

# RELIABILITY AND HAZARD FUNCTIONS

## 1.1 INTRODUCTION

One of the quality characteristics that consumers require from the manufacturer of products is reliability. Unfortunately, when consumers are asked what reliability means, the response is usually unclear. Some consumers may respond by stating that the product should always work properly without failure or by stating that the product will always function properly when required for use, while others will completely fail to explain what reliability means to them.

What is reliability from your viewpoint? Take, for instance, the example of starting your car. Would you consider your car reliable if it starts immediately? Would you still consider your car reliable if it takes you two times to turn on the ignition key for the car to start? How about three times? As you can see, without quantification, it becomes more difficult to define or measure reliability. We define reliability later in this chapter, but for now, to further illustrate the importance of reliability as a field of study and research, we present the following cases.

On April 9, 1963, the USS *Thresher*, a nuclear submarine, slipped beneath the surface of the Atlantic and began a run for deep waters (1000 feet below surface). *Thresher* exceeded its maximum test depth and imploded. Its hull collapsed, causing the death of 129 crewmembers and civilians. It should be noted that the *Thresher* had been the most advanced submarine of its day, with a destructive power beyond that of the Navy's entire submarine force in World War II. Though it was designed to sustain stresses at this depth, it failed catastrophically.

In 1979, a DC-10 commercial aircraft crashed, killing all passengers aboard. The cause of failure was poor maintenance procedure. The engineers specified that the engine should have been taken off before the engine mounting assembly, because of the excessive weight of the engines. Apparently, those guidelines were not followed when maintenance was conducted, causing excessive stresses and forces that cracked the engine mounts.

On December 2, 1982, a team of doctors and engineers at Salt Lake City, Utah, performed an operation to replace a human heart by a mechanical one—the Jarvik heart. Two days later, the patient underwent further operations due to a malfunction of the valve of the mechanical heart. Here, a failure of the system may directly affect one human life at a time. In January 1990, the Food and Drug Administration stunned the medical community by recalling the world's first artificial heart because of deficiencies in manufacturing quality, training, and other areas. This heart affected the lives of 157 patients over an eight-year period. Now, consider the following case, where the failures of the systems have a much greater effect.

On April 26, 1986, two explosions occurred at the newest of the four operating nuclear reactors at the Chernobyl site in the former USSR. It was the worst commercial disaster in the history of the nuclear industry. A total of 31 site workers and members of the emergency crew died as a result of the accident. About 200 people were treated for symptoms of acute radiation syndrome. Economic losses were estimated at \$3 billion, and the full extent of the long-term damage has yet to be determined.

More recently, on July 25, 2000, a Concorde aircraft while taking off at a speed of 175 knots ran over a strip of metal from a DC-10 airplane, which had taken off a few minutes before. This strip cut the tire on wheel No. 2 of the left landing gear resulting in one or more pieces of the tire, which were thrown against the underside wing fuel tank. This led to the rupture of the tank causing fuel leakage and consequently resulting in a fire in the landing gear system. Fire spread to both engines of the aircraft causing loss of power and crash of the aircraft. Clearly, such field condition was not considered in the design process. This type of failure has ended the operation of the Concorde fleet indefinitely.

The explosions of the space shuttle *Challenger* in 1986 and the space shuttle *Columbia* in 2003, as well as the loss of the two external fuel tanks of the space shuttle *Columbia* in an earlier flight (at a cost of \$25 million each), are other examples of the importance of reliability in the design, operation, and maintenance of critical and complex systems. Indeed, field conditions similar to those of the Concorde aircraft have led to the failure of the *Columbia*. The physical cause of the loss of *Columbia* and its crew was a breach in the Thermal Protection System of the leading edge of the left wing. The breach was initiated by a piece of insulating foam that separated from the left bipod ramp of the External Tank and struck the wing in the vicinity of the lower half of Reinforced Carbon-Carbon panel 8 at 81.9 seconds after launch. During the reentry, reheated air penetrated the leading-edge insulation and progressively melted the aluminum structure until increasing aerodynamic forces caused loss of control, failure of the wing, and breakup of the Orbiter (Walker and Grosch, 2004).

Reliability plays an important role in the service industry. For example, to provide virtually uninterrupted communications for its customers, American Telephone and Telegraph Company (AT&T) installed the first transatlantic cable with a reliability goal of a maximum of one failure in 20 years of service. The cable surpassed the reliability goal and was replaced by new fiber optic cables for economic reasons. The reliability goal of the new cables is one failure in 80 years of service!

Another example of the reliability role in structural design is illustrated by the Point Pleasant Bridge (West Virginia/Ohio border), which collapsed on December 15, 1967, causing the death of 46 persons and the injuries of several dozen persons. The failure was attributed to the metal fatigue of a crucial eyebar, which started a chain reaction of one structural member falling after another. The bridge failed before its designed life.

The failure of a system can have a widespread effect and a far reaching impact on many users and on the society as a whole. On August 14, 2003, the largest power blackout in North American history affected eight U.S. states and the Province of Ontario, leaving up to 50 million people with no electricity. Controllers in Ohio, where the blackout started, were overextended, lacked vital data, and failed to act appropriately on outages that occurred more than an hour before the blackout. When energy shifted from one transmission line to another, overheating caused lines to sag into a tree. The snowballing cascade of shunted power that rippled across the Northeast in seconds would not have happened had the grid not been operating so near to

its transmission capacity and assessment of the entire power network reliability when operating at its peak capacity were carefully estimated (The Industrial Physicist, 2003; U.S.-Canada Power System Outage Task Force, 2004).

Most of the above examples might imply that failures and their consequences are due to hardware. However, many systems' failures are due to human errors and software failures. For example, the Therac-25, a computerized radiation therapy machine, massively overdosed patients at least six times between June 1985 and January 1987. Each overdose was several times the normal therapeutic dose and resulted in the patient's severe injury or even death (Leveson and Turner, 1993). Overdoses, although they sometimes involved operator error, occurred primarily because of errors in the Therac-25's software and because the manufacturer did not follow proper software engineering practices. Other software errors might result from lack of validation of the input parameters. For example, in 1998, a crew member of the guided-missile cruiser USS Yorktown mistakenly entered a zero for a data value, which resulted in a division by zero. The error cascaded and eventually shut down the ship's propulsion system. The ship was dead in the water for several hours because a program did not check for valid input.

Another recent example of software reliability includes the Mars Polar Lander which was launched in January 1999 and was intended to land on Mars in December of that year. Legs were designed to deploy prior to landing. Sensors would detect touchdown and turn off the rocket motor. It was known and understood that the deployment of the landing legs generated spurious signals of the touchdown sensors. The software requirements, however, did not specifically describe this behavior and the software designers therefore did not account for it. The motor turned off at too high an altitude and the probe crashed into the planet at 50 mi/h and was destroyed. Mission costs exceeded \$120 million (Gruhn, 2004). Reliability also has a great effect on the consumers' perception of a manufacturer. For example, consumers' experiences with car recalls, repairs, and warranties will determine the future sales and survivability of that manufacturer. Most manufactures have experienced car recalls and extensive warranties that range from as low as 1.2% to 6% of the revenue. Some car recalls are extensive and costly such as the recall of 8.6 million cars due to the ignition causing small engine fires. In 2010, an extensive recall of several car models due to sudden acceleration resulted in the shutdown of the entire production system and hundreds of lawsuits. One of the causes of the recall is lack of thoroughness in testing new cars and car parts under varying weather conditions; the gas-pedal mechanism tended to stick more as humidity increased. Clearly, the number and magnitude of the recalls are indicative of the reliability performance of the car and potential survivability of the manufacturer.

## **1.2 RELIABILITY DEFINITION AND ESTIMATION**

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A formal definition of reliability is given as follows:

### **1.2.1 Reliability**

Reliability is the probability that a product will operate or a service will be provided properly for a specified period of time (design life) under the design operating conditions (such as temperature, load, volt ...) without failure.

In other words, reliability may be used as a measure of the system's success in providing its function properly during its design life. Consider the following.

Suppose  $n_o$  identical components are subjected to a design operating conditions test. During the interval of time  $(t - \Delta t, t)$ , we observed  $n_f(t)$  failed components, and  $n_s(t)$  surviving components [ $n_f(t) + n_s(t) = n_o$ ]. Since reliability is defined as the cumulative probability function of success, then at time  $t$ , the reliability  $R(t)$  is

$$R(t) = \frac{n_s(t)}{n_s(t) + n_f(t)} = \frac{n_s(t)}{n_o}. \tag{1.1}$$

In other words, if  $T$  is a random variable denoting the time to failure, then the reliability function at time  $t$  can be expressed as

$$R(t) = P(T > t). \tag{1.2}$$

The cumulative distribution function (CDF) of failure  $F(t)$  is the complement of  $R(t)$ , that is,

$$R(t) + F(t) = 1. \tag{1.3}$$

If the time to failure,  $T$ , has a probability density function (p.d.f.)  $f(t)$ , then Equation 1.3 can be rewritten as

$$R(t) = 1 - F(t) = 1 - \int_0^t f(\zeta) d\zeta. \tag{1.4}$$

Taking the derivative of Equation 1.4 with respect to  $t$ , we obtain

$$\frac{dR(t)}{dt} = -f(t). \tag{1.5}$$

For example, if the time to failure distribution is exponential with parameter  $\lambda$ , then

$$f(t) = \lambda e^{-\lambda t}, \tag{1.6}$$

and the reliability function is

$$R(t) = 1 - \int_0^t \lambda e^{-\lambda \zeta} d\zeta = e^{-\lambda t}. \tag{1.7}$$

From Equation 1.7, we express the probability of failure of a component in a given interval of time  $[t_1, t_2]$  in terms of its reliability function as

$$\int_{t_1}^{t_2} f(t) dt = R(t_1) - R(t_2). \tag{1.8}$$

We define the failure rate in a time interval  $[t_1, t_2]$  as the probability that a failure per unit time occurs in the interval given that no failure has occurred prior to  $t_1$ , the beginning of the interval. Thus, the failure rate is expressed as

$$\frac{R(t_1) - R(t_2)}{(t_2 - t_1)R(t_1)}. \quad (1.9)$$

If we replace  $t_1$  by  $t$  and  $t_2$  by  $t + \Delta t$ , then we rewrite Equation 1.9 as

$$\frac{R(t) - R(t + \Delta t)}{\Delta t R(t)}. \quad (1.10)$$

The hazard function is defined as the limit of the failure rate as  $\Delta t$  approaches zero. In other words, the hazard function or the instantaneous failure rate is obtained from Equation 1.10 as

$$h(t) = \lim_{\Delta t \rightarrow 0} \frac{R(t) - R(t + \Delta t)}{\Delta t R(t)} = \frac{1}{R(t)} \left[ -\frac{d}{dt} R(t) \right]$$

or

$$h(t) = \frac{f(t)}{R(t)}. \quad (1.11)$$

From Equations 1.5 and 1.11, we obtain

$$R(t) = e^{\left[ -\int_0^t h(\zeta) d\zeta \right]}, \quad (1.12)$$

$$R(t) = 1 - \int_0^t f(\zeta) d\zeta, \quad (1.13)$$

and

$$h(t) = \frac{f(t)}{R(t)}. \quad (1.14)$$

Equations 1.5, 1.12–1.14 are the key equations that relate  $f(t)$ ,  $F(t)$ ,  $R(t)$ , and  $h(t)$ .

The following example illustrates how the hazard rate,  $h(t)$ , and reliability are estimated from failure data.

### EXAMPLE 1.1

A manufacturer of light bulbs is interested in estimating the mean life of the bulbs. Two hundred bulbs are subjected to a reliability test. The bulbs are observed, and failures in 1000-h intervals are recorded as shown in Table 1.1.

Plot the failure density function estimated from data  $f_e(t)$ , the hazard-rate function estimated from data  $h_e(t)$ , the cumulative probability function estimated from data  $F_e(t)$ , and the reliability function estimated from data  $R_e(t)$ . The subscript  $e$  refers to *estimated*. Comment on the hazard-rate function.

**TABLE 1.1** Number of Failures in the Time Intervals

Time interval (hours)	Failures in the interval
0–1000	100
1001–2000	40
2001–3000	20
3001–4000	15
4001–5000	10
5001–6000	8
6001–7000	7
Total	200

**SOLUTION**

We estimate  $f_e(t)$ ,  $h_e(t)$ ,  $R_e(t)$ , and  $F_e(t)$  using the following equations:

$$f_e(t) = \frac{n_f(t)}{n_o \Delta t}, \quad (1.15)$$

$$h_e(t) = \frac{n_f(t)}{n_s(t) \Delta t}, \quad (1.16)$$

$$R_e(t) = \frac{f_e(t)}{h_e(t)} = \frac{n_s(t)}{n_o}, \quad (1.17)$$

and

$$F_e(t) = 1 - R_e(t). \quad (1.18)$$

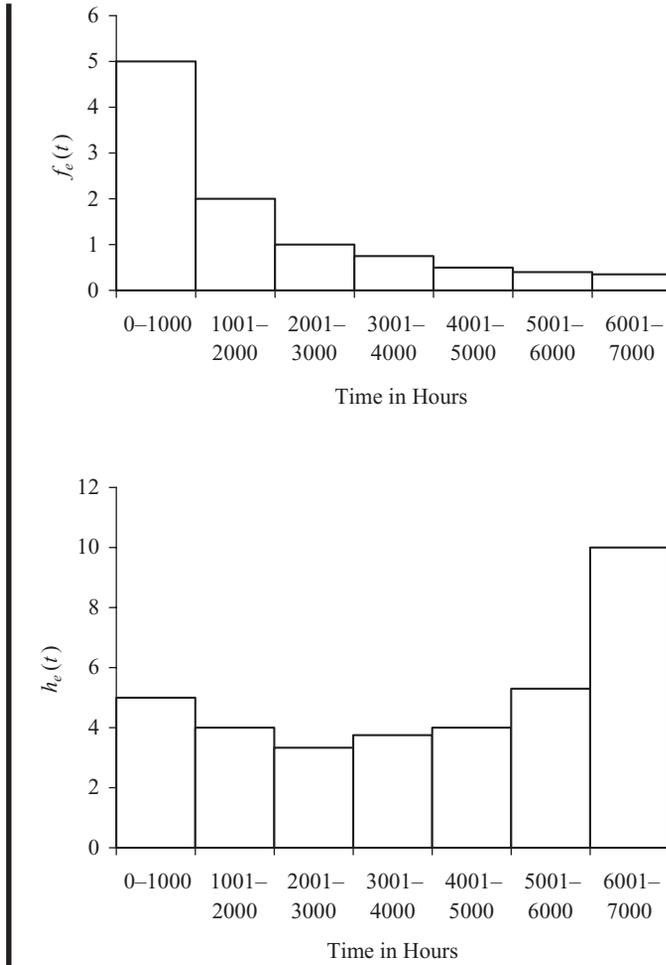
Note that  $n_s(t)$  is the number of surviving units at the beginning of the period  $\Delta t$ . Summaries of the calculations are shown in Tables 1.2 and 1.3. The plots are shown in Figures 1.1 and 1.2.

TABLE 1.2 Calculations of  $f_e(t)$  and  $h_e(t)$ 

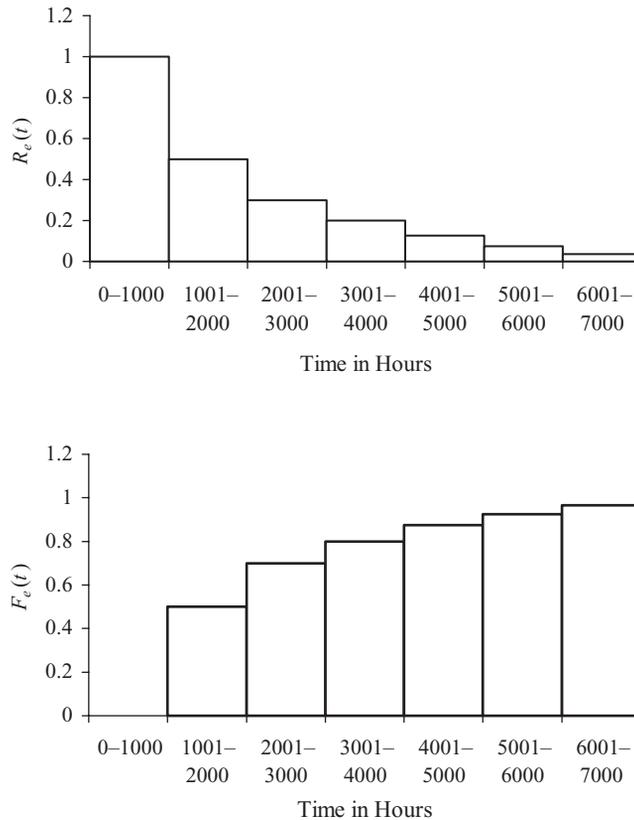
Time interval (h)	Failure density $f_e(t) \times 10^{-4}$	Hazard rate $h_e(t) \times 10^{-4}$
0–1000	$\frac{100}{200 \times 10^3} = 5.0$	$\frac{100}{200 \times 10^3} = 5.0$
1001–2000	$\frac{40}{200 \times 10^3} = 2.0$	$\frac{40}{100 \times 10^3} = 4.0$
2001–3000	$\frac{20}{200 \times 10^3} = 1.0$	$\frac{20}{60 \times 10^3} = 3.33$
3001–4000	$\frac{15}{200 \times 10^3} = 0.75$	$\frac{15}{40 \times 10^3} = 3.75$
4001–5000	$\frac{10}{200 \times 10^3} = 0.5$	$\frac{10}{25 \times 10^3} = 4.0$
5001–6000	$\frac{8}{200 \times 10^3} = 0.4$	$\frac{8}{15 \times 10^3} = 5.3$
6001–7000	$\frac{7}{200 \times 10^3} = 0.35$	$\frac{7}{7 \times 10^3} = 10.0$

TABLE 1.3 Calculations of  $R_e(t)$  and  $F_e(t)$ 

Time interval	Reliability $R_e(t) = f_e(t)/h_e(t)$	Unreliability $F_e(t) = 1 - R_e(t)$
0–1000	$\frac{5.0}{5.0} = 1.000$	0.000
1001–2000	$\frac{2.0}{4.0} = 0.500$	0.500
2001–3000	$\frac{1.0}{3.33} = 0.300$	0.700
3001–4000	$\frac{0.75}{3.75} = 0.200$	0.800
4001–5000	$\frac{0.5}{4.0} = 0.125$	0.875
5001–6000	$\frac{0.4}{5.3} = 0.075$	0.925
6001–7000	$\frac{0.35}{10.0} = 0.035$	0.965



**FIGURE 1.1** Plots of  $f_e(t) \times 10^{-4}$  and  $h_e(t) \times 10^{-4}$  versus time.



**FIGURE 1.2** Plots of  $R_e(t)$  and  $F_e(t)$  versus time.

As shown in Figure 1.1, the hazard rate is constant until time of 5000 h and then increases linearly with  $t$ . Thus,  $h_e(t)$  can be expressed as

$$h_e(t) = \begin{cases} \lambda_0 & 0 \leq t \leq 6,000 \\ \lambda_1 t & t > 6,000 \end{cases},$$

where  $\lambda_0$  and  $\lambda_1$  are constants. ■

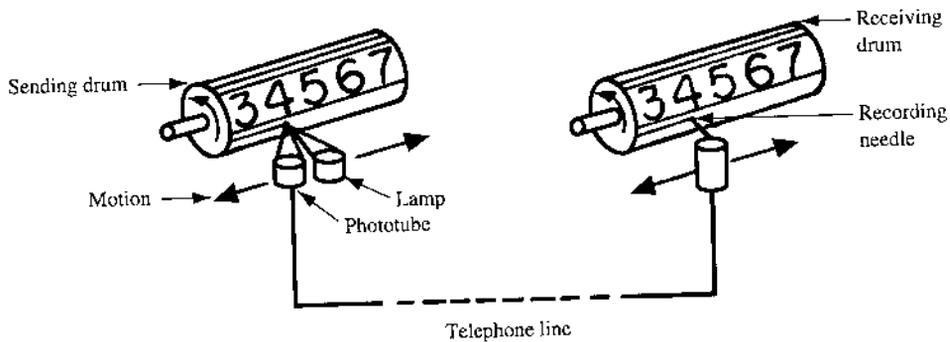
The above example shows the hazard-rate function is constant for a period of time and then linearly increases with time. In other situations, the hazard-rate function may be decreasing, constant, or increasing, and the rate at which the function decreases or increases may be constant, linear, polynomial, or exponential with time. The following example is an illustration of an exponentially increasing hazard-rate function.

**EXAMPLE 1.2**

Facsimile (fax) machines are designed to transmit documents, figures, and drawings between locations via telephone lines. The principle of a fax machine is shown in Figure 1.3. The document on the sending unit drum is scanned in both the horizontal and rotating directions. The document is divided into graphic elements, which are converted into electrical signals by a photoelectric reading head. The signals are transmitted via telephone lines to the receiving end where they are demodulated and reproduced by a recording head.

The quality of the received document is affected by the reliability of the photoelectric reading head in converting the graphic elements of the document being sent into proper electrical signals. A manufacturer of fax machines performs a reliability test to estimate the mean life of the reading head by subjecting 180 heads to repeated cycles of readings. The threshold times, at which the quality of the received document is unacceptable, are recorded in Table 1.4.

Estimate the hazard rate and reliability function of the machines.



**FIGURE 1.3** The principle of a fax machine.

**TABLE 1.4** Failure Data of the Facsimile Machines

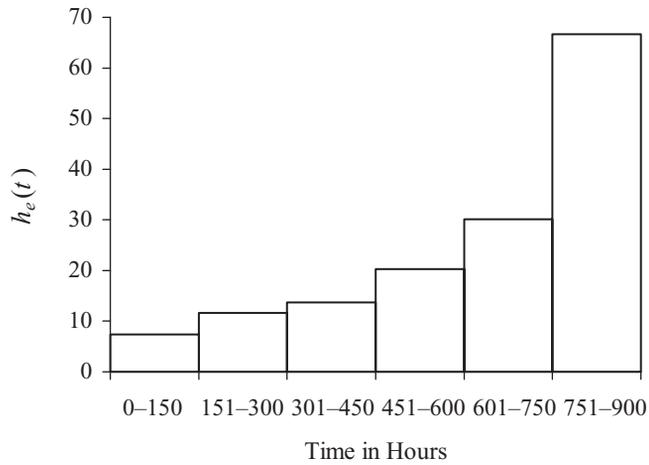
Time interval (hours)	0–150	151–300	301–450	451–600	601–750	751–900
Number of failures	20	28	27	32	33	40

**SOLUTION**

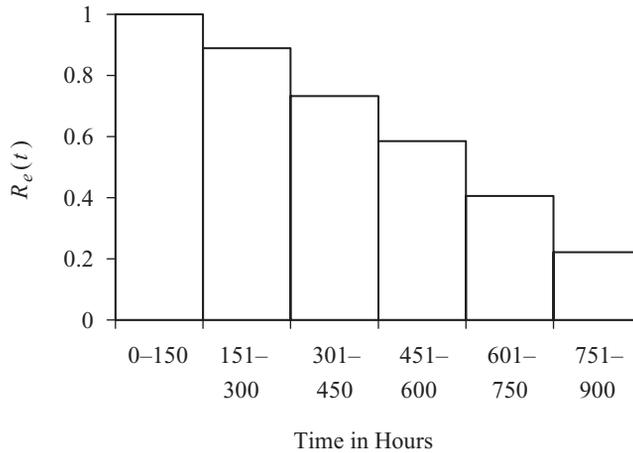
Using Equations 1.15–1.17, we calculate  $f_e(t)$ ,  $h_e(t)$ , and  $R_e(t)$  as shown in Table 1.5. Plots of the hazard rate and the reliability function are shown in Figures 1.4 and 1.5, respectively.

**TABLE 1.5** Calculations for  $f_e(t)$ ,  $h_e(t)$ , and  $R_e(t)$

$t$	$f_e(t) \times 10^{-4}$	$h_e(t) \times 10^{-4}$	$R_e(t)$
0-150	7.407	7.407	1.000
151-300	10.370	11.666	0.889
301-450	10.000	13.636	0.733
451-600	11.852	20.317	0.583
601-750	12.222	30.137	0.406
751-900	14.815	66.667	0.222



**FIGURE 1.4** Plot of the hazard-rate function versus time.



**FIGURE 1.5** Plot of the reliability function versus time.



In some situations, it is possible to observe the exact failure time of every unit (component). In such situations, we utilize order statistics to obtain a “distribution free” reliability function and its associated characteristics. There are several approaches to do so starting with the “naïve” estimator followed by median rank estimators. Since all these estimators utilize the order of the observations (failure times) only, we refer to them as ordered statistics, and the empirical estimate of  $F(t)$ , denoted as  $\hat{F}(t)$ , is referred to as rank estimator which is then used to generate the density plot and reliability function plot. We present the commonly used rank estimators (mean and median) as follows.

