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TRANSMISSION LINES: PHYSICAL DIMENSIONS VS. ELECTRIC DIMENSIONS

With the operating frequencies of today's high-speed digital and high-frequency analog systems continuing to increase into the GHz ($1\text{GHz} = 10^9$ Hz) range, previously-used lumped-circuit analysis methods such as Kirchhoff's laws *will no longer be valid* and will give *incorrect answers*. Physical dimensions of the system that are "electrically large" (greater than a tenth of a wavelength) *must be analyzed* using the transmission-line model. The wavelength, λ, of a single-frequency sinusoidal current or voltage wave is defined as $\lambda = v/f$ where v is the velocity of propagation of the wave on the system's conductors, and f is the cyclic frequency of the single-frequency sinusoidal wave on the conductor. Velocities of propagation on printed circuit boards (PCBs) lie between 60 and 100% of the speed of light in a vacuum, $v_0 = 2.99792458 \times 10^8 \text{m/s}$. A 1-GHz single-frequency sinusoidal wave on a pair of conductors of total length $\mathscr{L}$ will be one wavelength for

$$\mathscr{L} = 1\lambda = \frac{v_0 = 3 \times 10^8}{f = 1 \times 10^9} = 30\,\text{cm} \cong 11.8\,\text{in}$$

In this case the largest circuit that can be analyzed successfully using Kirchhoff's laws and lumped-circuit models is of length $\mathscr{L} = \frac{1}{10}\lambda = 3\,\text{cm}$

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= 1.18 in! In that case "electrically long" pairs of interconnect conductors ($\mathscr{L} > 3$ cm = 1.18 in) that interconnect the electronic modules must be treated as *transmission lines* in order to give correct answers.

The spectral (frequency) content of modern high-speed digital waveforms today as well as the operating frequencies of analog systems extend into the gigahertz regime. A digital clock waveform has a *repetitive* trapezoidal shape, as illustrated in Fig. 1.1. The period T of the periodic digital waveform is the reciprocal of the clock fundamental frequency, f_0, and the fundamental radian frequency is $\omega_0 = 2\pi f_0$. The rise and fall times are denoted τ_r and τ_f, respectively, and the pulse width (between 50% levels) is denoted τ. As the fundamental frequencies of the clocks, f_0, are increased, their period $T = 1/f_0$ decreases and hence the rise and fall times of the pulses must be reduced commensurately in order that pulses resemble a trapezoidal shape rather than a "sawtooth" waveform, thereby giving adequate "setup" and "hold" time intervals. Typically, the rise and fall times are chosen to be 10% of the period T to achieve this. Reducing the pulse rise and fall times has had the consequence of increasing the spectral content of the waveshape. Typically, this spectral content is significant up to the inverse of the rise and fall times, $1/\tau_r$. For example, a 1-GHz digital clock signal having rise and fall times of 100 ps (1 ps = 10^{-12} s) has significant spectral content at multiples (harmonics) of the basic clock frequency (1 GHz, 2 GHz, 3 GHz, ...) up to around 10 GHz. Since the digital clock waveform shown in Fig. 1.1 is a periodic, repetitive waveform, according to the Fourier series their time-domain waveforms can be viewed alternatively as being composed of an infinite number of harmonically related sinusoidal components as

$$\boxed{\begin{aligned} x(t) &= c_0 + c_1\cos(\omega_0 t + \theta_1) + c_2\cos(2\omega_0 t + \theta_2) + c_3\cos(3\omega_0 t + \theta_3) + \cdots \\ &= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n) \end{aligned}} \tag{1.1a}$$

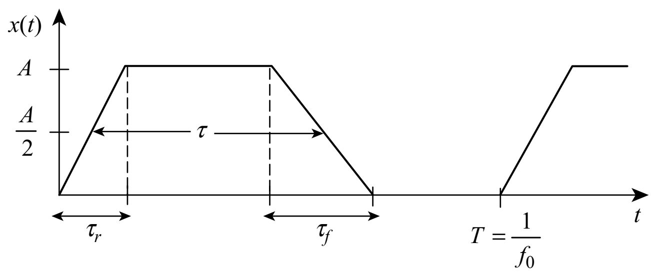


FIGURE 1.1. Typical digital clock/data waveform.

The constant component c_0 is the average (dc) value of the waveform over one period of the waveform,

$$\boxed{c_0 = \frac{1}{T}\int_{t_1}^{t_1+T} x(t)\,dt} \tag{1.1b}$$

and the other coefficients (magnitude and angle) are obtained from

$$\boxed{c_n \angle \theta_n = \frac{2}{T}\int_{t_1}^{t_1+T} x(t)e^{-jn\omega_0 t}\,dt} \tag{1.1c}$$

where $j = \sqrt{-1}$ and the exponential is complex-valued with a unit magnitude and an angle as $e^{-j\omega_0 t} = 1\angle -\omega_0 t$.

In the past, clock speeds and data rates of digital systems were in the low megahertz ($1\text{MHz} = 10^6$ Hz) range, with rise and fall times of the pulses in the nanosecond ($1\text{ ns} = 10^{-9}$s) range. Prior to that time, the "lands" (conductors of rectangular cross section) that interconnect the electronic modules on PCBs were "electrically short" and had little effect on the proper functioning of those electronic circuits. The time delays through the modules dominated the time delay imposed by the interconnect conductors. Today, the clock and data speeds have moved rapidly into the low gigahertz range. The rise and fall times of those digital waveforms have decreased into the picosecond ($1\text{ ps} = 10^{-12}$ s) range. The delays caused by the interconnects have become the dominant factor.

Although the "physical lengths" of the lands that interconnect the electronic modules on the PCBs have not changed significantly over these intervening years, their "electrical lengths" (in wavelengths) have increased dramatically because of the increased spectral content of the signals that the lands carry. Today these "interconnects" can have a significant effect on the signals they are carrying, so that just getting the systems to work properly has become a major design problem. Remember that it does no good to write sophisticated software if the hardware cannot execute those instructions faithfully. This has generated a new design problem referred to as *signal integrity*. Good signal integrity means that the interconnect conductors (the lands) should not adversely affect operation of the modules that the conductors interconnect. Because these interconnects are becoming "electrically long," lumped-circuit modeling of them is becoming inadequate and gives erroneous answers. Many of the interconnect conductors must now be treated as distributed-circuit *transmission lines*.

