exponential distribution PDF for varying values of λ.

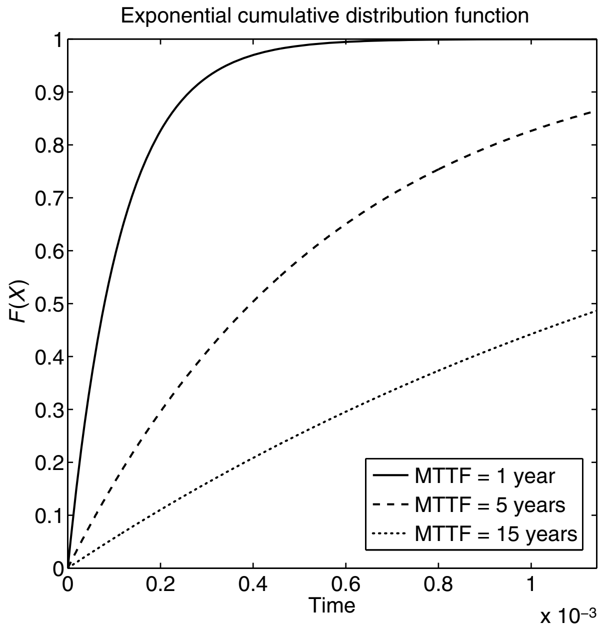

Figure 1.5. Exponential distribution CDF for varying values of λ.

Figure 1.5 plots the CDF for the same failure rates presented in Figure 1.4.

When using the exponential distribution to model the time to failure of an electronic component, there are several metrics of interest to be investigated. The mean time to failure (MTTF) of an exponential random variable is given by

$$\text{MTTF} = E[X] = \int_0^\infty x \cdot f(x)dx = \frac{1}{\lambda} \tag{1.20}$$

where $X \sim \text{EXP}(\lambda)$ with failure rate given by λ. Exponentially distributed random variables have several properties that greatly simplify analysis. Exponential random variables do not have a "memory." That is, the future behavior of a random variable is independent of past behavior. From a practical perspective, this means that if a component with an exponential time to failure fails and is subsequently repaired that repair places the component in "as good as new" condition.

The historical development of component modeling using exponential random variables is derived from the advent of semiconductors in electronic systems. Semiconductor components fit the steady-state constant failure rate model well. After an initial burn-in period exposes early failures, semiconductors exhibit a relatively constant failure rate for an extended period of time. This steady-state period can extend for many years in the case of semiconductor components. Early telecommunications systems consisted of circuit boards comprised of many discrete semiconductor components. As will be shown in Section 1.3, the failure rate of a serial system of many exponentially distributed semiconductor components is simply the sum of the individual component failure rates. Furthermore, since the sum of individual exponential random variables is an exponentially random variable, the failure rate of the resultant circuit board is exponentially distributed.

Modern telecommunications systems continue to use circuit boards comprised of many semiconductor devices. Modern systems use programmable components consisting of complex software modules. This software complicates analysis of telecommunications systems. Although the underlying components continue to exhibit exponentially distributed failure rates, the software operating on these systems is not necessarily exponentially distributed.

Although the exponential distribution is commonly used to model component repair, it is not well suited for this task. The repair of components typically is much more accurately modeled by normal, lognormal, or Weibull distributions. The reason that repair is typically modeled by an exponential random variable is due to the ease of analysis. As will be shown in Section 1.3, both the reliability block diagram (RBD) and Markov chain techniques of analysis rely upon the analyst assuming that repairs can be modeled by an exponential random variable. When the repair period of a system is very small with respect to the time between failures, this assumption is reasonable. When the repair period is not insignificant with respect to the time between failures, this assumption does not hold.

1.2.2 Normal and Lognormal Distributions

The normal (Gaussian) and lognormal distributions are continuous statistical distributions used to model a multitude of physical and abstract statistical systems. Both distributions can be used to model a large number of varying types of system repair behavior. In telecommunications systems, the failure can many times be well represented by the exponential distribution. Repair is more often well modeled by normal or lognormal random variables. System analysts or designers typically make assumptions or collect empirical data to support their system time to repair model selections. It is common to model the repair of a telecommunications system using a normal random variable since the normal distribution is completely defined by the mean and variance of that variable. These metrics are intuitive and useful when modeling system repair. In cases where empirical data is available, performing a best-fit statistical analysis to determine the best distribution for the time to repair model is recommended.

The PDF of the normal distribution (Bain and Englehardt, 1992) is given as

$$f(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{1.21}$$

The normal distribution should be familiar to readers. The mean value (μ) represents the average value of the distribution while the standard deviation (σ) is a measure of the variability of the random variable. The lognormal distribution is simply the distribution of a random variable whose logarithm is normally distributed. Figure 1.6 shows the PDF of a normal random variable with $\mu = 8$ h and $\sigma = 2$ h. These values of mean and standard deviation represent the time to repair for an arbitrary telecommunications system.

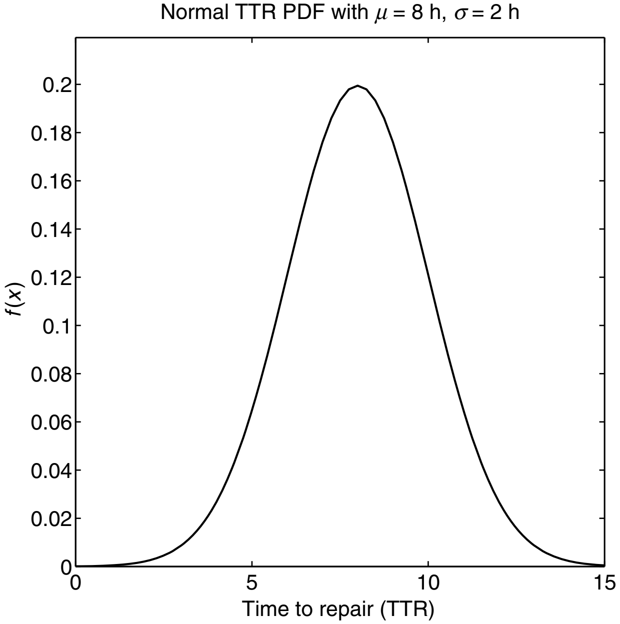

Figure 1.6. Normal distribution PDF of TTR, where $\mu = 8$ h and $\sigma = 2$ h.

The CDF of the normal distribution is given by a relatively complex expression involving the error function (erf) (Bain and Englehardt, 1992).

$$F(x,\ \mu, \sigma^2) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right) \tag{1.22}$$

Figure 1.7 shows the cumulative distribution function of the random variable in Figure 1.6. The CDF provides insight into the expected behavior of the modeled repair time. The challenge in application of normally distributed repair models comes from the combination of these random variables with exponentially distributed failure models. Neither the reliability block diagram nor the Markov chain techniques allow the analyst to use any repair distribution but the exponential distribution. The most practical method for modeling system performance using normal, lognormal, or Weibull distributions is to apply Monte Carlo methods. Reliability and failure rate calculations are not presented in this section as it would be very unusual to use a normally distributed random variable to model the time to failure of a component in a telecommunications system. Exceptions to this might occur in submarine cable systems or wireless propagation models.

1.2.3 Weibull Distribution

The Weibull distribution is an extremely flexible distribution in the field of reliability engineering. The flexibility of the Weibull distribution comes from the ability to model many different lifetime behaviors by careful selection of the shape (α) and scale (λ)

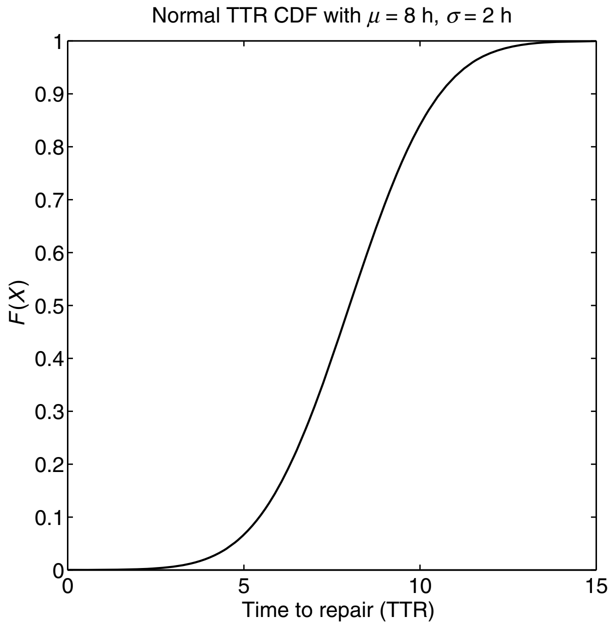

Figure 1.7. Normal distribution CDF of TTR, where $\mu = 8$ h and $\sigma = 2$ h.

parameters. Generally, all but the most sophisticated telecommunications systems failure performance models use exponentially distributed time to failure. The Weibull distribution gives the analyst a powerful tool for modeling the time to failure or time to repair of nonelectronic system components (such as fiber-optic cables or generator sets). Parameter selection for Weibull distributed random variables requires expert knowledge of component performance or empirical data to ensure that the model properly reflects the desired parameter.

The PDF of a Weibull distributed random variable $T \sim \text{Weibull}(\alpha, \lambda)$ with $\alpha > 0$ and $\lambda > 0$ is given by equation 1.23 while the CDF of the time to failure T is given by equation 1.24 (Bain and Englehardt, 1992).

$$f(t) = \begin{cases} \alpha\lambda^{\alpha} t^{\alpha-1} e^{-(\lambda t)^{\alpha}} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.23}$$

$$F(t) = Pr(T \leq t) = \begin{cases} 1 - e^{-(\lambda t)^{\alpha}} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.24}$$

The two parameters in the Weibull distribution are known as the scale (λ) and the shape (α). When the shape parameter $\alpha = 1$, the Weibull distribution is equal to the familiar exponential distribution where λ mirrors the failure rate as discussed in Section 1.2.1.

The reliability function of a Weibull distributed random variable can be calculated by applying the definition of reliability in terms of the distribution CDF. That is

$$R(t) = 1 - F(t) = Pr(T \geq t) = e^{-(\lambda t)^{\alpha}} \quad \text{for } t > 0 \tag{1.25}$$

Recalling that the failure rate of a random variable is given by

$$z(t) = \frac{f(t)}{R(t)} = \alpha\, \lambda^{\alpha} t^{\alpha-1} \quad \text{for } t > 0 \tag{1.26}$$

Empirical curve fitting or parameter experimentation are generally the best methods for selection of the shape and scale parameters for Weibull distributed random variables applied to telecommunications system models.

Figure 1.8 shows the PDF and CDF of a Weibull distributed random variable representing the time to repair of a submarine fiber-optic cable.

1.2.4 Other Distributions

The field of mathematical probability and statistics defines a very large number of statistical distributions. All of the statistical distributions defined in literature have potential for use in system models. The difficulty is in relating distributions and their parameters to physical systems.

System analysts and engineers must rely on academic literature, research, and expert knowledge to guide distribution selection for system models. This book focuses

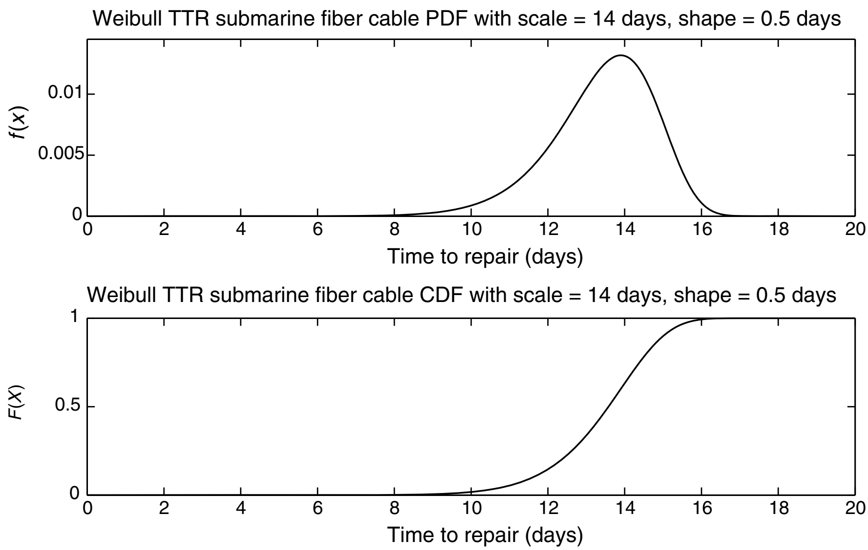

Figure 1.8. Weibull distributed random variable for submarine fiber-optic cable TTR.

on the presentation of relevant probability and statistics theory, and the concepts presented here have common practical application to telecommunications systems. More complex or less relevant statistical distributions not presented here are not necessarily irrelevant or inapplicable but rather must be used with care as they are not commonly used to model telecommunications systems.

On the most fundamental level, the entire behavior of a system or component is dictated by the analyst's selection of random variable distribution. As such, a significant amount of time and thought should be spent on the selection and definition of these statistical models. Care must be taken to ensure that the distribution selected is appropriate, relevant, and that it accurately reflects either the time to failure or time to repair behavior of the component of interest. Improper or incorrect distribution selection invalidates the entire model and the results produced by that model.

1.3 SYSTEM MODELING TECHNIQUES

Analysis of telecommunications systems requires accurate modeling in order to produce relevant, useful results. The metrics discussed in Section 1.1 are calculated by developing and analyzing system models. Many different reliability and availability modeling techniques exist. This book presents the methods and theories that are most relevant to the modeling and analysis of telecommunications systems. These techniques include RBD models, Markov chains, and Monte Carlo simulation. Each method has advantages and disadvantages. RBDs lend themselves to quick and easy results but sacrifice flexibility and accuracy, particularly when used with complex system topologies. Markov chain analysis provides higher accuracy but can be challenging to apply and requires models to use exponentially distributed random variables for both failure and repair rates. Monte Carlo simulation provides the ultimate in accuracy and flexibility but is the most complex and challenging to apply and is computationally intensive, even for modern computing platforms.

Availability is the most common metric analyzed in telecommunications systems design. Although reliability analysis can produce interesting and useful information, most systems are analyzed to determine the steady-state average (or mean) availability. RBDs and Markov chains presented in this chapter are limited to providing mean values of reliability or availability. Monte Carlo simulation techniques can be used to calculate instantaneous availabilities for components with nonconstant failure rates. The following sections present model theory and analysis techniques for each method discussed.