We begin by ordering the failure times in an increasing order such that  $t_1 \leq t_2 \leq \dots \leq t_{i-1} \leq t_i \leq t_{i+1} \leq \dots \leq t_{n-1} \leq t_n$  where  $t_i$  is the failure time of the  $i^{\text{th}}$  unit. Since we are interested in obtaining the naïve rank estimator  $\hat{F}(t)$ , we assign a probability mass of  $1/n$  to each of the  $n$  failure times and set  $\hat{F}(t_0) = 0$ . The naïve mean rank estimator  $\hat{F}(t)$  is expressed as

$$\hat{F}(t) = \frac{i}{n} \quad t_i \leq t \leq t_{i-1}.$$

This estimator has a deficiency in that, for  $t \geq t_n$ ,  $\hat{F}(t) = 1.0$ . Therefore, improvements are introduced that result in a more accurate estimate.

Among them is the most commonly used Herd–Johnson estimator (Herd, 1960; Johnson, 1964) which is expressed as

$$\hat{F}(t_i) = \frac{i}{n+1} \quad i = 0, 1, 2, \dots, n.$$

Others propose the use of the median rank instead. Several estimates of the median rank are commonly used; among them are Bernard’s median rank estimator (Bernard and Bosi-Levenbach, 1953) and Blom’s (1958) median rank estimator. They are expressed as

Bernard’s estimator of  $\hat{F}(t_i)$  is

$$\hat{F}(t_i) = \frac{i-0.3}{n+0.4} \quad i = 0, 1, 2, \dots, n.$$

Blom’s estimator is

$$\hat{F}(t_i) = \frac{i-3/8}{n+1/4} \quad i = 0, 1, 2, \dots, n.$$

The corresponding p.d.f., reliability function, and the hazard-rate function are derived as follows.

We consider the mean rank estimator (the approach is also valid for median rank estimators). The mean rank estimator is

$$\hat{F}(t_i) = \frac{i}{n+1} \quad i = 0, 1, 2, \dots, n.$$

The reliability expression is

$$R(t_i) = 1 - F(t_i) = \frac{n+1-i}{n+1} \quad t_i \leq t \leq t_{i+1} \quad i = 0, 1, 2, \dots, n.$$

Since the p.d.f. is the derivative of the CDF, then

$$\hat{f}(t_i) = \frac{\hat{F}(t_{i+1}) - \hat{F}(t_i)}{\Delta t_i} \quad \Delta t_i = t_{i+1} - t_i$$

or

$$\hat{f}(t_i) = \frac{1}{\Delta t_i \cdot (n+1)}.$$

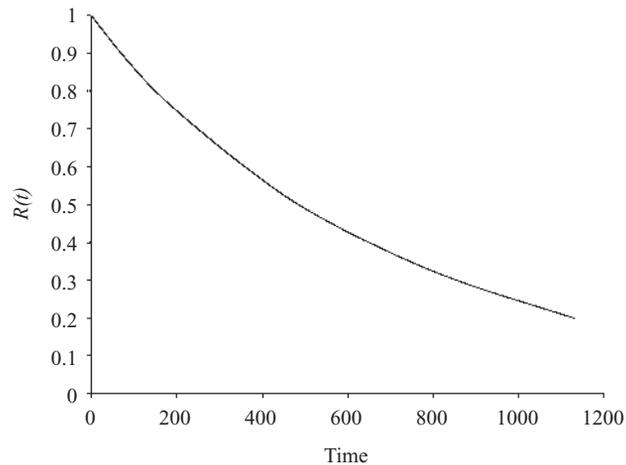
The hazard rate is  $h(t_i) = \frac{f(t_i)}{R(t_i)} = \frac{1}{\Delta t_i (n+1-i)}$ .

### EXAMPLE 1.3

Nine light bulbs are observed, and the exact failure time of each is recorded as 70, 150, 250, 360, 485, 650, 855, 1130, and 1540. Estimate the CDF, reliability function, p.d.f., and hazard-rate function. Plot these functions with time.

#### SOLUTION

Figures 1.6–1.8 show  $R(t)$ ,  $f(t)$ , and  $h(t)$  graphs, and the corresponding calculations are given in Table 1.6.



**FIGURE 1.6** Plot of the reliability function versus time.

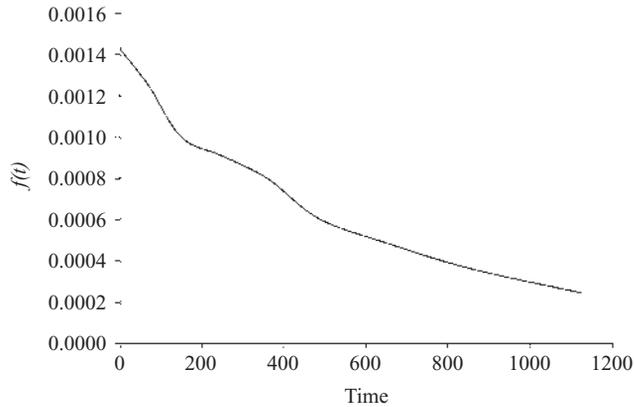


FIGURE 1.7 Plot of the probability density function versus time.

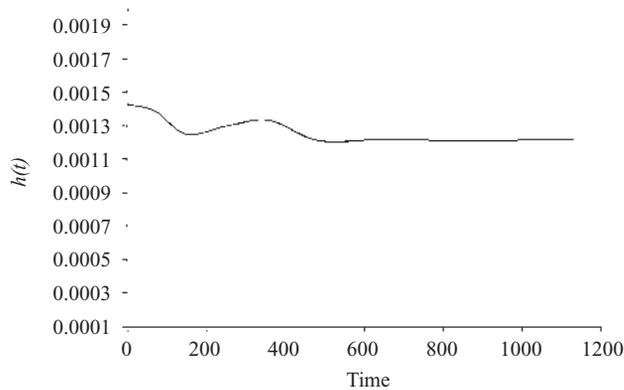


FIGURE 1.8 Plot of the hazard-rate function versus time.

TABLE 1.6  $F(t)$ ,  $R(t)$ ,  $f(t)$ , and  $h(t)$  Calculations

$i$	$t_i$	$t_{i+1}$	$\hat{F}(t_i) = \frac{i}{10}$	$R(t_i) = \frac{10-i}{10}$	$f(t_i) = \frac{1}{\Delta t_i(n+1)}$	$h(t_i) = \frac{1}{\Delta t_i(n+1-i)}$
0	0	70	0.0	1.0	0.001429	0.001429
1	70	150	0.1	0.9	0.001250	0.001389
2	150	250	0.2	0.8	0.001000	0.001250
3	250	360	0.3	0.7	0.000909	0.001299
4	360	485	0.4	0.6	0.000800	0.001333
5	485	650	0.5	0.5	0.000606	0.001212
6	650	855	0.6	0.4	0.000488	0.001220
7	855	1,130	0.7	0.3	0.000364	0.001212
8	1,130	1,540	0.8	0.2	0.000244	0.001220
9	1,540	—	0.9	0.1	—	—

The hazard-rate estimates using the mean rank and median rank are expressed respectively as

$$h_{\text{mean-rank}}(t_i) = \frac{1}{(n-i+1)(t_{i+1}-t_i)}$$

and

$$h_{\text{median-rank}}(t_i) = \frac{1}{(n-i+0.7)(t_{i+1}-t_i)}.$$

There are other estimates of the hazard rate such as Kaplan–Meier (to be discussed in Chapter 5) and Martz and Waller (1982) which is expressed as

$$h_{\text{Martz-Waller}}(t_i) = \frac{1}{(n-i+0.625)(t_{i+1}-t_i)}.$$

Martz and Waller’s estimate is suitable when the sample size is small. It should be noted that hazard rates estimated by the above three estimators differ only slightly especially when the number of observed failure-time data is large.

Analysis of the historical data of failed products, components, devices, and systems resulted in widely used expressions for  $h(t)$  and  $R(t)$ . We now consider the most commonly used expressions for  $h(t)$ .

### 1.3 HAZARD FUNCTIONS

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The *hazard function* or *hazard rate*  $h(t)$  is the conditional probability of failure in the interval  $t$  to  $(t + dt)$ , given that there was no failure at  $t$  divided by the length of the time interval  $dt$ . It is expressed as

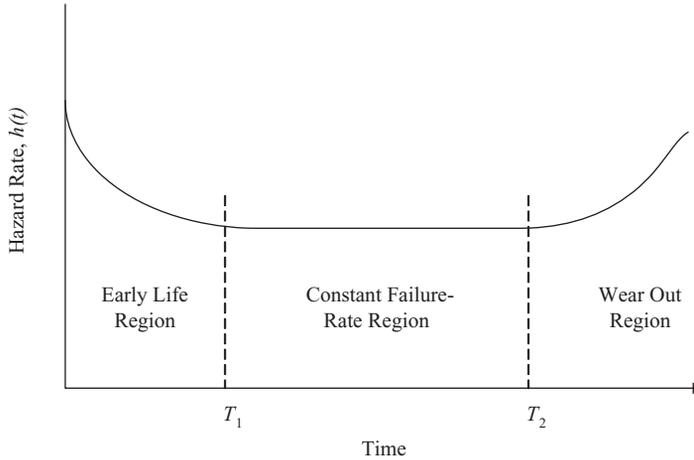
$$h(t) = \frac{f(t)}{R(t)}. \quad (1.19)$$

The *cumulative hazard function*  $H(t)$  is the conditional probability of failure in the interval 0 to  $t$ . It is also the total number of failures during the time interval 0 to  $t$ .

$$H(t) = \int_0^t h(\zeta) d\zeta. \quad (1.20)$$

The hazard rate is also referred to as the instantaneous failure rate. The hazard-rate expression is of the greatest importance for system designers, engineers, and repair and maintenance groups. The expression is useful in estimating the time to failure (or time between failures), repair crew size for a given repair policy, the availability of the system, and in estimating the warranty cost. It can also be used to study the behavior of the system’s failure with time.

As shown in Equation 1.19, the hazard rate is a function of time. One may ask what type of function does the hazard rate exhibit with time? The general answer to this question is the bathtub-shaped function as shown in Figure 1.9. To illustrate how this function is obtained, consider a population of identical components from which we take a large sample  $N$  and place it in operation at time  $T = 0$ . The sample experiences a high failure rate at the beginning of the operation time due to weak or substandard components, manufacturing imperfections, design



**FIGURE 1.9** The general failure curve.

errors, and installation defects. As the failed components are removed, the time between failures increases which results in a reduction in the failure rate. This period of decreasing failure rate (DFR) is referred to as the “infant mortality region,” the “shakedown” region, the “debugging” region, or the “early failure” region. This is an undesirable region from both the manufacturer and consumer viewpoints as it causes an unnecessary repair cost for the manufacturer and an interruption of product usage for the consumer. The early failures can be minimized by employing burn-in of systems or components before shipments are made (burn-in is a common process where the unit is subjected to a slightly severer stress conditions than those at normal operating conditions for a short period), by improving the manufacturing process, and by improving the quality control of the products. Time  $T_1$  represents the end of the early failure-rate region (normally this time is about  $10^4$  h for electronic systems).

At the end of the early failure-rate region, the failure rate will eventually reach a constant value. During the constant failure-rate region (between  $T_1$  and  $T_2$ ), the failures do not follow a predictable pattern, but they occur at random due to the changes in the applied load (the load may be higher or lower than the designed load). A higher load may cause overstressing of the component while a lower load may cause derating (application of a load in the reverse direction of what the component experiences under normal operating conditions) of the component and both will lead to failures. The randomness of the material flaws or manufacturing flaws will also lead to failures during the constant failure-rate region.

The third and final region of the failure-rate curve is the wear-out region, which starts at  $T_2$ . The beginning of the wear-out region is noticed when the failure rate starts to increase significantly more than the constant failure-rate value, and the failures are no longer attributed to randomness but are due to the age and wear of the components. Within this region, the failure rate increases rapidly as the product reaches its useful (designed) life. To minimize the effect of the wear-out region, one must use periodic preventive maintenance or consider replacement of the product.

Obviously, not all components exhibit the bathtub-shaped failure-rate curve. Most electronic and electrical components do not exhibit a wear-out region. Some mechanical compo-

nents may not show a constant failure-rate region but may exhibit a gradual transition between the early failure-rate and wear-out regions. The length of each region may also vary from one component (or product) to another. The estimates of the times at which the bathtub curve changes from one region to another have been of interest to researchers. They are referred to as the change-point estimates. One of the approaches for estimating the change point is to equate the estimated hazard rate at the end of the region to the estimated hazard rate at the beginning of the following region.

We now describe the failure-time distributions that exhibit one or more of the regions as follows.

### 1.3.1 Constant Hazard

Many electronic components—such as transistors, resistors, integrated circuits (ICs), and capacitors—exhibit constant failure rate (CFR) during their lifetimes. Of course, this occurs at the end of the early failure region, which usually has a time period of 1 year (8760 h). The early failure region is usually reduced by performing burn-in of these components. Burn-in is performed by subjecting components to stresses slightly higher than the expected operating stresses for a short period in order to weed out failures due to manufacturing defects. The constant hazard-rate function,  $h(t)$ , is expressed as

$$h(t) = \lambda, \quad (1.21)$$

where  $\lambda$  is a constant. The p.d.f.,  $f(t)$ , is obtained from Equation 1.19 as

$$f(t) = h(t) \exp \left[ - \int_0^t h(\zeta) d\zeta \right] \quad (1.22)$$

or

$$f(t) = \lambda e^{-\lambda t} \quad (1.23)$$

and

$$F(t) = \int_0^t \lambda e^{-\lambda \zeta} d\zeta = 1 - e^{-\lambda t}. \quad (1.24)$$

The reliability function,  $R(t)$ , is

$$R(t) = 1 - F(t) = e^{-\lambda t}. \quad (1.25)$$

Plots of  $h(t)$ ,  $f(t)$ ,  $F(t)$ , and  $R(t)$  are shown in Figures 1.10 and 1.11. At  $t = 1/\lambda$ ,  $f(1/\lambda) = \lambda/e$ ,  $F(1/\lambda) = 1 - 1/e = 0.632$ , and  $R(1/\lambda) = 1/e = 0.368$ . This is an important result since it states that the probability of failure of a product by its estimated mean time to failure (MTTF) ( $1/\lambda$ ) is 0.632. Also, note that the failure time for the constant hazard model is exponentially distributed.

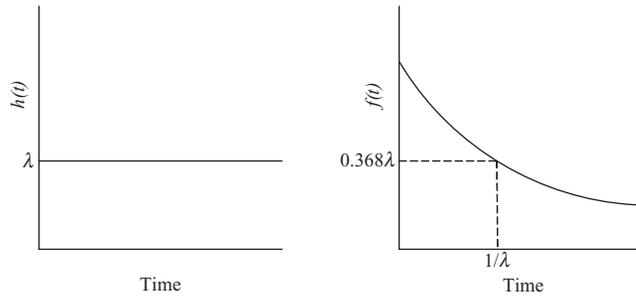


FIGURE 1.10 Plots of  $h(t)$  and  $f(t)$ .

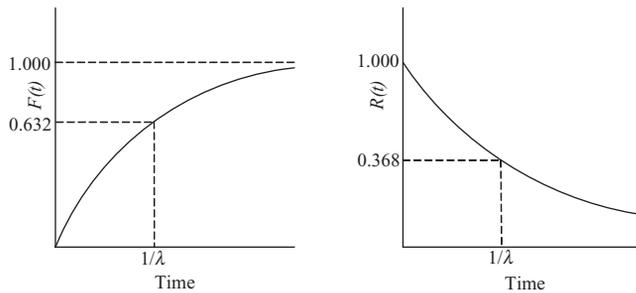


FIGURE 1.11 Plots of  $F(t)$  and  $R(t)$ .

#### EXAMPLE 1.4

A manufacturer performs an Operational Life Test (OLT) on ceramic capacitors and finds that they exhibit CFR (used interchangeably with hazard rate) with a value of  $3 \times 10^{-8}$  failures per hour. What is the reliability of a capacitor after  $10^4$  h? In order to accept a large shipment of these capacitors, the user decides to run a test for 5000 h on a sample of 2000 capacitors. How many capacitors are expected to fail during the test?

#### SOLUTION

Using Equations 1.21 and 1.25, we obtain

$$h(t) = 3 \times 10^{-8} \text{ failures per hour}$$

and

$$R(10^4) = e^{-3 \times 10^{-4}} = 0.99970.$$

To determine the expected number of failed capacitors during the test, we define the following:

$n_o$  number of capacitors under test,

$n_s$  expected number of surviving capacitors at the end of test, and

$n_f$  expected number of failed capacitors during the test.

Thus,

$$n_s = e^{-3 \times 10^{-8} \times 5000} \times 2000 = 1999 \text{ capacitors}$$

and

$$n_f = 2000 - 1999 = 1 \text{ capacitor.} \quad \blacksquare$$

### 1.3.2 Linearly Increasing Hazard

A component exhibits an increasing hazard rate when it either experiences wearout or when it is subjected to deteriorating conditions. Most of mechanical components—such as rotating shafts, valves, and cams—exhibit linearly increasing hazard rate due to wearout whereas components such as springs and elastomeric mounts exhibit linearly increasing hazard rate due to deterioration. Few electrical components such as relays exhibit linearly increasing hazard rate. The hazard-rate function is expressed as

$$h(t) = \lambda t, \quad (1.26)$$

where  $\lambda$  is constant. The p.d.f.,  $f(t)$ , is a Rayleigh distribution and is obtained as

$$f(t) = \lambda t e^{-\frac{\lambda t^2}{2}} \quad (1.27)$$

and

$$F(t) = 1 - e^{-\frac{\lambda t^2}{2}}. \quad (1.28)$$

The reliability function,  $R(t)$ , is

$$R(t) = e^{-\frac{\lambda t^2}{2}}. \quad (1.29)$$

Plots of  $h(t)$ ,  $f(t)$ ,  $R(t)$ , and  $F(t)$  are shown in Figures 1.12 and 1.13. It should be noted that the failure-time distribution of the linearly increasing hazard is a Rayleigh distribution. The mean (expected value) and the variance of the distribution are  $\sqrt{\pi/2\lambda}$  and  $2/\lambda(1 - \pi/4)$ , respectively.

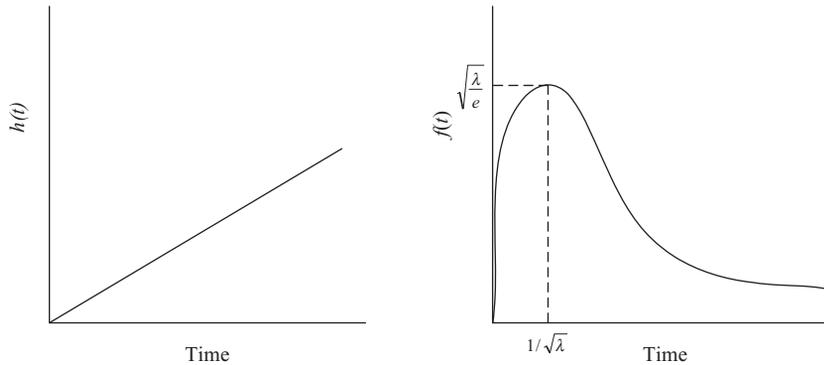


FIGURE 1.12 Plots of  $h(t)$  and  $f(t)$ .

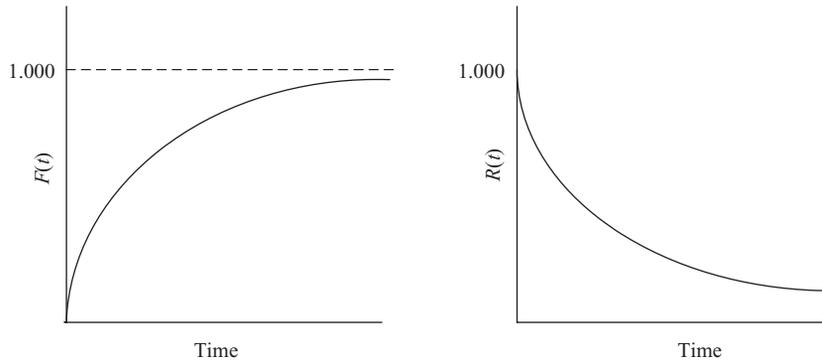


FIGURE 1.13 Plots of  $R(t)$  and  $F(t)$ .

### EXAMPLE 1.5

Rolling resistance is a measure of the energy lost by a tire under load when it resists the force opposing its direction of travel. In a typical car, traveling at 60 mi/h, about 20% of the engine power is used to overcome the rolling resistance of the tires. A tire manufacturer introduces a new material that, when added to the tire rubber compound significantly improves the tire rolling resistance but increases the wear rate of the tire tread. Analysis of a laboratory test of 150 tires shows that the failure rate of the new tire is linearly increasing with time (in hours). It is expressed as  $h(t) = 0.50 \times 10^{-8} t$ .

Determine the reliability of the tire after 1 year of use. What is the mean time to replace the tire?

### SOLUTION

Using Equation 1.29 we obtain the reliability after 1 year as

$$R(8,760) = e^{-\frac{0.5 \times 10^{-8} \times (8,760)^2}{2}} = 0.825.$$

The mean time to replace the tire is

$$\text{Mean time} = \sqrt{\frac{\pi}{2\lambda}} = \sqrt{\frac{\pi}{2 \times 0.5 \times 10^{-8}}} = 17,724 \text{ h,}$$

and the standard deviation of the time to tire replacement is

$$\sigma = \sqrt{\frac{2}{\lambda} \left(1 - \frac{\pi}{4}\right)} = 9,265 \text{ h.}$$

### 1.3.3 Linearly Decreasing Hazard

Most components (both mechanical and electrical) show decreasing hazard rates during their early lives. The hazard rate decreases linearly or nonlinearly with time. In this section, we shall consider linear hazard functions while nonlinear functions will be considered in the next section. The linearly decreasing hazard-rate function is expressed as

$$h(t) = a - bt \quad (1.30)$$

and

$$a \geq bt,$$

where  $a$  and  $b$  are constants. Similar to the linearly increasing hazard-rate function, we can obtain expressions for  $f(t)$ ,  $R(t)$ , and  $F(t)$ . The failure model and the reliability of a component exhibiting such hazard function depend on the values of  $a$  and  $b$ .