1.1 WAVES, TIME DELAY, PHASE SHIFT, WAVELENGTH, AND ELECTRICAL DIMENSIONS

In the analysis of electric circuits using Kirchhoff's voltage and current laws and lumped-circuit models, we *ignored* the connection leads attached to the lumped elements. When is this permissible? Consider the lumped-circuit element having attachment leads of total length $\mathscr{L}$ shown in Fig. 1.2. *Single-frequency sinusoidal* currents along the attachment leads are, in fact, *traveling waves*, which can be written in terms of position z along the leads and time t as

$$\boxed{i(t,z) = I\cos(\omega t - \beta z)} \tag{1.2}$$

where the radian frequency ω is written in terms of cyclic frequency f as $\omega = 2\pi f$ rad/s and β is the *phase constant* in units of rad/m. (Note that the argument of the cosine must be in radians and not degrees.) To observe the movement of these current waves along the connection leads, we observe and track the movement of a point on the wave in the same way that we observe the movement of an ocean wave at the seashore. Hence the argument of the cosine in (1.2) must remain constant in order to track the movement of a point on the wave so that $\omega t - \beta z = C$, where C is a constant. Rearranging this as $z = (\omega/\beta)t - C/\beta$ and differentiating with

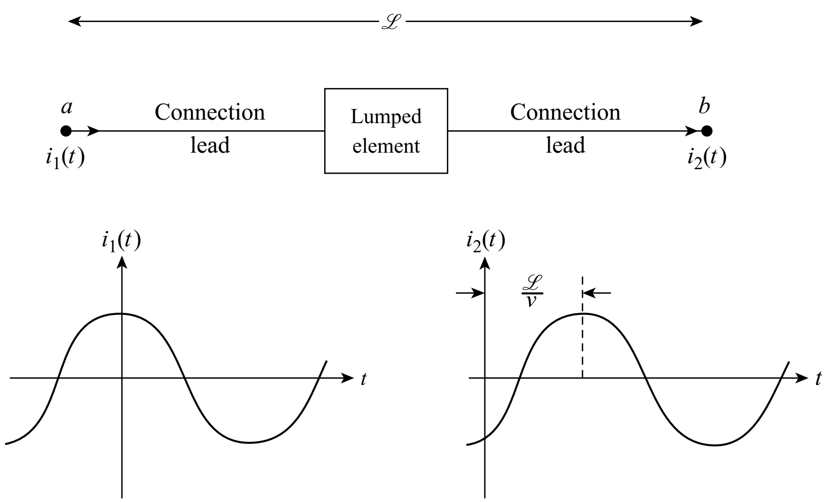


FIGURE 1.2. Current waves on connection leads of lumped-circuit elements.

respect to time gives the velocity of propagation of the wave as

$$\boxed{v = \frac{\omega}{\beta} \qquad \frac{\text{m}}{\text{s}}} \tag{1.3}$$

Since the argument of the cosine, $\omega t - \beta z$, in (1.2) must remain a constant in order to track the movement of a point on the wave, as time t increases, so must the position z. Hence the form of the current wave in (1.2) is said to be a *forward-traveling wave*, since it must be traveling in the $+z$ direction in order to keep the argument of the cosine constant for increasing time. Similarly, a *backward-traveling wave* traveling in the $-z$ direction would be of the form $i(t, z) = I \cos(\omega t + \beta z)$, since as time t increases, position z must decrease to keep the argument of the cosine constant and thereby track the movement of a point on the waveform. Since the current is a *traveling wave*, the current entering the leads, $i_1(t)$, and the current exiting the leads, $i_2(t)$, are separated in time by a *time delay* of

$$\boxed{T_D = \frac{\mathscr{L}}{v} \qquad \text{s}} \tag{1.4}$$

as illustrated in Fig. 1.2. These single-frequency waves suffer a *phase shift* of $\phi = \beta z$ radians as they propagate along the leads. Substituting (1.3) for $\beta = \omega/v$ into the equation of the wave in (1.2) gives an equivalent form of the wave as

$$\boxed{i(t, z) = I \cos\left[\omega\left(t - \frac{z}{v}\right)\right]} \tag{1.5}$$

which indicates that *phase shift is equivalent to a time delay*. Figure 1.2 plots the current waves *versus time*. Figure 1.3 plots the current wave *versus position in space at fixed times*.

As we will see, the critical property of a traveling wave is its wavelength, denoted λ. *A wavelength is the distance the wave must travel in order to shift its phase by* 2π *radians or* 360°. Hence $\beta\lambda = 2\pi$ or

$$\boxed{\lambda = \frac{2\pi}{\beta} \qquad \text{m}} \tag{1.6}$$

Alternatively, the wavelength is the distance between the same adjacent points on the wave: for example, between adjacent wave crests, as illustrated in

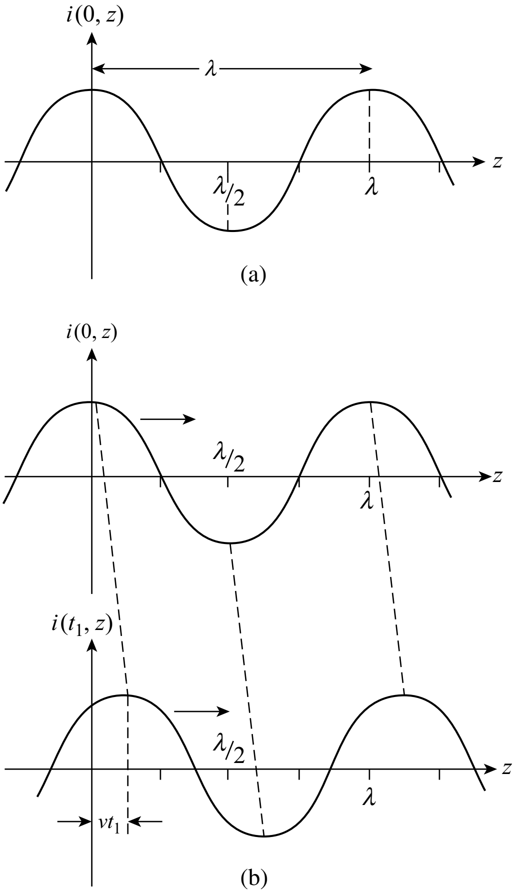


FIGURE 1.3. Waves in space and wavelength.

Fig. 1.3. Substituting the result in (1.3) for β in terms of the wave velocity of propagation v gives an alternative result for computing the wavelength:

$$\boxed{\lambda = \frac{v}{f} \qquad \text{m}} \tag{1.7}$$

Table 1.1 gives the wavelengths of single-frequency sinusoidal waves in free space (essentially, air) where $v_0 \cong 3 \times 10^8 \text{m/s}$. (The velocities of propagation of current waves on the lands of a PCB are less than in free space, which is due to the interaction of the electric fields with the board material. Hence wavelengths on a PCB are shorter than they are in free space.) Observe that a wave of frequency 300 MHz has a wavelength of 1 m. Note that the product of the frequency of the wave and its wavelength equals the velocity of

TABLE 1.1. Frequencies of Sinusoidal Waves in Free Space (Air) and Their Corresponding Wavelengths

Frequency, f	Wavelength, λ
60 Hz	3107 mil (5000 km)
3 kHz	100 km
30 kHz	10 km
300 kHz	1 km
3 MHz	100 m ($\approx$ 300 f)
30 MHz	10 m
300 MHz	**1 m** ($\approx$ 3 f)
3 GHz	10 cm ($\approx$ 4 in)
30 GHz	1 cm
300 GHz	0.1 cm

propagation of the wave, $f\lambda = v$. Wavelengths scale linearly with frequency. As frequency decreases, the wavelength increases, and vice versa. For example, the wavelength of a 7-MHz wave is easily computed as

$$\lambda|_{@7\text{ MHz}} = \frac{300\text{ MHz}}{7\text{ MHz}} \times 1\text{ m} = 42.86\text{ m}$$

Similarly, the wavelength of a 2-GHz cell phone wave is 15 cm, which is approximately 6 in.