1.3.1 System Reliability

Analysis of system reliability requires the evaluation of interacting component random variables used to model failure performance of a system. This analysis is performed by evaluating the state of n discrete binary state variables $X_i(t)$, where $i = 1, 2, \ldots, n$.

Recall that the reliability function of a component is the probability that the component survives for a time t. Thus, the reliability of each component state variable $X_i(t)$ can be written as[1]

$$E[X_i(t)] = 0 \times Pr(X_i(t) = 0) + 1 \times Pr(X_i(t) = 1) = R_i(t) \text{ for } i = 1, 2, \ldots, n. \quad (1.27)$$

Equation 1.27 can be extended to the system case by applying[1]

$$R_S(t) = E[S(t)] \quad (1.28)$$

where $S(t)$ is the structure function of the component state vector $\mathbf{X}(t) = [X_1, X_2, \ldots, X_n]$. If we assume that the components of the system are statistically independent, then it can be shown that:[1]

$$R_S(t) = h(R_1(t), R_2(t), \ldots, R_n(t)) = h(R(t)) \quad (1.29)$$

1.3.2 Reliability Block Diagrams

RBDs are a common method for modeling the reliability of systems in which the order of component failure is not important and for which no repair of the system is considered. Many telecommunications engineers and analysts incorrectly apply parallel and serial reliability block diagram models to systems in which repair is central to the system's operation. Results obtained by applying RBD theory to availability models can produce varying degrees of inaccuracy in the output of the analysis. RBDs are success-based networks of components where the probability of mission success is calculated as a function of the component success probabilities. RBD theory can be understood most easily by considering the concept of a structure function. Figure 1.9 shows the reliability block diagram for both a series and a parallel combination of two components.

1.3.2.1 Structure Functions. Consider a system comprised of n independent components each having an operational state x_i. We can write the state of the ith component as shown in equation 1.30.[1] This analysis considers the component x_i to be a binary variable taking only one of two states (working or failed).

$$x_i = \begin{cases} 1 & \text{if the component is working} \\ 0 & \text{if the component has failed} \end{cases} \quad (1.30)$$

Thus, the system state vector $\mathbf{x}$ can be written as $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$. If we assume that knowledge of the individual states of x_i in $\mathbf{x}$ implies knowledge of the state of $\mathbf{x}$, we can write the structure function $S(\mathbf{x})$[1]

$$S(\mathbf{x}) = \begin{cases} 1 & \text{if the system is working} \\ 0 & \text{if the system has failed} \end{cases} \quad (1.31)$$

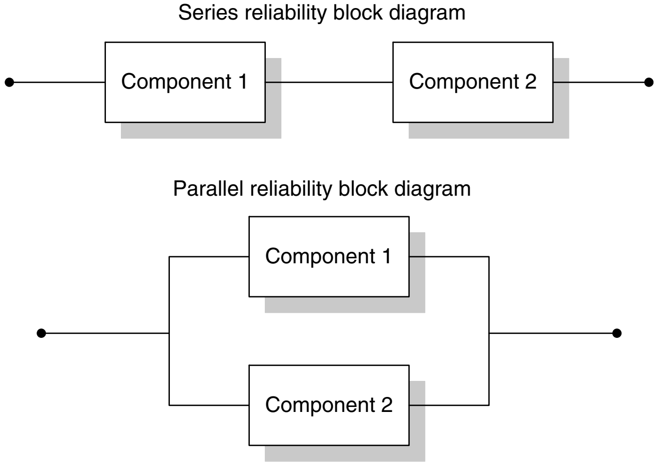

Figure 1.9. Series and parallel reliability block diagrams.[1]

where $S(\mathbf{x})$ is given by[1]

$$S(\mathbf{x}) = S(x_1,\ x_2,\ x_3 \ldots,\ x_n) \tag{1.32}$$

Thus, the structure function provides a resultant output state as a function of individual component states. It is important to note that reliability block diagrams are success-based network diagrams and are not always representative of system functionality. Careful development of RBDs requires the analyst to identify components and subsystems that can cause the structure function to take on the "working" or "failed" system state. In many cases, complex systems can be simplified by removing components from the analysis that are irrelevant. Irrelevant components are those that do not change the system state regardless of their failure condition.

RBDs can be decomposed into one of two different constituent structure types (series or parallel). It is instructive to analyze both of these system structures in order to develop an understanding of system performance and behavior. These RBD structures will form the basis for future reliability and availability analysis discussions.

1.3.2.2 Series Structures. Consider a system of components for which success is achieved if and only if all the components are working. This component configuration is referred to as a series structure (Figure 1.10). Consider a series combination of n components. The structure function for this series combination of

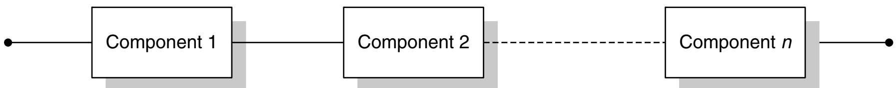

Figure 1.10. Series structure reliability block diagram.

components can be written as shown in equation 1.33, where x_n is the state variable for the nth component.[1]

$$S(\mathbf{x}) = x_1 \times x_2 \times \ldots \times x_n = \prod_{i=1}^{n} x_i \tag{1.33}$$

Series structures of components are often referred to as "single-thread" systems in telecommunications networks and designs. Single-thread systems are so named because all of the components in the system must be functioning in order for the system to function. Single-thread systems are often deployed in circumstances where redundancy is either not required or not practical. Deployment of single-thread systems in telecommunications applications often requires a trade-off analysis to determine the benefits of single-thread system simplicity versus the increased reliability of redundant systems.

The reliability of series structures can be computed by inserting equation 1.33 into equation 1.29 as shown below[1]

$$S(\boldsymbol{X}(t)) = \prod_{i=1}^{n} X_i(t) \tag{1.34}$$

$$R(S(t)) = E\left[\prod_{i=1}^{n} X_i(t)\right] = \prod_{i=1}^{n} E[X_i(t)] = \prod_{i=1}^{n} R_i(t) \tag{1.35}$$

It is worth noting that the reliability of the system is at most as reliable as the least reliable component in the system[1]

$$R(S(t)) \leq \min(R_i(t)) \tag{1.36}$$

Figure 1.11 shows a single-thread satellite link RF chain and the reliability block diagram for that system. The reliability of the overall system is calculated below.

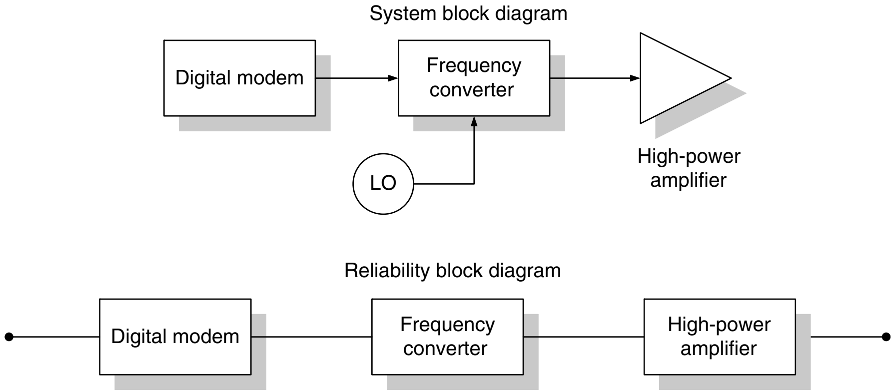

Figure 1.11. Single-thread satellite link RF chain.

Table 1.1. RF Chain Model Component Performance

Frequency Converter	Digital Modem	High-Power Amplifier
MTBF = 95,000 h R(4,380 h) = 95.5%	MTBF = 120,000 h R(4,380 h) = 96.4%	MTBF = 75,000 h R(4,380 h) = 94.3%

Assume that the components of the RF chain have the following representative failure metrics (Table 1.1). We will calculate the probability that system survives 6 months of operation ($t = (365 \times 24/2) = 4380$ h).

If we apply equation 1.35, we find that the system reliability is given by:

$$R(S(t)) = \prod_{i=1}^{n} R_i(t) = R_{\text{converter}} \times R_{\text{modem}} \times R_{\text{SSPA}} = 86.8\%$$

Although the relative reliabilities of the frequency converter, modem, and SSPA components are similar, the serial combination of the three elements results in a much lower predicted system reliability. Note that the frequency converter reliability includes the local oscillator.

1.3.2.3 Parallel Structures. Consider a system of components for which success is achieved if any of the components in the system are working. This component configuration is referred to as a parallel structure as shown in Figure 1.12. Consider a parallel combination of n components. The structure function for this parallel combination of components can be written as shown in equation 1.37, where x_n is the state variable for the nth component.[1]

$$S(\boldsymbol{x}) = 1 - (1 - x_1) \times (1 - x_2) \times \ldots (1 - x_n) = 1 - \prod_{i=1}^{n} (1 - x_i) \tag{1.37}$$

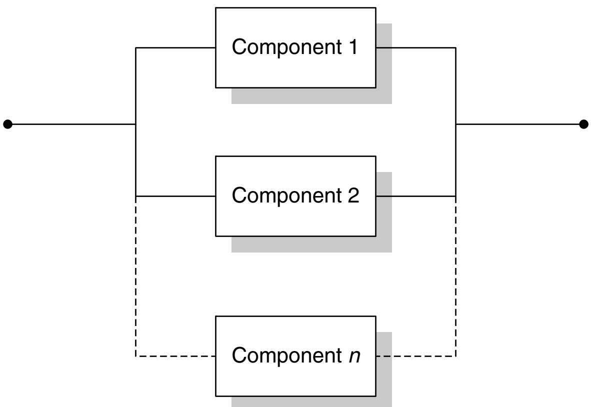

Figure 1.12. Parallel structure reliability block diagram.[1]

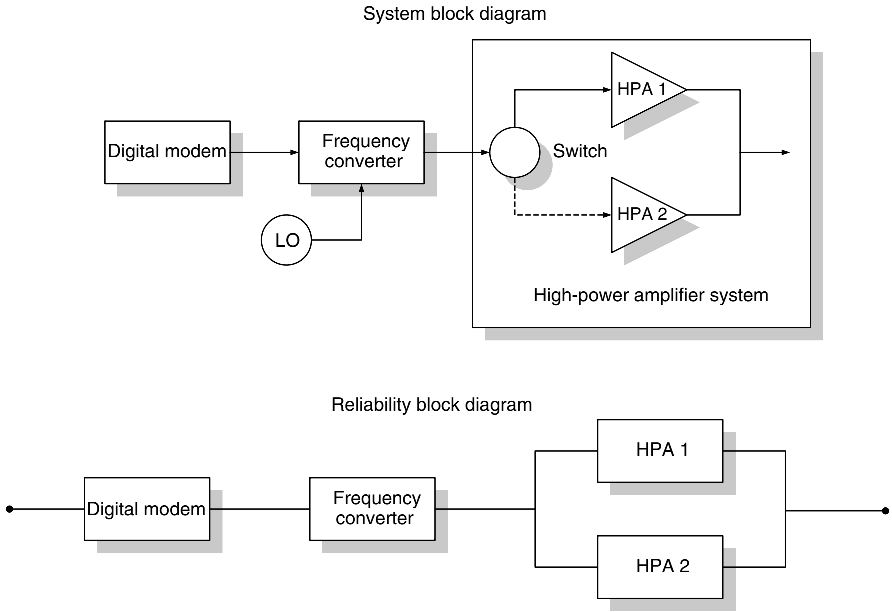

Figure 1.13. Parallel satellite RF chain system.

Parallel structures of components are often referred to as one-for-one or one-for-*n* redundant systems in telecommunications networks and designs. Redundant systems require the operation of only one of the components in the system for success. Figure 1.13 is a graphical depiction of a redundant version of the high power amplifier system portion of the satellite RF chain shown in Figure 1.11. This configuration of components dramatically increases the reliability of the RF chain but requires increased complexity for component failure switching. For the purposes of this simple example, the failure rate of the high-power amplifier switching component will be assumed to have a negligible impact on the overall system reliability.

Parallel system reliability is calculated by applying equation 1.37 to equation 1.29. The calculation follows the same procedure as shown in equations 1.34 and 1.35:

$$S(\boldsymbol{X}(t)) = 1 - \prod_{i=1}^{n}(1 - X_i(t)) \tag{1.38[1]}$$

where $S(X(t))$ is the redundancy structure function for the HPA portion of the RF chain.

$$R(S(t)) = E\left[1 - \prod_{i=1}^{n}(1 - X_i(t))\right] = 1 - \prod_{i=1}^{n}(1 - R_i(t)) \tag{1.39[1]}$$

Examining the reliability improvement obtained by implementing redundant high-power amplifiers, we find that the previously low reliability of the single-thread amplifier now far exceeds that of the reliability of the single-thread modem and frequency converter. It can generally be found that the most significant improvement in system reliability performance can be obtained by adding redundancy to critical system components. Inclusion of secondary or tertiary redundancy systems continues to improve performance but does not provide the same initially dramatic increase in reliability that is observed by the addition of redundancy to a component or a subsystem.

$$R(S(t)) = R_{\text{converter}} \times R_{\text{modem}} \times R_{\text{HPA}}$$

In this case, R_{HPA} is a redundant system

$$R_{\text{SSPA}_{\text{System}}} = 1 - \prod_{i=1}^{2}(1 - R_{\text{SSPA}_i}) = 1 - (1 - R_{\text{SSPA}}) \times (1 - R_{\text{HPA}}) = \ 99.7\%$$

Thus, the total system reliability is now

$$R(S(t)) = R_{\text{converter}} \times R_{\text{modem}} \times R_{\text{HPA}_{\text{system}}} = 91.8\%$$

1.3.2.4 k-Out-of-n Structures. The k-out-of-n structure is a system of components for which success is achieved if k or more of the n components in the system are working (this text assumes the "k-out-of-n: working" approach. A second approach is published in literature (Way and Ming, 2003), where success is achieved if k-out-of-n of the system components have failed "k-out-of-n: failed." This approach is not discussed here although the mathematics of this approach is very similar). This component configuration is referred to as a k-out-of-n structure. The parallel structure presented is a special case of the k-out-of-n structure, where $k = 1$ and $n = 2$ (one out of two). The structure function for this redundant combination of components can be written as shown in equation 1.40, where x_n is the state variable for the nth component (Way and Ming, 2003).

$$S(\boldsymbol{X}) = \begin{cases} 1 & \text{if} \sum_{i=1}^{n} x_i \geq k \\ 0 & \text{if} \sum_{i=1}^{n} x_i < k \end{cases} \tag{1.40}$$

k-out-of-n structures occur commonly in telecommunications systems. They are implemented in multiplexer systems, power rectification and distribution and RF power amplifiers among other systems. The advantage of implementing a k-out-of-n redundancy structure is cost savings. For example, a one-for-two redundancy configuration has $k = 2$ and $n = 3$. The one-for-two redundancy configuration is common in solid state power amplifier systems and power rectification systems, where modularity allows for expansion and cost savings. In this configuration, one of the three modules is redundant and thus $k = 2$. The cost savings that are obtained in this configuration can be substantial.