### 1.3.4 Weibull Model

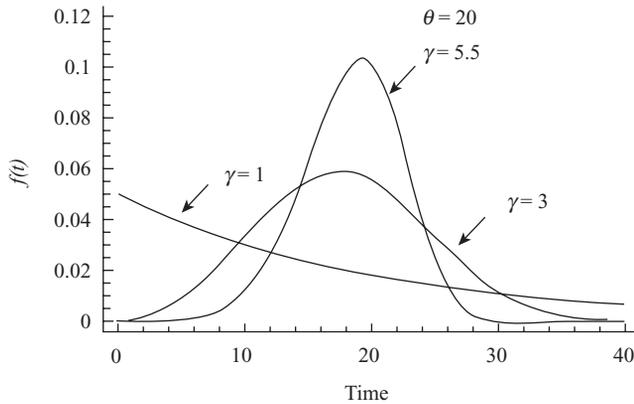
A nonlinear expression for the hazard-rate function is used when it clearly cannot be represented linearly with time. A typical expression for the hazard function (decreasing or increasing) under this condition is

$$h(t) = \frac{\gamma}{\theta} \left(\frac{t}{\theta}\right)^{\gamma-1}. \quad (1.31)$$

This model is referred to as the Weibull model, and its  $f(t)$  is given as

$$f(t) = \frac{\gamma}{\theta} \left(\frac{t}{\theta}\right)^{\gamma-1} e^{-\left(\frac{t}{\theta}\right)^\gamma} \quad t > 0, \quad (1.32)$$

where  $\theta$  and  $\gamma$  are positive and are referred to as the characteristic life and the shape parameter of the distribution, respectively. For  $\gamma = 1$  this  $f(t)$  becomes an exponential density. When  $\gamma = 2$ , the density function becomes a Rayleigh distribution. It is also well known that the Weibull



**FIGURE 1.14** The Weibull p.d.f. for different  $\gamma$ .

p.d.f. approximates to a normal distribution if a suitable value for the shape parameter  $\gamma$  is chosen. Makino (1984) approximated the Weibull distribution to a normal using the mean hazard rate and found that the shape parameter that approximates the two distributions is  $\gamma = 3.43927$ . This value of  $\gamma$  is near to the value  $\gamma = 3.43938$ , which is the value of the shape parameter of the Weibull distribution at which the mean is equal to the median. The p.d.f.'s of the Weibull distribution for different  $\gamma$ 's are shown in Figure 1.14. The distribution and reliability functions of the Weibull distribution  $F(t)$  and  $R(t)$  are given in Equations 1.34 and 1.35, respectively.

$$F(t) = \int_0^t \frac{\gamma}{\theta} \left( \frac{\zeta}{\theta} \right)^{\gamma-1} e^{-\left(\frac{\zeta}{\theta}\right)^\gamma} d\zeta \quad (1.33)$$

or

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^\gamma} \quad t > 0 \quad (1.34)$$

and

$$R(t) = e^{-\left(\frac{t}{\theta}\right)^\gamma} \quad t > 0 \quad (1.35)$$

The Weibull distribution is widely used in reliability modeling since other distributions such as exponential, Rayleigh, and normal are special cases of the Weibull distribution. Again, the hazard-rate function follows the Weibull model:

$$h(t) = \frac{f(t)}{1 - F(t)} = \frac{\gamma}{\theta} \left( \frac{t}{\theta} \right)^{\gamma-1}. \quad (1.36)$$

When  $\gamma > 1$ , the hazard rate is a monotonically increasing function with no upper bound that describes the wear-out region of the bathtub curve. When  $\gamma = 1$ , the hazard rate becomes constant (constant failure-rate region), and when  $\gamma < 1$ , the hazard-rate function decreases with time (the early failure-rate region). This enables the Weibull model to describe the failure rate of many failure data in practice. The mean and variance of the Weibull distribution are

$$E[T \text{ (time to failure)}] = \theta \Gamma\left(1 + \frac{1}{\gamma}\right), \quad (1.37)$$

$$\text{Var}[T] = \theta^2 \left\{ \Gamma\left(1 + \frac{2}{\gamma}\right) - \left[ \Gamma\left(1 + \frac{1}{\gamma}\right) \right]^2 \right\}, \quad (1.38)$$

where  $\Gamma(n)$  is the gamma function

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

and

$$\int_0^{\infty} x^{n-1} e^{-x/\theta} dx = \Gamma(n)\theta^n.$$

### EXAMPLE 1.6

To determine the fatigue limit of specially treated steel bars, the Prot method (Collins, 1981) for performing fatigue test is utilized. The test involves the application of a steadily increasing stress level with applied cycles until the specimen under test fails. The number of cycles to failure is observed to follow a Weibull distribution with  $\theta = 5$  (measurements are in  $10^3$  cycles) and  $\gamma = 2$ .

1. What is the reliability of a bar at  $10^6$  cycles? What is the corresponding hazard rate?
2. What is the expected life (in cycles) for a bar of this type?

### SOLUTION

Since the shape parameter  $\gamma$  equals 2, the Weibull distribution becomes a Rayleigh distribution, and we have a linearly increasing hazard function. Its p.d.f. is given by Equation 1.32.

1. The reliability expression for the Weibull model is given by Equation 1.35:

$$\begin{aligned} R(10^6) &= e^{-(10^6/5 \times 10^3)^2} \\ &= e^{-40,000} = 0. \end{aligned}$$

The hazard rate at  $10^6$  cycles is

$$h(t) = \frac{\gamma}{\theta} \left( \frac{t}{\theta} \right)^{\gamma-1} = \frac{2}{5000} \times \left( \frac{10^6}{5000} \right)$$

or  $h(10^6) = 0.08$  failures/cycle.

2. The expected life of a bar is

$$\begin{aligned} E[T(\text{cycles to failure})] &= \theta \Gamma \left( 1 + \frac{1}{\gamma} \right) = (5 \times 10^3) \Gamma \left( \frac{3}{2} \right) \\ &= (5000) \left( \frac{1}{2} \right) \sqrt{\pi} = 4431. \end{aligned}$$

The expected life of a bar from this steel is 4431 cycles. ■

In the above example, the Weibull model became a Rayleigh model since the failure rate is linearly increasing with time. In the following example, we consider the situation when the failure rate is nonlinearly increasing with time.

It is assumed that the failure time follows a known Weibull distribution and that the parameters of the distribution are known. In actual situations, the actual failure-time observations are the only known information. In this case, the failure-time data are used to obtain the failure-time distribution by fitting the data to the appropriate probability distribution. This can be achieved by plotting the frequency of failure times in a histogram and fitting a curve to them. The fitted curve is then used as a basis to select appropriate probability distribution that fits the data. The latter step is accomplished using standard software or probability papers. The following example illustrates these procedures.

### EXAMPLE 1.7

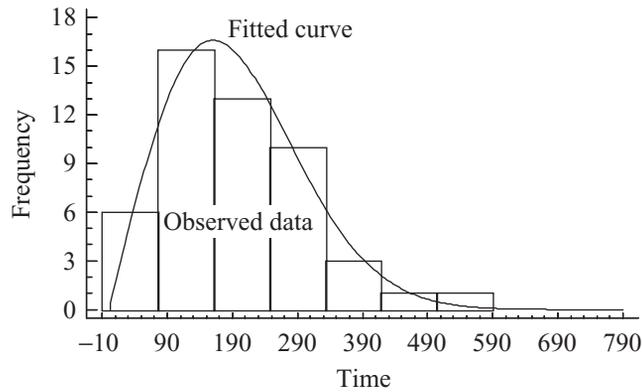
A manufacturer of a tungsten-carbide cutting tool for highly abrasive rubber materials conducted a tool life experiments on 50 tools. The times to tool failure are given as

17	31	58	66	73	73	97	108	111	117
132	132	138	140	143	143	145	147	150	157
158	161	164	168	171	177	182	185	187	196
197	202	223	242	246	249	260	269	276	287
298	308	312	314	316	338	349	354	423	529

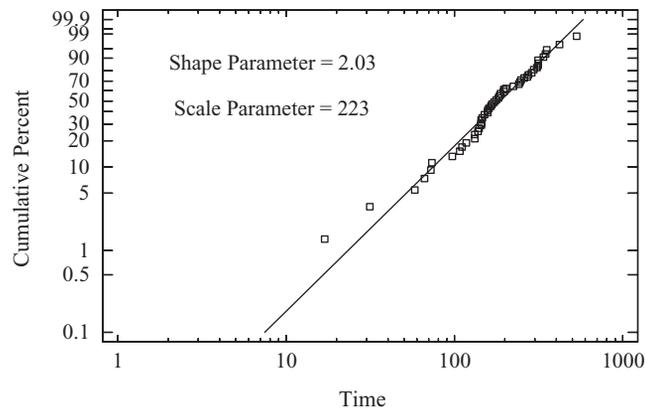
Use probability plot and fit the data with an appropriate probability distribution.

### SOLUTION

Using a standard software such as STATGRAPHICS™ or SAS™, obtain a frequency distribution as shown in Figure 1.15. The fitted curve indicates an increasing hazard rate similar to the Weibull model discussed earlier in this chapter. Since Weibull is one of the most widely used distributions for analyzing reliability data, a Weibull probability plot is shown in Figure 1.16.



**FIGURE 1.15** Frequency distribution of the failure times.



**FIGURE 1.16** Probability plot of the failure times.

The straight line indicates that Weibull distribution is appropriate to describe the failure times. The parameters of the model are estimated as  $\gamma = 2.03$  and  $\theta = 223$  (using the software). Several methods for estimating these parameters are described in Chapter 5. ■

Alternatively, the parameters of the Weibull model can be obtained by using one of the approaches discussed above for estimating the CDF  $F(t)$  from failure data and fitting a linear regression model as described below.

The CDF is expressed as

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^\gamma} \quad t > 0.$$

Taking the natural logarithm of  $1 - F(t)$  results in

$$\ln(1 - F(t)) = -\left(\frac{t}{\theta}\right)^\gamma.$$

Taking the logarithm one more time, we obtain

$$\ln\left[\ln\left(\frac{1}{1 - F(t)}\right)\right] = \gamma \ln t - \gamma \ln \theta.$$

Fitting a linear regression model to the left hand side of the above expression and  $\ln t$  we can then easily obtain the parameters of the Weibull model.

### EXAMPLE 1.8

A manufacturing engineer observes the wear-out rate of a milling machine tool insert and fits a Weibull hazard model to the tool wear data. The parameters of the model are  $\gamma = 2.25$  and  $\theta = 30$ . Determine the reliability of the tool insert after 10 h, the expected life of the insert, and the standard deviation of the mean life.

#### SOLUTION

The reliability after 10 h of operation is

$$R(10) = e^{-\left(\frac{10}{30}\right)^{2.25}} = 0.919.$$

The mean life of the insert is

$$\begin{aligned} \text{Mean life} &= \theta \Gamma\left(1 + \frac{1}{\gamma}\right) \\ &= 30 \Gamma\left(1 + \frac{1}{2.25}\right), \end{aligned}$$

or Mean life =  $30 \Gamma(1.444) = 26.572$  h.

The value of  $\Gamma(1.444)$  is obtained from the tables of the gamma function given in Appendix A.

Using Equation 1.38, we obtain the variance of the life as

$$\begin{aligned} \text{Variance} &= \theta^2 \left\{ \Gamma\left(1 + \frac{2}{\gamma}\right) - \left[ \Gamma\left(1 + \frac{1}{\gamma}\right) \right]^2 \right\} \\ &= 30^2 \left\{ \Gamma(1.888) - [\Gamma(1.444)]^2 \right\} \end{aligned}$$

or

Variance = 156.140, and the standard deviation of the life is 12.50 h. ■

### 1.3.5 Mixed Weibull Model

This model is applicable when components or products experience two or more failure modes. For example, a mechanical component, such as a load-carrying bearing or a cutting tool, may fail due to wearout or when the applied stress exceeds the design strength of component material resulting in catastrophic failure (catastrophic failure is a failure that destroys the system, such as a missile failure). Each type of these failures may be modeled by a separate simple Weibull model. Since the component or the tool can fail in either of the failure modes, it is then appropriate to describe the hazard rate by a mixed Weibull model. It is expressed as

$$f(t) = p \frac{\gamma_1}{\theta_1} \left( \frac{t}{\theta_1} \right)^{\gamma_1 - 1} e^{-\left( \frac{t}{\theta_1} \right)^{\gamma_1}} + (1-p) \frac{\gamma_2}{\theta_2} \left( \frac{t}{\theta_2} \right)^{\gamma_2 - 1} e^{-\left( \frac{t}{\theta_2} \right)^{\gamma_2}} \quad (1.39)$$

for  $\theta_1, \theta_2 > 0$ .

The quantity  $p$  ( $0 \leq p \leq 1$ ) is the probability that the component or the tool fails in the first failure mode, and  $1-p$  is the probability that it fails in the second failure mode. Clearly, if a product experiences more than two failure modes, the model given by Equation 1.39 can be expanded to include all failure modes and associated probabilities such that  $\sum_{i=1}^n p_i = 1$  where  $p_i$  is the probability that the product fails in the  $i$ th failure mode, and  $n$  is the total number of failure modes.

Following Kao (1959), the time  $t_e$  at which the proportion of the catastrophic failure is equal to that of wear-out failure is obtained as

$$1 - e^{-\left( \frac{t_e}{\theta_1} \right)^{\gamma_1}} = 1 - e^{-\left( \frac{t_e}{\theta_2} \right)^{\gamma_2}}$$

or

$$t_e = \left( \frac{\theta_2^{\gamma_2}}{\theta_1^{\gamma_1}} \right)^{\frac{1}{\gamma_2 - \gamma_1}} = \exp \left( \frac{\gamma_2 \ln \theta_2 - \gamma_1 \ln \theta_1}{\gamma_2 - \gamma_1} \right). \quad (1.40)$$

The reliability expression of the mixed Weibull model is

$$R(t) = 1 - p \left[ 1 - e^{-\left( \frac{t}{\theta_1} \right)^{\gamma_1}} \right] - (1-p) \left[ 1 - e^{-\left( \frac{t}{\theta_2} \right)^{\gamma_2}} \right]. \quad (1.41)$$

Clearly, if the second failure mode occurs after a delay time  $\delta$ , from the first failure mode, we rewrite Equations 1.39 and 1.41 as follows:

$$f_d(t) = p \frac{\gamma_1}{\theta_1} \left( \frac{t}{\theta_1} \right)^{\gamma_1 - 1} e^{-\left( \frac{t}{\theta_1} \right)^{\gamma_1}} + (1-p) \frac{\gamma_2}{\theta_2} \left( \frac{t-\delta}{\theta_2} \right)^{\gamma_2 - 1} e^{-\left( \frac{t-\delta}{\theta_2} \right)^{\gamma_2}} \quad (1.42)$$

and

$$R_d(t) = 1 - p \left[ 1 - e^{-\left( \frac{t}{\theta_1} \right)^{\gamma_1}} \right] - (1-p) \left[ 1 - e^{-\left( \frac{t-\delta}{\theta_2} \right)^{\gamma_2}} \right], \quad (1.43)$$

where the subscript  $d$  denotes delay.

### 1.3.6 Exponential Model (the Extreme Value Distribution)

The extreme value distribution is closely related to the Weibull distribution. It is useful in modeling cases when the hazard function is initially constant and then begins to increase rapidly with time.

The distribution is used to describe the failure time of products (or components) that will operate properly at normal operating conditions and will fail owing to a secondary cause of failure (such as overheating or fracture) when subjected to extreme conditions. In other words, the interest is in the tails of the failure distribution. Here, the hazard-rate function, the failure-time density function, and the reliability function are expressed as

$$h(t) = be^{\alpha t} \quad (1.44)$$

$$f(t) = be^{\alpha t} e^{-\int_0^t h(\zeta) d\zeta} \quad (1.45)$$

$$f(t) = be^{\alpha t} e^{-\frac{b}{\alpha}(e^{\alpha t} - 1)} \quad (1.46)$$

$$R(t) = e^{-\frac{b}{\alpha}(e^{\alpha t} - 1)}, \quad (1.47)$$

where  $b$  is a constant and  $e^{\alpha}$  represents the increase in failure rate per unit time. For example, if it is found that the failure rate of a component increases about 10% each year, then  $h(t) = b(1.1)^t$  where  $\alpha = \ln(1.1) = 0.0953$ . The function  $f(t)$  as given by Equation 1.46 is also known as the *Gompertz distribution*.

Plots of the hazard rate and the reliability functions of the *extreme value distribution* for different values of  $\alpha$  and  $b$  are shown in Figure 1.17. Some electronic components show such a hazard function. There are mechanical assemblies that exhibit extreme value hazard functions when subjected to high stresses. An example of such assemblies is a gearbox that operates properly at the recommended speeds. Excessive speeds may cause wearout of bearings that result in misalignments of shafts and an eventual failure of the assembly.

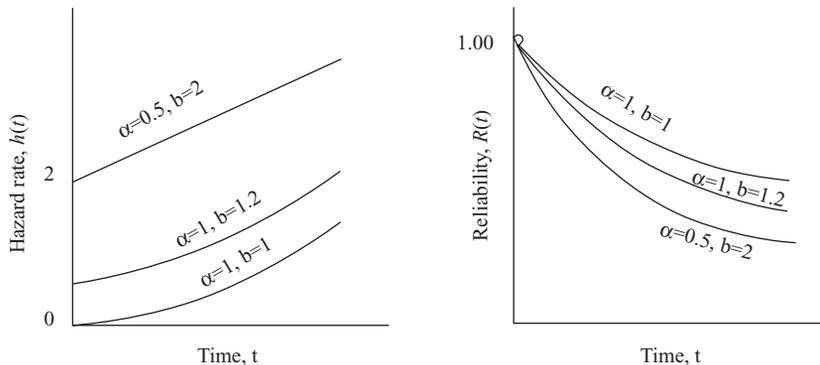


FIGURE 1.17 Plots of  $h(t)$  and  $R(t)$ .

**EXAMPLE 1.9**

Excessive vibrations due to high speed cutting on a computer numerical control (CNC) machine may lead to the failure of the cutting tool. The failure time of the tool follows an extreme value distribution. The failure rate increases about 15% per hour. Assuming that  $b = 0.01$ , calculate the reliability of the tool at  $t = 10$  h.

**SOLUTION**

Since the failure rate increases by 15% per hour, then  $\alpha = \ln(1.15) = 0.1397$ . Substituting the parameters  $\alpha$  and  $b$  into Equation 1.47, we obtain

$$R(10) = e^{-\frac{0.01}{0.1397}(e^{0.1397 \times 10} - 1)}$$

$$R(10) = 0.8042. \quad \blacksquare$$

**1.3.7 Normal Model**

There are many practical situations where the failure time of components (or parts) can be described by a normal distribution. For example, most of the mechanical components that are subjected to repeated cyclic loads, such as a fatigue test, exhibit normal hazard rates. Unlike other continuous probability distributions, there are no closed form expressions for the reliability or hazard-rate functions. The CDF of the life of a component is given by

$$F(t) = P[T \leq t] = \int_{-\infty}^t \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\tau - \mu}{\sigma}\right)^2\right] d\tau, \quad (1.48)$$

and

$$R(t) = 1 - F(t),$$

where  $\mu$  and  $\sigma$  are the mean and the standard deviation of the distribution. Unlike other distributions, the integral of the cumulative distribution cannot be evaluated in a closed form. However, the standard normal distribution ( $\sigma = 1$  and  $\mu = 0$ ) can be utilized in evaluating the probabilities for any normal distribution. The p.d.f. for the standard normal distribution is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \quad -\infty < z < \infty, \quad (1.49)$$

where

$$z = \frac{\tau - \mu}{\sigma}.$$

The CDF is

$$\Phi(\tau) = \int_{-\infty}^{\tau} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz. \quad (1.50)$$

Therefore, when the failure time of a component is expressed as a normally distributed random variable  $T$ , with mean  $\mu$  and standard deviation  $\sigma$ , one can easily determine the probability that

the component will fail by time  $t$  (i.e., the unreliability of the component) by using the following equation:

$$P(T \leq t) = P\left(\frac{T - \mu}{\sigma} \leq \frac{t - \mu}{\sigma}\right) = \Phi\left(\frac{t - \mu}{\sigma}\right). \quad (1.51)$$

The right side of Equation 1.51 can be evaluated using the standard normal tables. The hazard function,  $h(t)$ , of the normal distribution is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\phi\left(\frac{t - \mu}{\sigma}\right)/\sigma}{R(t)}. \quad (1.52)$$

It can be shown that the hazard function for a normal distribution is a monotonically increasing function of  $t$ ,

$$\begin{aligned} h(t) &= \frac{f(t)}{1 - F(t)} \\ h'(t) &= \frac{(1 - F)f' + f^2}{(1 - F)^2}. \end{aligned} \quad (1.53)$$

The denominator is nonnegative for all  $t$ . Hence, it is sufficient to show that the numerator of Equation 1.53 is  $\geq 0$ :

$$(1 - F)f' + f^2 \geq 0. \quad (1.54)$$

The p.d.f. of the normal distribution is

$$f(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(t-\mu)^2/2\sigma^2}, \quad -\infty < t < \infty,$$

and Equation 1.54 can be rewritten as

$$R(t) \frac{d}{dt} f(t) + f^2(t) \geq 0.$$

Now, the derivative term is

$$\begin{aligned} \frac{d}{dt} f(t) &= \frac{1}{\sqrt{2\pi\sigma^2}} \frac{d}{dt} e^{-(t-\mu)^2/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{-(t-\mu)}{\sigma^2} e^{-(t-\mu)^2/2\sigma^2} \\ &= \frac{-(t-\mu)}{\sigma^2} f(t), \end{aligned}$$

so now the condition that must be satisfied is

$$f(t) \left( \frac{-(t-\mu)}{\sigma^2} R(t) + f(t) \right) \geq 0.$$

Since  $f(t) \geq 0$  by definition and  $R(t) = \int_t^\infty f(x)dx$ , we may use the condition

$$\frac{(t-\mu)}{\sigma^2} \int_t^\infty f(x)dx \leq \int_t^\infty \frac{(x-\mu)}{\sigma^2} f(x)dx = \int_t^\infty -df(x) = f(t)$$

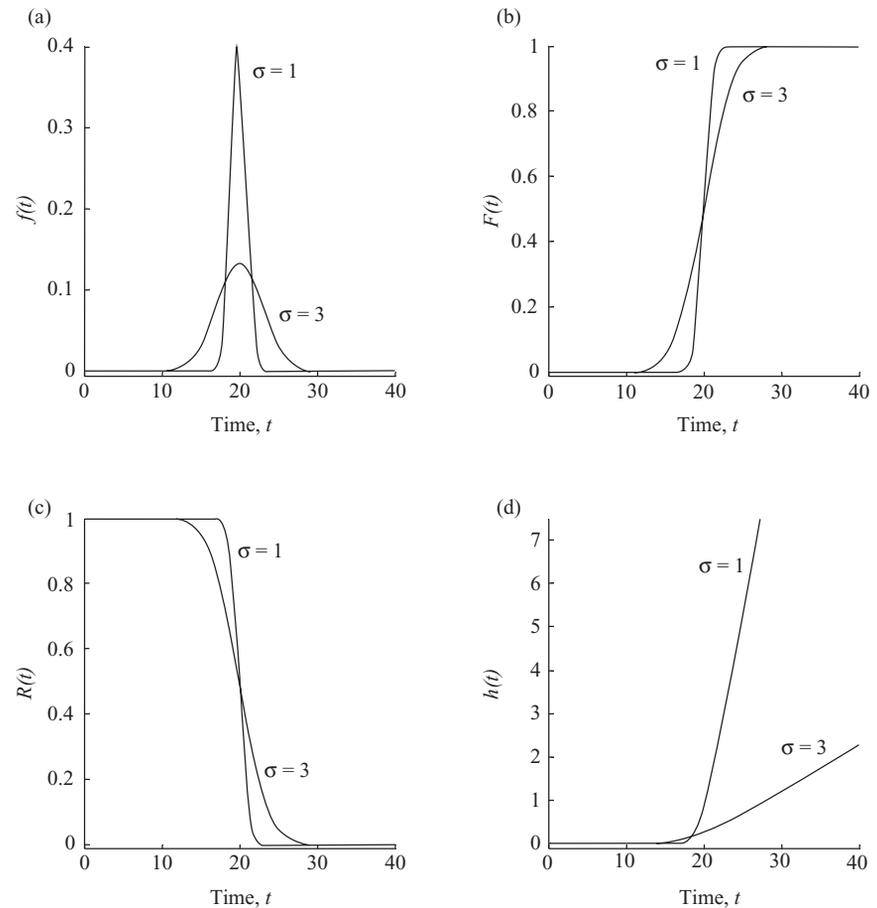
to obtain

$$f(t) \geq \frac{t-\mu}{\sigma^2} \int_t^\infty f(x)dx$$

So

$$f(t) \left( f(t) - \frac{(t-\mu)}{\sigma^2} \int_t^\infty f(x)dx \right) \geq 0,$$

and therefore the Gaussian hazard function is a monotonically increasing function of time. The plots of  $f(t)$ ,  $F(t)$ ,  $R(t)$ , and  $h(t)$  for  $\mu = 20$  are shown in Figure 1.18.



**FIGURE 1.18**  $f(t)$ ,  $F(t)$ ,  $R(t)$ , and  $h(t)$  for the normal model.

**EXAMPLE 1.10**

A component has a normal distribution of failure times with  $\mu = 40,000$  cycles and  $\sigma = 2000$  cycles. Find the reliability and hazard function at 38,000 cycles.

**SOLUTION**

The reliability function is

$$\begin{aligned} R(t) &= P\left(z > \frac{t - \mu}{\sigma}\right) \\ R(38,000) &= P\left(z > \frac{38,000 - 40,000}{2,000}\right) \\ &= P[z > -1.0] = \Phi(1.0) \\ &= 0.8413. \end{aligned}$$

The value of  $h(38,000)$  is

$$\begin{aligned} h(38,000) &= \frac{f(38,000)}{R(38,000)} = \frac{\phi\left(\frac{38,000 - 40,000}{2000}\right)/2000}{R(38,000)} \\ &= \frac{\phi(-1.0)}{2000 \times 0.8413} = \frac{0.2420}{2000 \times 0.8413} \\ &= 0.0001438 \text{ failures per cycle.} \end{aligned}$$

**1.3.8 Lognormal Model**

One of the most widely used probability distributions in describing the life data resulting from a single semiconductor failure mechanism or a closely related group of failure mechanisms is the lognormal distribution. It is also used in predicting reliability from accelerated life test data. The p.d.f. of the lognormal distribution is

$$f(t) = \frac{1}{\sigma t \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{\ln t - \mu}{\sigma}\right)^2\right] \quad -\infty < \mu < \infty, \sigma > 0, t > 0.$$

Figure 1.19 shows the p.d.f. of the lognormal distribution for different  $\mu$  and  $\sigma$ .