Now we turn to the important criterion of physical dimensions in terms of wavelengths: that is, "electrical dimensions." To determine a physical dimension, $\mathscr{L}$, in terms of wavelengths (its "electrical dimension") we write $\mathscr{L} = k\lambda$ and determine the length in wavelengths as

$$k = \frac{\mathscr{L}}{\lambda} = \frac{\mathscr{L}}{v}f$$

where we have substituted the wavelength in terms of the frequency and velocity of propagation as $\lambda = v/f$. Hence we obtain an important relation for the electrical length in terms of frequency and time delay:

$$\boxed{\begin{aligned}\frac{\mathscr{L}}{\lambda} &= f\frac{\mathscr{L}}{v}\\ &= fT_D\end{aligned}} \tag{1.8}$$

so that a dimension is one wavelength, $\mathscr{L}/\lambda = 1$, at a frequency that is the inverse of the time delay:

$$\boxed{f|_{\mathscr{L}=1\lambda} = \frac{1}{T_D}} \tag{1.9}$$

A *single-frequency sinusoidal* wave shifts phase as it travels a distance $\mathscr{L}$ of

$$\boxed{\begin{aligned} \phi &= \beta\mathscr{L} \\ &= 2\pi\frac{\mathscr{L}}{\lambda} \qquad \text{rad} \\ &= \frac{\mathscr{L}}{\lambda}\times 360^\circ \qquad \text{deg} \end{aligned}} \tag{1.10}$$

Hence if a wave travels a distance of one wavelength, $\mathscr{L} = 1\lambda$, it shifts phase by $\phi = 360^\text{o}$. If the wave travels a distance of one-half wavelength, $\mathscr{L} = \frac{1}{2}\lambda$, it shifts phase by $\phi = 180^\text{o}$. This can provide for cancellation, for example, when two antennas that are separated by a distance of one-half wavelength transmit the same frequency signal. Along a line containing the two antennas, the two radiating waves being of opposite phase cancel each other, giving a result of zero. This is essentially the reason that antennas have "patterns" where a null is produced in one direction, whereas a maximum is produced in another direction. Using this principle, phased-array radars "steer" their beams electronically rather than by rotating the antennas mechanically. Next consider a wave that travels a distance of one-tenth of a wavelength, $\mathscr{L} = \frac{1}{10}\lambda$. The phase shift incurred in doing so is only $\phi = 36^\circ$, and a wave that travels one-one hundredth of a wavelength, $\mathscr{L} = \frac{1}{100}\lambda$, incurs a phase shift of $\phi = 3.6^\circ$. Hence we say that

> *for any distance less than, say, $\mathscr{L} < \frac{1}{10}\lambda$, the phase shift is said to be negligible and the distance is said to be* electrically short.

For electric circuits whose physical dimension is electrically short, $\mathscr{L} < \frac{1}{10}\lambda$, Kirchhoff's voltage and current laws and other lumped-circuit analysis solution methods work very well.

> For physical dimensions that are NOT electrically short, Kirchhoff's laws and lumped-circuit analysis methods *give erroneous answers*!

For example, consider an electric circuit that is driven by a 10-kHz sinusoidal source. The wavelength at 10 kHz is 30 km (18.641 mi)! Hence at this frequency any circuit having a dimension of less than 3 km (1.86 mi) can be analyzed successfully using Kirchhoff's laws and lumped-circuit analysis methods. Electric power distribution systems operating at 60 Hz can be analyzed using Kirchhoff's laws and lumped-circuit analysis principles as long as their physical dimensions, such as transmission-line length, are less than some 310 mi! Similarly, a circuit driven by a 1-MHz sinusoidal source can be analyzed successfully using lumped-circuit analysis methods if its physical dimensions are less than 30 m! On the other hand, connection conductors in cell phone electronic circuits operating at a frequency of around 2 GHz cannot be analyzed using lumped-circuit analysis methods unless their dimensions are less than around 1.5 cm or about 0.6 in! We can, alternatively, determine the frequency where a dimension is electrically short in terms of the time delay from (1.8):

$$\boxed{f\big|_{\mathscr{L}=(1/10)\lambda} = \frac{1}{10T_D}} \tag{1.11}$$

Substituting $\lambda f = v$ into the time-delay expression in (1.4) gives the time delay as a portion of the period of the sinusoid, T:

$$\boxed{\begin{aligned} T_D &= \frac{\mathscr{L}}{v} \\ &= \frac{\mathscr{L}}{\lambda}\frac{1}{f} \\ &= \frac{\mathscr{L}}{\lambda}T \end{aligned}} \tag{1.12}$$

where the period of the sinusoidal wave is $T = 1/f$. This shows that if we plot the current waves in Fig. 1.2 that enter and leave the connection leads versus time *t on the same time plot*, they will be displaced in time by a fraction of the period, $\mathscr{L}/\lambda$. If the length of the connection leads $\mathscr{L}$ is electrically short at this frequency, the two current waves will be displaced from each other *in time* by an inconsequential amount of less than $T/10$ and may be considered to be coincident in time. This is the reason that Kirchhoff's laws and lumped-circuit analysis methods work well only for circuits whose physical dimensions are "electrically small."

Waves propagated along transmission lines and radiated from antennas are of the same mathematical form as the currents on the connection leads of an element shown in (1.2). These are said to be *plane waves* where the electric and magnetic field vectors lie in a plane *transverse* or perpendicular to the direction of propagation of the wave, as shown in Fig. 1.4. These are said to be *transverse electromagnetic* (TEM) *waves*.

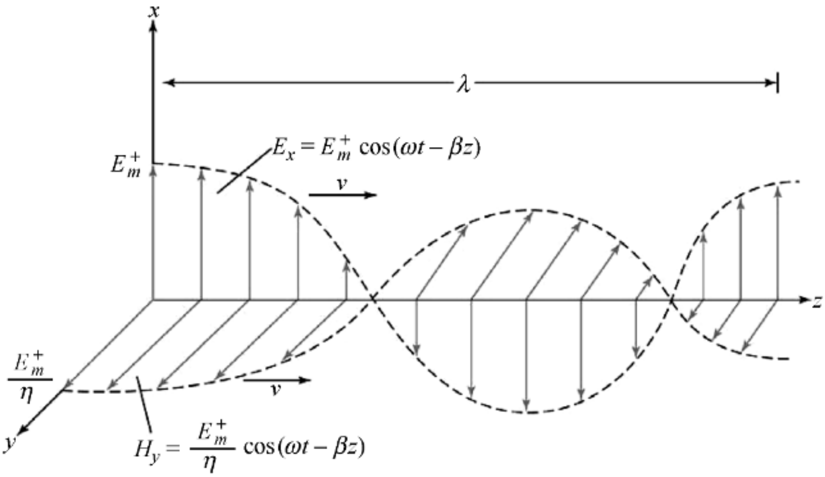


FIGURE 1.4. Electric and magnetic fields of plane waves on transmission lines and radiated by antennas.

This has demonstrated the following important principle in electromagnetics:

In electromagnetics, "physical dimensions" of structures don't matter; their "electrical dimensions in wavelengths" are important.