Consider the system where parallel or one-for-one redundancy is implemented.

$$\text{Total Modules Required} = (2\,\text{working modules}) + (2\,\text{protection modules}) = 4\,\text{modules}$$

Now consider the one-for-two configuration.

$$\text{Total Modules Required} = (2\,\text{working modules}) + (1\,\text{protection module}) = 3\,\text{modules}$$

The trade-off in this configuration is cost versus failure performance. The one-for-one configuration represents a 33% increase in component count over the one-for-two system. As will be shown through system reliability analysis, the reduction in reliability is relatively small and is often determined to be a reasonable sacrifice.

Calculation of k-out-of-n system reliability can be performed by observing that since the component failure events are assumed to be independent, we find that summation of the component states $S(\boldsymbol{X})$ is a binomially distributed random variable[1]

$$S(\boldsymbol{X}) = \sum_{i=1}^{n} X_i(t) \rightarrow S(\boldsymbol{X}) \sim \text{bin}(n, R(t)) \tag{1.41}$$

Note this treatment assumes that all of the components in the redundant system are identical. Recalling the probability of a specific binomial combination event[1]

$$Pr(S(\boldsymbol{X}) = y) = \binom{n}{y} R(t)^y (1 - R(t))^{n-y} \tag{1.42}$$

In the working k-out-of-n case, we are interested in the probability of the summation $S(\boldsymbol{X}) \geq k$.[1]

$$Pr(S(\boldsymbol{X}) \geq k) = \sum_{y=k}^{n} \binom{n}{y} R(t)^y (1 - R(t))^{n-y} \tag{1.43}$$

Equation 1.43 simply sums all of the discrete binomial probabilities for states in which the system is working.

Examination of the previously discussed HPA redundant system shows that the two-out-of-three configuration results in a relatively small reduction in reliability performance with a large cost savings (see Figure 1.14 for a system block diagram and the associated reliability block diagram for the 1:2 HPA system).

$$Pr(S(\boldsymbol{X}) \geq 2) = \sum_{y=2}^{3} \binom{3}{y} R_{\text{HPA}}{}^{y} (1 - R_{\text{HPA}})^{3-y}$$

$$Pr(S(\boldsymbol{X}) \geq 2) = \binom{3}{2} 0.943^2 (1 - 0.943)^1 + \binom{3}{3} 0.943^3 \approx 99.1\%$$

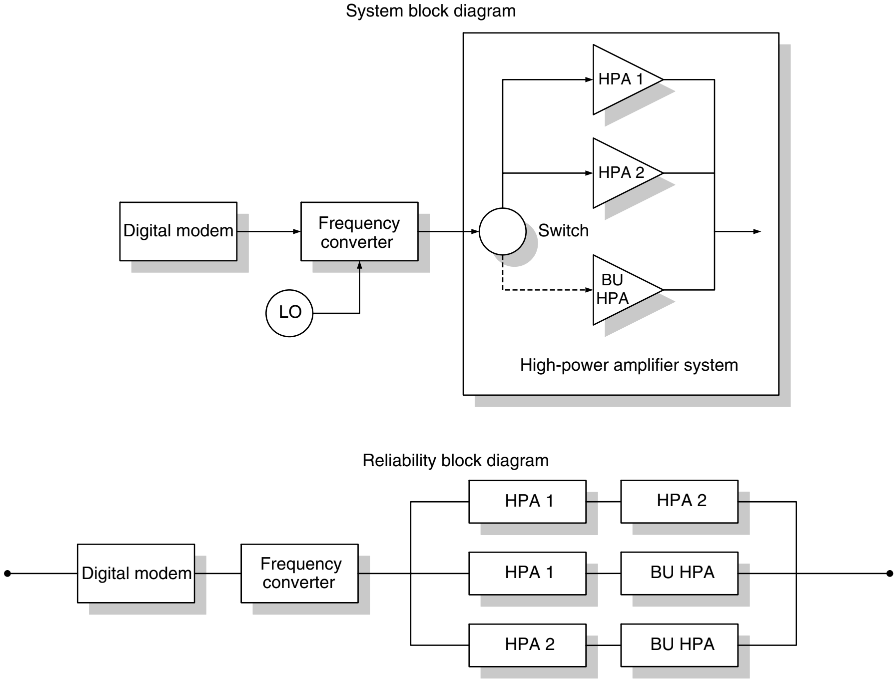

Figure 1.14. One-for-two (1:2) redundant HPA system block diagram.

1.4 SYSTEMS WITH REPAIR

The discussion of system failure performance up to this point has only examined systems in which repair is not possible or is not considered. Specifically, the term "reliability block diagram" refers to the success-based network of components from which system *reliability* is calculated. It is instructive to recall here that the definition of reliability is "the probability that an item can perform its intended function for a specified interval under stated conditions," as stated in Section 1.1. Thus, by definition, the behavior of the component or system following the first failure is not considered. In a reliability analysis, only the performance prior to the first failure is calculated.

For components (and systems of components), subject to repair after failure, different system modeling techniques must be used to obtain accurate estimates of system performance. The most common system performance metric used in repairable systems is availability. Availability is often used as a key performance indicator in telecommunications system design and frequently appears in contract service-level agreement (SLA) language. By specifying availability as the performance metric of interest, the analyst immediately implies that the system is repairable. Furthermore, by specifying availability, the applicability of RBDs as an analysis approach must be immediately discounted.

This section presents several key concepts related to repairable system analysis. These concepts include system modeling approaches, repair period models, and equipment sparing considerations. Each of these concepts plays an important role in the development of a complete and reasonable system model.

Two distinct modeling approaches are presented: Markov chain modeling and Monte Carlo simulation. Markov chain modeling is a state-based approach used to build a model in which the system occupies one of n discrete states at a time t. The probability of being in any one of the n states is calculated, thus resulting in a measure of system performance based on state occupation. Markov chain modeling is an extensively treated topic in literature and is useful in telecommunications system modeling of relatively simple system topologies. This book presents only a simple, abbreviated treatment of Markov chain analysis and interested readers are encouraged to do further research. More complex system configurations are better suited to Monte Carlo simulation-based models. Monte Carlo simulation refers to the use of numerical (typically computer-based) repetitive simulation of system performance. The Monte Carlo model is "simulated" for a specific system life many times and the lifetime failure statistics across many simulation "samples" are compiled to produce system performance statistics. Many performance metrics that can be easily derived from simulation results include failure frequency, time to failure (both mean and standard deviation), and availability among others. Although powerful results are available from applying Monte Carlo simulation, the development and execution of these models can be complex and tedious. An expert knowledge of reliability theory is often required to obtain confident results. Monte Carlo concepts and basic theory are presented in this section.

Repairable system models rely not only upon the assumed component TTF distributions (typically exponential) but also on the time to repair (TTR) distributions. It is shown in this chapter that one of the major drawbacks of applying Markov chain analysis techniques is that the TTR distribution must be exponential. This severely limits the models flexibility. In cases of electronic components or systems with TTF $\gg$ TTR, this assumption is often reasonable. It should be clear to the reader that an exponentially distributed random variable is an inherently poor model of telecommunications system repairs. Unfortunately, it is often the case in telecommunications systems that the TTF $\gg$ TTR assumption does not necessarily hold. This section presents the limitations and drawbacks of assuming an exponentially distributed TTR. In addition to the exponentially distributed time to repair, this section discusses Weibull, normal, and lognormal repair distributions and provides applications for these models.

The last section of this chapter presents the topic of system sparing. The concept of sparing in telecommunications systems should be familiar to anyone working in the field. Although the importance of sparing is typically recognized, it is often under-analyzed. Calculation of required sparing levels based on return material authorization (RMA) or component replacement period is presented. Cost implications and geographic considerations are also discussed. Component sparing can have significant impacts on the availability of a system but because it is not typically considered as part of the total system model, it is often overlooked and neglected.

1.5 MARKOV CHAIN MODELS

Consider a system consisting of a number of discrete *states* and *transitions* between those *states*. A Markov chain is a stochastic process (stochastic processes have behavior that is intrinsically nondeterministic) possessing the Markov property. The Markov property is simply the absence of "memory" within the process. This means that the current state of the system is the only state that has any influence on future events. All historical states are irrelevant and have no influence on future outcomes. For this reason, Markov processes are said to be "memory-less." It should be noted that Markov chains are not an appropriate choice for modeling systems where previous behavior has an affect on future performance.

To form a mathematical framework for the Markov chain, assume a process $\{X(t),\ t \geq 0\}$ with continuous time and a state space $\chi = \{0, 1, 2, \ldots, r\}$. The state of the process at a time s is given by $X(s) = i$, where i is the ith state in state space χ. The probability that the process will be in a state j at time $t + s$ is given by[1]

$$Pr(X(t+s) = j \mid X(s) = i),\ X(u) = x(u), 0 \leq u < s) \tag{1.44}$$

where $\{x(u), 0 \leq u < s\}$ denotes the processes "history" up to time s. The process is said to possess the Markov property if[1]

$$\begin{aligned} &Pr(X(t+s) = j \mid X(s) = i),\ X(u) = x(u),\\ &0 \leq u < s) = Pr(X(t+s) = j \mid X(s) = i) \quad \text{for all}\, x(u),\ 0 \leq u < s \end{aligned} \tag{1.45}$$

Processes possessing the behavior shown in equation 1.45 are referred to as Markov processes. The Markov process treatment presented in this book assumes time-homogeneous behavior. This means that system global time does not affect the probability of transition between any two states i and j. Thus[1]

$$Pr(X(t+s) = j|X(s) = i) = Pr(X(t) = j \mid X(0) = i) \quad \text{for all}\, s, t \tag{1.46}$$

Stated simply, equation 1.46 indicates that the probability of moving between states i and j is not affected by the current elapsed time. All moments in time result in the same probability of transition.

One classic telecommunications system problem is the calculation of availability for the one-for-one redundant system. Several different operational models exist in the one-for-one redundant system design. The system can be designed for hot-standby, cold-standby, or load-sharing operation. Each of these system design choices has an impact on the achievable system availability and the maintainability of the system. The Markov chain modeling technique is well suited to model systems of this type as long as the repair period is much shorter than the interfailure period (time to failure). This redundancy problem will be used to demonstrate the application and use of Markov chains in system modeling for the remainder of this section.

1.5.1 Markov Processes

Assume that a system can be modeled by a Markov process $\{X(t), t \geq 0\}$ with state space $\chi = \{0, 1, 2, \ldots, r\}$. Recall that the probability of transition between any two states i and j is time independent (stationary). The probability of a state transition from i to j is given by[1]

$$P_{ij}(t) = Pr(X(t) = j | X(0) = i) \quad \text{for all } i, j \in \chi \tag{1.47}$$

That is, P_{ij} is the probability of being in state j given that the system is in state i at time $t = 0$. One of the most powerful implications of the Markov process technique is the ability to represent these state transition probabilities in matrix form

$$P(t) = \begin{pmatrix} P_{00}(t) & \cdots & P_{0r}(t) \\ P_{10}(t) & & P_{1r}(t) \\ \vdots & \ddots & \vdots \\ P_{r0}(t) & \cdots & P_{rr}(t) \end{pmatrix} \tag{1.48}$$

Since the set of possible states $\chi = \{0, 1, 2, \ldots, r\}$ is finite and $i, j \in \chi$, for all $t \geq 0$, we find that the sum of all matrix row transition probabilities must necessarily be equal to unity.

$$\sum_{j=0}^{r} P_{ij}(t) = 1 \quad \text{for all } i \in \chi \tag{1.49}$$

The rows in the transition matrix represent the probability of a transition *out of* state i (where $i \neq j$) while the columns of the matrix represent the probability of transition *into* state j (where $i \neq j$).

From a practical perspective, the definition of a model using the Markov chain theory is relatively straightforward and simple. The approach presented here forgoes a number of mathematical subtleties in the interest of practical clarity. Readers interested in a more mathematical (and rigorous) treatment of the Markov chain topic are referred to Rausand and Høyland (2004).

As shown in equation 1.48, the Markov chain can be represented as a matrix of values indicating the probability of either entering or leaving a specific state in the state space χ. We introduce the term "sojourn time" to indicate the amount of time spent in any particular state i. It can be shown that the mean sojourn time in state i can be expressed as[1]

$$E(\tilde{T}_\iota) = \frac{1}{\propto_i} \tag{1.50}$$

where α_i is the rate of transition from state i to another state in the state space (rate out of state i). Since the process is a Markov chain, it can also be shown that the sojourn time (and thus the transition rate α_i) must be exponentially distributed and that all sojourn times must be independent. These conditions ensure that the Markov chain's memory-less property is maintained. Analyses presented here assume $0 \leq \alpha_i \leq \infty$.

This assumption implies that no states are instantaneous ($\alpha_i \to \infty$) or absorbing ($\alpha_i \to 0$). Instantaneous states have a sojourn time equal to zero while absorbing states have an infinite sojourn time. We only consider states with finite sojourn durations.