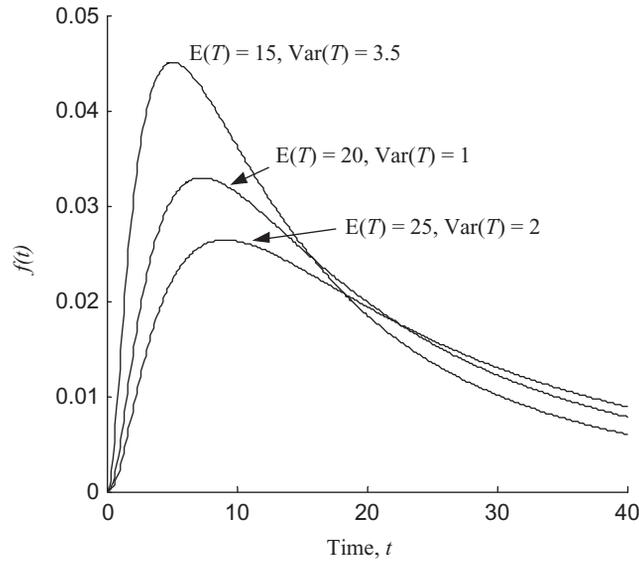
If a random variable  $X$  is defined as  $X = \ln T$ , where  $T$  is lognormal, then  $X$  is normally distributed with mean  $\mu$  and standard deviation  $\sigma$ .

$$E[X] = E[\ln(T)] = \mu$$

$$\text{Var}[X] = \text{Var}[\ln(T)] = \sigma^2.$$

Since  $T = e^X$ , then the mean of the lognormal can be found by using the normal distribution:

$$E(T) = E(e^X) = \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} \exp\left[x - \frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2\right] dx$$



**FIGURE 1.19**  $f(t)$  of the lognormal distribution for different  $\mu$  and  $\sigma$ .

$$E(T) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x - (\mu + \sigma^2))^2\right] dx.$$

The mean of the lognormal is

$$E(T) = \exp\left[\mu + \frac{\sigma^2}{2}\right].$$

The second moment is obtained as

$$E(T^2) = E[e^{2X}] = \exp\left[2\left(\mu + \frac{\sigma^2}{2}\right)\right],$$

and the variance of the lognormal is

$$\text{Var}(T) = [e^{2\mu + \sigma^2}] [e^{\sigma^2} - 1].$$

The distribution function of the lognormal is

$$F(t) = \int_0^t \frac{1}{\tau\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln\tau - \mu}{\sigma}\right)^2\right] d\tau$$

or

$$F(t) = P(T \leq t) = P\left[z \leq \frac{\ln t - \mu}{\sigma}\right].$$

The reliability is

$$R(t) = P[T > t] = P\left[z > \frac{\ln t - \mu}{\sigma}\right]. \quad (1.55)$$

Thus, the hazard function is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\phi\left(\frac{\ln t - \mu}{\sigma}\right)}{t\sigma R(t)}. \quad (1.56)$$

Figure 1.20 shows the reliability and the hazard-rate functions of the lognormal distribution for different values of  $\mu$  and  $\sigma$ .

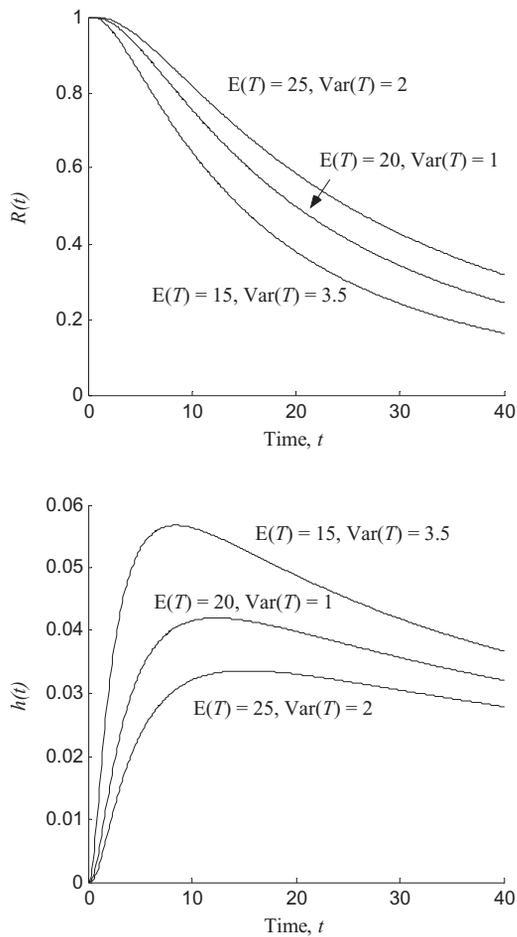


FIGURE 1.20  $R(t)$  and  $h(t)$  for the lognormal model.

**EXAMPLE 1.11**

The failure time of a component is lognormally distributed with  $\mu = 6$  and  $\sigma = 2$ . Find the reliability of the component and the hazard rate for a life of 200 time units.

SOLUTION

$$R(200) = P\left[z > \frac{\ln 200 - 6}{2}\right] = P[z > -0.350] = 0.6386.$$

The hazard function is

$$\begin{aligned} h(200) &= \frac{\phi\left(\frac{\ln 200 - 6}{2}\right)}{200 \times 2 \times 0.6386} \\ &= \frac{\phi(-0.350)}{200 \times 2 \times 0.6386} = \frac{0.3752}{200 \times 2 \times 0.6386} \\ &= 0.00147 \text{ failures per unit time.} \end{aligned}$$

**1.3.9 Gamma Model**

Like the Weibull model, the gamma model covers a wide range of the hazard-rate functions: decreasing, constant, or increasing hazard rates. The gamma distribution is suitable for describing the failure time of a component whose failure takes place in  $n$  stages or the failure time of a system that fails when  $n$  independent subfailures have occurred.

The gamma distribution is characterized by two parameters: shape parameter  $\gamma$  and scale parameter  $\theta$ . When  $0 < \gamma < 1$ , the failure rate monotonically decreases from infinity to  $1/\theta$  as time increases from 0 to infinity. When  $\gamma > 1$ , the failure rate monotonically increases from  $1/\theta$  to infinity. When  $\gamma = 1$ , the failure rate is constant and equals  $1/\theta$ .

The p.d.f. of a gamma distribution is

$$f(t) = \frac{t^{\gamma-1}}{\theta^\gamma \Gamma(\gamma)} e^{-\frac{t}{\theta}}. \quad (1.57)$$

When  $\gamma > 1$ , there is a single peak of the density function at time  $t = \theta(\gamma - 1)$ . The CDF,  $F(t)$ , is

$$F(t) = \int_0^t \frac{\tau^{\gamma-1}}{\theta^\gamma \Gamma(\gamma)} e^{-\frac{\tau}{\theta}} d\tau.$$

Substituting  $\tau/\theta = \mu$ , we obtain

$$F(t) = \frac{1}{\Gamma(\gamma)} \int_0^{t/\theta} \mu^{\gamma-1} e^{-\mu} d\mu$$

or

$$F(t) = I\left(\frac{t}{\theta}, \gamma\right),$$

where  $I(t/\theta, \gamma)$  is known as the incomplete gamma function and is tabulated in Pearson (1957). The reliability function  $R(t)$  is

$$R(t) = \int_t^{\infty} \frac{1}{\theta \Gamma(\gamma)} \left(\frac{\tau}{\theta}\right)^{\gamma-1} e^{-\frac{\tau}{\theta}} d\tau. \quad (1.58)$$

When the shape parameter  $\gamma$  is an integer  $n$ , the gamma distribution becomes the well-known Erlang distribution. In this case, the CDF is written as

$$F(t) = 1 - e^{-\frac{t}{\theta}} \sum_{k=0}^{n-1} \frac{\left(\frac{t}{\theta}\right)^k}{k!} \quad (1.59)$$

and the reliability function is

$$R(t) = e^{-\frac{t}{\theta}} \sum_{k=0}^{n-1} \frac{\left(\frac{t}{\theta}\right)^k}{k!}. \quad (1.60)$$

The hazard rate of the gamma model, when  $\gamma$  is an integer  $n$ , is obtained by dividing Equation 1.57 by Equation 1.60:

$$h(t) = \frac{\frac{1}{\theta} \left(\frac{t}{\theta}\right)^{n-1}}{(n-1)! \sum_{k=0}^{n-1} \frac{\left(\frac{t}{\theta}\right)^k}{k!}}. \quad (1.61)$$

Figures 1.21–1.23 show the gamma density function, the reliability function, and the hazard rate for different  $\gamma$  values and a constant  $\theta = 20$ .

The mean and variance of the gamma distribution are obtained as

$$\begin{aligned} \text{Mean life} &= \int_{-\infty}^{\infty} t f(t) dt \\ &= \int_0^{\infty} t t^{\gamma-1} \frac{1}{\Gamma(\gamma)\theta^\gamma} e^{-\frac{t}{\theta}} dt \\ &= \frac{1}{\Gamma(\gamma)\theta^\gamma} \int_0^{\infty} t^\gamma e^{-\frac{t}{\theta}} dt \end{aligned}$$

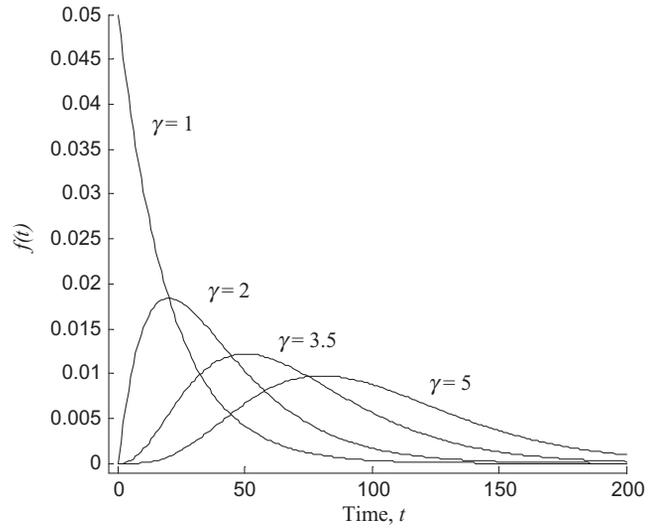


FIGURE 1.21 Gamma density function with different  $\gamma$  values,  $\theta = 20$ .

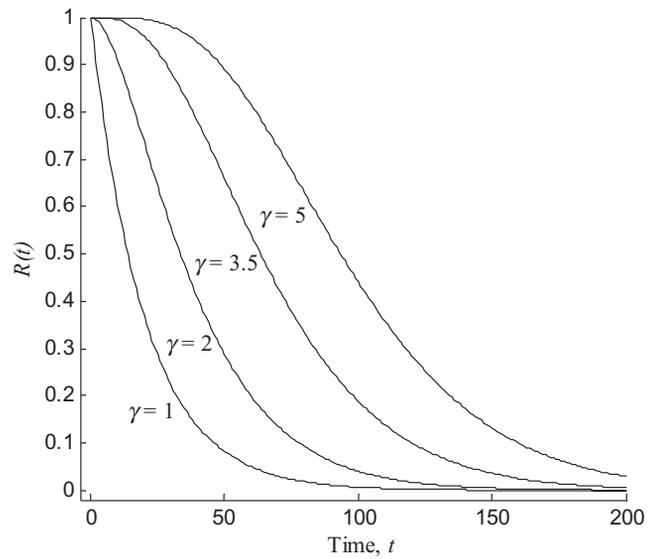
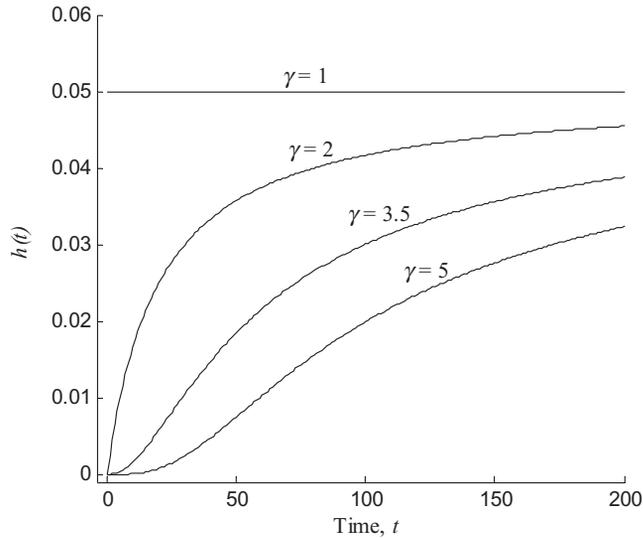


FIGURE 1.22 Gamma reliability function for different  $\gamma$  values,  $\theta = 20$ .



**FIGURE 1.23** Gamma hazard rate for different  $\gamma$  values,  $\theta = 20$ .

or

$$\text{Mean life} = \frac{1}{\Gamma(\gamma)\theta^\gamma} \Gamma(\gamma+1)\theta^{\gamma+1} = \gamma\theta.$$

Similar manipulations yield  $E[T^2] = \gamma(\gamma+1)\theta^2$ , and the variance of the life is  $\text{Var}(T) = \gamma(\gamma+1)\theta^2 - \gamma^2\theta^2 = \gamma\theta^2$ .

### EXAMPLE 1.12

A mechanical system requires a constant supply of electric current, which is provided by a main battery having life length  $T_1$  with an exponential distribution of mean 120 h. The main battery is supported by two identical backup batteries with mean lives of  $T_2$  and  $T_3$ . When the main unit fails, the first backup battery provides the necessary current to the system. The second backup battery provides the current when the first backup unit fails. In other words, the batteries provide the current independently but sequentially.

Determine the reliability and the hazard rate of the mechanical system at  $t = 280$  h. What is the mean life of the system?

### SOLUTION

Since the life lengths of the batteries are independent exponential random variables each with mean 120, the distribution of the total life of the mechanical system,  $T_1$ ,  $T_2$ , and  $T_3$  has a gamma distribution with  $\gamma = 3$  and  $\theta = 120$ . Using Equation 1.60 we obtain

$$R(280) = e^{-\frac{280}{120}} \sum_{k=0}^2 \frac{\left(\frac{280}{120}\right)^k}{k!} = 0.58723.$$

The hazard rate at 280 h is obtained by substituting into Equation 1.61

$$h(280) = \frac{1}{120} \frac{\left(\frac{280}{120}\right)^2}{2!(6.055)} = 0.003746 \text{ failures per hour.}$$

The mean life of the mechanical system is given by

$$\text{mean life} = \gamma\theta = 3(120) = 360 \text{ h.} \quad \blacksquare$$

The above results can also be obtained using Special Erlang distribution as follows.

The Erlang distribution is the convolution of  $n$  identical units (times) each follows the exponential distribution with parameter  $\lambda$ . Since  $T_1$ ,  $T_2$ , and  $T_3$  are equal, then we can express the density function of the Erlang distribution for  $n$  units as

$$f_n(t) = \frac{\lambda^n e^{-\lambda t} t^{n-1}}{(n-1)!}$$

and

$$F_n(t) = 1 - \sum_{i=0}^{n-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}.$$

The reliability function of Erlang distribution is

$$R_n(t) = \sum_{i=0}^{n-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}.$$

The reliability and hazard-rate values obtained for this density function are identical to the values above.

### 1.3.10 Log-Logistic Model

If  $T > 0$  is a random variable representing the failure time of a system and  $t$  represents a typical time instant in its range, we use  $Y \equiv \log T$  to represent the log failure time (Kalbfleisch and Prentice, 2002). The log-logistic distribution for  $T$  is obtained if we express  $Y = \alpha + \sigma W$  and  $W$  has the logistic density

$$f(w) = \frac{e^w}{(1 + e^w)^2}. \quad (1.62)$$

The logistic density is symmetric with mean = 0 and variance =  $\pi^2/3$  with slightly heavier tails than the normal density function (Kalbfleisch and Prentice, 2002). The p.d.f. of the failure time  $t$  is

$$f(t) = \lambda p (\lambda t)^{p-1} [1 + (\lambda t)^p]^{-2}, \quad (1.63)$$

where  $\lambda = e^{-\alpha}$  and  $p = 1/\sigma$ .

The reliability and hazard functions of the log-logistic model are

$$R(t) = \frac{1}{1 + (\lambda t)^p} \quad (1.64)$$

and

$$h(t) = \frac{\lambda p (\lambda t)^{p-1}}{1 + (\lambda t)^p}. \quad (1.65)$$

This model has the same advantage as both the Weibull and exponential models; it has simple expressions for  $R(t)$  and  $h(t)$ .

Examination of Equation 1.65 reveals that the hazard function is monotonically decreasing when  $p = 1$ . If  $p > 1$ , the hazard rate increases from 0 to a peak at  $t = (p - 1)^{1/p}/\lambda$  and then decreases with time thereafter. The hazard rate is monotonically decreasing if  $p < 1$ . Figures 1.24 and 1.25 show the reliability function and the hazard rate for different values of  $p$  and a constant  $\lambda = 20$ .

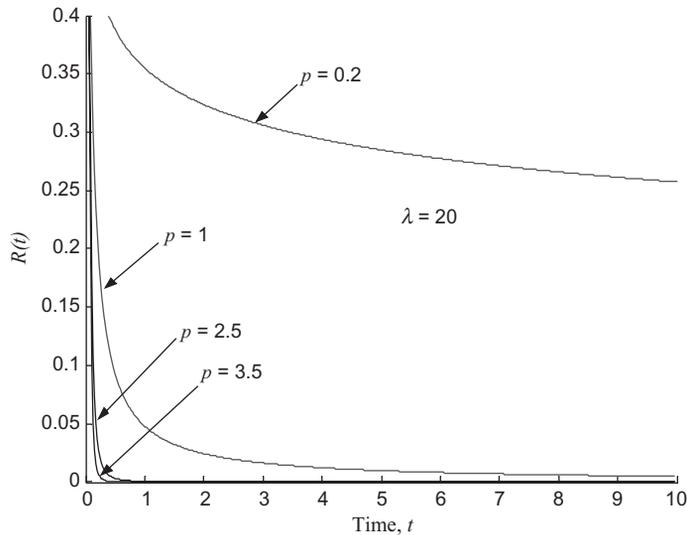


FIGURE 1.24 Reliability function for the log-logistic distribution.

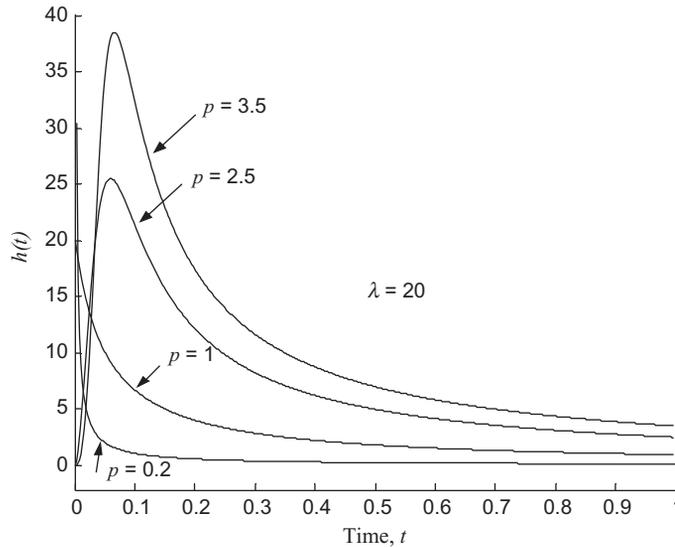


FIGURE 1.25 Hazard rate for the log-logistic distribution.

### 1.3.11 Beta Model

The hazard function models discussed thus far are defined as nonzero functions over the time range of zero to infinity. However, the life of some products or components may be constrained to a finite interval of time. In such cases, the beta model is the most appropriate model that can describe the reliability behavior of the product during the constrained interval  $(0, 1)$ . Clearly, any finite interval can be transformed to a  $(0, 1)$  interval.

Like other distributions that describe three types of hazard functions—decreasing, constant, and increasing hazard rates—the two parameters of the beta model make it flexible to describe the above hazard rates. The standard form of the density function of the beta model is

$$f(t) = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1} (1-t)^{\beta-1} & 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases} \quad (1.66)$$

The parameters  $\alpha$  and  $\beta$  are positive. Since

$$\int_0^1 f(t) dt = 1,$$

then,

$$\int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \quad (1.67)$$

for positive  $\alpha$  and  $\beta$ .

In general, there is no closed form expression for the cumulative distribution or the hazard-rate function. However, if  $\alpha$  or  $\beta$  is a positive integer, a binomial expansion can be used to obtain  $F(t)$  and consequently  $h(t)$ .  $F(t)$  will be a polynomial in  $t$ , and the powers of  $t$  will be, in general, positive real numbers ranging from 0 through  $\alpha + \beta - 1$ .

The mean and variance of the beta distribution are

$$\text{Mean} = \frac{\alpha}{\alpha + \beta}$$

$$\text{Variance} = \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}.$$

### 1.3.12 The Inverse Gaussian Model

In most of the models presented so far, the reliability model is often selected based on how well the data appear to be fitted by the model. Clearly, incorporating the failure mechanism or the characteristics of the components (temperature effect, electric field effect, fatigue and cumulative damage effect, etc.) in the model will result in a more realistic model for the system. In other words, it is desirable to use the physical description of a failure to make a choice of distribution accordingly. This is demonstrated further in Chapter 6.