1.2 SPECTRAL (FREQUENCY) CONTENT OF DIGITAL WAVEFORMS AND THEIR BANDWIDTHS

A periodic waveform of fundamental frequency f_0 such as the digital clock waveform in Fig. 1.1 can be represented equivalently as an infinite summation of harmonically related sinusoids with the Fourier series shown in (1.1). The coefficients in the Fourier series are obtained for a digital clock waveform shown in Fig. 1.1, where the rise and fall times, τ_r and τ_f, are equal: $\tau_r = \tau_f$ (which digital clock waveforms approximate) as:

$$\begin{aligned} c_0 &= A\frac{\tau}{T} \\ c_n \angle \theta_n &= 2A\frac{\tau}{T}\frac{\sin(n\pi\tau/T)}{n\pi\tau/T}\frac{\sin(n\pi\tau_r/T)}{n\pi\tau_r/T} \angle -n\pi\frac{\tau+\tau_r}{T} \qquad \tau_r = \tau_f \end{aligned} \tag{1.13}$$

This result is in the form of the product of two $\sin(x)/x$ expressions, with the first depending on the ratio of the pulse width to the period, τ/T (also called the *duty cycle* of the waveform, $D = \tau/T$), and the second depending on the ratio of the pulse rise or fall time to the period, τ_r/T. [The *magnitude* of the coefficient, denoted as c_n, must be a positive number. Hence there may be an additional $\pm 180^\circ$ added to the angle shown in (1.13), depending on the signs of each $\sin(x)$ term.] If, in addition to the rise and fall times being equal, the duty cycle is 50%, that is, the pulse is "on" for half the period and "off" for the other half of the period (which digital waveforms also tend to approximate), $\tau = \frac{1}{2}T$, the result for the coefficients given in (1.13) simplifies to

$$
\begin{aligned}
c_0 &= \frac{A}{2} \\
c_n \angle \theta_n &= A \frac{\sin(n\pi/2)}{n\pi/2} \frac{\sin(n\pi\tau_r/T)}{n\pi\tau_r/T} \angle -n\pi\left(\frac{1}{2} + \frac{\tau_r}{T}\right) \qquad \tau_r = \tau_f, \tau = \frac{T}{2}
\end{aligned}
$$

Note that the first $\sin(x)/x$ function is zero for n even, so that *for equal rise and fall times and a 50% duty cycle the even harmonics are zero and the spectrum consists of only odd harmonics*. By replacing n/T with the smooth frequency variable f, $\boxed{n/T \rightarrow f}$, we obtain the *envelope* of the magnitudes of these discrete frequencies:

$$
\boxed{c_n = 2A\frac{\tau}{T}\left|\frac{\sin(\pi f \tau)}{\pi f \tau}\right|\left|\frac{\sin(\pi f \tau_r)}{\pi f \tau_r}\right| \qquad \tau_r = \tau_f, \frac{n}{T} \rightarrow f} \tag{1.14}
$$

In doing so, remember that the spectral components occur only at the discrete frequencies $f_0,\ 2f_0,\ 3f_0, \ldots$.

Observe some important properties of the $\sin(x)/x$ function:

$$
\underbrace{\lim}_{x \to 0} \frac{\sin(x)}{x} = 1
$$

which relies on the property that $\sin(x) \cong x$ for small x (or using l'Hôpital's rule) and

$$
\left|\frac{\sin(x)}{x}\right| \le \begin{cases} 1 & x \le 1 \\ \dfrac{1}{x} & x \ge 1 \end{cases}
$$

The second property allows us to obtain a *bound* on the magnitudes of the c_n coefficients and relies on the fact that $|\sin(x)| \le 1$ for all x.

A *square wave* is the trapezoidal waveform where the rise and fall times are *zero*:

$$c_0 = A\frac{\tau}{T}$$
$$c_n \angle \theta_n = 2A\frac{\tau}{T}\frac{\sin(n\pi\tau/T)}{n\pi\tau/T} \angle - n\pi\frac{\tau}{T} \qquad \tau_r = \tau_f = 0$$

If the duty cycle of the square wave is 50%, this result simplifies to

$$c_0 = \frac{A}{2}$$
$$c_n \angle \theta_n = \begin{cases} \dfrac{2A}{n\pi} \angle -\dfrac{\pi}{2} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \qquad \tau_r = \tau_f = 0,\ \tau = \frac{T}{2}$$

Figure 1.5 shows a plot of the magnitudes of the c_n coefficients for a square wave where the rise and fall times are zero, $\tau_r = \tau_f = 0$. The spectral components appear only at *discrete* frequencies, f_0, $2f_0$, $3f_0$, ... The envelope is shown with a dashed line. Observe that the envelope goes to zero where the argument of $\sin(\pi f \tau)$ becomes a multiple of π at $f = 1/\tau, 2/\tau, \ldots$.

A more useful way of plotting the envelope of the magnitudes of the spectral coefficients is by plotting the horizontal frequency axis logarithmically and, similarly, plotting the magnitudes of the coefficients along the vertical axis in decibels as $|c_n|_{\text{dB}} = 20\log_{10}|c_n|$. The envelope as well as the bounds of the magnitudes of the sin(x)/x function are shown in Fig. 1.6.

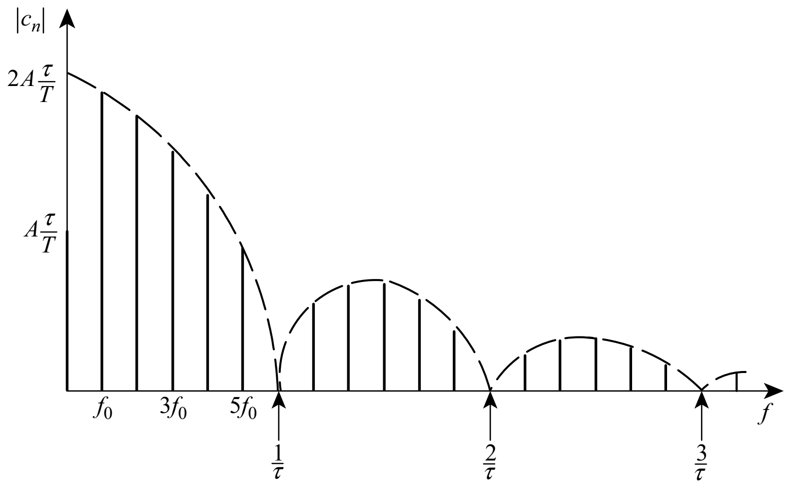


FIGURE 1.5. Plot of the magnitudes of the c_n coefficients for a square wave, $\tau_r = \tau_f = 0$.

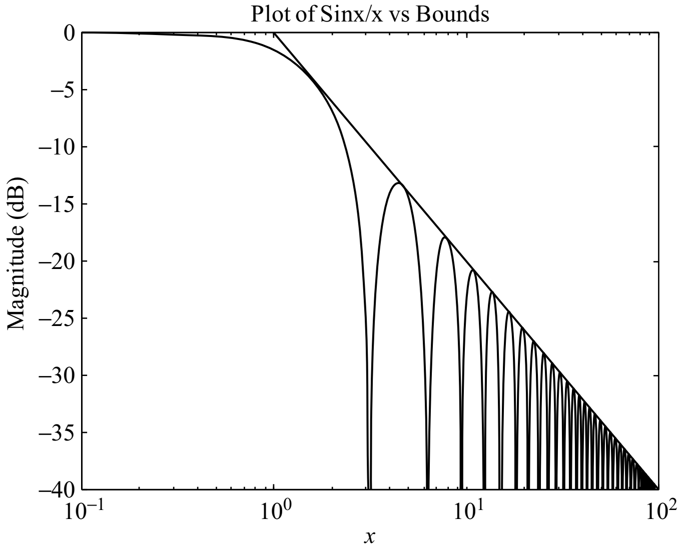


FIGURE 1.6. The envelope and bounds of the sin(x)/x function are plotted with logarithmic axes.