Let the variable a_{ij} be the rate at which the process leaves state i and enters state j. Thus, the variable a_{ij} is the transition rate from i to j.[1]

$$a_{ij} = \propto_i \times P_{ij} \quad \text{for all } i \neq j \tag{1.51}$$

Recall that α_i is the rate of transition out of state i and P_{ij} is the probability that the process enters state j after exiting state i. It is intuitive that when leaving state i, the process must fall into one of the r available states, thus[1]

$$\propto_i = \sum_{\substack{j=0 \\ j\neq i}}^{r} a_{ij} \tag{1.52}$$

Since the coefficients a_{ij} can be calculated for each element in a matrix **A** by applying equation 1.51, we can define the transition rate matrix as shown in equation 1.53.[1]

$$\mathbb{A} = \begin{pmatrix} a_{00} & \cdots & a_{0r} \\ a_{10} & & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r0} & \cdots & a_{rr} \end{pmatrix} \tag{1.53}$$

The sum of all transition probabilities P_{ij} for each row must be equal to one, thus we can write the diagonal elements of $\mathbb{A}$ as[1]

$$a_{ii} = -\propto_i = -\sum_{\substack{j=0 \\ j\neq i}}^{r} a_{ij} \tag{1.54}$$

The diagonal elements of $\mathbb{A}$ represent the sum of the departure and arrival rates for a state i. Markov processes can be visualized using a state transition diagram. This diagram provides an intuitive method for developing the transition rate matrix for a system model. It is common in state transition diagrams to represent system states by circles and transitions between states as directed segments. Figure 1.15 shows a state transition diagram for a one-for-one redundant component configuration. If both of the redundant components in the system are identical, the transition diagram can be further simplified (see Figure 1.16).

The procedure for establishing a Markov chain model transition rate matrix $\mathbb{A}$ involves several steps.

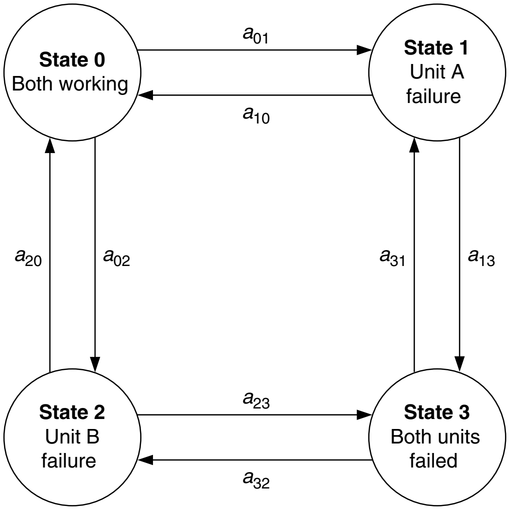

Figure 1.15. Redundant Markov chain state diagram.

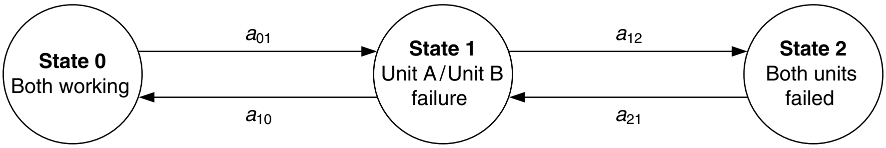

Figure 1.16. Redundant Markov chain state diagram, identical components.

Step 1. The first step in developing the transition rate matrix is to identify and describe all of the system states relevant to operation. Recall that relevant states are those that can affect the operation of the system. Irrelevant states are system states that do not affect system operation or failure, regardless of the condition. The identified relevant system states are then given an integer state identifier

$$S_i \in \chi, \quad \text{where } \chi = \{0,\ 1, \ldots, r\} \tag{1.55}$$

Step 2. Having identified the system states to be modeled, the transition rates to and from each state must be determined. In basic reliability analysis, these transition rates will almost always correspond to a component failure or repair rate. Component failure rates can typically be derived from system documentation, empirical data, or expertise in a field of study. Component repair rates are often based on assumptions, experience, or system requirements. In Figure 1.15, the transition rates $\{a_{01}, a_{02}, a_{13}, a_{23}\}$ all represent component failure transition rates while the rates $\{a_{10}, a_{20}, a_{32}, a_{31}\}$ represent repair transition rates. Table 1.2 shows a tabulation of the transition rate, the common nomenclature used to represent each rate, and representative values for a 1:1 redundant high-power amplifier system.

Table 1.2. Markov Chain Transition Rate Matrix Table Example

Transition Rate	Commonly Used Term	Example Value (Failures/h)
$a_{01}, a_{02}, a_{13}, a_{23}$	λ_{HPA}	1.33×10^{-5}
$a_{10}, a_{20}, a_{32}, a_{31}$	μ_{HPA}	8.33×10^{-2}

Step 3. The values in Table 1.2 are inserted into the transition rate matrix $\mathbb{A}$ in their appropriate positions as shown below.

$$\mathbb{A} = \begin{pmatrix} \boldsymbol{a_{00}} & a_{01} & a_{02} & a_{03} \\ a_{10} & \boldsymbol{a_{11}} & a_{12} & a_{13} \\ a_{20} & a_{21} & \boldsymbol{a_{22}} & a_{23} \\ a_{30} & a_{31} & a_{32} & \boldsymbol{a_{33}} \end{pmatrix}$$

The tabulated failure and repair rates for each transition replace the interstate coefficients.

$$\mathbb{A} = \begin{pmatrix} \boldsymbol{a_{00}} & \lambda_{\text{HPA}} & \lambda_{\text{HPA}} & 0 \\ \mu_{\text{HPA}} & \boldsymbol{a_{11}} & 0 & \lambda_{\text{HPA}} \\ \mu_{\text{HPA}} & 0 & \boldsymbol{a_{22}} & \lambda_{\text{HPA}} \\ 0 & \mu_{\text{HPA}} & \mu_{\text{HPA}} & \boldsymbol{a_{33}} \end{pmatrix}$$

Step 4. The diagonal elements of the transition rate matrix are populated by applying equation 1.54 along each row. The resultant, completed transition rate matrix is shown below.

$$\mathbb{A} = \begin{pmatrix} -(\lambda_{\text{HPA}} + \lambda_{\text{HPA}}) & \lambda_{\text{HPA}} & \lambda_{\text{HPA}} & 0 \\ \mu_{\text{HPA}} & -(\lambda_{\text{HPA}} + \mu_{\text{HPA}}) & 0 & \lambda_{\text{HPA}} \\ \mu_{\text{HPA}} & 0 & -(\lambda_{\text{HPA}} + \mu_{\text{HPA}}) & \lambda_{\text{HPA}} \\ 0 & \mu_{\text{HPA}} & \mu_{\text{HPA}} & -(\mu_{\text{HPA}} + \mu_{\text{HPA}}) \end{pmatrix}$$

Careful consideration of the relevant states in Step 1 of the transition rate matrix definition can result in simplifications. Consider the system diagram shown in Figure 1.16. If the redundant components shown were assumed to be identical (as presented in Table 1.2), the system model could be shown as having three distinct states instead of four.

The transition rate matrix for Markov chain in Figure 1.16 is given by

$$\mathbb{A} = \begin{pmatrix} -2\lambda_{\text{HPA}} & 2\lambda_{\text{HPA}} & 0 \\ \mu_{\text{HPA}} & -(\mu_{\text{HPA}} + \lambda_{\text{HPA}}) & \lambda_{\text{HPA}} \\ 0 & \mu_{\text{HPA}} & -\mu_{\text{HPA}} \end{pmatrix}$$

As Figure 1.16 shows, the complexity of the system model is greatly reduced with no loss of accuracy in the case where the two redundant components are identical.

1.5.2 State Equations

In order to solve the Markov chain for the relative probabilities of occupation for each system state, we must apply two sets of equations. Through analysis of the Chapman-Kolmogorov equations, it can be shown[1] that the following differential equation can be derived.

$$\dot{\mathbb{P}}(t) = \mathbb{P}(t) \cdot \mathbb{A} \tag{1.56}$$

where $\mathbb{P}(t)$ is the time-dependent state transition probability matrix and $\mathbb{A}$ is the transition rate matrix. The set of equations resulting from the matrix in Equation 1.56 are referred to as the Kolmogorov forward equations.

Assuming that the Markov chain is defined to occupy state 0 at time $t = 0$, $X(0) = i$ and $P_i(0) = 1$ while all other probabilities $P_k(0) = 0$ for $k \neq i$. This simply means that by defining the system to start in state i at time $t = 0$, we have forced the probability of occupation for state i at time $t = 0$ to be unity while the probability of being in any other state is zero. By defining the starting state, we can simplify equation 1.56 to the following form.

$$\begin{pmatrix} a_{00} & \cdots & a_{0r} \\ a_{10} & & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r0} & \cdots & a_{rr} \end{pmatrix} \cdot \begin{bmatrix} P_0(t) \\ P_1(t) \\ \vdots \\ P_r(t) \end{bmatrix} = \begin{bmatrix} \dot{P}_0(t) \\ \dot{P}_1(t) \\ \vdots \\ \dot{P}_r(t) \end{bmatrix} \tag{1.57}$$

Equation 1.57 does not have a unique solution but by applying the initial condition ($P_i(0) = 1$) and recalling that the sum of each column is equal to one, we can often find a solution to the set of equations. In practical problems, it is rare that the system of equations does not result in a real, finite solution.

Solutions to equation 1.57 are time dependent. Analyses performed on telecommunications systems are often interested in the steady-state solution to equation 1.57. In these circumstances, we can further simplify our problem by examining the behavior of equation 1.57 as $t \to \infty$. It can be shown[1] that after a long time ($t \to \infty$), the probability of occupation for a particular system state is not dependent on the initial system state. Furthermore, if the probability of state occupation is constant, it is clear that the derivative of that probability is necessarily zero.

$$\lim_{t \to \infty} P_j(t) = P_j \quad \text{for } j = 1, 2, \ldots, r \tag{1.58}$$

$$\lim_{t \to \infty} \dot{P}_j(t) = 0 \quad \text{for } j = 1, 2, \ldots, r \tag{1.59}$$

Thus, we can rewrite equation 1.57 as

$$\begin{pmatrix} a_{00} & \cdots & a_{0r} \\ a_{10} & & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r0} & \cdots & a_{rr} \end{pmatrix} \cdot \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{1.60}$$

Solution of equation 1.60 for each P_j relies upon use of linear set of algebraic equations and the column sum for each column j.

$$\sum_{j=0}^{r} P_j = 1 \tag{1.61}$$

1.5.3 State Equation Availability Solution

The system *availability* or *unavailability* is easily calculated once the state equations have been solved for the vector $\mathbb{P}$.

Define the set of all possible system states $S = \{S_0, S_1, \ldots, S_r\}$. Define a set W as the subset of S containing only the states in S where the system is working. Define another set F as the subset of S containing only those states where the system has failed. The availability of the system is the sum of all probabilities in W.[1]

$$A = \sum_{j \in W} P_j, \quad \text{where } W \in S \tag{1.62}$$

The unavailability of the system is likewise the sum of P_j over all states where the system has failed. Alternatively, the unavailability can be calculated by recognizing that the sum of the availability and unavailability must be unity.

$$1 = \sum_{j \in W} P_j + \sum_{j \in F} P_j \tag{1.63}$$

Replacing the sum in equation 1.63 and rearranging[1]

$$1 - A = \sum_{j \in F} P_j \tag{1.64}$$

Thus, calculating the availability immediately provides us with the unavailability as well.

1.6 PRACTICAL MARKOV SYSTEM MODELS

Markov system models have been used extensively in many industries to model the reliability of a variety of systems. Within the field of telecommunications and with the advent of modern computing techniques, the application of Markov chain modeling methods in telecommunications systems models is limited to a few special cases.

Markov models can provide quick, accurate assessments of redundant system availabilities for relatively simple topologies. Systems in which the time-to-repair distribution is not exponential or where the redundancy configuration is complex are not good candidates for practical Markov models. Complex mathematics and sophisticated matrix operations in those types of models should lead the engineer to consider Monte Carlo simulation in those circumstances.

The Markov chain modeling technique is well suited to redundancy models consisting of a small number of components. The mathematics of analyzing 1:1 or

1:2 redundancies remains manageable and typically do not require numerical computation or computer assistance. For this reason, the engineer can usually obtain results much more quickly than would be possible using a Monte Carlo analysis approach. In many cases, a full-blown system model is not required and only general guidelines are desired in the decision-making process.

Although the scope of practical Markov system models is somewhat limited, the types of problems that are well suited for Markov analysis are common and practical. This section will present the Markov model for the following system types.

1. Single-component system model
2. Hot-standby redundant system model
3. Cold-standby redundant system model

Each of the models listed above represent a common configuration deployed in modern telecommunications systems. These models apply to power systems, multiplexing systems, amplifier systems, and so on.

1.6.1 Single-Component System Model

The simplest Markov chain model is the model for a single component. This model consists of two system states $S = \{S_0, S_1\}$.

Let S_0 be the working component state and S_1 be the failed component state. Figure 1.17 is the Markov state transition diagram for this system model.

The transition rate matrix is very straightforward, consisting of four coefficients. If we let the failure rate of a component be defined as λ and the repair rate of the component be defined as μ, we have (by applying the steps listed previously)

$$\mathbb{A} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

Applying equation 1.60, we can solve the state equations to determine the probabilities of state occupation $\mathbb{P} = [P_0 \ P_1]$.

$$\mathbb{P} \cdot \mathbb{A} = \bar{0} = [P_0 \quad P_1] \cdot \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

The set of linear equations is thus:

$$\begin{aligned} -\lambda P_0 + \mu P_1 &= 0 \quad (1) \\ \lambda P_0 - \mu P_1 &= 0 \quad (2) \\ P_0 + P_1 &= 1 \quad (3) \end{aligned}$$

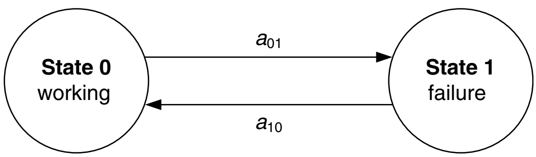

Figure 1.17. Single-component Markov state transition diagram.

Solving the equations using (1) and (3), we obtain

$$P_0 = \frac{\mu}{\mu + \lambda} \quad \text{and} \quad P_1 = \frac{\lambda}{\mu + \lambda}$$

Calculation of the component availability is straightforward once the individual state probabilities have been determined. Let the set $W = \{S_0\}$ and $F = \{S_1\}$. Thus, the availability of the system is simply equal to

$$A = \sum_{j \in W} P_j = P_0$$

Recall that we earlier made the assumption (in order to preserve the Markov property) that the state transition probabilities were exponentially distributed random variables and thus the transition rates λ and μ are constant, so we can write

$$\text{MTBF} = \frac{1}{\lambda}, \quad \text{MTTR} = \frac{1}{\mu}$$

If we rewrite the expression for P_0 in terms of MTBF and MTTR, we find

$$P_0 = \frac{\frac{1}{\text{MTTR}}}{\frac{1}{\text{MTTR}} + \frac{1}{\text{MTBF}}} = \frac{\text{MTBF}}{\text{MTBF} + \text{MTTR}}$$

This is the same result that was obtained in Section 1.1.