The Inverse Gaussian (IG) distribution is applicable when there is a high occurrence of early failures. Its failure rate is nonmonotonic; initially it increases and then decreases with a nonzero asymptotic value at the end. In effect the IG distribution is suitable for modeling the first two regions of the bathtub curve. Examples of its application are found in accelerated life testing and repair time situations whenever early failures dominate the lifetime distribution. The lognormal distribution could be used instead except when the asymptotic value of the failure rate is zero (Watson and Wells, 1961). However, there is difficulty in justifying the use of the lognormal distribution on physical basis (Chhikara and Folks, 1977). The physical aspect of Brownian motion or any Gaussian process gives rise to the IG as the first passage time distribution which implies its applicability in studying life testing or lifetime phenomenon (Cox and Miller, 1965). Like both the normal and lognormal distributions, the IG has two parameters  $\mu$  and  $\lambda$ . The p.d.f. is

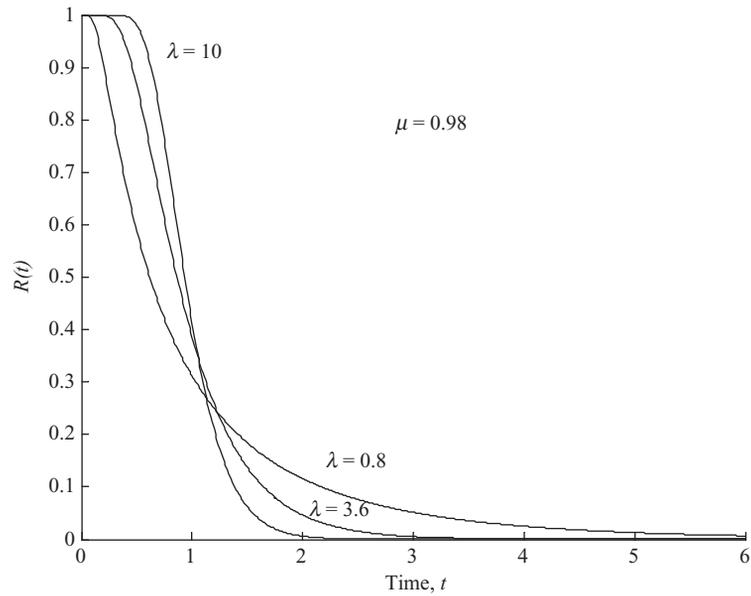
$$f(t; \mu, \lambda) = \sqrt{\lambda/2\pi t^3} \exp(-\lambda(t - \mu)^2 / 2\mu^2 t), \quad t > 0 \quad (1.68)$$

where  $\mu$  and  $\lambda$  are assumed to be positive and are referred to as the mean and the shape parameters of the distribution. The variance is  $\mu^3/\lambda$  and the p.d.f. is unimodal and skewed. The reliability function  $R(t)$  and the hazard-rate function  $h(t)$  are given by Equations 1.69 and 1.70 and are shown in Figures 1.26 and 1.27, respectively:

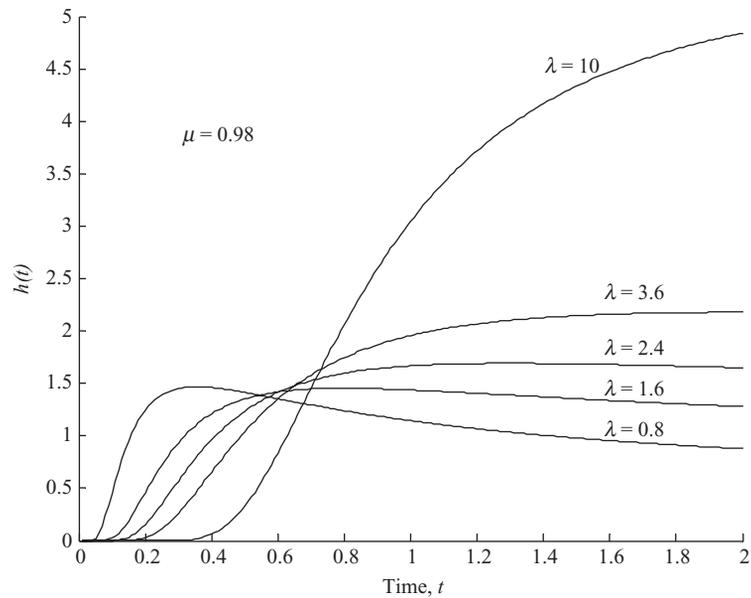
$$R(t) = \Phi\left(\sqrt{\frac{\lambda}{t}}\left(1 - \frac{t}{\mu}\right)\right) - e^{2\lambda/\mu} \Phi\left(-\sqrt{\frac{\lambda}{t}}\left(1 + \frac{t}{\mu}\right)\right) \quad (1.69)$$

$$h(t) = \frac{\sqrt{\lambda/2\pi t^3} \exp(-\lambda(t - \mu)^2 / 2\mu^2 t)}{\Phi\left(\sqrt{\lambda/t}(1 - t/\mu)\right) - e^{2\lambda/\mu} \Phi\left(-\sqrt{\lambda/t}(1 + t/\mu)\right)}, \quad (1.70)$$

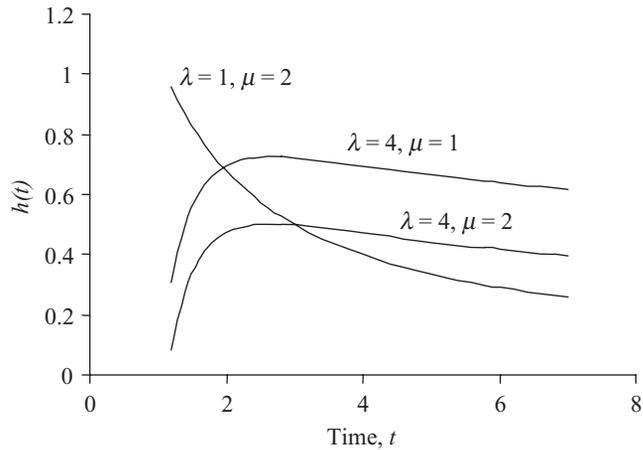
where  $\Phi$  denotes the CDF of the standard normal distribution.



**FIGURE 1.26** Reliability function for the Inverse Gaussian distribution.



**FIGURE 1.27** Hazard-rate function for the Inverse Gaussian distribution.



**FIGURE 1.28** Maximum values of  $h(t)$  for different  $\mu$  and  $\lambda$ .

As shown in Figure 1.27, the failure rate is not monotonic for all  $\mu$  and  $\lambda$ . However, the failure rate might be monotonic for some parameter values. It is also important to note that there exists a nonzero asymptotic value of  $h(t)$  unlike the failure rate of the lognormal, which approaches zero asymptotically. Since the failure rate might increase then decrease with time, not common in practice, it becomes desirable to determine the time at which the failure rate is maximum in order to assess the system performance at the worst conditions and when it will occur. The maximum value of  $h(t)$  can be found by differentiating  $\log h(t)$  with respect to  $t$  as given in Equation 1.71:

$$\begin{aligned} \frac{d}{dt} \log h(t) &= \frac{d}{dt} \log f(t) + \frac{f(t)}{R(t)} \\ &= -\frac{\lambda}{2\mu^2} - \frac{3}{2t} + \frac{\lambda}{2t^2} + h(t). \end{aligned} \quad (1.71)$$

The maximum value of  $h(t)$  is obtained at  $t^*$  by setting Equation 1.71 to zero. Figure 1.28 shows the maximum values of  $h(t)$  for different values of the distribution parameters.

### EXAMPLE 1.13

The following failure data are reported on failure times (hours) of electronic capacitors under an accelerated test conditions. 1.0, 1.5, 2.5, 2.5, 2.5, 2.5, 3.0, 3.0, 3.5, 3.5, 3.5, 4.0, 4.0, 5.0, 5.0, 5.0, 5.0, 5.5, 6.5, 7.5, 7.5, 7.5, 7.5, 7.5, 10.0, 10.0, 11.0, 12.5, 13.5, 15.0, 15.0, 16.5, 16.5, 20.0, 20.0, 22.5, 23.5, 25.0, 27.0, 27.0, 35.0, 37.5, 44.0, 45.0, 51.5, 110.0, 122.5. Estimate the parameters of the IG distribution.

## SOLUTION

Let  $T_i (i = 1, 2, \dots, n)$  be a random sample from an IG distribution. The maximum likelihood estimate (MLE's) of  $\mu$  and  $\lambda$  are

$$\hat{\mu} = \bar{T} = \frac{1}{n} \sum_{i=1}^n T_i$$

$$\hat{\lambda}^{-1} = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{T_i} - \frac{1}{\bar{T}} \right)$$

The MLE of the variance is given by

$$\hat{\sigma}^2 = \mu^3 / \hat{\lambda}$$

$$= \frac{1}{n} \left[ \sum_{i=1}^n \frac{\bar{T}^3}{T_i} - n\bar{T}^2 \right]$$

Using the above expressions, we obtain  $\hat{\mu} = 18.032,61$ ,  $\hat{\lambda} = 8.11398$ , and  $\hat{\sigma}^2 = 722.67301$ . ■

### 1.3.13 The Frechet Model

The Frechet distribution is the only distribution defined on the nonnegative real numbers that is a well-defined limiting distribution for the maxima of random variables. Let  $\{t_i; 1 \leq i \leq n\}$  be a collection of independent and identically distributed random variables characteristic of a critical variable in an engineering or physical application. Often the essence of the application is dependent upon the statistical behavior of the maximum  $M_n = \max\{t_i; 1 \leq i \leq n\}$  or the  $m_n = \min\{t_i; 1 \leq i \leq n\}$ , especially for large  $n$ . Classical extreme value theory is concerned with the distributions for  $M_n$  and  $m_n$ , when  $n$  is large. Of all possible nondegenerate limiting distributions, only Frechet distribution for  $M_n$  and the Weibull distribution for  $m_n$  are concentrated on the nonnegative real numbers (Harlow, 2001). This is useful in reliability applications when, for example, we are interested in estimating the time that a characteristic such as crack length will reach a maximum length that will cause failure (Lorén, 2003). The two-parameter Frechet p.d.f. (Kotz and Nadarajah, 2000) is

$$f(t) = \frac{\gamma}{\theta} \left( \frac{t}{\theta} \right)^{-(\gamma+1)} e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}, \quad t \geq 0, \gamma > 0, \theta > 0 \quad (1.72)$$

and its hazard rate  $h(t)$  is given as

$$h(t) = \frac{\frac{\gamma}{\theta} \left( \frac{t}{\theta} \right)^{-(\gamma+1)} e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}{1 - e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}, \quad t \geq 0, \gamma > 0, \theta > 0, \quad (1.73)$$

where  $\theta$  and  $\gamma$  are positive and are referred to as the characteristic scale and the shape parameters of the distribution, respectively. The p.d.f.'s and hazard function of the Frechet distribution with different  $\gamma$ 's are shown in Figures 1.29 and 1.30, respectively.

The distribution and reliability functions of the Frechet distribution  $F(t)$  and  $R(t)$  are given by Equations 1.74 and 1.75, respectively.

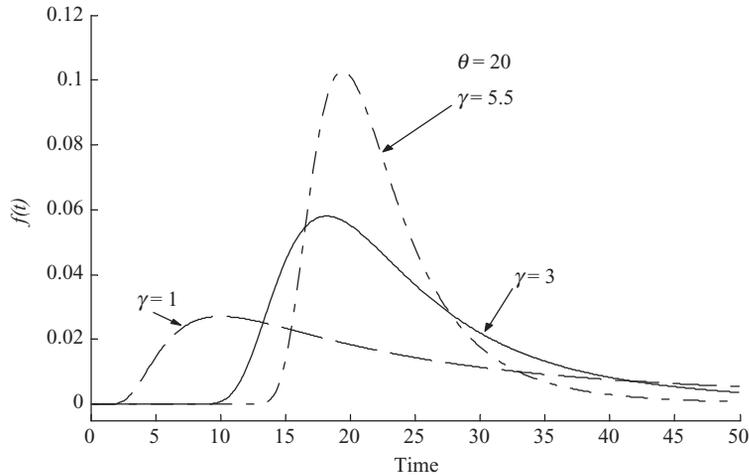


FIGURE 1.29 The Frechet p.d.f. for different  $\gamma$ .

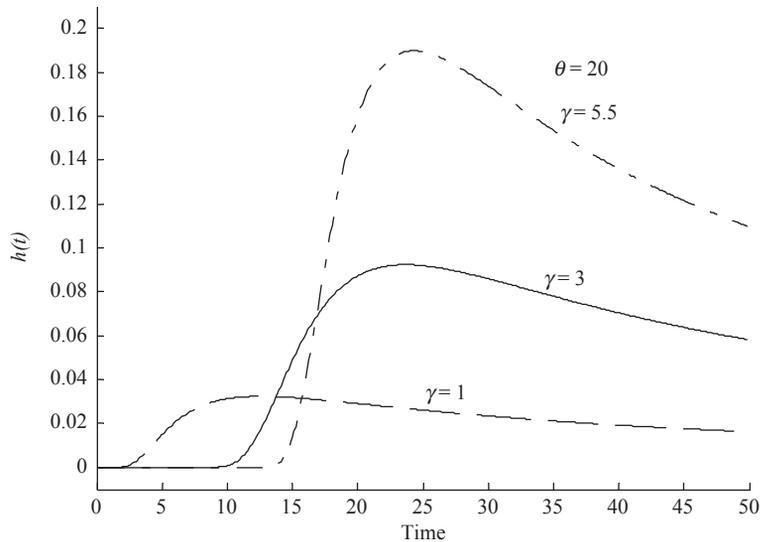


FIGURE 1.30 The hazard function of Frechet distribution for different  $\gamma$ .

$$F(t) = e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}, \quad t > 0 \quad (1.74)$$

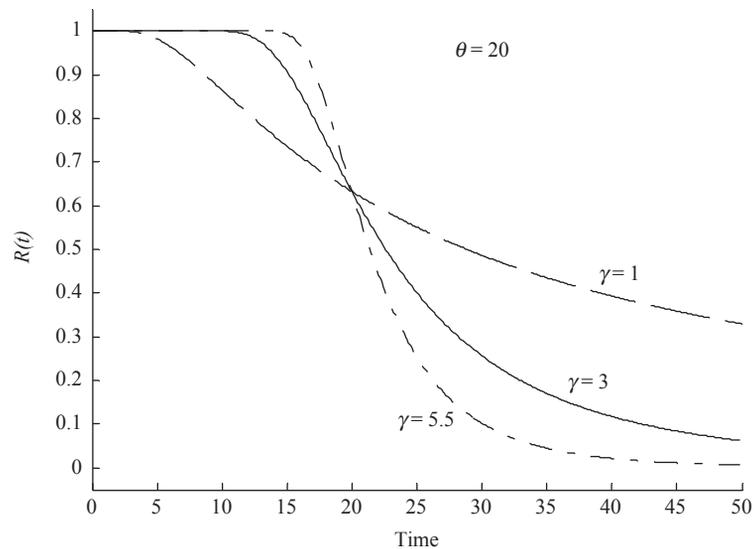
$$R(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}, \quad t > 0. \quad (1.75)$$

Equation 1.74 is also referred to as CDF of the Inverse Weibull distribution. The reliability and distribution function of the Frechet distribution with different  $\gamma$ 's are shown in Figures 1.31 and 1.32, respectively.

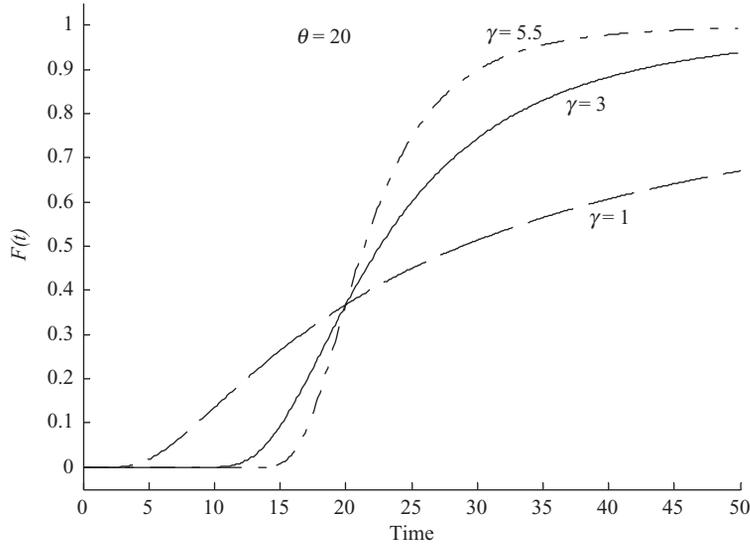
Again, the hazard rate of the Frechet distribution is given as

$$h(t) = \frac{\frac{\gamma}{\theta} \left(\frac{t}{\theta}\right)^{-(\gamma+1)} e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}{1 - e^{-\left(\frac{t}{\theta}\right)^{-\gamma}}}, \quad t \geq 0, \gamma > 0, \theta > 0,$$

which is not monotonic. It initially increases to a maximum value and subsequently decreases. It can be shown that the maximum is unique, but its value must be determined numerically. Thus, like the IG distribution, the Frechet distribution may not be appropriate to describe the failure rate of many components or systems in the classical reliability modeling. However, it is commonly used in modeling the inclusion size distribution (inclusions are nonmetallic particles) to determine the mechanical properties of hard and clean metals. It is also used to model the extreme bursts (large file size, sudden increase in traffic) in network traffic.



**FIGURE 1.31** The reliability function of Frechet distribution for different  $\gamma$ .



**FIGURE 1.32** The CDF of Frechet distribution for different  $\gamma$ .

The  $k$ th moment of the Frechet distribution is given as

$$E[T^k] = \int_0^{\infty} t^k f(t) dt = \theta^k \Gamma\left(1 - \frac{k}{\gamma}\right),$$

where  $\Gamma(x)$  is the gamma function. Notice that  $E[T^k]$  only exists if  $k < \gamma$ . In particular, the mean and variance, and coefficient of variation  $CV$  could be derived as follows:

$$E[T(\text{time to failure})] = \theta \Gamma\left(1 - \frac{1}{\gamma}\right),$$

$$\text{Var}[T] = \theta^2 \left[ \Gamma\left(1 - \frac{2}{\gamma}\right) - \Gamma^2\left(1 - \frac{1}{\gamma}\right) \right],$$

$$CV = \sqrt{\frac{\Gamma\left(1 - \frac{2}{\gamma}\right) - \Gamma^2\left(1 - \frac{1}{\gamma}\right)}{\Gamma^2\left(1 - \frac{1}{\gamma}\right)}}.$$

Again,  $E[T(\text{time to failure})]$  may be estimated from the above equation if  $\gamma > 1$ , and likewise for  $\text{Var}[T]$  and  $CV$  if  $\gamma > 2$ . Using simple curve fitting, the  $CV$  is well approximated by

$$CV \approx 1/[1.55(\gamma - 2)^{0.7}], \quad \gamma > 2.$$

Since  $CV$  depends on  $\gamma$  only, it is indicative of variability. As  $\gamma$  increases, the scatter decreases, and vice versa. If  $\gamma$  is sufficiently large,  $\theta$  is approximately equal to the mean  $E[T]$ .

### 1.3.14 Birnbaum–Saunders (BS) Distribution

In some engineering applications, it is observed that the failure rate increases with time until it reaches a peak value then it begins to decrease; that is, it is unimodal. This type of behavior was observed by Birnbaum and Saunders (1969) who noted that the failure of units subject to fatigue stresses occurs when the crack length reaches a prespecified limit. It is assumed that the  $j$ th fatigue cycle increases the crack length by  $x_j$ . The cumulative growth in the crack length after  $n$  cycles is  $\sum_{j=1}^n x_j$  which follows a normal distribution with mean  $n\mu$  and variance  $n\sigma^2$ . The probability that the crack does not exceed a critical length  $\omega$  is expressed as

$$\Phi\left(\frac{\omega - n\mu}{\sigma\sqrt{n}}\right) = \Phi\left(\frac{\omega}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right). \quad (1.76)$$

Assume that the unit fails when the crack length exceeds  $\omega$  and that the lifetime is  $T$  (expressed either in time or number of fatigue cycles). The reliability at time  $t$  is then expressed as

$$R(t) = P(T < t) \approx 1 - \Phi\left(\frac{\omega}{\sigma\sqrt{t}} - \frac{\mu\sqrt{t}}{\sigma}\right) = \Phi\left(\frac{\mu\sqrt{t}}{\sigma} - \frac{\omega}{\sigma\sqrt{t}}\right). \quad (1.77)$$

Substituting  $\beta = \omega/\mu$  and  $\alpha = \sigma/\sqrt{\omega\mu}$ , Equation 1.77 can be written as

$$R(t) = 1 - \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right] \quad 0 < t < \infty \quad \alpha, \beta > 0. \quad (1.78)$$

Where  $\Phi(\cdot)$  is the CDF of the standard normal,  $\alpha$  is the shape parameter and  $\beta$  is the scale parameter. Following Kundu et al. (2008), the p.d.f. of the two-parameter BS random variable  $T$  corresponding to the complementary CDF of Equation 1.78 is

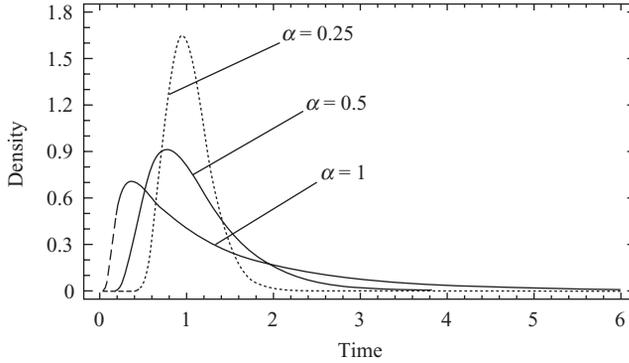
$$f(t; \alpha, \beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta} \left[ \sqrt{\frac{\beta}{t}} + \left(\frac{\beta}{t}\right)^{3/2} \right] \exp\left[-\frac{1}{2\alpha^2} \left(\frac{t}{\beta} + \frac{\beta}{t}\right) - 2\right], \quad (1.79)$$

$$0 < t < \infty, \quad \alpha, \beta > 0$$

This distribution is used to model situations when the maximum hazard rate occurs after several years of operations and then it decreases slowly over a fixed period. It is also applicable for modeling self-healing material or systems where its hazard rate increases up to a point of time then slowly decreases. The p.d.f.'s for different values of  $\alpha$  and  $\beta = 1$  are shown in Figure 1.33.

Kundu et al. (2008) consider the following transformation of a random variable  $T$  that follows BS ( $\alpha, \beta$ )

$$X = \frac{1}{2} \left[ \left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{-\frac{1}{2}} \right],$$



**FIGURE 1.33** Probability density function of BS distribution.

which is equivalent to

$$T = \beta(1 + 2X^2 + 2X(1 + X^2)^{\frac{1}{2}}).$$

Then  $X$  is normally distributed with mean zero and variance  $(\alpha^2/4)$ . The above relationship is utilized to obtain several characteristics of the BS distribution (Johnson et al., 1995). They are

$$E(T) = \beta \left( 1 + \frac{1}{2} \alpha^2 \right). \quad (1.80)$$

$$V(T) = (\alpha\beta)^2 \left( 1 + \frac{5}{4} \alpha^2 \right). \quad (1.81)$$

$$\text{Coefficient of Skewness} = \frac{16\alpha^2(11\alpha^2 + 6)}{(5\alpha^2 + 4)^3}. \quad (1.82)$$

$$\text{Coefficient of Kurtosis} = 3 + \frac{6\alpha^2(93\alpha^2 + 41)}{(5\alpha^2 + 4)^2}. \quad (1.83)$$

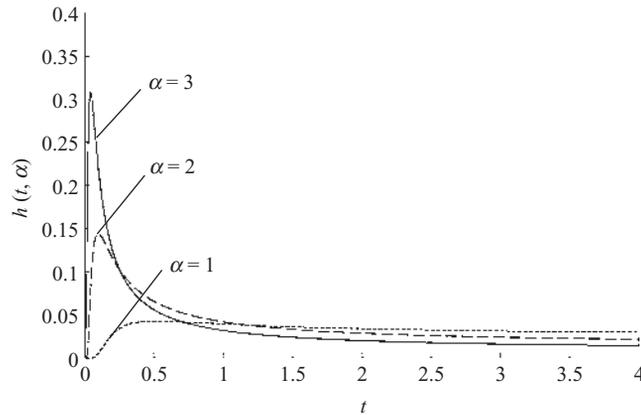
Note that Equation 1.80 is the mean life (or MTTF).

The hazard rate  $h(t)$  is obtained by dividing Equation 1.79 by Equation 1.78. There is no closed form for  $h(t)$ , but it can be estimated numerically. Figure 1.34 shows the hazard-rate function for different values of  $\alpha$ .

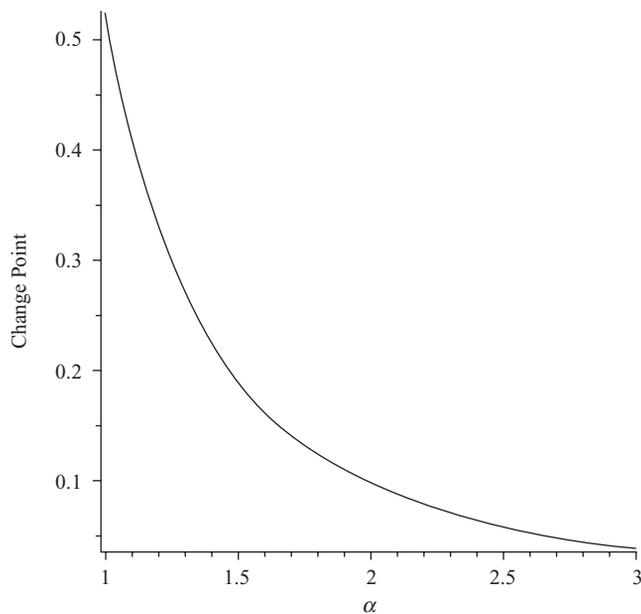
Kundu et al. (2008) show that the hazard rate is unimodal, and it increases to a peak value then slowly decreases with time. Assuming  $\beta = 1$ , they show that the change point of the hazard rate occurs approximately at

$$c(\alpha) = \frac{1}{(-0.4604 + 1.8417\alpha)^2}.$$

This approximation is for  $\alpha > 0.25$  and works quite well for  $\alpha > 0.60$ . The change point moves closer to zero as the shape parameter increases which implies that the units exhibit a decreasing



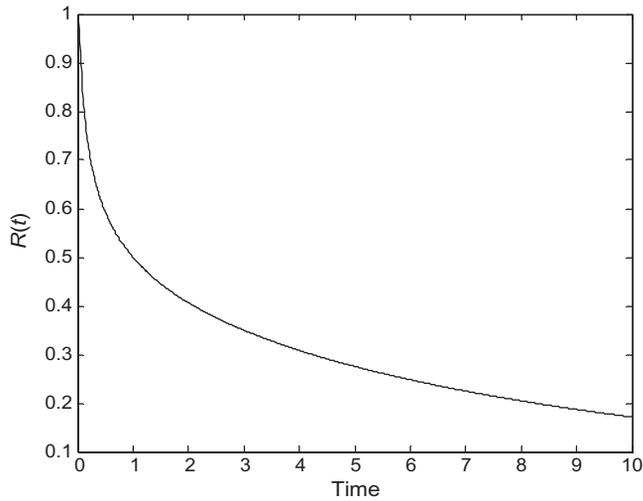
**FIGURE 1.34** The hazard-rate function of the BS distribution.