Observe that the actual result is bounded by 1 for $x \leq 1$ and decreases at a rate of -20 dB/decade for $x \geq 1$. This rate is equivalent to a 1/x decrease. Also note that the magnitudes of the actual spectral components go to zero where the argument of $\sin(x)$ goes to a multiple of π or $x = \pi,\ 2\pi,\ 3\pi, \ldots$.

The amplitude of the spectral components of a trapezoidal waveform where $\tau_r = \tau_f$ given in (1.14) is the product of two sin(x)/x functions: $\sin(x_1)/x_1 \times \sin(x_2)/x_2$. When log-log axes are used, this gives the result for the bounds on the amplitudes of the spectral coefficients shown in Fig. 1.7. Note that the bounds are constant (0 dB/decade) out to the first breakpoint of $f_1 = 1/\pi\tau = f_0/\pi D$, where the *duty cycle* is $D = \tau/T = \tau f_0$. Above this they decrease at a rate of -20 dB/decade out to a second breakpoint of $f_2 = 1/\pi\tau_r$ and decrease at a rate of -40 dB/decade above that. This plot shows the important result that *the high-frequency spectral content of the trapezoidal clock waveform is determined by the pulse rise and fall times*. Longer rise and fall times push the second breakpoint lower in frequency, thereby reducing the high-frequency spectral content. Shorter rise and fall times push the second breakpoint higher in frequency, thereby increasing the high-frequency spectral content.

How do we quantitatively determine the *bandwidth* of a periodic clock waveform? Although the Fourier series in (1.1) requires that we sum an *infinite* number of terms, as a practical matter we use NH terms (harmonics) as an approximate finite-term approximation: $\tilde{x}(t) = c_0 + \sum_{n=1}^{\mathrm{NH}} c_n \cos(n\omega_0 t + \theta_n)$. The *pointwise approximation error* is $x(t) - \tilde{x}(t)$. The logical definition of

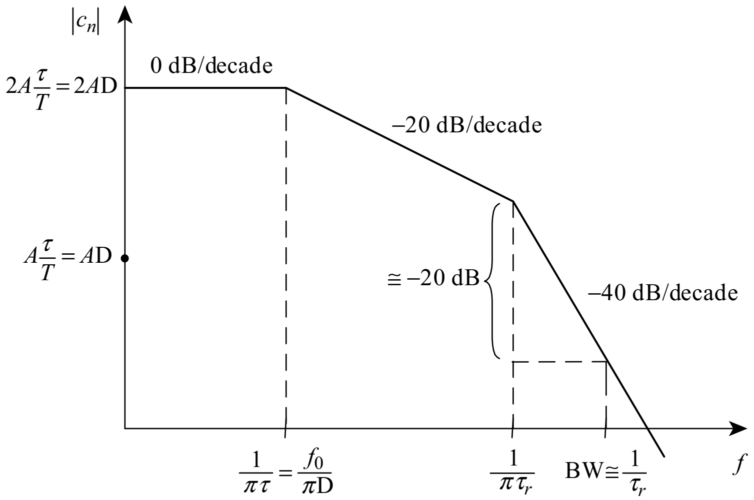


FIGURE 1.7. Bounds on the spectral coefficients of the trapezoidal pulse train for equal rise and fall times $\tau_r = \tau_f$.

the *bandwidth* (BW) of the waveform is that the BW should be the *significant spectral content of the waveform.* In other words,

> *the BW should be the minimum number of harmonic terms required to reconstruct the original periodic waveform such that adding more harmonics gives a negligible reduction in the pointwise error, whereas using less harmonics gives an excessive pointwise reconstruction error.*

If we look at the plot of the bounds on the magnitude spectrum shown in Fig. 1.7, we see that above the second breakpoint, $f_2 = 1/\pi\tau_r$, the levels of the harmonics are rolling off at a rate of -40 dB/decade. If we go past this second breakpoint by a factor of about 3 to a frequency that is the inverse of the rise and fall time, $f = 1/\tau_r$, the levels of the component at the second breakpoint will have been reduced further, by around 20 dB. Hence above this frequency the remaining frequency components are probably so small in magnitude that they do not provide any substantial contribution to the shape of the resulting waveform. Hence we might define the *bandwidth* of the trapezoidal clock waveform (and other data waveforms of similar shape) to be

$$\boxed{\mathrm{BW} \cong \frac{1}{\tau_r}} \tag{1.15}$$

TABLE 1.2. Spectral (Frequency) Components of a 5-V, 1-GHz, 50% Duty Cycle, 100-ps Rise/Fall Time Digital Clock Signal

Harmonic	Frequency (GHz)	Wavelength, λ (cm)	Level (V)	Angle (deg)
1	1	30	3.131	−108
3	3	10	0.9108	−144
5	5	6	0.4053	−180
7	7	4.29	0.1673	144
9	9	3.33	0.0387	108

The bandwidth in (1.15) obviously does not apply to a square wave, $\tau_r = \tau_f = 0$, since that would imply that its BW would be infinite. But an ideal square wave cannot be constructed in practice.

For a 1-GHz clock waveform having a 5-V amplitude, a 50% duty cycle, and 100-ps rise and fall times, the bandwidth by this criterion is 10 GHz. Table 1.2 shows the first nine coefficients for this digital waveform. Observe that the ninth harmonic of 9 GHz has a wavelength of 3.33 cm. Using Kirchhoff's voltage and current laws and lumped-circuit analysis principles to analyze a circuit driven by this frequency would require that the largest dimension of the circuit be less that 3.33 mm (0.131 in)! Similarly, to analyze a circuit that is driven by the fundamental frequency of 1 GHz whose wavelength is 30 cm using Kirchhoff's laws and lumped-circuit analysis methods would restrict the maximum circuit dimensions to being less than 3 cm or about 1 in (2.54 cm)! This shows that the use of lumped-circuit analysis methods to analyze a circuit having a physical dimension of, say, 1 in that is driven by this clock waveform would result in erroneous results for all but perhaps the fundamental frequency of the waveform! Figure 1.8 shows the bounds and envelope of the spectrum for this waveform. The first breakpoint of $f_1 = 1/\pi\tau = f_0/\pi D = 636.6$ MHz is not shown because it falls below the fundamental frequency of 1 GHz.

Figure 1.9(a) to (d) show the approximation to the clock waveform achieved by adding the dc component and the first three harmonics, the first five harmonics, the first seven harmonics, and the first nine harmonics, respectively. This increasing convergence of these partial sums to the true waveform supports the idea that using only the first 10 harmonic components as its BW, $\text{BW} = 1/\tau_r = 1/0.1$ ns $= 10$ GHz, gives a reasonable representation of the actual waveform.