1.6.2 Hot-Standby Redundant System Model

Consider a system consisting of two identical components that are both operating continuously. This particular system does not implement load sharing but rather one of the two components carries the entire load at any given time. Upon failure of one of the components, the system immediately switches from the primary module to the backup (redundant) module.

In our hot-standby model (Figure 1.18), we have three system states $S = \{S_0, S_1, S_2\}$. Define the systems states as described in Table 1.3.

One of the disadvantages of the hot-standby redundancy configuration is that during operation, the backup module accumulates life-cycle operational hours that ultimately lead to the failure of that module. The module is in operation only to ensure that the system continues to operate if the primary module fails. In a cold-standby system, the backup module is not operated until such time that the primary module fails. This "saves" the operational hours of the backup modules for use when the component is doing real work.

Definition of the transition rate matrix follows the same procedure used previously.

$$\mathbb{A} = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu + \lambda) & \lambda \\ \mu & 0 & -\mu \end{pmatrix}$$

Table 1.3. Hot-Standby System State Descriptions

State	System Operating Condition	Description
S_0	Working	Both modules working, system in nominal condition
S_1	Working	Single module failure, system operating with one module failed
S_2	Failure	Dual module failure, system failure

Applying the state equation matrix definition to determine the linear algebraic equations in terms of state occupation probabilities, $\mathbb{P} = [P_0\ P_1\ P_2]$.

$$\mathbb{P} \cdot \mathbb{A} = \bar{0} = [P_0 \quad P_1 \quad P_2] \cdot \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ \mu & 0 & -\mu \end{pmatrix}$$

$$\begin{aligned} -2\lambda P_0 + \mu P_1 + \mu P_2 &= 0 \\ 2\lambda P_0 - (\mu+\lambda) P_1 &= 0 \\ \lambda P_1 - \mu P_2 &= 0 \\ P_0 + P_1 + P_2 &= 1 \end{aligned}$$

Solving the simultaneous equations for $\mathbb{P}$, we find

$$P_0 = \frac{\mu}{2\lambda + \mu}$$

$$P_1 = \frac{2\lambda\mu}{(\lambda+\mu)(2\lambda+\mu)}$$

$$P_2 = \frac{2\lambda^2}{(\lambda+\mu)(2\lambda+\mu)}$$

We now define the subsets of S for which the system is working and failed. In the working case, we have $W = \{S_0, S_1\}$ and for the failed case, we have $F = \{S_2\}$. Thus, we

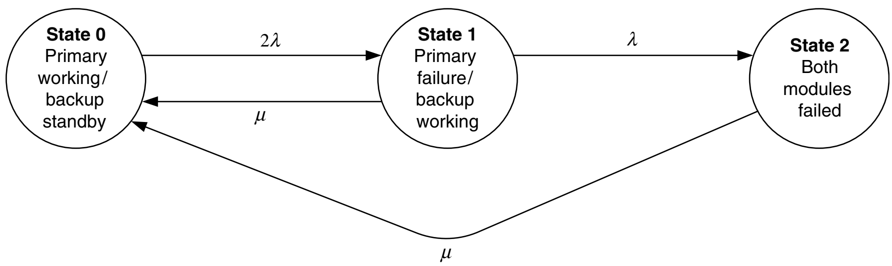

Figure 1.18. Hot-standby redundant Markov state transition diagram.

can calculate the availability of the system to be

$$A = \sum_{j \in W} P_j = P_0 + P_1 = \frac{\mu^2 + 3\lambda\mu}{(\lambda + \mu)(2\lambda + \mu)}$$

1.6.3 Cold-Standby Redundant Model

Analysis of the cold-standby redundant model follows the same process that was used in the hot-standby model. In this case, the assumptions are changed slightly, resulting in a modified state transition diagram and a different overall result. The diagram shown in Figure 1.19 shows the modified state transition diagram.

Note that in this case, we have assumed that a failure of both units will force a repair that places the working module back into operation and simultaneously repairs the standby module making it ready for service once again. Also note that during normal operation, only one of the two modules is accumulating operational hours ($a_{01} = \lambda$).

Continuing with the same analysis procedure used in the hot-standby case, we define each of the system states $S = \{S_0, S_1, S_2\}$ as in Table 1.4.

The transition rate matrix is given by

$$\mathbb{A} = \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu + \lambda) & \lambda \\ \mu & 0 & -\mu \end{pmatrix}$$

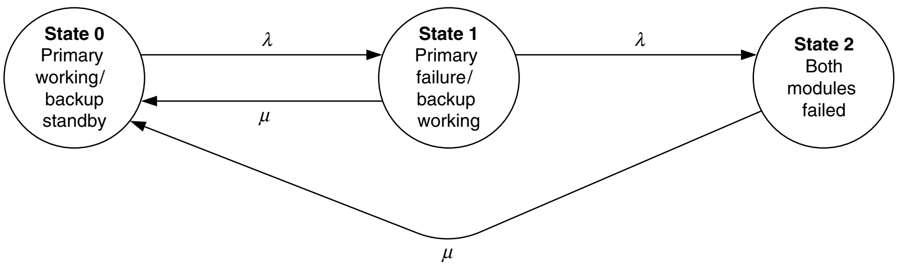

Figure 1.19. Cold-standby Markov state transition diagram.

Table 1.4. Cold-Standby System State Descriptions

State	System Operating Condition	Description
S_0	Primary working Backup standby	Primary module working, backup in standby mode, system working
S_1	Primary failed Backup working	Primary module failure, backup module operating, system working
S_2	Primary failed Backup failed	Primary module failure, backup module failure, system failure

Applying the state equation matrix definition to determine the linear algebraic equations in terms of state occupation probabilities, $\mathbb{P} = [P_0\ P_1\ P_2]$.

$$\mathbb{P} \cdot \mathbb{A} = \bar{0} = [P_0 \quad P_1 \quad P_2] \cdot \begin{pmatrix} -\lambda & \lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ \mu & 0 & -\mu \end{pmatrix}$$

Solving the simultaneous equations for $\mathbb{P}$, we find

$$P_0 = \frac{\mu}{\lambda + \mu}$$

$$P_1 = \frac{\lambda\mu}{\lambda^2 + 2\lambda\mu + \mu^2}$$

$$P_2 = \frac{\lambda^2}{\lambda^2 + 2\lambda\mu + \mu^2}$$

As previously mentioned, the availability is calculated by defining the subsets of S for which the system is working and failed. In the working case, we have $W = \{S_0, S_1\}$ and for the failed case, we have $F = \{S_2\}$. Thus, we can calculate the availability of the system to be

$$A = \sum_{j \in W} P_j = P_0 + P_1 = \frac{\mu^2 + 2\lambda\mu}{\lambda^2 + 2\lambda\mu + \mu^2}$$

As a comparison of relative performance between the hot-standby and cold-standby availabilities, consider a generator system in which the MTBF of a particular generator set is approximately 8000 h (about 1 year). Assume that the MTTR for the system is approximately 24 h. In the hot-standby case, we find that the availability is

$$A_{\text{hot}} = \frac{\mu^2 + 3\lambda\mu}{2\lambda^2 + 3\lambda\mu + \mu^2} \approx 99.9982\%$$

where we have calculated the values for μ and λ by applying

$$\mu = \frac{1}{\text{MTTR}} \quad \text{and} \quad \lambda = \frac{1}{\text{MTBF}}$$

The cold-standby case provides an increase in availability performance since the standby component is not operational until the primary unit fails. Even when the backup unit is called upon to operate, its time in service is very short compared with the primary unit.

$$A_{\text{cold}} = \frac{\mu^2 + 2\lambda\mu}{\lambda^2 + 2\lambda\mu + \mu^2} \approx 99.9991\%$$

It should be noted that in electronic telecommunications systems, the MTBF is generally very large (typically greater than 150,000 h) and the MTTR is often less than 8 h. It should be clear that the cold-standby redundancy configuration is preferable, particularly in systems where the failure rate is significantly increased in hot-standby operation. Within the hot-standby redundancy configuration is an approach known as monitored hot standby (MHSB). MHSB systems are often preferred when component or system MTBF values are large because the operator has confidence that the backup system will be operational when called upon (because it is monitored and has been known to be operational). A cold standby may not operate and may in fact fail when called upon suddenly to operate (e.g., when say a high-voltage supply power is applied), particularly after long durations without in-service testing. Additionally, cold standby systems may have a “warm-up” time, and thus may not provide uninterrupted service.

1.7 MONTE CARLO SIMULATION MODELS

All of the models presented thus far have assumed that both the TTF and the TTR of an item or a system follow an exponential distribution. The exponentially distributed random variable assumption lends mathematical simplicity to both the reliability block diagram and the Markov chain models. In both cases, the mathematics of analysis is sufficiently simple that quick results are possible. The results obtained are often useful for *what if* analyses and for small system designs.

Unfortunately, the limitations imposed by assuming exponentially distributed time to failure and time to repair for a system can lead to unrealistic or inaccurate results in many telecommunications systems. It is in these cases that Monte Carlo simulation is beneficial. Most system models produced using Monte Carlo simulation involve many hours of model development and implementation. Engineers considering use of Monte Carlo simulation for reliability/availability analysis on a particular project should consider the following questions in order to determine whether Monte Carlo simulation is the best fit.

1. What is the purpose of the analysis?
2. What is the budget of the project? Can it support the labor costs associated with the Monte Carlo simulation approach?
3. What is the expertise of the analyst and project team? Will there be sufficient knowledge to derive the maximum benefit from a sophisticated analysis?
4. What are the specific reasons that reliability block diagrams and/or Markov chain analysis are not sufficient to meet the analysis requirements?
5. Does the project involve repairs that are not reasonably approximated by an exponentially distributed random variable?

In many cases, it is possible to make simplifying assumptions in the system model that allow reasonable results to be obtained without embarking on a full Monte Carlo system simulation. This section discusses the theory of Monte Carlo simulation.

1.7.1 System Modeling

Analysis using Monte Carlo simulation has the advantage of allowing the analysis of a system with different failure and repair distributions, thereby creating a more accurate model and better representation of system availability performance. In Monte Carlo simulation, a computer is used to generate and evaluate random variable models. In the approach presented here, a system is modeled for the duration of its life (or longer, if necessary, to obtain accurate results). This life simulation is performed for many trials to obtain a statistical result. This statistical result represents the performance of the system. Monte Carlo simulation is computationally intensive and requires significant computing power to complete all but the simplest simulations in a timely manner. Fortunately, modern computing has advanced to a point where significant computing power is readily available in off-the-shelf desktop computer platforms. The Monte Carlo simulation algorithm consists of three major steps.

1. *Simulate the State of Individual Components.* In this step, the life-cycle state of each component is modeled. The TTF and TTR are computed as random variables until the system life has been reached. This model results in a time series representing each component as a working or failed state for each sample in time.
2. *Evaluate the System State from Individual Component States.* A logic function is developed and applied to the system components to determine the operational state of the system for each time series sample. An output time series is produced representing the system state for each time series sample.
3. *Compute the Desired System Metrics (Availability, Reliability, MTTF, MTTR, etc.) from Output System States.* Performance metrics for the system are calculated using the system state time series. Metrics such as availability, reliability, MTTF, and MTTR are easily computed from the system state time series.

The simulation algorithm uses the output of each step as an input to the next step to help modularize the process. Figure 1.20 shows an overview of the algorithm process for Monte Carlo simulation.

1.7.2 Individual Component Models

Modeling of a system requires the simulation of each individual system component. Each relevant system component must be represented in order to provide an accurate assessment of system performance. Recall that relevant components are defined as those components that impact system performance when a failure occurs. Irrelevant components are those whose state does not impact the performance of the system.

1.7.2.1 Step 1. Component Description. Individual components are modeled by representing the component as a combination of two discrete random variables.

Random variable TTF simulates the time to failure of the component. Although this variable can take on any random process distribution, the failure of electronic components is typically modeled as an exponential random process. The exponential distribution is completely defined by the parameter value λ. The value λ represents the

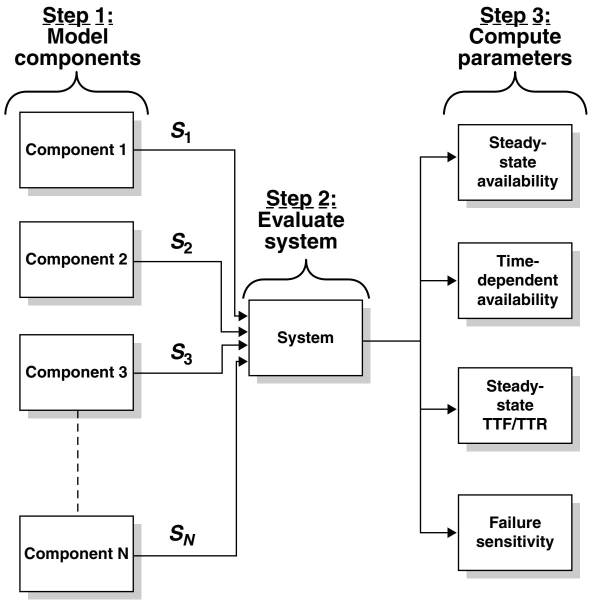

Figure 1.20. Monte Carlo system analysis algorithm.

component failure rate and is usually provided by the manufacturer or vendor of the equipment. This parameter can be specified in units of failures per hour (failure rate), failures per 10^9 h (FITs), or hours per failure (MTBF). The TTF random variable is thus expressed mathematically as

$$\text{TTF} \sim \text{EXP}(\lambda) \tag{1.65}$$

Random variable TTR simulates the time to repair of the component that follows the failure of that component. This variable can also take on any random process distribution. In system analysis, the proper selection of the repair distribution and its parameters is crucial to obtaining an accurate simulation. Many different techniques for simulating equipment repair exist. These techniques are discussed in Section 1.6.

The failure and repair random processes are sampled to produce a time domain representation of the system state based on the sampled values of TTF and TTR. The algorithm for translating these sampled values into a time domain state vector is presented in the next section.

1.7.3 Time Series Creation

Component life is modeled over a time period defined by the simulation duration requirement and is represented by the variable t_{end}. The simulation duration must be long enough to accurately assess the component availability. Highly available components can require a simulation duration longer than the system life to produce reliable

Figure 1.21. Component model.

statistics. A measurable number of failures must have occurred in order for an accurate availability to be obtained.