**FIGURE 1.35** Effect of  $\alpha$  on the change point of the hazard-rate function.

hazard rate as the shape parameter increases and the BS distribution might not be appropriate to model such hazard function. Indeed, a Weibull model with shape parameter less than one will result in a better fit. On the other hand, the distribution tends to normality as  $\alpha$  tends to zero. The relationship between  $\alpha$  and the change point is shown in Figure 1.35.

One of the interesting properties of the BS ( $\alpha, \beta$ ) is that  $T^{-1}$  also follows a BS distribution with parameters  $\alpha$  and  $\beta^{-1}$ . The reliability function of BS (3,1) is shown in Figure 1.36.



**FIGURE 1.36** Reliability function of BS (3,1).

Assume that  $n$  time observations  $(t_1, t_2, \dots, t_n)$  corresponding to crack growth are recorded until the crack length reaches a critical threshold. These observations follow a BS distribution, and its parameters are estimated as follows (Kundu et al., 2008).

Let  $s$  and  $r$  denote the arithmetic mean and harmonic mean of the observations, respectively:

$$s = \frac{1}{n} \sum_{i=1}^n t_i$$

and

$$r = \left[ \frac{1}{n} \sum_{i=1}^n \frac{1}{t_i} \right]^{-1}.$$

The modified moment estimator of the distribution parameters are

$$\hat{\alpha} = \left( 2 \left[ \left( \frac{s}{r} \right)^2 - 1 \right] \right)^{1/2} \quad (1.84)$$

and

$$\hat{\beta} = (sr)^{1/2} \quad (1.85)$$

Due to the bias of the sample size, Kundu et al. (2008) obtain the bias-corrected modified moment estimators as

$$\tilde{\alpha} = \left( \frac{n}{n-1} \right) \hat{\alpha}$$

$$\tilde{\beta} = \left(1 + \frac{\tilde{\alpha}^2}{4n}\right)^{-1} \hat{\beta}.$$

The reliability function and the hazard rate can be readily obtained.

#### EXAMPLE 1.14

An engineer conducts an axial fatigue test on a sample of alloy steel and measures the crack growth. The incremental increases in the length are set to equal values, and the corresponding times are recorded as follows:

200, 300, 390, 485, 560, 635, 695, 755, 810, 860, 905, 945, 985, 1020, 1053, 1100, 1150, 1200, 1280, 1370, 1400, 1600

Assume that a BS distribution fits these data. Determine the parameters of the distribution and plot the reliability function.

#### SOLUTION

The parameters of the distribution are obtained using Equation 1.84 and 1.85. The shape and scale parameters are  $\hat{\alpha} = 1.1845$  and  $\hat{\beta} = 783.94$ . The unbiased estimates are  $\tilde{\alpha} = 1.2409$  and  $\tilde{\beta} = 89.93$ .

Using the unbiased estimates, we obtain the reliability function shown in Figure 1.37.

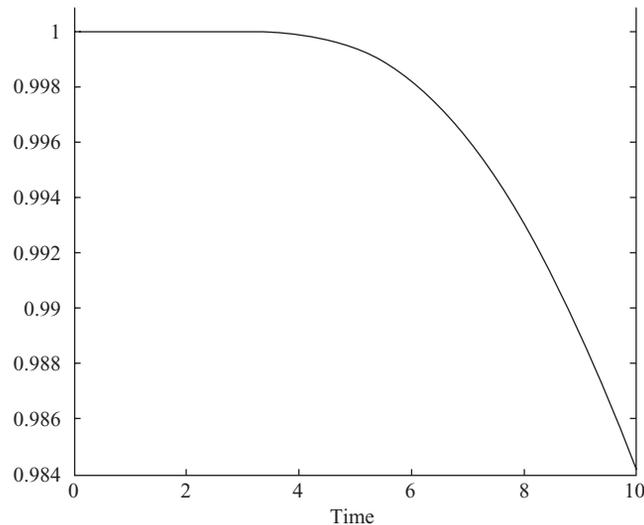


FIGURE 1.37 The reliability function of Example 1.14.

### 1.3.15 Other Forms

**1.3.15.1 The Generalized Pareto Model** When the hazard rate is either monotonically increasing or monotonically decreasing, it can be described by a three-parameter distribution with a hazard-rate function of the form

$$h(t) = \alpha + \frac{\beta}{t + \lambda}, \quad (1.86)$$

where  $\alpha$ ,  $\beta$ , and  $\lambda$  are the parameters of the model.

**1.3.15.2 The Gompertz–Makeham Model** This is a generalized model of the Gompertz hazard model with hazard rate

$$h(t) = \rho_0 + \rho_1 e^{\rho_2 t}, \quad (1.87)$$

where  $\rho_0$ ,  $\rho_1$ , and  $\rho_2$  are the parameters of the model.

**1.3.15.3 The Power Series Model** There are many practical situations where none of the above-mentioned models is suitable to accurately fit the hazard-rate values. In such a case, a general power series model can be used to fit the hazard-rate values. Clearly, the number of terms in the power series model relates to the desired level of fitness of the model to the empirical data. A good measure for the appropriateness of fitting the model to the data is the mean squared error between the hazard values obtained from the model and the actual data. The hazard-rate function of the power series model is

$$h(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n. \quad (1.88)$$

The reliability function,  $R(t)$ , is

$$R(t) = \exp \left[ - \left( a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \dots + \frac{a_n t^{n+1}}{n+1} \right) \right]. \quad (1.89)$$

#### EXAMPLE 1.15

Electromigration is a common failure mechanism in semiconductor devices. It is a phenomenon whereby a metal line in a device “grows” a link to another line or creates an open condition, due to movement (migration) of metal ions toward the anode at high temperatures or current densities (Comeford, 1989). Two hundred ICs are subjected to an elevated temperature of 250°C to accelerate their failures. The number of failures observed due to electromigration during the test intervals are given in Table 1.7.

Assume that the hazard-rate function is expressed as a power series function. Determine the hazard rate and the reliability after 10 h of operation at the same elevated temperature.

**TABLE 1.7 Failure Data for the Integrated Circuits**

Time interval (hours)	Failures in the interval
0–100	10
101–200	20
201–300	35
301–400	40
401–500	45
501–600	50
Total	200

**SOLUTION**

We calculate the hazard rate from the data as shown in Table 1.8.

We use the above hazard-rate data in Table 1.8 to fit the model given by Equation 1.88 using the least squares method to obtain

$$h(t) = 3.653 \times 10^{-3} - 0.171 \times 10^{-4}t + 4.86 \times 10^{-8}t^2$$

$$h(10 \text{ h}) = 3.484 \times 10^{-3}.$$

The reliability is obtained using Equation 1.89 as

$$R(10) = \exp \left[ - \left( 3.653 \times 10^{-2} - \frac{0.171}{2} \times 10^{-2} + \frac{4.86}{3} \times 10^{-5} \right) \right]$$

$$= 0.9649.$$

**TABLE 1.8 Hazard-Rate Calculation for Example 1.15**

Time interval (hours)	Failures in the interval	Hazard rate $\times 10^{-3}$
0–100	10	10/(200 $\times$ 100) = 0.50
101–200	20	20/(190 $\times$ 100) = 1.05
201–300	35	35/(170 $\times$ 100) = 2.05
301–400	40	40/(135 $\times$ 100) = 2.92
401–500	45	45/(95 $\times$ 100) = 4.73
501–600	50	50/(50 $\times$ 100) = 10.00

## 1.4 MULTIVARIATE HAZARD RATE

When a system is composed of two or more components, the joint life lengths are described by a multivariate distribution whose nature depends on the individual component life length. For example, consider a two-component system connected in parallel with each component having an exponentially distributed life length. The system fails when the two components fail.

When the effect of the operating conditions is accounted for, the joint life lengths of the components are shown to have a bivariate distribution whose marginals are univariate Paretos.

Assume that  $\lambda_i$  is the parameter of component  $i$  ( $i = 1, 2$ ). If the lives of the two components are assumed to be independent, then the reliability of the system is

$$R(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t}.$$

Suppose that the operating conditions affect the parameter  $\lambda_i$  by a common positive factor  $\eta$ . Then the system reliability is expressed as

$$R(t) = e^{-\eta\lambda_1 t} + e^{-\eta\lambda_2 t} - e^{-\eta(\lambda_1 + \lambda_2)t}.$$

Following Lindley and Singpurwalla (1986), if  $\eta$  is an unknown quantity whose uncertainty is described by the distribution function  $G(\eta)$ , then the system reliability becomes

$$R(t) = G^*(\lambda_1 t) + G^*(\lambda_2 t) - G^*[(\lambda_1 + \lambda_2)t],$$

where

$$G^*(y) = \int \exp(-\eta y) dG(\eta)$$

is the Laplace transform of  $G$ .

When  $G(\eta)$  is a gamma distribution with density,

$$g(\eta) = \beta^{\alpha+1} \frac{\eta^\alpha}{\alpha!} e^{-\eta\beta}, \quad (1.90)$$

where  $\alpha > -1$  and  $\beta > 0$ , then

$$R(t) = \left( \frac{\beta}{\lambda_1 t + \beta} \right)^{\alpha+1} + \left( \frac{\beta}{\lambda_2 t + \beta} \right)^{\alpha+1} - \left( \frac{\beta}{(\lambda_1 + \lambda_2)t + \beta} \right)^{\alpha+1}. \quad (1.91)$$

The joint density of  $T_1$  and  $T_2$ , the times to failure of the two components at  $t_1$  and  $t_2$ , respectively, is

$$f(t_1, t_2, \lambda_1, \lambda_2, \alpha, \beta) = \frac{\lambda_1 \lambda_2 (\alpha + 1)(\alpha + 2) \beta^{\alpha+1}}{(\lambda_1 t_1 + \lambda_2 t_2 + \beta)^{\alpha+3}}. \quad (1.92)$$

Plots of Equation 1.92 for different values of  $\lambda_1$ ,  $\lambda_2$ ,  $\alpha$ , and  $\beta$  are shown in Figures 1.38 and 1.39.

The bivariate hazard rate of the system is

$$h(t_1, t_2, \lambda_1, \lambda_2, \alpha, \beta) = \frac{(\alpha + 1)(\alpha + 2) \lambda_1 \lambda_2}{(\beta + \lambda_1 t_1 + \lambda_2 t_2)^2}. \quad (1.93)$$

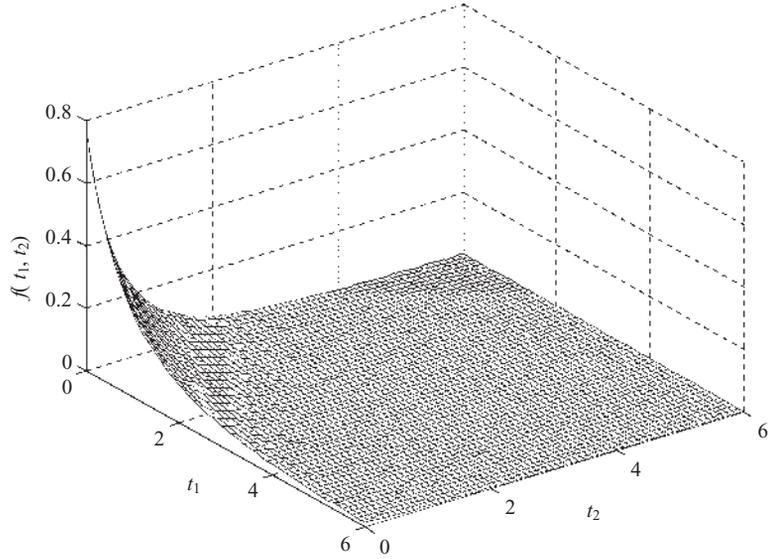


FIGURE 1.38 Plot of the bivariate gamma density ( $\lambda_1 = 0.5, \lambda_2 = 0.3, \alpha_1 = 0.6, \beta = 0.9$ ).

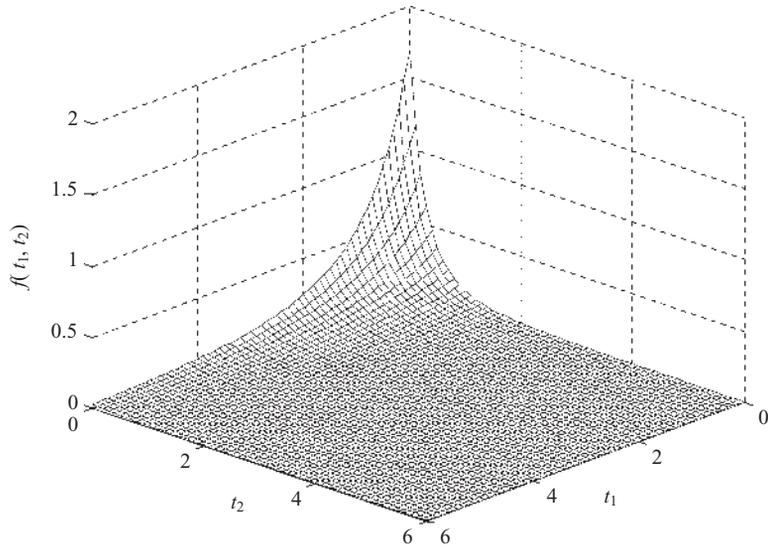
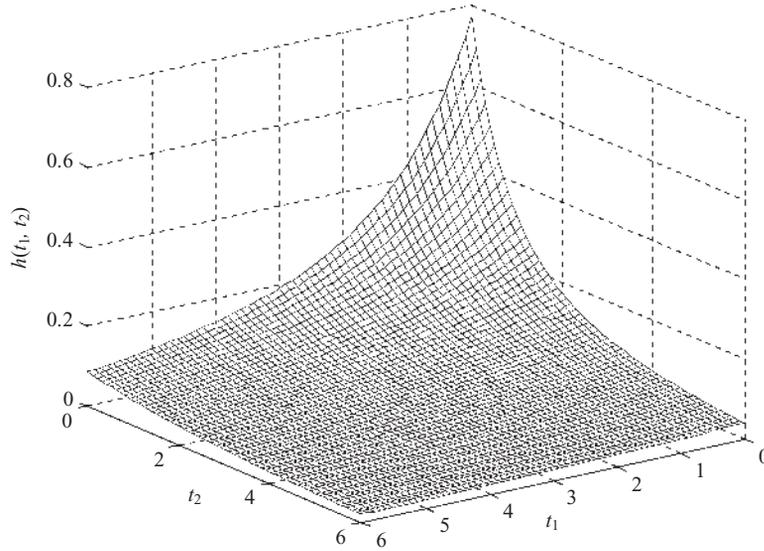
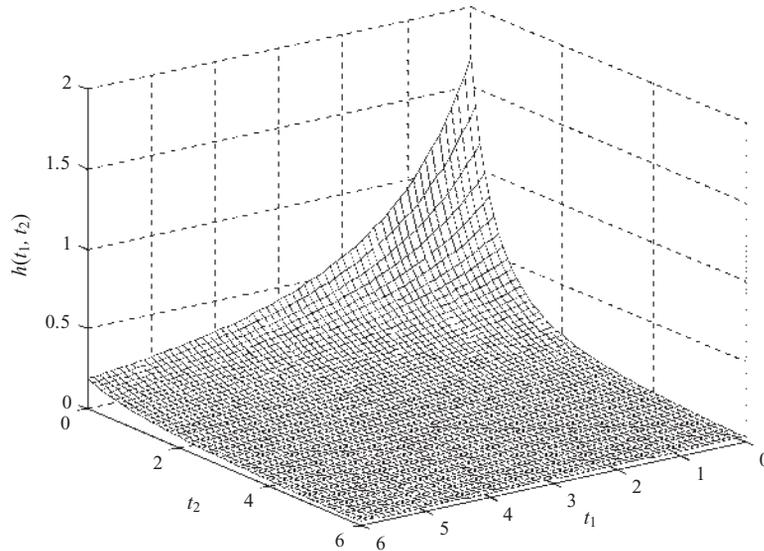


FIGURE 1.39 Plot of the bivariate gamma density ( $\lambda_1 = 0.9, \lambda_2 = 0.3, \alpha = 0.3, \beta = 0.9$ ).



**FIGURE 1.40** Plot of the bivariate hazard rate ( $\lambda_1 = 0.5$ ,  $\lambda_2 = 0.3$ ,  $\alpha = 0.6$ ,  $\beta = 0.9$ ).



**FIGURE 1.41** Plot of the bivariate hazard rate ( $\lambda_1 = 0.9$ ,  $\lambda_2 = 0.3$ ,  $\alpha = 0.8$ ,  $\beta = 0.9$ ).

The plots of the bivariate hazard rates for different  $\lambda_1$ ,  $\lambda_2$ ,  $\alpha$ , and  $\beta$  are shown in Figures 1.40 and 1.41. Like univariate hazard rates, the bivariate hazard exhibits similar shapes—decreasing, constant, and increasing hazard rate.

The marginal density function of  $t_1$  is obtained by integrating Equation 1.92 with respect to  $t_2$  which yields

$$f(t_1, \lambda_1, \alpha, \beta) = \frac{\lambda_1 (\alpha + 1) \beta^{\alpha+1}}{(\lambda_1 t_1 + \beta)^{\alpha+2}}. \quad (1.94)$$

The density function given by Equation 1.94 is a Pearson Type VI whose mean and variance exist only for certain values of the shape parameter  $\alpha$ . This distribution is also referred to as the ‘‘Pareto distribution of the second kind’’ (Lindley and Singpurwalla, 1986). Johnson and Kotz (1972) refer to Equation 1.94 as the *Lomax distribution*.

## 1.5 COMPETING RISK MODEL AND MIXTURE OF FAILURE RATES

Sometimes the failure data cannot be modeled by a single failure-time distribution. This is common in situations when a unit fails in different failure modes due to different failure mechanisms. For example, it has been shown that humidity has detrimental effects on semiconductor devices as it could induce failures due to large increases in threshold current in lasers (Osenbach et al., 1995; Chand et al., 1996; Osenbach and Evanosky, 1996; Osenbach et al., 1997) or could induce mechanical stresses due to polymeric layers’ volume expansion in micromechanical devices (Buchhold et al., 1998). Humidity in silver-based metallization in microelectronic interconnects has caused metal corrosion and dendrites due to migration (Manepalli et al., 1999). In such situations, the failure data can be modeled using competing risk models or mixture of failure-rates models. We now discuss the necessary conditions for using either type of models.

### 1.5.1 Competing Risk Model

The competing failure model (also known as compound model, series system model, or multi-risk model) plays an important role in reliability engineering as it can be used to model failure of units with several failure causes. There are three necessary conditions for this model: (1) failure modes are independent of each other, (2) the unit fails when the first of all failure mechanisms reaches the failure state, and (3) each failure mode has its own failure-time distribution. The model is constructed as follows,

Consider a unit that exhibits  $n$  failure modes and that the time to failure  $T_i$  due to failure mechanism  $i$  is distributed according to  $F_i(t)$ ,  $i = 1, 2, \dots, n$ . The failure time of the unit is the minimum of  $\{T_1, T_2, \dots, T_n\}$  and the distribution function  $F(t)$  is

$$F(t) = 1 - [1 - F_1(t)][1 - F_2(t)] \dots [1 - F_n(t)]. \quad (1.95)$$

The reliability function is

$$R(t) = \prod_{i=1}^n R_i(t) \quad (1.96)$$

and the hazard function is

$$h(t) = \sum_{i=1}^n h_i(t). \quad (1.97)$$

To illustrate the application of the competing risk model we consider a product that experiences two different failure modes and each follows a Weibull distribution. The reliability of the product is

$$R(t) = R_1(t)R_2(t) = e^{-\left(\frac{t}{\theta_1}\right)^{\gamma_1}} e^{-\left(\frac{t}{\theta_2}\right)^{\gamma_2}}, \quad (1.98)$$

where  $\theta_i$  and  $\gamma_i$  are the scale and shape parameters, respectively, of failure mode  $i$ . Upon differentiation we obtain the density function as (Jiang and Murthy, 1997)

$$\begin{aligned} f(t) &= R_1(t)f_2(t) + R_2(t)f_1(t) \\ &= R(t) \left[ \frac{\gamma_1}{\theta_1} \left(\frac{t}{\theta_1}\right)^{\gamma_1-1} + \frac{\gamma_2}{\theta_2} \left(\frac{t}{\theta_2}\right)^{\gamma_2-1} \right] \end{aligned} \quad (1.99)$$

and the hazard-rate function is

$$h(t) = h_1(t) + h_2(t) = \gamma_1 \theta_1^{-\gamma_1} t^{\gamma_1-1} + \gamma_2 \theta_2^{-\gamma_2} t^{\gamma_2-1}. \quad (1.100)$$

The characteristics of the resultant  $f(t)$  and  $h(t)$  depend on the values of the parameters  $\theta_1$ ,  $\theta_2$ ,  $\gamma_1$ , and  $\gamma_2$ . Of course, the hazard rate  $h(t)$  exhibits different characteristics: decreasing, constant, and increasing depending on the values and relationships among these parameters.

### EXAMPLE 1.16

Consider a product that fails in two failure modes. Each failure is characterized independently by a Weibull model, and the parameters of failure mode 1 are  $\theta_1 = 10,000$  and  $\gamma_1 = 2.0$  and the parameters of the failure mode 2 are  $\theta_2 = 15,000$  and  $\gamma_2 = 2.5$ . Plot the reliability function based on the competing risk model and compare it with the reliability function of each failure mode independently.

#### SOLUTION

The reliability function based on the competing risk model is (Fig. 1.42)

$$R(t) = R_1(t)R_2(t) = e^{-\left(\frac{t}{10,000}\right)^2} e^{-\left(\frac{t}{15,000}\right)^{2.5}}.$$

It is obvious that the competing risk model results in more accurate reliability estimates than modeling each failure mode separately.

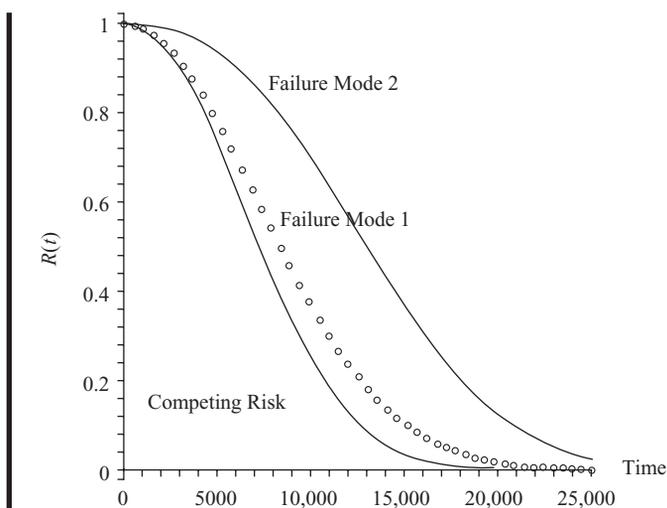


FIGURE 1.42 Reliability of the competing risk model. ■

## 1.5.2 Mixture of Failure-Rates Model

It is obvious that the mixtures of distributions with decreasing failure rates (DFRs) are always DFR. On the other hand, it may be intuitive to assume that the mixtures of distributions with increasing failure rates (IFRs) are also IFR. Unfortunately, some mixtures of distributions with IFR may exhibit DFR. In this section we discuss the conditions that guarantee that mixtures of IFR distributions will exhibit a DFR.

This is very important since, in practice, different IFR distributions are usually pooled in order to enlarge the sample size. In doing so, the analysis of data may actually reverse the IFR property of the individual samples to a DFR property for the mixture. Proschan (1963) shows that the mixture of two exponential distributions (each has a CFR) exhibits the DFR property.

Based on the work of Gurland and Sethuraman (1993), we consider mixtures of two arbitrary IFR distribution functions  $F_i(t)$ ,  $i = 1, 2$ . The pooled distribution function of the mixture of the two distributions is  $F_p(t) = p_1F_1(t) + p_2F_2(t)$  where  $p = (p_1, p_2)$  with  $0 \leq p_1, p_2 \leq 1$ , and  $p_1 + p_2 = 1$  is a mixing vector.

We use the notation

$$h'_i(t) = H'_i(t) \quad \text{and} \quad \mathfrak{R}_i(t) = p_i R_i(t), \quad i = 1, 2,$$

where  $h_i(t)$ ,  $H_i(t)$ , and  $R_i(t)$  are the hazard-rate function, the cumulative hazard function, and the reliability function of component  $i$  at time  $t$ . From Section 1.2,  $R_i(t) = 1 - F_i(t)$ ,  $H_i(t) = -\ln R_i(t)$  and  $h_i(t) = H'_i(t)$ .