This Fourier representation of a periodic waveform, such as a digital waveform, as a *summation* of single-frequency sinusoidal basic components as in (1.1) provides a useful and simple method for approximately solving a linear system indirectly. Consider the single-input, $x(t)$, single-output, $y(t)$,

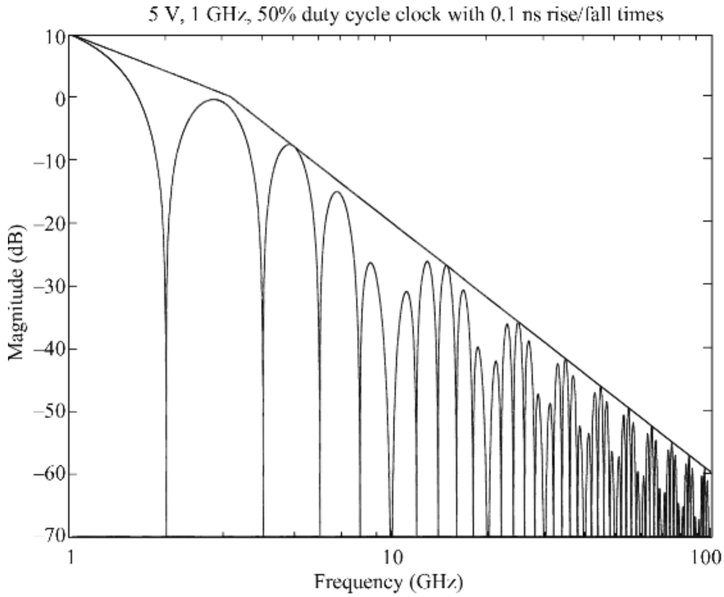


FIGURE 1.8. Plot of the spectrum of a 5-V, 1-GHz clock waveform having a 50% duty cycle and rise and fall times of 0.1 ns.

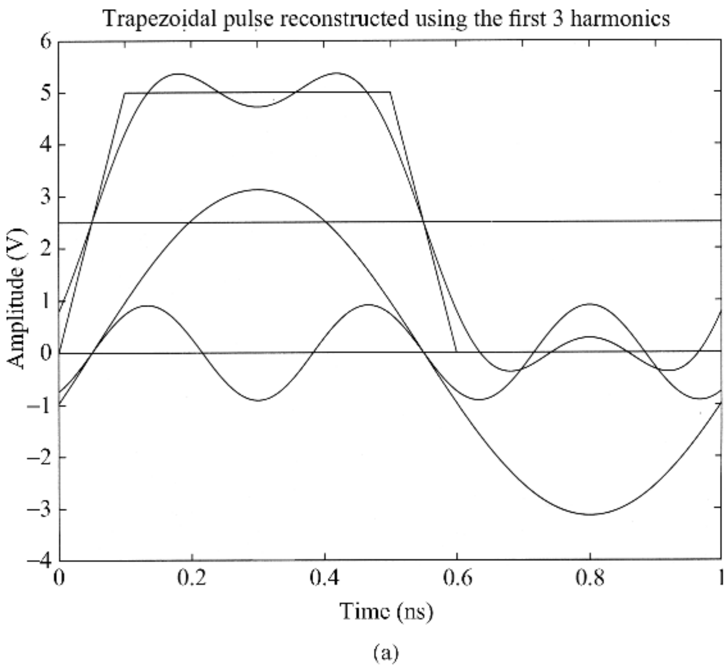
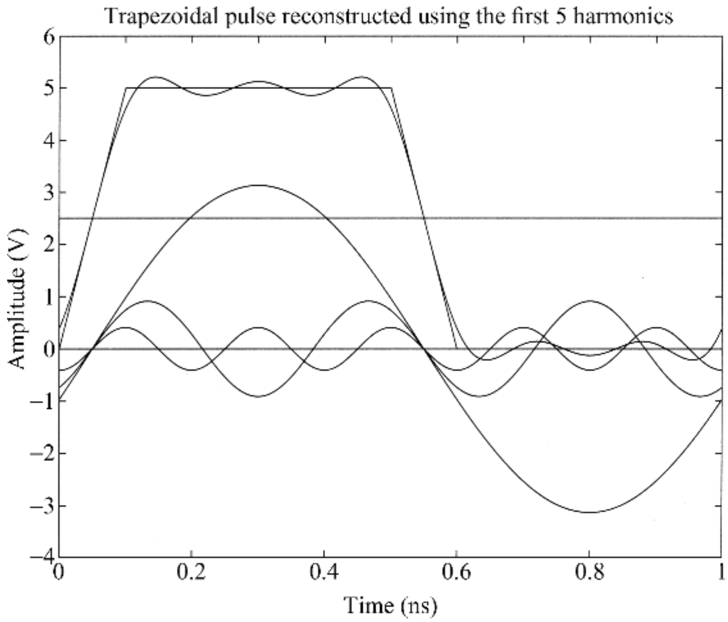
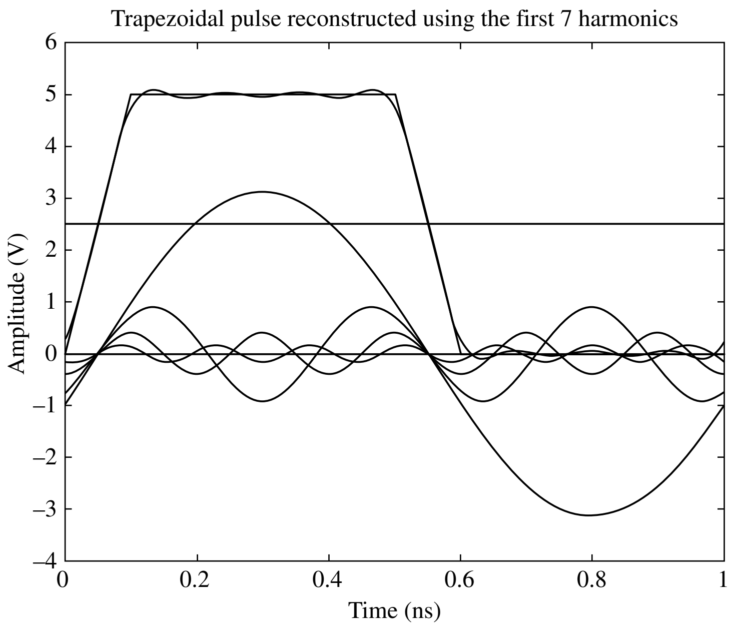


FIGURE 1.9. Approximating the clock waveform using (a) the first three harmonics, (b) the first five harmonics, (c) the first seven harmonics, and (d) the first nine harmonics.