Figure 1.21 shows the component as a block that accepts one input and produces one output. The input to the block is a discrete time series sampled at the interval t_{sample}.

The individual samples of the times series $\mathbb{T}$ are given by

$$t_i = i \times t_{\text{sample}} \tag{1.66}$$

These samples are placed into a time series vector

$$\mathbb{T} = [\, t_0 \;\; t_1 \;\; t_2 \ldots \; t_N] \tag{1.67}$$

where N is the total number of samples and is defined by the required simulation duration and the sample period

$$N = \frac{t_{\text{end}}}{t_{\text{sample}}} \tag{1.68}$$

The output of the block is a component state vector **S** representing the state of the component for each time series sample

$$\mathbb{S} = [S_0 \; S_1 \; S_2 \ldots S_N] \tag{1.69}$$

The algorithm for creating the state vector output as a function of time for each component is as follows.

1. Sample TTF from failure distribution.
2. Create "system working" samples.
3. Sample TTR from repair distribution.
4. Create "system failed" samples.
5. Repeat Steps 1–4 until system life ($t \geq t_{\text{end}}$) is simulated.

The selection of t_{sample} must be such that the sampling period is sufficient to resolve all failures and repairs of the component. This sampling requirement is given by the Nyquist relation

$$t_{\text{sample}} \leq \frac{1}{2}\text{MIN(TTF, TTR)} \tag{1.70}$$

Evaluating the minimum values of TTF and TTR requires knowledge of the distribution for the random process associated with those variables. In practice, TTF $\gg$ TTR and only TTR statistics need to be analyzed. Calculation of the minimum sampled value for TTR can be performed by analyzing the distribution for TTR and selecting an appropriate sample period.

The value for t_{sample} must be selected such that

$$\text{TTR} \geq 2 \cdot t_{\text{sample}} \tag{1.71}$$

In terms of the TTR random variable PDF

$$Pr(\text{TTR} \geq 2 \cdot t_{\text{sample}}) = P \tag{1.72}$$

where P is the probability that the value of TTR will be sufficiently large to be resolved by the sampling period t_{sample}. Numerical methods can be used to determine the value of t_{sample} required to provide the desired probability P. In practice, P should be chosen such that the probability of not resolving a repair is unlikely. A poorly selected value of t_{sample} will result in a sampling error that skews the system availability to an artificially higher value. This skew is due to the unresolved repairs that do not appear as failures in the component output.

1.7.4 State Vector Creation

The state vector creation algorithm takes the time series as an input and creates a component state sample for each time series sample. The state vector defined in this procedure is a binary vector, taking on values of one and zero. The working state is given a numerical value of one and the failed state is given a numerical value of zero.

$$\begin{aligned} S(\text{working state}) &\equiv 1 \\ S(\text{failed state}) &\equiv 0 \end{aligned} \tag{1.73}$$

Figure 1.22 provides a flow chart diagram of the algorithm implementation. The details of each step are provided below.

1. *Algorithm Start.* Set the current time value to 0. This step takes as its input the time series T and the simulation duration t_{end}.
2. *Sample TTF.* Select a random value from the failure distribution model. This value represents the component time to failure.
3. *Sample TTR.* Select a random value from the repair distribution model. This value represents the component repair time. It includes fault diagnosis and repair.
4. *Current Time Iteration (TTF).* The current time t_{curr} is set to the cumulative (elapsed) time value. The cumulative time is then incremented by the failure value TTF selected in Step 2.
5. *State Vector Assignment (TTF).* The value of the state vector for all samples lying between the current time t_{curr} and the cumulative time t_{cum} is assigned the working state value $S(\text{working state}) = 1$.
6. *Current Time Iteration (TTR).* The current time t_{curr} is set to the cumulative (elapsed) time value. The cumulative time is then incremented by the repair value TTR selected in Step 3.

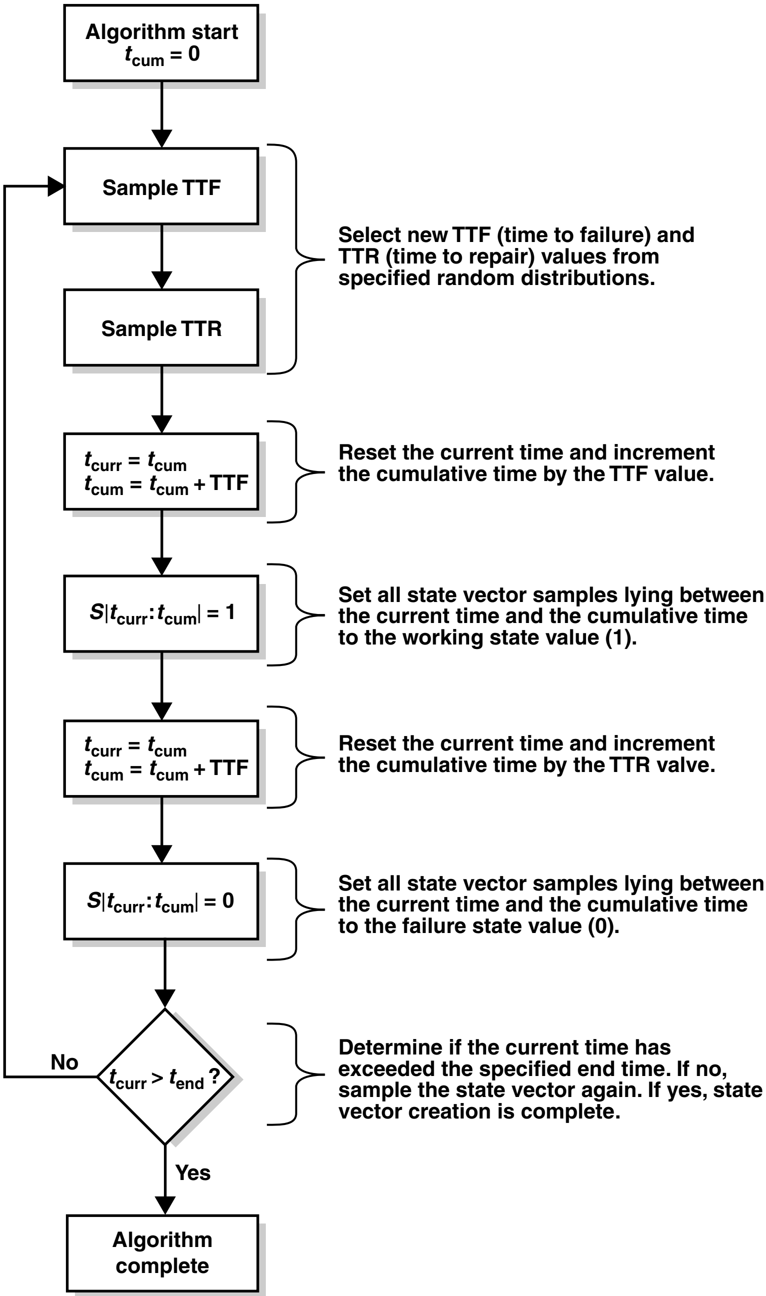

Figure 1.22. State vector algorithm flow chart.

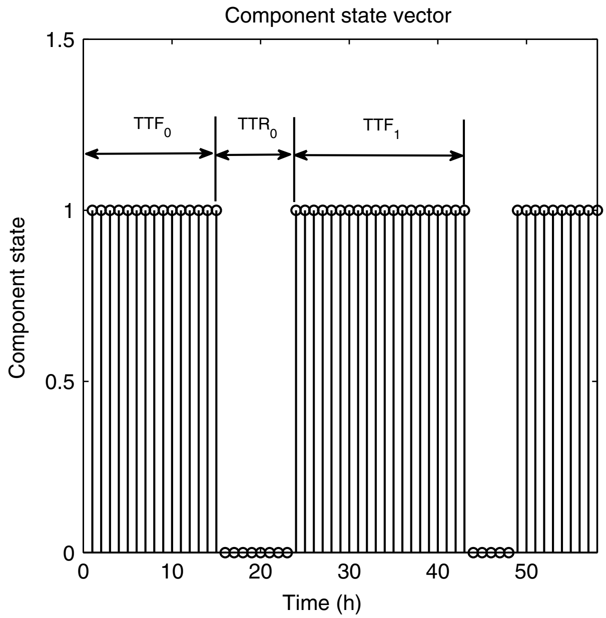

Figure 1.23. Sample state vector algorithm output.

7. *State Vector Assignment (TTR).* The value of the state vector for all samples lying between the current time t_{curr} and the cumulative time t_{cum} is assigned the failed state value $S(\text{failed state}) = 0$.
8. *Evaluate Cumulative Time.* The cumulative time value t_{cum} is compared with the end time value tend supplied to the algorithm. If the end value has not been exceeded, the procedure is repeated from Step 2. If the end time value has been exceeded, the process is complete and the algorithm ends.

A sample output (generated in MATLAB) of the state vector algorithm is shown in Figure 1.23.

1.7.5 Steady-State Availability Assessment

Steady-state availability assessment of the state vector is straightforward. The calculation can be performed directly from the output state vector. Availability is defined as the probability that a system (or in this case a component) is operating at any instant in time (see Section 1.1). The steady-state availability is the average value of the availability over the system life. This can be expressed mathematically as previously shown:

$$A = \frac{\text{item uptime}}{\text{item operational time}}$$

Numerically, the component model state vector is a binary vector in which a value of 1 represents the "component working" condition and a value of 0 represents the "component failed" condition. The availability of the component can be calculated from the state vector by applying

$$A = \frac{\sum_{N}^{i=0} S_i}{N} \tag{1.74}$$

where N is the total number of state vector samples. This analysis does not require knowledge of the time series vector, since availability is a ratio of "working" samples to the total number of samples.

1.7.6 Time-Dependent Availability

The variation of component availability with time can be determined from the time series vector and the state vector constructed by the model. The time-dependent availability is determined by calculating the availability of the system for each time series sample.

$$A(t_n) = \frac{\sum_{n}^{i=0} S_i}{n} \tag{1.75}$$

where n is the number of samples present on the time interval $[0, t_n]$. Calculating $A(t_n)$ for each sample results in the array

$$\mathbb{A} = [A(t_0)\,A(t_1)\ldots A(t_N)] \tag{1.76}$$

where the steady-state availability is the Nth (last) term in the array.

1.7.7 Time-to-Failure/Time-to-Repair Calculations

The TTF and TTR can be calculated for a component or a system using the time series and the state vector arrays. Calculation of these values is performed by counting the number of samples for each discrete system event. The steps involved in this algorithm are as follows.

1. Partition working and failure blocks into discrete bins.
2. Sum the number of samples in each bin.
3. Multiply the summation of those samples by t_{sample}.

Partitioning of the failures and repairs is the most difficult task in the implementation of this algorithm and will depend on programming style and the programming language chosen. Once the state sample sets have been partitioned, the TTF and TTR values are calculated by applying the equations

$$\text{TTF}_i = \sum_{n=j}^{k} (S_n = 1) \times t_{\text{sample}} \tag{1.77}$$

$$\text{TTR}_i = \sum_{n=j}^{k} (S_n = 0) \times t_{\text{sample}} \tag{1.78}$$

where j and k are the start and finish indices of the partitioned bins. The number of TTF and TTR values will vary by simulation. Averaging over the trial set is recommended to obtain an accurate assessment of the time to failure and time to repair for the components.

1.7.8 System Analysis with Multiple Components

The method for translating a set of component state vectors representing a system into a single system state vector is presented in this section. The algorithm for computing the system state vector is as follows.

1. Simulate system components using the individual component model.
2. Create a sample vector consisting of individual component sample states for time t_n.
3. Evaluate the system state for each sample state vector.
4. Calculate the availability of the system from system state vector.

This procedure assumes a general system comprised of N components. Each of the N discrete components must be modeled using the same sample period t_s so that the component state samples are correlated in time. Each component has a state vector

$$\begin{aligned} \mathbb{S}_0(t) &= [S_0(t_0)\ S_0(t_1)\ S_0(t_2)\ \ldots\ S_0(t_M)] \\ \mathbb{S}_1(t) &= [S_1(t_0)\ S_1(t_1)\ S_1(t_2)\ \ldots\ S_1(t_M)] \\ &\vdots \\ \mathbb{S}_N(t) &= [S_N(t_0)\ S_N(t_1)\ S_N(t_2) \ldots S_N(t_M)] \end{aligned} \tag{1.79}$$

where M is the number of samples in the system life. All values of **S** are binary (1 or 0). The system state is assessed for each sample in time. That is

$$\mathbb{S}(t_n) = [S_0(t_n)S_1(t_n)S_2(t_n)\ldots\ S_N(t_n)] \tag{1.80}$$

Thus, the system state is a function of time

$$\mathbb{S}_{\text{system}}(t_n) = F(\mathbb{S}(t_n)) \tag{1.81}$$

where $F(\mathbb{S}(t_n))$ is the rule set function and is applied to the sample set $\mathbb{S}(t_n)$.

1.7.9 System State Synthesis

Determining the state of the system based on the individual component states requires the development of a rule set that defines the state of the system for all possible sample sets. This rule set function $F()$ can be defined by developing a flow diagram relationship between the component states and the system state. In the case of simple component combinations, a mathematical relationship between the component states and the system state may be possible. Use of the flow diagram approach for more complicated systems simplifies this process as the component count grows and the interactions of the components become more complex. Definition of the rule set used for a system is highly dependent on the system and is specific to each system being modeled. As such, two simple cases are presented in the next sections. The serial combination and parallel combinations of components can be assessed using mathematical relations. More complicated systems require the flow diagram approach.

The serial and parallel cases are presented in both mathematical and flow diagram form to demonstrate the procedure.

1. *Serial Components.* The rule set for serially connected components put into words is "if any one component fails, the system has failed". Since the system working condition is defined with a numerical value of one, the state of a serially connected system is the product of the N component states. Consider a system with N individual components. The system state for these N components would be

$$\mathbb{S}_{\text{system}}(t_n) = \prod_{i=0}^{N} S_i(t_n) \tag{1.82}$$

2. *Parallel Components.* The rule set for parallel component configurations is more complicated, since many different types of configurations of component redundancy exist. For the simple case of two components where one is required for system operation and both operate continuously, the system state rule is "if either of the components is working, the system is working." This can be expressed mathematically as the logical OR operation.

$$\mathbb{S}_{\text{system}}(t_n) = \text{OR}(S_1(t_n),\ S_2(t_n)) \tag{1.83}$$

3. *Arbitrary Component Configuration.* The rule set for an arbitrary system made up of N different components can be analyzed by developing a system state flow chart that details failure flow. Although not technically required, the process for developing the system state flow chart for the serial and parallel configurations is demonstrated here for clarity. Figure 1.24 shows the system state flow chart for serially connected components.