The reliability function of the mixture of the two IFR distributions is

$$R_p(t) = p_1 R_1(t) + p_2 R_2(t).$$

But

$$\begin{aligned} H_p(t) &= -\ln R_p(t) \\ H_p(t) &= -\ln[p_1 R_1(t) + p_2 R_2(t)] \end{aligned}$$

and

$$\begin{aligned} h_p(t) &= H'_p(t) = \frac{p_1 R_1(t) h_1(t) + p_2 R_2(t) h_2(t)}{p_1 R_1(t) + p_2 R_2(t)} \\ &= \frac{\mathfrak{R}_1(t) h_1(t) + \mathfrak{R}_2(t) h_2(t)}{\mathfrak{R}_1(t) + \mathfrak{R}_2(t)}. \end{aligned} \quad (1.101)$$

A hazard-rate function  $h_p(t)$  is a DFR if  $h'_p(t) \leq 0$ . Therefore, we take the derivative of Equation 1.101 with respect to  $t$  to obtain

$$\begin{aligned} (\mathfrak{R}_1(t) + \mathfrak{R}_2(t))^2 h'_p(t) &= [\mathfrak{R}_1(t) + \mathfrak{R}_2(t)] \{ [\mathfrak{R}_1(t) h'_1(t) + \mathfrak{R}_2(t) h'_2(t)] \\ &\quad + [-\mathfrak{R}_1(t) h_1^2(t) - \mathfrak{R}_2(t) h_2^2(t)] \} + [\mathfrak{R}_1(t) h_1(t) + \mathfrak{R}_2(t) h_2(t)]^2 \\ &= (\mathfrak{R}_1(t) + \mathfrak{R}_2(t)) (\mathfrak{R}_1(t) h'_1(t) + \mathfrak{R}_2(t) h'_2(t)) - \mathfrak{R}_1(t) \mathfrak{R}_2(t) (h_1(t) - h_2(t))^2. \end{aligned} \quad (1.102)$$

Using the fact that  $\mathfrak{R}'_i(t) = -\mathfrak{R}_i(t) h_i(t)$  in the above equation, we show that the necessary and sufficient condition for  $h'_p(t) \leq 0$  and thus, for the mixture  $F_p(t)$  to be DFR is

$$[\mathfrak{R}_1(t) + \mathfrak{R}_2(t)] [\mathfrak{R}_1(t) h'_1(t) + \mathfrak{R}_2(t) h'_2(t)] \leq \mathfrak{R}_1(t) \mathfrak{R}_2(t) [h_1(t) - h_2(t)]^2. \quad (1.103)$$

### EXAMPLE 1.17

The failure-time distribution of a failure mode of a system is described by a truncated extreme distribution whose failure rate is  $h_1(t) = \theta e^t$ . Another mode of the system's failure exhibits a CFR  $h_2(t) = \lambda$ . Although one failure mode of the system exhibits IFR while the other is a CFR if treated separately, the analyst pools the data from both failure modes to obtain a pooled hazard-rate function. Prove that the pooled hazard rate is a DFR.

#### SOLUTION

The reliability functions of the failure modes of the system are

$$R_1(t) = e^{-\theta(e^t - 1)}$$

and

$$R_2(t) = e^{-\lambda t}.$$

The corresponding hazard rates are

$$h_1(t) = \theta e^t$$

and

$$h_2(t) = \lambda.$$

Let  $F_p(t) = (1 - p)F_1(t) + pF_2(t)$ . Then, the failure rate of the pooled data is

$$h_p(t) = \frac{(1-p)R_1(t)h_1(t) + pR_2(t)h_2(t)}{(1-p)R_1(t) + pR_2(t)}.$$

The necessary and sufficient condition that makes  $h_p(t)$  a DFR function is given by Equation 1.102. Substituting the parameters of the individual distributions and  $\mathfrak{R}_1(t) = (1-p)e^{-\theta(e^t-1)}$  and  $\mathfrak{R}_2(t) = pe^{-\lambda t}$ , it is easy to check that there is a  $t_0(p)$  such that the derivative of the pooled hazard rate with respect to  $t$  is negative for  $t \geq t_0(p)$  for each value of  $p$ . Thus, the mixture is DFR. ■

The class of IFR distributions that, when mixed with an exponential, becomes DFR is large; this is referred to as a mixture-reversible by exponential (MRE) distribution. It includes, for example, the Weibull, truncated extreme, gamma, truncated normal, and truncated logistic distributions. This phenomenon of the reversal of IFRs could be troublesome in practice when much of the data conform to an IFR distribution and the remainder (perhaps a small amount) of the data conform to an exponential distribution, and yet the overall pooled data would conform to a DFR distribution (Gurland and Sethuraman, 1994, 1995). For example, consider a mixture of an IFR gamma distribution with

$$f(t) = \frac{t^{\gamma-1}}{\theta^\gamma \Gamma(\gamma)} e^{-\frac{t}{\theta}}$$

where  $\gamma > 1$  and  $t > 0$  with an exponential distribution with parameter  $\lambda$  which satisfies the necessary conditions (Eq. 1.102) when  $1/\theta > \lambda$ . Thus, the IFR Gamma is MRE.

We note that the mixture failure rate for two populations is extensively studied. Gupta and Warren (2001) show that the mixture of two gamma distributions with IFRs (but have the same scale parameter) can result either in the increasing mixture failure rate or in the modified bathtub (MBT) mixture failure rate (the failure rate initially increases and then behaves like a bathtub failure rate). Jiang and Murthy (1998) show that the failure rate of the mixtures of two Weibull distributions with IFRs is similar to the failure rate of the mixture of two gamma distributions with IFRs. Likewise, Navarro and Hernandez (2004) state that the mixture failure rate of two truncated normal distributions depending on parameters involved can also be increasing, bathtub shaped, or MBT-shaped. Block et al. (2003) obtain explicit conditions for possible shapes of the mixture failure rate for two increasing linear failure rates.

Before concluding the presentation of the hazard functions, it is important to mention that some recent work argue that the bathtub curve is not a general failure-rate function that describes the failure rate of most, if not all, components. For example, Wong (1989) claims that the “roller-coaster” hazard-rate curve is more appropriate to describe the hazard rate of electronic systems than the bathtub curve. It is shown that semiconducting devices exhibit a generally decreasing hazard-rate curve with one or more humps on the curve. Data from a burn-in test of some electronic board assemblies demonstrate the trimodal (hump) characteristic on the cumulative failure rate. The wear-out (IFR) region starts immediately at the end of the decreasing failure-rate region without experiencing the constant failure-rate region, a main characteristic of the bathtub curve.

## 1.6 DISCRETE PROBABILITY DISTRIBUTIONS

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Before we conclude the continuous probability distributions, we briefly present and discuss the use of discrete probability distributions in the reliability engineering area.

As presented so far, reliability is considered a continuous function of time. However, there are situations when systems, units, or equipment are only used on demand such as missiles that are normally stored and used when needed. Likewise, when systems operate in cycles, only the number of cycles before failure is observed. In such situations, the reliability and system performance are normally described by discrete reliability distributions. In this section, we briefly describe relevant distributions for reliability modeling.

### 1.6.1 Basic Reliability Definition

Assume that a discrete lifetime is the number  $K$  of system demands until the first failure. Then,  $K$  is a random variable defined over the set  $N$  of positive integers (Bracquemond and Gaudoin, 2003). The probability function and CDF are expressed, respectively, as  $p(k) = P(K = k)$  and  $F(k) = P(K \leq k) = \sum_{i=1}^k p(i) \quad \forall k \in N$ . Consequently,

The reliability of a discrete lifetime distribution is

$$R(k) = P(K \geq k) = \sum_{i=k+1}^N p(i) \quad \forall k \in N.$$

The MTTF is the expectation of the random variable  $K$  expressed as

$$\text{MTTF} = E(K) = \sum_{i=1}^{\infty} ip(i).$$

Similar to the continuous time case, we define the failure rate as the ratio of the probability function and the reliability function, thus

$$h(k) = \frac{P(K = k)}{P(K \geq k)} = \frac{p(k)}{R(k)}.$$

Likewise, we express other reliability characteristics such as the *mean residual life* (MRL) function,  $L(k)$  as described in Section 1.8:

$$L(k) = E(K - k | K > k).$$

Of course, this can be generalized for the corresponding continuous time distributions. Since the practical use of such discrete lifetime distributions is limited, we show the above expressions for the geometric distribution case. Other distributions are found in (Bracquemond and Gaudoin, 2003).

### 1.6.2 Geometric Distribution

This distribution exhibits the memoryless property of the exponential distribution, and the system failure probabilities for each event (demand or request for use) are independent and all equal to  $p$ . In other words, the failure rate is constant or  $P(K > i + k | k > i) = P(K > k)$ . The probability of failure, reliability, and failure rate respectively, are

$$\begin{aligned} p(k) &= p(1-p)^{k-1}, \\ R(k) &= (1-p)^k, \\ h(k) &= \frac{p}{1-p}. \end{aligned}$$

Other use of discrete probability distributions arise when modeling system reliability, such as in the case of a four-engine aircraft, where its reliability is defined as the probability of at least two out of four engines function properly, and modeling the number of incidences (failures) of some characteristic in time as well as modeling warranty policies. We describe two commonly used distributions.

### 1.6.3 Binomial Distribution

In many situations, the reliability engineer might be interested in assessing system reliability by determining the probability that the system functions when  $k$  or more units out of  $n$  units function properly such as the case of the number of wires in a strand. This can be estimated using a binomial distribution. Let  $p$  be the probability that a unit is working properly;  $n$  is the total number of units; and  $k$  is the minimum number of units for the system to function properly. The probability of  $k$  units operating properly is

$$f(k) = \frac{n!}{k!(n-k)!} p^k q^{n-k} \quad k = 0, 1, \dots, n \quad q = 1 - p.$$

The reliability of the system is then the sum of the probabilities that  $k, k + 1, \dots, n$  units operate properly, that is,

$$\text{Reliability} = \sum_{i=k}^n \binom{n}{i} p^i q^{n-i},$$

where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}.$$

The expectation of the distribution is

$$E(K) = \sum_{k=1}^n \left[ \frac{kn!}{k(k-1)!(n-k)!} p^k q^{n-k} \right] = np.$$

The variance is

$$V(K) = E(K^2) - [E(K)]^2 = n^2 p^2 + np(1-p) - (np)^2 = np(1-p).$$

### 1.6.4 Poisson Distribution

Poisson distribution describes the probability that an event occurs in time  $t$ . The event may represent the number of defectives in a production process or the number of failures of a system or group of components. The Poisson distribution is derived based on the binomial distribution. This is achieved by taking the limit of the binomial distribution as  $n \rightarrow \infty$  with  $p = \lambda/n$ . Substitution  $p = \lambda/n$  in the binomial distribution results in

$$p(k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}.$$

Taking limit as  $n \rightarrow \infty$

$$\begin{aligned} \lim_{n \rightarrow \infty} p(k) &= \lim_{n \rightarrow \infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \frac{n(n-1)\dots(n-k+1)(n-k)!}{n^k(n-k)!} \left(1 - \frac{\lambda}{n}\right)^n \left(\frac{n-\lambda}{n}\right)^{-k}, \end{aligned}$$

which is reduced to

$$\lim_{n \rightarrow \infty} p(k) = \frac{\lambda^k}{k!} \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = \frac{e^{-\lambda} \lambda^k}{k!} \quad k = 0, 1, 2, \dots$$

Thus, the probability function of the Poisson distribution is

$$f(k) = \frac{e^{-\lambda} \lambda^k}{k!} \quad k = 0, 1, 2, \dots$$

Its expectation is

$$E(K) = \sum_{y=0}^{\infty} k \left[ \frac{e^{-\lambda} \lambda^k}{k!} \right] = \lambda.$$

The variance is

$$V(K) = E(K^2) - [E(K)]^2 = \lambda.$$

### 1.6.5 Hypergeometric Distribution

The hypergeometric distribution is used to model systems when successive events must occur before the failure of a system. Consider, for example, a system which is configured with implicit redundancy which requires the failure of two consecutive components for the system to fail. In this case, the reliability of the system is assessed using a hypergeometric distribution. Consider a population of size  $N$  with  $k$  working devices. A sample of size  $n$  is taken from the population; the number of working devices in the sample ( $y$ ) is a random variable  $Y$ , and its probability function is

$$p(y) = \frac{\binom{k}{y} \binom{N-k}{n-y}}{\binom{N}{n}} \quad y = 0, 1, \dots, \min(n, k).$$

The expectation and variance are

$$E(Y) = n \binom{k}{N}$$

$$V(Y) = n \binom{k}{N} \binom{N-k}{N} \binom{N-n}{N-1}.$$

## 1.7 MEAN TIME TO FAILURE

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One of the measures of the systems' reliability is the MTTF. It should not be confused with the mean time between failures (MTBF). We refer to the expected time between two successive failures as the MTTF when the system is nonrepairable. Meanwhile, when the system is repairable we refer to it as the MTBF.

Now, let us consider  $n$  identical nonrepairable systems and observe the time to failure for them. Assume that the observed times to failure are  $t_1, t_2, \dots, t_n$ . The mean time to failure,  $\overline{MTTF}$ , is

$$\widehat{MTTF} = \frac{1}{n} \sum_{i=1}^n t_i. \quad (1.104)$$

Since  $t_i$  is a random variable, then its expected value can be determined by

$$MTTF = \int_0^{\infty} t f(t) dt. \quad (1.105)$$

But  $R(t) = 1 - F(t)$  and  $f(t) = dF(t)/dt = -dR(t)/dt$ . Substituting in Equation 1.105, we obtain

$$\begin{aligned} MTTF &= -\int_0^{\infty} t \frac{dR(t)}{dt} dt \\ &= -\int_0^{\infty} t dR(t) \\ &= -tR(t)|_0^{\infty} + \int_0^{\infty} R(t) dt. \end{aligned}$$

Since  $R(\infty) = 0$  and  $R(0) = 1$ , then the first part of the above equation is 0 and the MTTF is

$$MTTF = \int_0^{\infty} R(t) dt. \quad (1.106)$$

The MTTF for a constant hazard-rate model is

$$MTTF = \int_0^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda}. \quad (1.107)$$

The MTTF of a linearly increased hazard-rate model is

$$MTTF = \int_0^{\infty} e^{-\frac{\lambda t^2}{2}} dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{\frac{\lambda}{2}}} = \sqrt{\frac{\pi}{2\lambda}}. \quad (1.108)$$

Similarly, the MTTF for the Weibull model is

$$MTTF = \int_0^{\infty} e^{-\left(\frac{t}{\theta}\right)^{\gamma}} dt.$$

Substituting  $x = \left(\frac{t}{\theta}\right)^{\gamma}$ , the above equation becomes

$$\begin{aligned} MTTF &= \frac{\theta}{\gamma} \int_0^{\infty} e^{-x} x^{\frac{1}{\gamma}-1} dx \\ &= \frac{\theta}{\gamma} \Gamma\left(\frac{1}{\gamma}\right) \\ &= \theta \Gamma\left(1 + \frac{1}{\gamma}\right). \end{aligned} \quad (1.109)$$

**EXAMPLE 1.18**

The MTTF for a robot controller that will be operating in different stress conditions is specified to be warranted for 20,000 h. The hazard-rate function of a typical controller is found to fit a Weibull model with  $\theta = 3000$  and  $\gamma = 1.5$ . Does the controller meet the warranty requirement? If not, what should the value of  $\theta$  be to meet the requirement (measurements are in hours)?

**SOLUTION**

Substituting  $\theta = 3000$  and  $\gamma = 1.5$  in Equation 1.109, we obtain the MTTF as

$$MTTF = (3000)\Gamma\left(1 + \frac{1}{1.5}\right) = 2700.8.$$

Thus, the MTTF is 2700.8 h. The MTTF does not meet the warranty requirement. The characteristic life that meets the requirement is calculated as  $20,000 = \theta\Gamma(1.666)$ .

Thus,  $\theta$  should equal 22,155. ■

**EXAMPLE 1.19**

The failure time of an electronic device is described by a Pearson type V distribution. The density function of the failure time is

$$f(t) = \begin{cases} \frac{t^{-(\alpha+1)} e^{-\beta/t}}{\beta^{-\alpha}\Gamma(\alpha)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The shape parameter  $\alpha = 3$  and the scale parameter  $\beta = 4000$  h. Determine the MTTF of the device.

**SOLUTION**

Using Equation 1.105, we obtain

$$\begin{aligned} MTTF &= \int_0^{\infty} \frac{t^{-\alpha} e^{-\beta/t}}{\beta^{-\alpha}\Gamma(\alpha)} dt \\ &= \frac{1}{\beta^{-\alpha}\Gamma(\alpha)} \int_0^{\infty} t^{-\alpha} e^{-\beta/t} dt \end{aligned}$$

or

$$MTTF = \frac{\beta}{\alpha - 1} = \frac{4,000}{3 - 1} = 2,000 \text{ h.} \quad \blacksquare$$

## 1.8 MEAN RESIDUAL LIFE (MRL)

A measure of the reliability characteristic of a product, component, or a system is the *MRL* function,  $L(t)$ . It is defined as

$$L(t) = E[T - t | T \geq t], \quad t \geq 0. \quad (1.110)$$

In other words, the mean residual function is the expected remaining life,  $T - t$ , given that the product, component, or a system has survived to time  $t$  (Leemis, 1995).

The conditional p.d.f. for any time  $\tau \geq t$  is

$$f_{T|T \geq t}(\tau) = \frac{f(\tau)}{R(t)} \quad \tau \geq t. \quad (1.111)$$

The conditional expectation of the function given in Equation 1.111 is

$$E[T | T \geq t] = \int_t^{\infty} \tau f_{T|T \geq t}(\tau) d\tau = \int_t^{\infty} \tau \frac{f(\tau)}{R(t)} d\tau. \quad (1.112)$$

Since the component, product, or system has survived up to time  $t$ , the MRL is obtained by subtracting  $t$  from Equation 1.112, thus

$$\begin{aligned} L(t) &= E[T - t | T \geq t] \\ &= \int_t^{\infty} (\tau - t) \frac{f(\tau)}{R(t)} d\tau = \int_t^{\infty} \tau \frac{f(\tau)}{R(t)} d\tau - t \end{aligned}$$

or

$$L(t) = \frac{1}{R(t)} \int_t^{\infty} \tau f(\tau) d\tau - t. \quad (1.113)$$

### EXAMPLE 1.20

A manufacturer uses rotary compressors to provide cooling liquid for a power-generating unit. Experimental data show that the failure times (between 0 and 1 year) of the compressors follow beta distribution with  $\alpha = 4$  and  $\beta = 2$ . What is the MRL of a compressor given that the compressor has survived 5 months?

#### SOLUTION

The p.d.f. of the failure time is

$$f(t) = \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1} (1-t)^{\beta-1} & 0 < t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

or

$$\begin{aligned} f(t) &= \frac{\Gamma(6)}{\Gamma(4)\Gamma(2)} t^3(1-t) \\ &= 20(t^3 - t^4). \end{aligned}$$

$$\text{But } R(t) = 1 - F(t) = 1 - \int_0^t 20(\tau^3 - \tau^4) d\tau.$$

The value of  $t$  corresponding to 5 months is  $5/12 = 0.416$ , thus

$$R(0.416) = 1 - 20 \int_0^{0.416} (t^3 - t^4) dt = 0.900.$$

Using Equation 1.113, we obtain the MRL of a compressor that survived 5 months as

$$L(0.416) = \frac{20}{0.900} \int_{0.416}^1 t(t^3 - t^4) dt - 0.416 = 0.288$$

or the MRL is 3.46 months. ■

## 1.9 TIME OF FIRST FAILURE

The advances in the design and production of medical devices, sensors, and nonmanufacturing have resulted in a wide range of medical devices and implants. Most of the implants are metallic due to their superior mechanical properties, such as hardness and fatigue strength, but one of their drawbacks is that electrochemical reactions take place on metallic surfaces in the human body which causes corrosion and degradation of the implants that might lead to extreme consequences. This has generated the interest in a different measure of reliability for such devices. One such measure is the time to first failure of  $N$  devices. In other words, we are interested in determining the time when the first failure occurs.

Consider a batch of  $N$  devices and assume that the failure time of a single device follows an exponential distribution. Let  $f(t)$  be the *p.d.f.* for a single device, that is,

$$f(t) = \frac{dF(t)}{dt} = \frac{1}{T} e^{-\frac{t}{T}}, \quad (1.114)$$

where  $T$  is the design life (duration of interest). We are interested in determining  $dF_1(t)/dt$  that the first failure in a batch of  $N$  devices occurs in  $[t, t + dt]$ . This can be expressed as

$$f_1(t) = \frac{dF_1(t)}{dt} = N f(t) \left( \int_t^{\infty} f(t') dt' \right)^{N-1}, \quad (1.115)$$

where  $f(t)$  is the probability that a device fails in  $[t, t + dt]$  and  $\left( \int_t^{\infty} f(t') dt' \right)^{N-1}$  is the probability that  $N - 1$  devices fail in  $[t, \infty]$ . Note that  $N$  is a combinatorial factor giving a number of choices

to the devices which fail in  $[t, t + dt]$  (Elsen and Schätzel, 2005). Normalization of Equation 1.115 yields the mean time of first failure as

$$\int_0^{\infty} t \left( \frac{dF_1(t)}{dt} \right) dt = \int_0^{\infty} t N f(t) \left( \int_t^{\infty} f(t') dt' \right)^{N-1} dt = \frac{T}{N} \quad (1.116)$$

The probability of the first failure  $f_1(t)$  for given  $N$  and  $T$  can be obtained using Equation 1.115 and the mean time of the first failure is obtained from Equation 1.116.

### EXAMPLE 1.21

Historical data show that most transistors exhibit CFR and are widely used in many applications. Consider the case where a manufacturer has the choice of releasing a batch of 100 or 200 devices that include one of the transistors and observe the time of the first failure of each batch for 5000 h. Show the failure-time distributions.

#### SOLUTION

Using Equation 1.115, we obtain the p.d.f of the first failure in a group of  $N$  devices over a period of time  $T$  as

$$f_1(t) = \frac{N}{T} e^{-\frac{tN}{T}}$$

The failure-time distributions are shown in Figure 1.43.

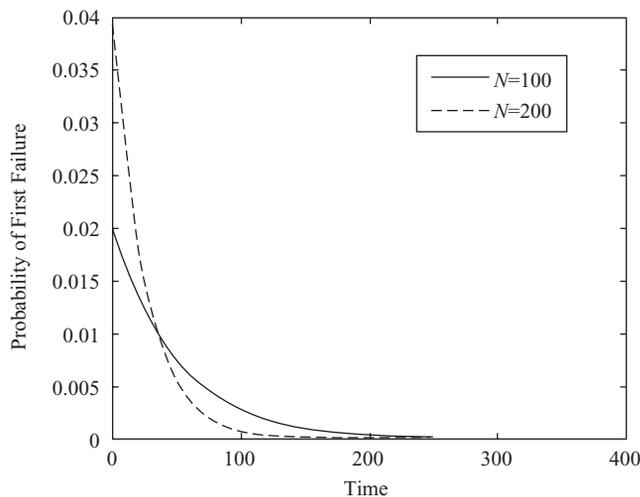


FIGURE 1.43 Time of first failure distributions. ■

Equation 1.115 can be generalized to obtain the time of the  $j$ th failure for  $N$  components. For example, we calculate the probability  $dF_2(t)/dt$  that the second failure in a batch of  $N$  devices occurs in  $[t, t + dt]$  as

$$f_2(t) = \frac{dF_2(t)}{dt} = N(N-1)f(t) \int_0^t f(x)dx \left( \int_t^\infty f(y)dy \right)^{N-2}. \tag{1.117}$$

The time to the second failure is the expectation of  $f_2(t)$ .

We conclude this chapter by providing a summary of the hazard-rate functions and their corresponding parameters, as shown in Table 1.9.

Table 1.9 summarizes the characteristics of the hazard functions discussed in this chapter.