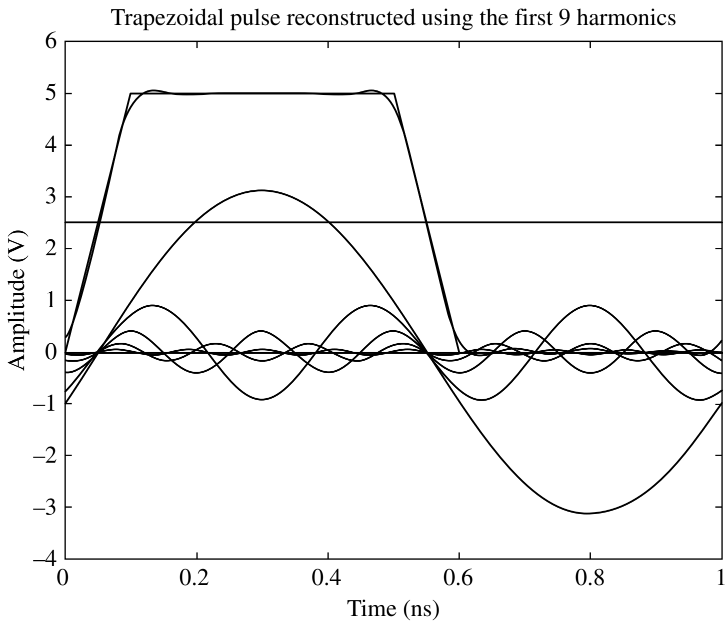


(b)



(c)

FIGURE 1.9. (*Continued*)



(d)

FIGURE 1.9. (*Continued*)

linear system illustrated in Fig. 1.10. A linear system is one for which the principle of *superposition* applies. In other words, the system is *linear* if $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, then (1) $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$ and (2) $kx(t) \rightarrow ky(t)$. The output is related to the input with a differential equation:

$$\frac{d^n y(t)}{dt} + a_1 \frac{d^{(n-1)} y(t)}{dt} + \cdots + a_n y(t) = b_0 \frac{d^m x(t)}{dt} + b_1 \frac{d^{(m-1)} x(t)}{dt} + \cdots + b_m x(t)$$

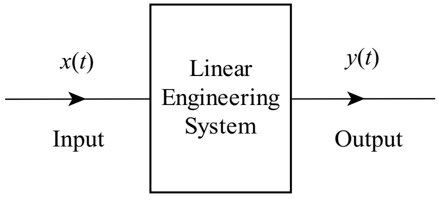


FIGURE 1.10. Single-input, single-output linear system.

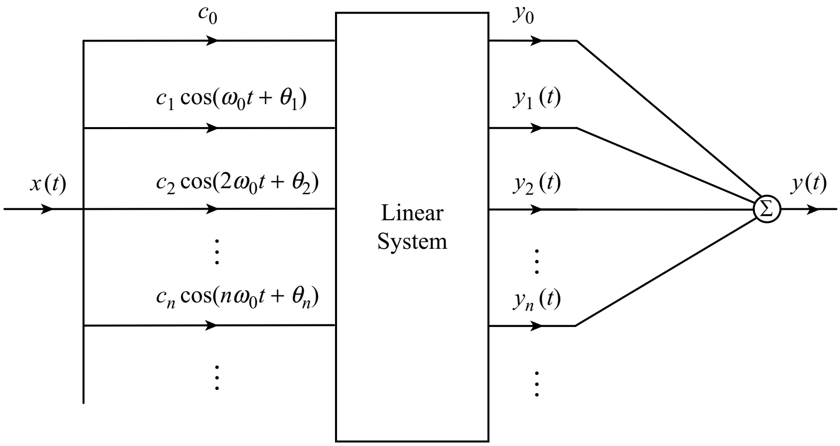


FIGURE 1.11. Using superposition to determine the (steady-state) response of a linear system to a waveform by passing the individual Fourier components through the system and summing their responses at the output.

The differential equation relating the input and output (sometimes referred to as the *transfer function*) can be solved for the waveform of the output, $y(t)$. But this can be a difficult and tedious task.

A simpler but approximate solution method is represented in Fig. 1.11. Decompose the input waveform, $x(t)$, into its Fourier components and pass each one through the system, giving a response to that component. Then sum all these responses in *time*, which gives an approximate solution to $y(t)$. This is, for several reasons a much simpler solution process than the direct solution of the differential equation relating the input and output. The basic functions in the Fourier series are the sinusoids: $c_n \cos(n\omega_0 t + \theta_n)$. It is usually much easier to determine the response to each of these sinusoids (referred to as the *frequency domain*). Then these responses are *summed in time* to give an approximation to the output, $y(t)$. An important restriction to this method is that it neglects any transient part of the solution and gives only the steady-state response.

As an example of this powerful technique, consider an RC circuit that is driven by a periodic square-wave voltage source as shown in Fig. 1.12. The square wave has an amplitude of 1 V, a period of 2 s, and a pulse width of 1 s (50% duty cycle). The RC circuit, which consists of the series connection of $R = 1\ \Omega$ and $C = 1$ F has a time constant of $RC = 1$s, and the voltage across the capacitor is the desired output voltage of this linear "system." The nodes of the circuit are numbered in preparation for using the SPICE circuit analysis program (or the personal computer version, PSPICE) to analyze it and plot the exact solution. The Fourier series of the input, $V_S(t)$, using only the first seven harmonics, is ($\omega_0 = 2\pi/T = \pi$)

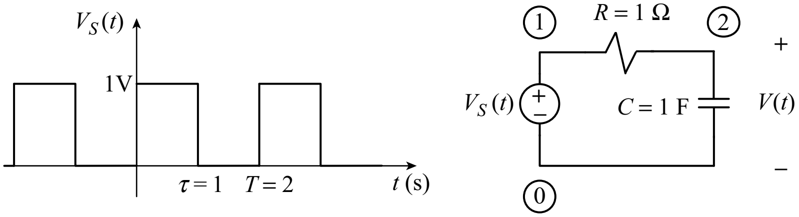


FIGURE 1.12. Example of using superposition of the Fourier components of a signal in obtaining the (steady-state) response to that signal.

$$
\begin{aligned}
V_S(t) &= c_0 + c_1\cos(\omega_0 t + \theta_1) + c_3\cos(3\omega_0 t + \theta_3) + c_5\cos(5\omega_0 t + \theta_5) \\
&\quad + c_7\cos(7\omega_0 t + \theta_7) \\
&= \frac{1}{2} + \frac{2}{\pi}\cos(\pi t - 90^\circ) + \frac{2}{3\pi}\cos(3\pi t - 90^\circ) + \frac{2}{5\pi}\cos(5\pi t - 90^\circ) \\
&\quad + \frac{2}{7\pi}\cos(7\pi t - 90^\circ)
\end{aligned}
$$

To determine the Fourier series of the output we first determine the response to a single-frequency input, $x(t) = c_n\cos(n\omega_0 t + \theta_n)$. The ratio of the output and this single-frequency sinusoidal input is referred to as the *transfer function* of the linear system response to this single-frequency input. The phasor (sinusoidal steady-state) *transfer function* of this linear system is the ratio of the output and input (magnitude and phase):

$$
\begin{aligned}
\hat{H}(jn\omega_0) &= \frac{\hat{V}}{\hat{V}_S} \\
&= \frac{1}{1 + jn\omega_0 RC} \\
&= \frac{1}{1 + jn\pi} \\
&= \frac{1}{\sqrt{1 + (n\pi)^2}} \angle - \tan^{-1}(n\pi) \\
&= H_n \angle \phi_n
\end{aligned}
$$

The phasor (sinusoidal steady state) voltages and currents will be denoted with carets and are complex valued, having a magnitude and an angle: $\hat{V} = V \angle \theta_V$ and $\hat{I} = I \angle \theta_I$. The output of this "linear system" is the voltage across the capacitor, $V(t)$, whose Fourier coefficients are obtained as $c_n H_n \angle(\theta_n + \phi_n)$

giving the Fourier series of the time-domain output waveform as

$$
\begin{aligned}
V(t) &= c_0 H_0 + \sum_{n=1}^{7} c_n H_n \cos[n\omega_0 t + \angle(\theta_n + \phi_n)] \\
&= 0.5 + 0.1931\cos(\pi t - 162.34^\circ) + 0.0224\cos(3\pi t - 173.94^\circ) \\
&\quad + 0.0081\cos(5\pi t - 176.36^\circ) + 0.0041\cos(7\pi t - 177.4^\circ)
\end{aligned}
$$

Figure 1.13 shows the approximation to the output waveform for $V(t)$ obtained by summing in time the steady-state responses to only the dc component and the first seven harmonics of $V_S(t)$.