In the case of two parallelly connected components, the flow diagram shown in Figure 1.25 is applied to determine the system state.

As can be seen in Figure 1.25, the benefit of using the flow diagram approach quickly becomes evident. Since only the outcome of the system state is of interest, the state of component 2 can be ignored if component 1 is working. This benefit is multiplied many times as the system becomes more complex. This approach implicitly applies *don't care* conditions to many component state combinations. Care must be taken such that actual failure modes are not neglected in the flow chart development.

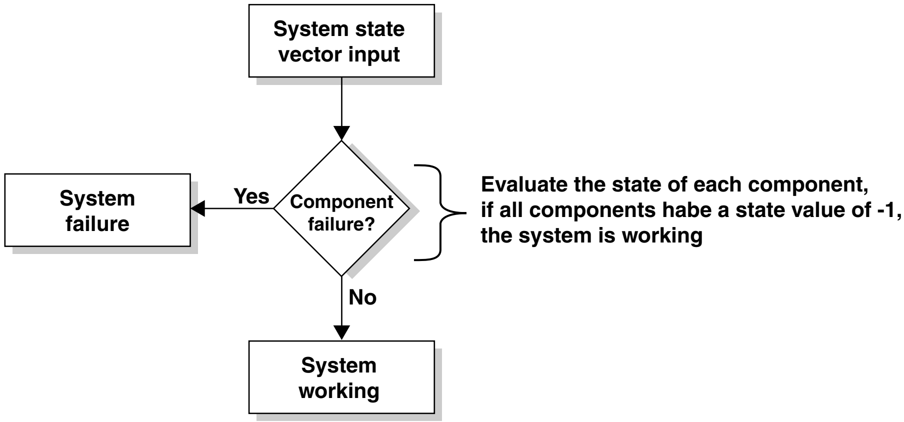

Figure 1.24. Serial component state assessment flow diagram.

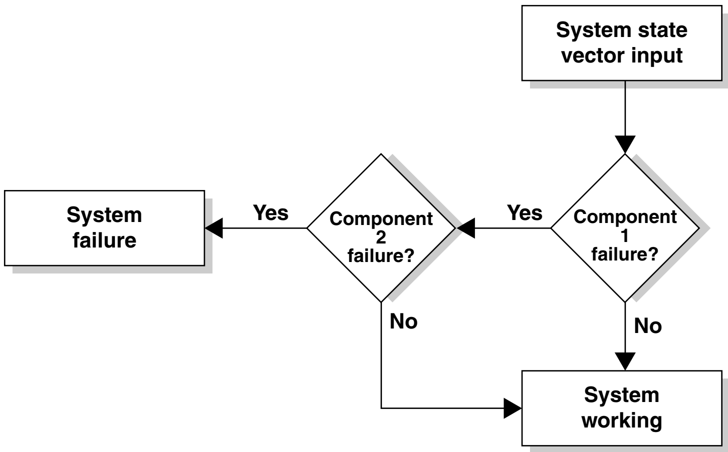

Figure 1.25. Parallel component state assessment flow diagram.

1.7.10 Failure Mode Sensitivity

When availability analysis is performed during the design phase of a project, it is desirable to know which components contribute most significantly to the unavailability of the system. A technique is presented here for quantifying that contribution. Development of the system state rule set establishes conditions on which "system working" or "system failed" decisions are made. During evaluation of these conditions, the numerical count of samples corresponding to the different failure modes can be summed. For example, in the parallel redundancy configuration presented in Figure 1.25, only one failure mode exists. This failure mode, labeled "*Component 2 Failure?*" is what causes a system failure. While this condition is being evaluated in the simulation software, whenever this condition evaluates true, a failure mode counter associated with the system failure is caused by "*Component 2 Failure*" increments. In the case where only one failure mode exists, this value will mirror the availability calculation. In cases where multiple failure modes exist, a chart can be constructed that displays the failures that occur most frequently and how many outage hours they contribute to the system's total outage hours over the simulation life.

1.8 REPAIR PERIOD MODELS

Modeling the time to repair for a component or system is an important part of the development of a reliability or availability model. As mentioned in previous sections of this chapter, the exponentially distributed random variable is often used to model the time to repair out of necessity. In the case of reliability block diagrams and Markov chain analyses, an exponentially distributed time-to-repair model is the only option due to the requirement for the memory-less condition to be met.

This section discusses methods to model time to repair and their implications on model accuracy. It should be obvious to the reader that the exponential distribution, while being a good fit for electronic component failure modeling, is not particularly well suited to model the repair of those components. When reliability models require accurate modeling of system repair, Monte Carlo simulation is often the only feasible option.

1.8.1 Downtime

Downtime is the total time period that a component or system is not functioning following a failure. The downtime of a component or system consists of a number of constituent elements. Some of these elements are often overlooked in availability analysis. Let the variable D represent the total downtime of a component or a system following a failure event. We can write D as

$$D = D_{\text{identify}} + D_{\text{dispatch}} + D_{\text{repair}} + D_{\text{close-out}} \tag{1.84}$$

When considering the downtime of a component, it is important to review and understand the service-level agreement associated with the service or system being analyzed. The total system downtime consists of at least the following elements:

D_{identify}. Downtime associated with the identification of a failure. In telecommunications systems, the time associated with identifying a failure may depend on human, electronic, or a combination of human and electronic factors. This value can be as little as seconds in the case of an electronically alarmed network operations center or could be hours for a service that requires customer feedback to identify a failure.

D_{dispatch}. Dispatch downtime is the outage time associated with travel to the location of the failure. Telecommunications systems are typically implemented with a network operation center contacting a field technician or an engineer to repair a failed component or system. The time to dispatch after the failure that has been identified can vary dramatically. In urban environments, with well-staffed technician resources, the dispatch time might be an hour or less. In remote or rural environments, the time to dispatch can be a day or more when fly outs or rural travel are required.

D_{repair}. The repair downtime is often confused with the total downtime or time to repair. Specifically, the repair downtime refers to the downtime associated with the actual repair activity. This could be the replacement of an interface module, repair of a fiber-optic cable break, or the bypass of a service to a backup configuration. The amount of time required to effect a specific type of repair can often be modeled accurately but careful consideration is in order.

$D_{\text{close-out}}$. Close-out downtime refers to the amount of time required to relay repair messages back to the appropriate parties. This downtime may be very small in systems that electronically log system up and down events. In cases where manual outage logs are contractually required, this time may have a finite and measureable effect on the total downtime. Normally, a system is returned to service immediately upon completion of repair. Examples where $D_{\text{close-out}}$ must be considered and included are, for example, the time to move traffic back to the primary system if the traffic was manually routed to alternate path. It may also be the time to achieve customer acceptance that the system is in fact repaired (the customer may want to test the repaired system and concur that it is indeed meeting performance requirements).

The four downtime elements listed above are not meant to represent a comprehensive list of all possible contributions to downtime for a system or a component. Rather, these elements are common to most repairs following the failure of a telecommunications system or a component. Each element provides a distinct contribution to the total downtime and can be modeled using a different statistical distribution (in the case of Monte Carlo model). Of particular note is the opportunity to analyze sensitivity of system performance to changes in downtime element. For example, by varying the dispatch downtime portion of the total downtime in a Monte Carlo simulation, one can glean insight into the effect of operational improvements on downtime performance.

1.8.2 Statistical Models

This chapter presented a number of different statistical models that can be used to model either the time to failure, time to repair, or both for a component or a system.

In order to better understand how to select an appropriate downtime or time-to-repair model, we will present an example.

Consider a single component where the downtime to be modeled takes on a one of four different distributions. Assume that through empirical data collection and process analysis, the following time-to-repair observations are made.

1. Mean downtime is 8 h.
2. Downtime variance is 2 h.
3. Downtime never exceeds 24 h.
4. Downtime is always greater than 1 h.

The exponential distribution is completely defined by a single parameter. The field of reliability analysis typically refers to the repair rate of an item as μ.

The PDF and CDF for an exponentially distributed time to repair with MTTR = 8 h is shown in Figure 1.26. The PDF and CDF for a normal distributed random variable with an MTTR = 8 h and variance = 2 h are shown in Figure 1.27. Recall that the MTTR is equal to $1/\mu$ for exponential random variables.

If we compare the time-to-repair models in Figures 1.26 and 1.27 to our model criteria, we find that although the mean value is a good fit, the other criteria are not a good match. Specifically, neither the target for a not to exceed value of 24 h nor the must be greater than value of 1 h are both missed. Unfortunately, in the case of the exponential distribution, one often has to modify the mean value assumptions if the not to exceed or greater than criteria are particularly important and an exponential distribution is a requirement.

The exponential distribution model for the time to repair in this example would therefore not be a particularly good fit. It may be desirable in some circumstances to proceed with the analysis but having performed the comparison shown in Table 1.5, the

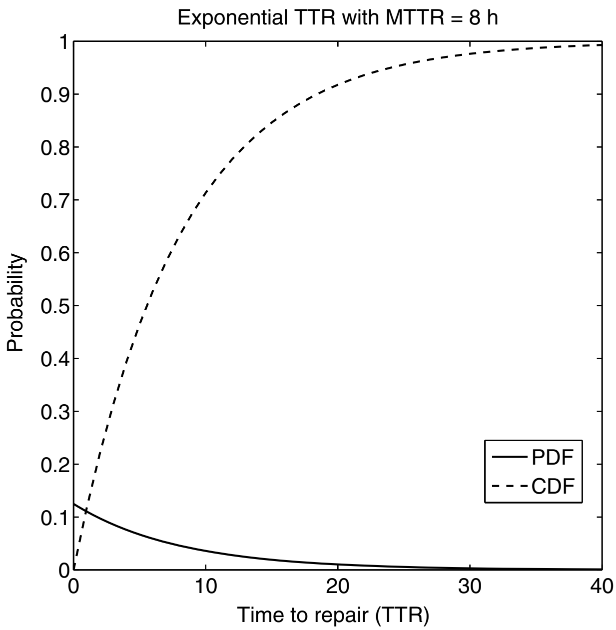

Figure 1.26. Exponentially distributed TTR with MTTR = 8 h.

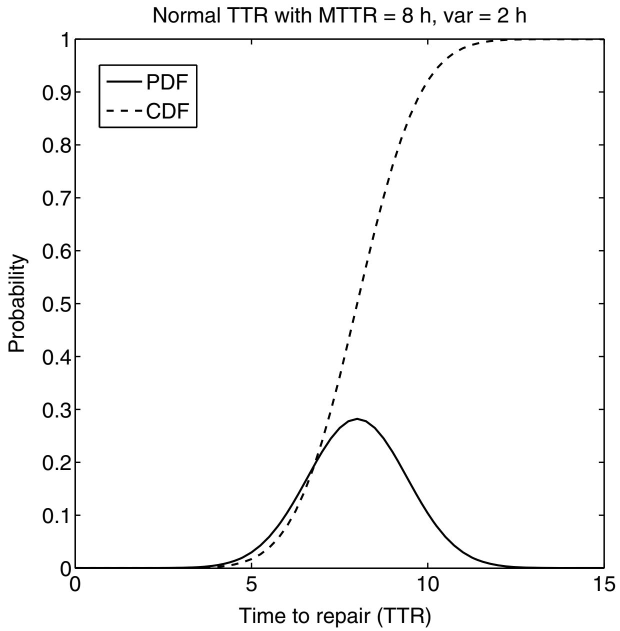

Figure 1.27. Normal distributed TTR with MTTR = 8 h, variance = 2 h.

Table 1.5. Exponential Time-to-Repair Criteria Versus Model

Criteria	Exponential Model Value	Normal Model Value
Mean downtime is 8 h	Mean downtime is 8 h	Mean downtime is 8 h
Downtime variance is 2 h	Not applicable	Variance is 2 h
Downtime never exceeds 24 h	95% of values are less than 24 h	True
Downtime is always greater than 1 h	11% of values are less than 1 h	True

limitations of this model have been clearly identified and the analyst should proceed with caution.

1.9 EQUIPMENT SPARING

The concept of equipment sparing is central to operation of telecommunications networks. In consideration of the importance of equipment sparing, it would seem obvious that careful attention should be paid to both equipment spares placement and quantities available for repair. It is unfortunate that equipment sparing design is often neglected in telecommunications systems. In many cases, sparing levels are determined by historic “experience” and are not based on quantitative analysis.

This section discusses optimization of equipment sparing levels, the impact of sparing levels, and RMA on system downtime and considerations for geographic placement of spares in long-haul systems.

1.9.1 Sparing-Level Optimization

Selection of sparing levels in telecommunications systems can be a difficult problem when quantitative analysis tools are not used. Optimization of the spares pool level for system components has important financial and logistical implications. Selecting the minimum spares pool quantity for any particular component minimizes the logistical impact of storage and management of hardware on warehouse staff while also minimizing capital or operational expenditures.

Consider a system consisting of n discrete, identical components such that $S = \{1, 2, \ldots, n\}$ is the set of all components in the system. Assume that the n components in the system S are in operation at a time $t = 0$ and operate for a duration T. Thus, the total operational time for all components is

$$T_{\text{total}} = N \times T \tag{1.87}$$

Calculation of the sparing level requires knowledge of the failure characteristics of each component. Replaceable items within the system must be identified and analyzed to determine the distribution of failures, failure rate ($z(t)$), or MTBF. Any of these three metrics can be utilized for analysis. It is most desirable to use a combination of empirically collected field data in conjunction with calculated failure rates. This provides the best combination of academic and empirical experience. Without knowledge of the failure behavior of system components, it is impossible to determine optimal sparing levels.

Assume that the MTBF of each component is given by M. The predicted average failure count for the system in time period T can thus be calculated as

$$F_{\text{sys}} = \frac{T_{\text{total}}}{M} \tag{1.88}$$

where M is the empirical or calculated mean time between failures. The average failure count can be used to determine the predicted number of spares required for each time period T. It should be noted that many telecommunications utilize maintenance agreements in which failed components are repaired by a vendor at a rate determined within a prearranged contract. This maintenance agreement can complicate sparing-level determination. The maintenance agreement contract must have specific provisions for turnaround time period in equipment repair. This turnaround time T_{vendor} must be weighed against the time between system events. The failure rate of that system can be calculated as

$$f = \frac{F_{\text{sys}}}{T} \tag{1.89}$$

where f is expressed in failures per hour. With knowledge of the failure rate f and the maintenance agreement turnaround time T_{vendor}, we can calculate the required spares.