**TABLE 1.9 Characteristics of the Hazard Functions**

Hazard function	$h(t)$	$f(t)$	$R(t)$	Parameters
Constant	$\lambda$	$\lambda e^{-\lambda t}$	$e^{-\lambda t}$	$\lambda$
Linearly increasing	$\lambda t$	$\lambda t e^{-\frac{\lambda t^2}{2}}$	$e^{-\frac{\lambda t^2}{2}}$	$\lambda$
Weibull	$\frac{\gamma}{\theta} \left( \frac{t}{\theta} \right)^{\gamma-1}$	$\frac{\gamma}{\theta} \left( \frac{t}{\theta} \right)^{\gamma-1} e^{-\left( \frac{t}{\theta} \right)^\gamma}$	$e^{-\left( \frac{t}{\theta} \right)^\gamma}$	$\gamma, \theta$
Exponential	$b e^{at}$	$b e^{at} e^{-\frac{b}{a}(e^{at}-1)}$	$e^{-\frac{b}{a}(e^{at}-1)}$	$a, b$
Normal	$\frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{\sigma R(t)}$	$\frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$	$1 - \int_{-\infty}^t \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} d\tau$	$\mu, \sigma$
Lognormal	$\frac{\phi\left(\frac{\ln t - \mu}{\sigma}\right)}{t\sigma R(t)}$	$\frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln t - \mu}{\sigma}\right)^2}$	$1 - \int_0^t \frac{1}{\tau\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln \tau - \mu}{\sigma}\right)^2} d\tau$	$\mu, \sigma$
Gamma	$\frac{f(t)}{R(t)}$	$\frac{t^{\gamma-1}}{\theta^\gamma \Gamma(\gamma)} e^{-\frac{t}{\theta}}$	$\int_t^\infty \frac{1}{\theta \Gamma(\gamma)} \left( \frac{\tau}{\theta} \right)^{\gamma-1} e^{-\frac{\tau}{\theta}} d\tau$	$\theta, \gamma$
Log-logistic	$\frac{\lambda p (\lambda t)^{p-1}}{1 + (\lambda t)^p}$	$\frac{\lambda p (\lambda t)^{p-1}}{[1 + (\lambda t)^p]^2}$	$\frac{1}{1 + (\lambda t)^p}$	$\lambda, p$

## PROBLEMS

1.1 Determine the mean and the variance of a uniform random variable  $X$  whose p.d.f. is

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

1.2 Determine the first and second moments for a normal distribution with parameters  $\mu$  and  $\sigma^2$ .

- 1.3 The p.d.f. of the lognormal distribution is given by

$$f(t) = \frac{1}{\sigma t \sqrt{2\pi}} e^{-\frac{1}{2} \left[ \frac{\ln t - \mu}{\sigma} \right]^2}.$$

Determine the variance and the median. (Hint: Median is defined as  $\int_{med}^{\infty} f(x) dx = 1/2$ ).

- 1.4 A mechanical fatigue test is conducted on 100 specimens of a new polymer. The applied stress is identical for all specimens. The number of cycles observed and the corresponding numbers of failed specimens are given in Table 1.10.
- Plot graphs for  $f_e(t)$ ,  $R_e(t)$ ,  $h_e(t)$ , and  $F_e(t)$ .
  - Comment on the above results.
  - Derive an analytical expression for  $h_e(t)$  and estimate the MTTF of a bar made of the same material and is subjected to the same loading conditions.

**TABLE 1.10 Fatigue Test Results**

Number of cycles $\times 10^5$	Cumulative number of failed specimens
10	35
20	59
30	72
40	84
50	93
60	100

- 1.5 The reliability of disk drives can be predicted by increasing the operational machine hours accumulated in the field or in the laboratory as part of the initial design process. The failures have been accumulated and given in Table 1.11.
- Plot graphs for  $f_e(t)$ ,  $R_e(t)$ ,  $h_e(t)$ , and  $F_e(t)$ .
  - Comment on the above results.

**TABLE 1.11 Failure Data for Problem 1.5**

Hour of operation $\times 10^3$	Number of failed disks
0–10.0	0
10.1–14.0	10
14.1–18.0	15
18.1–22.0	18
22.1–26.0	20
26.1–30.0	16
30.1–34.0	22
34.1–38.0	20

- c. Derive an analytical expression for  $h_c(t)$  and estimate the MTTF of a bar made of the same material and is subjected to the same loading conditions.
  - d. Would you buy a disk produced by the above manufacturer? Why?
- 1.6 One of the modern methods for stress screening is called highly accelerated stress screening (HASS), which use the highest possible stresses (well beyond the normal operating level) to attain time compression on the screens. The HASS exhibits an exponential acceleration of screen strength with stress level. A manufacturer employs a HASS test on newly designed leaf springs for light trucks. A cyclic load was applied on a number of springs and the failure times are recorded in Table 1.12.
- a. Fit a nonlinear polynomial hazard function to describe the hazard rate of the springs.
  - b. What is the reliability at  $t = 8$ ?
  - c. Assume that we obtained 500 springs that require testing under the same conditions. What is the expected time to failure? What is the least time needed to ensure that all units fail under test?

**TABLE 1.12 Failure Data for Problem 1.6**

Time interval (minutes)	Number of failed units
0–1.999	10
2–3.999	15
4–5.999	22
6–7.999	34
8–9.999	49
10–11.999	63
12–14	70

- 1.7 A reliability engineer subjected 10 steel specimens to High-Cycle Fatigue (HCF) that occurs at relatively large numbers of cycles and is caused by high frequency vibrations in both static and rotating hardware. The number of cycles to failure is recorded for each specimen and is reported as follows:  
 200,000, 250,000, 280,000, 300,000, 350,000, 370,000, 380,000, 400,000, 420,000, 460,000
- a. Use the improved mean rank to obtain the p.d.f.,  $R(t)$  and  $h(t)$ .
  - b. Use two median rank approaches to obtain the p.d.f.,  $R(t)$  and  $h(t)$ .
  - c. Compare the results obtained from (a) with those obtained from (b).
- 1.8 Show that the variance of a component whose hazard rate can be described by  $h(t) = \gamma\theta(t/\theta)^{\gamma-1}$  is

$$\text{Var}[T] = \theta^2 \left\{ \Gamma\left(1 + \frac{2}{\gamma}\right) - \left[ \Gamma\left(1 + \frac{1}{\gamma}\right) \right]^2 \right\},$$

where

$$\Gamma(n) = \int_0^\infty \tau^{n-1} e^{-\tau} d\tau$$

and

$$\int_0^\infty \tau^{n-1} e^{-\tau/\theta} d\tau = \Gamma(n)\theta^n.$$

- 1.9 Use the Weibull graph paper to estimate the parameters of a Weibull distribution that fits the data given in Problem 1.6.
- 1.10 Plot  $h(t)$  and  $R(t)$  for  $t = 0$  to 1000, for different shape parameters of 0.5–3.5 with an increment of 0.5 and for different characteristic lives of 200–300 with an increment of 25. What is the effect of the characteristic life on the hazard-rate function? What is the best combination of shape parameter and characteristic life that results in the highest reliability at  $t = 1000$ ? (Weibull distribution).
- 1.11 Dhillon (1979) proposes a hazard-rate model given by

$$h(t) = k\lambda ct^{c-1} + (1-k)bt^{b-1}\beta e^{\beta t^b}$$

for

$$b, c, \beta, \lambda > 0 \quad 0 \leq k \leq 1 \quad t \geq 0,$$

where

$b, c$  = shape parameters,  
 $\beta, \lambda$  = scale parameters, and  
 $t$  = time.

Derive the reliability function and determine the conditions that make the hazard rate increasing, decreasing, or constant.

- 1.12 A rolling bearing rotating under load may ultimately suffer from material fatigue. Typically, fatigue damage is characterized by a small piece of material breaking away from the raceway leaving a cavity. This cavity may then propagate into a crack and the bearing will fail. If a large batch of identical bearings is run under the same conditions until 10% of the batch has failed from the material fatigue damage, then the batch is said to have attained its  $L_{10}$  life. In other words, the remaining 90% of the bearings in the batch will survive for periods longer than the  $L_{10}$  life. Consider a rolling bearing which has a hazard-rate function in the form

$$h(t) = \frac{\frac{1}{\theta} \left( \frac{t}{\theta} \right)^{n-1}}{(n-1)! \sum_{k=0}^{n-1} \frac{(t/\theta)^k}{k!}},$$

where  $n = 3$  and  $\theta = 290$  h. Determine the reliability of the bearing at  $t = 100$  h. Assuming  $L_{10} = 100$  h, determine the MRL of the bearing.

- 1.13 Find  $f(t)$ ,  $h(t)$ ,  $R(t)$ , and MTTF, assuming

$$F(t) = 1 - \frac{8}{7}e^{-t} + \frac{1}{7}e^{-8t}.$$

- 1.14 Find  $f(t)$ ,  $F(t)$ ,  $R(t)$ , and MTTF, assuming

$$h(t) = \frac{1}{25}t^{-1/4}.$$

If 200 units are placed in operation at the same time, how many failures are expected during 1 year of operation?

- 1.15** The failure rate of a brake system is found to be  $h(t) = 0.006(1.5 + 2t + 3t^2)$  failures per year.
- What is the reliability at  $t = 10^4$  h?
  - If 20 systems are subjected to a test at the same time, how many would have survived at time  $t = 10^3$  h? What is the expected number of failures in 1 year of operation?
- 1.16** The failure rate of a hydraulic system is found to be  $h(t) = 0.003(1 + 2.5e^{-3t} + e^{-t/50})$  failures per year.
- What is the reliability at  $t = 10^5$  h?
  - What is the MTTF?
  - If 10 systems are subjected to a test at the same time, how many would have survived at time  $t = 10^3$  h. What is the expected number of failures in 1 year of operation?
- 1.17** Consider the general hazard failure rate (Hjorth, 1980) that is given by  $h(t) = \delta t + \theta/(1 + \beta t)$ . Special cases are

$\theta = 0$  The Rayleigh distribution,

$\delta = \beta = 0$  The exponential distribution,

$\delta = 0$  DFR,

$\delta \geq \theta\beta$  IFR, and

$0 \leq \delta \leq \theta\beta$  The bathtub curve.

The reliability function corresponding to this general hazard rate is

$$R(t) = \frac{e^{-\delta t^2/2}}{(1 + \beta t)^{\theta/\beta}}, \quad t \geq 0.$$

Let  $T$  have the above reliability function, and define

$$I(a, b) = \int_0^{\infty} \frac{e^{-at^2/2}}{(1+t)^b} dt.$$

Find the mean and the variance of  $T$ . Plot the hazard rate for different values of the parameters.

- 1.18** The viscosity of a lubricant used in a heavy machinery (at 70°C) is measured in centipoise at equal intervals of times (days) as shown in Table 1.13. The lubricant needs to be replaced when the threshold value of the viscosity is 1400 centipoise. Assuming that the measurements follow a BS distribution, determine its parameters and plot the reliability function with time. Determine the change point of the hazard-rate function.
- 1.19** The p.d.f. of the early failure times of the circuit boards used in high-speed modems is found to follow a Pearson type V distribution given by

$$f(t) = \begin{cases} \frac{t^{-(\alpha+1)} e^{-\beta/t}}{\beta^{-\alpha} \Gamma(\alpha)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

where  $\alpha$  and  $\beta$  are the shape and scale parameters, respectively. Find the reliability function, the hazard rate, and the MTTF for the special case when  $\beta = 1$  and  $\alpha = 3$ . Is the hazard rate increasing, decreasing, or constant?

**TABLE 1.13** Viscosity Data for Problem 1.18

11	28	43	56	84	108	129	170	238	354
15	31	44	58	86	109	141	175	246	383
15	34	46	59	89	109	146	177	261	396
15	36	47	61	90	115	155	177	264	417
22	36	47	62	95	119	161	177	272	425
23	37	47	62	97	119	162	180	281	448
24	38	49	68	97	123	168	184	283	472
24	41	50	68	98	127	169	196	301	646
25	42	50	79	106	127	169	227	303	777
27	42	56	83	108	129	170	238	318	1181

- 1.20 Let  $t$  denote the time to failure of a component whose p.d.f. is given by

$$f(t) = \frac{1}{\ln 2} \frac{1}{t}, \quad 25,000 < t < 50,000 \text{ h.}$$

- Verify that  $f$  is a density for a continuous random variable.
  - What is the hazard function of this component?
  - What is the expected life of the component?
- 1.21 A manufacturer of medical equipment introduces three different prototype machines, Machine A, Machine B, and Machine C, all capable of sensing contrast or saline pooling under a patient's skin during a chemotherapy procedure. This task approximately equals one unit of time for every patient. The manufacturer records the incidents of each machine in terms of the number of patients served before the machine fails. Assume that when the machine fails it is repaired to be as good as new. The data are shown in Tables 1.14–1.16.

**TABLE 1.14** Failure Data for Machine A

Incident #	Cumulative patients	Patients between failures
1	1	1
2	7	6
3	94	87
4	193	99
5	217	24
6	367	150
7	390	23
8	411	21
9	654	243
10	779	125
11	1,016	237
12	1,035	19
13	1,038	3
14	1,074	36

**TABLE 1.15 Failure Data for Machine B**

<b>Incident #</b>	<b>Cumulative patients</b>	<b>Patients between failures</b>
1	13	13
2	20	7
3	59	39
4	67	8
5	71	4
6	91	20
7	123	32
8	128	5
9	129	1
10	140	11
11	155	15
12	166	11
13	192	26
14	203	11
15	241	38
16	253	12
17	255	2
18	282	27
19	305	23
20	344	39
21	356	12
22	413	57
23	432	19
24	485	53
25	498	13
26	501	3
27	518	17
28	565	47
29	631	66
30	651	20
31	672	21
32	718	46
33	761	43
34	865	104
35	876	11
36	913	37
37	946	33
38	978	32
39	1,045	67

TABLE 1.16 Failure Data for Machine C

Incident #	Cumulative patients	Patients between failure
1	67	67
2	178	111
3	240	62
4	411	171
5	427	16
6	445	18
7	454	9
8	457	3
9	464	7
10	482	18
11	524	42
12	529	5
13	698	169
14	706	8
15	744	38
16	757	13
17	780	23
18	791	11
19	802	11
20	815	13
21	830	15
22	853	23
23	860	7
24	874	14
25	918	44
26	935	17
27	957	22
28	1,016	59
29	1,034	18
30	1,071	37
31	1,075	4
32	1,084	9

- a. Analyze the failure data and compare the hazard-rate functions for the three machines.
- b. Plot the reliability functions and estimate the MTTF for each machine.
- c. What are your suggestions to the manufacturer?

**1.22** In most electronic manufacturing operations, the role of process control has traditionally fallen to automated board-test systems. These systems are typically placed at the end of the manufacturing line in order to monitor fault trends and thus help control the process. The failure data collected at a board-test system show that the failure time follows a triangular distribution with the following p.d.f.

$$f(t) = \begin{cases} \frac{2(t-a)}{(b-a)(c-a)} & \text{if } a \leq t \leq c \\ \frac{2(b-t)}{(b-a)(c-a)} & \text{if } c < t \leq b \\ 0 & \text{otherwise.} \end{cases}$$

where  $a$ ,  $b$ , and  $c$  are real numbers with  $a < c < b$ .  $a$  is a location parameter,  $b - a$  is a scale parameter,  $c$  is a shape parameter. Assume that  $a = 2$ ,  $b = 4$ , and  $c = 3$ . What is the expected MTTF? What is its variance?

- 1.23** A manufacturer intends to introduce a new product. Five products are subjected to a reliability test. The mean of the failure times is 300 h and the variance is 90,000 h<sup>2</sup>. Since the number of failure data is limited, it is difficult to determine with an acceptable confidence level the type of the failure-time distribution.
- What is the expected number of failures at 500 h?
  - The similarity between this product and another product that has already been in the market for the last 10 years indicates that the failure-time distribution is likely to follow gamma distribution. What is the expected number of failures under these conditions at 500 h? Compare the results with (a) above. What do you conclude?
- 1.24** The failure time of a new brake drum design is observed to follow a gamma distribution with a p.d.f. of

$$f(t) = \frac{\lambda(\lambda t)^{\gamma-1} e^{-\lambda t}}{\Gamma(\gamma)}.$$

For  $\gamma = 2$  and  $\lambda = 0.0002$ , determine

- The expected number of failures in 1 year of operation,
  - The MTTF, and
  - The reliability at  $t = 1000$  h.
- 1.25** Solve the above problem when  $\gamma = 3$  and  $\lambda = 0.0002$ . Compare the results. Which brake system is better? Why?
- 1.26** Most fractional horsepower motor controllers use silicon-controlled rectifier (SCR) to vary the power applied to the motor and thereby control armature voltage and thus the motor's speed. The SCR is made of different layers of semiconductor materials. The heat dissipation from the motor increases the failure rate of the SCR. Failure data from the field show that the failure time follows a beta distribution with the following p.d.f.

$$f(t) = \begin{cases} \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} t^\alpha (1-t)^\beta & 0 < t < 1, \alpha > -1, \beta > -1 \\ 0 & \text{otherwise.} \end{cases}$$

Assuming that  $\alpha = 1.8$  and  $\beta = 4.7$ , what is the expected MTTF? What is its variance? What is the expected number of failures at  $t = 2.5$ ?

- 1.27 Consider the case where the failure time of components follows a logistic distribution with p.d.f. of 0

$$f(t) = \frac{(1/\beta)e^{-(t-\alpha)/\beta}}{(1+e^{-(t-\alpha)/\beta})^2}, \quad -\infty < \alpha < \infty, \beta > 0, -\infty < t < \infty.$$

Determine the expected number of failures in the interval  $[t_1, t_2]$ .

- 1.28 In order for a manufacturer to determine the length of the warranty period for newly developed ICs, 100 units are placed under test for 5000 h. The hazard-rate function of the units is  $h(t) = 5 \times 10^{-9}t^{0.9}$ .

What is the expected number of failures at the end of the test? Should the manufacturer make the warranty period longer or shorter if the ICs were redesigned and its new hazard-rate function became  $h(t) = 6 \times 10^{-8}t^{0.75}$ ?

- 1.29 The manufacturer of diodes subjects 100 diodes to an elevated temperature testing for a 2-year period. The failed units are found to follow a Weibull distribution with parameters  $\theta = 50$  and  $\gamma = 2$  (in thousands of hours). What is the expected life of the diodes? What is the expected number of failures in a 2-year period?
- 1.30 In Problem 1.29, if a diode survives 1 year of operation, what is its MRL?
- 1.31 The hazard-rate function of a manufacturer's jet engines is a function of the amount of silver and iron deposits in the engine oil. If the metal deposit readings are "high," the engine is removed from the aircraft and overhauled. The hazard-rate function (Jardine and Buzacott, 1985) is

$$h(t; z(t)) = \frac{5.335}{3255.19} \left( \frac{t}{3255.19} \right)^{4.335} \exp[0.506 z_1(t) + 1.25 z_2(t)],$$

where  $t$  = flight hours,

$z_1(t)$  = iron deposits in parts per million at time  $t$ , and

$z_2(t)$  = silver deposits in parts per million at time  $t$ .

Analysis of the deposits over time shows that

$$z_1(t) = 0.0005 + 0.00006t$$

$$z_2(t) = 0.00008t + 8 \times 10^{-8}t^2.$$

Plot the reliability of the engine against flying hours. What is the MTTF?

- 1.32 A mixture model of the Inverse Gaussian (IG) and the Weibull (W) distributions, called the IG-W model, is capable of covering six different combinations of failure rates: one of the components has an upside-down bath tub failure rate (UBTFR) or IFR and the other component has a DFR, CFR, or IFR (Al-Hussaini and Abd-el-Hakim, 1989). The mixture density function of the IG-W model is

$$f(t) = p f_1(t) + q f_2(t),$$

where  $p$  is the mixing proportion,  $0 \leq p \leq 1$  and  $q = 1 - p$ . The density functions  $f_1(t)$  and  $f_2(t)$  are those of the Inverse Gaussian  $IG(\mu, \lambda)$  and the Weibull  $W(\theta, \beta)$  having the respective forms

$$f_1(t) = \sqrt{\lambda/2\pi t^3} \exp(-\lambda(t-\mu)^2/2\mu^2 t), \quad t > 0, \mu, \lambda > 0$$

$$f_2(t) = \left(\frac{\beta}{\theta}\right) \left(\frac{t}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{t}{\theta}\right)^\beta\right], \quad t > 0, \beta > 0, \theta > 0.$$

The reliability function  $R(t)$  of the mixed model is

$$R(t) = p R_1(t) + q R_2(t).$$

The hazard-rate function,  $h(t)$  of the mixed model is

$$h(t) = \frac{f(t)}{R(t)} = \frac{p f_1(t) + q f_2(t)}{p R_1(t) + q R_2(t)} = r(t)h_1(t) + (1-r(t))h_2(t)$$

where

$$r(t) = \frac{1}{1+g(t)}, \quad g(t) = \frac{q R_2(t)}{p R_1(t)}.$$

Investigate the necessary conditions for an IFR, CFR, and DFR.

- 1.33** A beginner reliability engineer did not realize that the failures of the system should be grouped by type instead of having them in one group. The system was observed to fail because of two types of failures: electrical ( $E$ ) and mechanical ( $M$ ). The failure data for  $E$  are

316, 138, 87, 923, 921, 1113, 1152, 577, 480, 1401

The data for  $M$  are

746, 1281, 1304, 1576, 1386, 671, 2106, 660, 1149, 425

The true data for  $E$  comes from an exponential distribution with mean = 1000 h, and the data for  $M$  comes from Weibull with  $\gamma = 2$  and  $\theta = 1000$ .

- What is the reliability expression for the true distribution?
- What is the reliability expression for the combined failures?
- Is the analysis of the engineer correct? Why?

- 1.34** Determine the mean life and the variance of a component whose failure time is expressed by

$$f(t) = \sum_{i=1}^n p_i \frac{\gamma_i}{\theta_i} \left(\frac{t}{\theta_i}\right)^{\gamma_i-1} e^{-\left(\frac{t}{\theta_i}\right)^{\gamma_i}},$$

where  $\sum_{i=1}^n p_i = 1$ .

- 1.35** Assume that the mean hazard rate is given by

$$E[h(T)] = \int_0^{\infty} h(t)f(t)dt$$

and the MTTF  $E[T]$  is

$$E[T] = \int_0^{\infty} R(t) dt.$$

Prove that  $\{E[h(t)] \cdot E[T]\}$  is an increasing function of the shape parameter of the Weibull model.

- 1.36** Consider a Weibull distribution with a reliability function  $R(t) = \exp(-\theta \lambda t^\gamma)$  for  $t \geq 0$ . For  $\gamma > 1$ ,  $\theta > 0$ , and  $\lambda > 0$ , the Weibull density becomes an IFR distribution (the wear-out region of the bathtub curve). Suppose that the values of  $\lambda$  follow a gamma distribution with p.d.f.  $f(\lambda)$  given by

$$f(\lambda) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha\lambda} \lambda^{\beta-1} \quad \alpha > 0, \beta > 0, \lambda > 0.$$

The reliability function of the mixture is given by

$$R_{\text{mixture}}(t) = \int_0^{\infty} R(t)f(\lambda)d\lambda.$$

- a.** Show that the failure-rate function of the mixture is as given by (Gurland and Sethuraman, 1994),

$$h_{\text{mixture}}(t) = \beta \frac{\theta \gamma t^{\gamma-1}}{\alpha + \theta t^\gamma}.$$

- b.** Plot  $h_{\text{mixture}}(t)$  for large values of  $t$ . What do you conclude?  
**c.** Plot the hazard rate for different values of  $\alpha$ ,  $\beta$ ,  $\theta$ , and  $\gamma$ . What are the conditions at which  $h_{\text{mixture}}(t)$  is an IFR function? A DFR function? A CFR function?
- 1.37** Data from a linearly increasing failure-rate distribution is mixed with some data from a constant failure-rate distribution. Assume that the linearly increasing failure rate is a Rayleigh distribution with  $R_R(t) = e^{-\lambda t^2/2}$ , where  $\lambda$  is a constant, and the reliability function of the CFR is  $R_c(t) = e^{-\theta t}$ . Investigate  $h(t)$  of the mixture of the distributions.
- 1.38** The failure time of a component follows a Pareto distribution with a p.d.f. of

$$f(t) = \frac{\gamma \lambda^\gamma}{t^{\gamma+1}}, \quad \lambda > 0, \gamma > 1, \lambda < t < \infty.$$

Determine the MTTF of the component and its MRL function.

- 1.39** Derive an expression for the probability that the first failure in a batch of  $N$  devices in  $[t, t + dt]$  when every device has the same Weibull failure-time distribution. Estimate the mean time of the first failure. Plot the probability distribution for 200 devices with shape parameter of 2.5 and scale parameter of 4000.
- 1.40** The probability  $dF_2(t)/dt$  that the second failure in a batch of  $N$  devices occurs in  $[t, t + dt]$  is expressed as

$$f_2(t) = \frac{dF_2(t)}{dt} = N(N-1)f(t) \int_0^t f(x)dx \left( \int_t^\infty f(y)dy \right)^{N-2}.$$

Generalize the above expression for the  $j$ th failure in a batch of  $N$  devices.

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