Consider a system F_{vendor} failure events per time period T_{vendor}

$$F_{vendor} = f \times T_{vendor} \tag{1.90}$$

The number of required spares N_{spares} must be greater than or equal to the number of failures expected to occur during the vendor repair period.

$$N_{spares} \geq F_{vendor} \tag{1.91}$$

In cases where a vendor maintenance agreement does not exist, the sparing levels must be selected such that a sufficient number of spares is purchased so that all failures can be repaired within a give period of time T. In this case, the calculation of N_{spares} is straightforward:

$$N_{spares} \geq F_{sys} \tag{1.92}$$

where F_{sys} is the number of predicted failures in the system over a time period T.

As an example, consider a wireless cellular network consisting of 100 base station transceiver elements. Through empirical analysis and vendor interaction, it is determined that the MTBF for the base station transceiver element is approximately 55,000 h of continuous operation. It is desired to analyze the sparing levels required for both design in which no maintenance agreement is assumed and for a system with a maintenance agreement where the turnaround time $T_{vendor} = 8$ weeks for a 1-year period (8760 h). First, we will calculate the total number of operational hours in the system consisting of 100 base station transceiver elements. The total system operational time is

$$T_{total} = N \times T = 100 \times 8760 = 876000 \text{ h}$$

The predicted average number of annual failures can thus be calculated as

$$F_{sys} = \frac{T_{total}}{M} = \frac{876000}{55000} \approx 15.9$$

Thus, the number of expected annual failures per year is approximately 16 under steady-state operation. The number of required spares for the case in which spares are annually purchased and allocated is given by

$$N_{spares} \geq F_{sys} \rightarrow N_{spares} \geq 16$$

Thus, the number of spares required for steady-state operation is 16. It should be noted that this analysis assumed average behavior. It is always good practice to select sparing levels such that anomalies can be accommodated. A reasonable sparing compliment for one year on the system above might be a value $16 \leq N_{spares} \leq 20$. Because the failure rate of the system F_{sys} is a statistical value, the number of failures in any given year can vary. The number of spares purchased in one year may be insufficient while another year it may be too great. This variation tends to disappear as the number of deployed components becomes large and the statistics become stationary in time.

In the case where a maintenance agreement exists with $T_{vendor} = 1344$ h, we must first calculate the system failure rate f as

$$f = \frac{F_{sys}}{T} = \frac{15.9}{8760} = 1.8 \times 10^{-3} \text{per h}$$

The number of failures that might occur during a vendor repair or replacement period is thus

$$F_{vendor} = f \times T_{vendor} = \left(1.8 \times 10^{-3}\right) \times 1344 \approx 2.4$$

By applying the spares count rule, we find that

$$N_{spares} \geq F_{vendor} \rightarrow N_{spares} \geq 3$$

Clearly, in the maintenance agreement model, the number of spares required is significantly smaller than in the self-repaired case. The trade-off analysis between maintenance agreement costs and the equipment costs is now easy.

Assume that the base station transceiver element has an equipment cost of $40,000 per element and that the annual maintenance agreement cost is $350,000 (Table 1.6).

The maintenance agreement approach to this particular problem is clearly the less-expensive solution. Although a telecommunications provider may opt to select a self-repaired model for finance or business reasons, it is easy to see the cost trade-offs after the sparing analysis is complete.

1.9.2 Geographic Considerations for Spares Placement

Analysis of the geographic placement of spare components is often required in order to achieve the required time to repair for systems covering large geographic areas or having very difficult terrain.

Telecommunications networks generally cover large geographic areas due to the nature of their mission. Whether the system is a long-haul submarine fiber-optic network, a backbone microwave system, or an urban cellular wireless network, the area being served typically covers a large geographic region. Because of this large area being served, it is important to consider the optimal sparing levels and placement to ensure that both the time to repair and the number of spares available maintain the necessary levels.

Table 1.6. Spares Cost Comparison Between Self-Repaired and Vendor-Repaired Models

	Self-Repaired System (No Maintenance Agreement)	Vendor-Repaired System (Maintenance Agreement)
Spares cost	$16 \times \$40,000 = \$640,000$	$3 \times \$40,000 = \$120,000$
Maintenance agreement cost	N/A	$350,000
Total annual cost	$640,000	$470,000

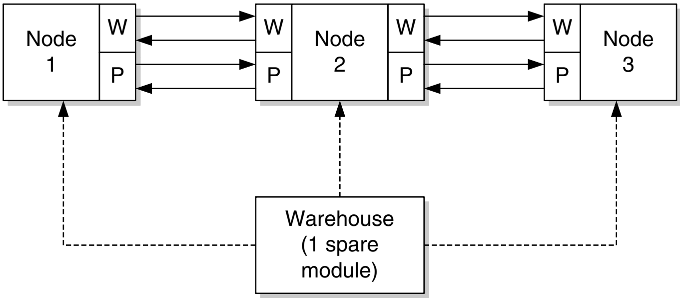

Figure 1.28. Centralized warehousing and dispatch sparing approach.

A number of different approaches for locating spare equipment are provided below:

1. *Centralized Warehousing and Dispatch.* In a centralized warehousing approach (Figure 1.28), all system spares are located in a central warehouse or depot and are picked up or shipped from this location in the event of a failure. For systems implementing full redundancy, this approach is often the most convenient since the time to repair can be relaxed enough (because of component redundancy) to support the logistics time required to place a spare unit on site. In cases where the shipping or logistics time causes the time to repair to exceed the requirement, this approach may be unacceptable.

   Systems implementing a relatively small number of deployed components can also benefit from a reduced spare equipment count. Consider a fiber-optic network consisting of a total of eight optical interface modules (four working, four protection). Assume that the sparing level analysis results in a requirement for one spare optical interface module. In the centralized sparing model, only one spare interface module would be purchased and placed in the warehouse. This module would be deployed when any failure occurs in the system.

2. *Territorial Warehousing and Dispatch.* Territorial warehousing places spare equipment at strategically selected locations, reducing the logistics time to place units on site while keeping the spare unit costs at a reasonable level. In the case of a unit failure in the system, the spare unit would be dispatched from a predetermined location that provides the minimal logistics dispatch time.

   Examination of the system presented above using a territorial approach to sparing results in an increased spare unit requirement of one additional spare is shown in Figure 1.29. Warehouses A and B would both store one spare optical interface module each. If Node 1 was at a significant geographic distance from Nodes 2 and 3 (e.g., in a submarine fiber-optic network), this approach would represent a good compromise of performance versus cost.

3. *On-Site Sparing.* The last sparing approach to be considered is the on-site sparing model (Figure 1.30). In this model, every site houses the spares required to restore the system in the case of an outage or failure. This approach to sparing

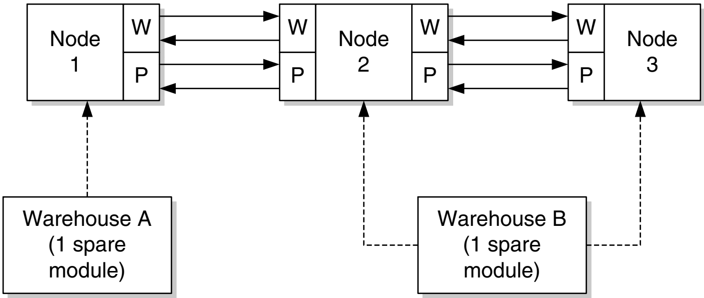

Figure 1.29. Territorial warehousing and dispatch sparing approach.

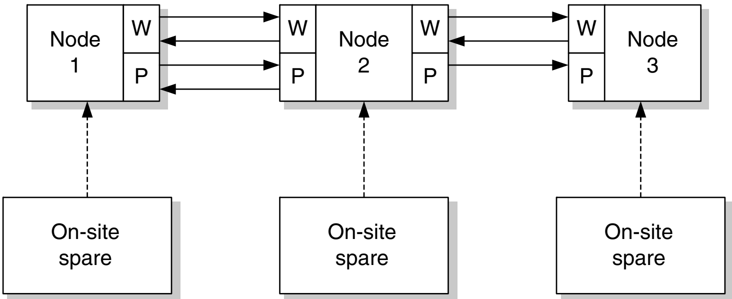

Figure 1.30. On-site sparing approach.

provides the highest attainable performance since the logistics time to place a spare on site is zero. On-site sparing comes at the highest cost as well. It is common in systems operating with on-site spares to see a dramatically increased sparing cost because of spares redundancy required to place spare equipment on site. In the case of the example presented here, on-site sparing would result in three spare interface modules (one at each location). This is three times the sparing level calculated due to expected failures.

QUESTIONS

1.1. Create a flow chart that graphically depicts the development of reliability engineering in the twentieth century.

1.2. Modern reliability engineering analysis utilizes what type of mathematics for analysis? What is the purpose of this type of mathematical analysis?

1.3. What role does empirical data play in modern reliability analysis? What specific implications does reliability engineering have on telecommunications systems?

1.4. Enumerate your goals as a reader of this book. What benefits do you hope to derive from the study of telecommunications reliability analysis?

1.5. Define the term *reliability* and give three practical examples of reliability applications. Ensure that the examples provide both the duration and the conditions for which the reliability is defined.

1.6. Describe the time to failure and its relationship with the reliability function.

1.7. Define *availability* and give three examples of its use in telecommunications design, operation, and business.

1.8. Explain the difference between average and instantaneous availability. Provide examples where the average and instantaneous availability are the same and are different.

1.9. A particular network element's datasheet indicates an MTBF of 65,000 h. Calculate the network element availability (in percent) if the expected mean downtime is 12 h.

1.10. If 25 of the network elements described in Q1.9 are placed into service at time $t = 0$, what is the expected number of element failures annually?

1.11. Explain *maintainability* in terms of an operational telecommunications system. Why is the maintainability metric a critical performance measure? Describe qualitatively how downtime and maintainability are related.

1.12. A vendor provides an MTBF in their equipment cutsheet indicating a value of 125,000 h. Convert the MTBF to both failure rate (in failures/h) and FITS.

1.13. A system of 100 telecommunications nodes is deployed and operates for 5 years. The table below enumerates the annual failures per year. Calculate the annual failure rate (in failures/h and FITS) for each year and the average failure rate (in failures/h and FITS) for the 5-year period.

Year	Failures
1	6
2	4
3	8
4	7
5	3

1.14. An interface card for a multiplexer has an MTBF of 95,000 h as defined on a vendor datasheet. Assuming that the TTF for the card is exponentially distributed, write the TTF, PDF, and CDF functions. Plot the PDF and CDF functions using a graphing calculator or computer analysis tool.

1.15. Why is an exponentially distributed random variable beneficial for analyzing telecommunications systems hardware? What is the failure rate of an exponentially distributed random variable.

1.16. Why is the exponential distribution poorly suited for modeling time to repair? What distributions are well suited to model system downtime?

1.17. The time to repair of a network is characterized by a mean value of 24 h. Assuming that 90% of the variability of the TTR is contained with the range of 12–36 h, develop normal, lognormal, and Weibull distribution models for the TTR. Plot the PDF of each distribution.

1.18. Empirical data collection has tabulated the date and downtime for repair of a system. Develop a Weibull TTR model for the data shown in the table below. Plot the CDF and PDF of the model developed. Calculate the MTBF of the system.

Date	Downtime (h)
6/15/2001	11
9/3/2002	3
12/5/2002	14
7/5/2003	5
11/2/2003	9
2/6/2004	20
4/29/2004	2
8/4/2004	7
10/21/2004	3
12/16/2004	13

1.19. What flexibility does the use of Monte Carlo simulation provide in system analysis? What are the advantages of using reliability block diagrams or Markov chains?

1.20. Define "relevance" as it relates to a reliability block diagram analysis. What is the impact of a relevant component on the reliability performance of a system? What impact does an irrelevant component have on system performance?

1.21. A telecommunications network consists of three discrete components that are combined to form a single-thread network. If the reliability of each constituent component is 99.9%, 99.99%, and 99.95%, respectively, what is the maximum achievable reliability of the system (based only on observation of the constituent component reliabilities and without performance a calculation)?

1.22. Applying the serial combination structure function definition for system reliability, calculate the actual reliability of the single-thread system described in Q1.21.

1.23. Calculate the reliability of the following two system designs.

a. Serial combination of 1:1 redundant components in Q1.21.

b. Parallel combination of serial components in Q1.21.

1.24. Redundancy is being considered for a telecommunications subsystem. If the modular system costs $15,000 per module and four active modules are required, calculate the following (assume that each module has an MTBF of 40,000 h).

a. The cost difference between a 1:1 and a 1:4 system design.

b. The reliability after 2 years for each system.

1.25. Explain why reliability is not applicable as a performance metric in repairable system analysis. For what types of systems is reliability a good metric?

1.26. Describe Monte Carlo simulation and give a specific example of a simulation.

1.27. Describe the Markov chain analysis technique in your own words. What condition must exist in order for a process to possess the "Markov property." For what conditions is the Markov chain analysis technique best suited within a telecommunications environment?

1.28. Develop a Markov transition diagram for a system consisting of two redundant components operating in a hot-standby configuration with the same failure rate. Assume that both components have a failure rate of $\lambda = 4 \times 10^{-5}$ failures/h. Repair of each component takes 16 h on average. Indicate the failure rate and repair rate of each transition. Assume that a repair of a system failure returns the system to fully redundant operation.

1.29. Write the transition rate matrix for the transition diagram developed in Q1.28.

1.30. Assuming a steady-state solution, solve the Chapman-Kolmogorov equations to determine the probability of state occupation for the states identified in Q1.28.

1.31. Determine the availability and unavailability of the system described in Q1.28 using the results of Q1.30.

1.32. Develop a Monte Carlo simulation for the system identified in Q1.28. Assume a system life of 8 y for the simulation. Model the repair as a random variable TTR $\sim$ NORM(16, 2). Provide an analysis algorithm overview indicating the system components, evaluation logic, and metrics to be computed.

1.33. Develop a system state flow diagram for the operation of the system model in Q1.32. Implement logic to compute the state of the system for the two input system.

1.34. Simulate the system in Q1.33 for 5000 sample life cycles. Compute the life-cycle availability. Provide a histogram plot of availability.

1.35. Compare the Monte Carlo and Markov chain results. What are the simulation differences and similarities?