The exact result for $V(t)$ is obtained with PSPICE and shown in Fig. 1.14. The PSPICE program is

```
EXAMPLE
VS 1 0 PULSE(0 1 0 0 0 1 2)
RS 1 2 1
C 2 0 1
.TRAN 0.01 10 0 0.01
.PRINT TRAN V(2)
.PROBE
.END
```

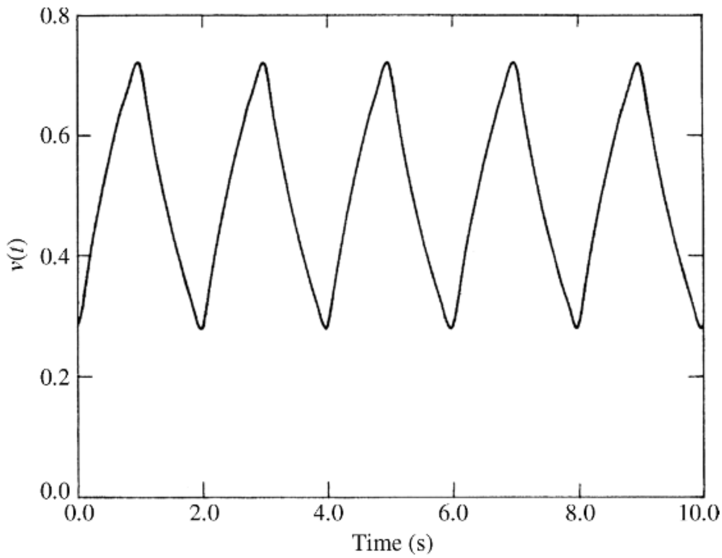


FIGURE 1.13. Voltage waveform across the capacitor of Fig. 1.12 obtained by adding the (steady-state) responses of the dc component and the first seven harmonics of the Fourier series of the square wave.

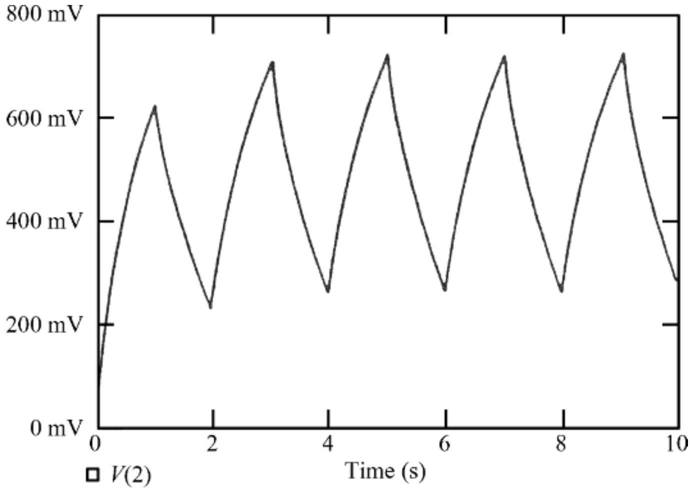


FIGURE 1.14. PSPICE solution for $V(t)$ for the circuit in Fig. 1.12.

Note that there is an initial transient part of the solution over the first 2 or 3 s due to the capacitor being charged up to its steady-state voltage. These results make sense because as the square wave transitions to 1 V, the voltage across the capacitor increases according to $1 - e^{-t/RC}$. Since the time constant is $RC = 1$ s, the voltage has not reached steady state (which requires about five time constants to have elapsed) when the square wave turns off at $t = 1$s. Then the capacitor voltage begins to discharge. But when the square wave turns on again at $t = 2$ s, the capacitor has not fully discharged and begins recharging. This process and the resulting output voltage waveform repeats with a period of 2 s. The transitions in the exact waveform of the output voltage in Fig. 1.14 are sharper than the corresponding transitions in the approximate waveform in Fig. 1.13 obtained by summing the responses to the first seven harmonics of the Fourier series of the input waveform. This is due to neglecting the responses to the high-frequency components of the input waveform and is a general property. The initial transient response in the exact PSPICE solution in Fig. 1.14 is absent from the Fourier method in Fig. 1.13 since the Fourier method only obtains the steady-state response.

1.3 THE BASIC TRANSMISSION-LINE PROBLEM

The basic transmission-line problem connects a *source* to a *load* with a *transmission line* as shown in Fig. 1.15(a). The transmission line consists of a parallel pair of conductors of total length $\mathscr{L}$ having uniform cross sections along its length. The objective will be to determine the time-

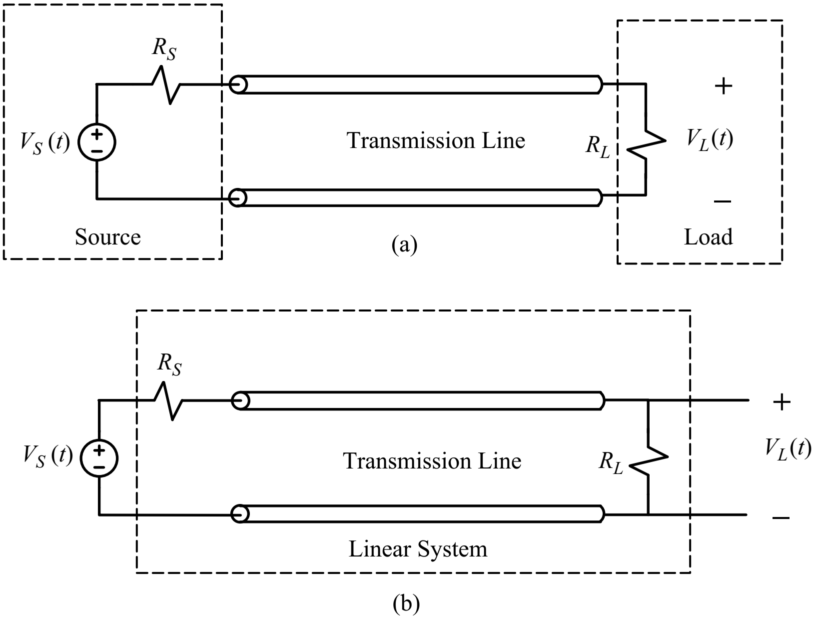


FIGURE 1.15. Basic transmission-line problem.

domain response waveform of the output voltage of the line, $V_L(t)$, given the termination impedances, R_S and R_L, the source voltage waveform, $V_S(t)$, and the properties of the transmission line. If the source and termination impedances are linear, we may alternatively view the transmission-line problem as a linear system having an input $V_S(t)$ and an output $V_L(t)$ by embedding the terminations and the transmission line into one system, as shown in Fig. 1.15(b).

We first determine the *frequency-domain* response of the system as shown in Fig. 1.16. A single-frequency sinusoidal source, $V_S(t) = V_S \cos(\omega t + \theta_S)$, produces a similar form of a sinusoidal load voltage: $V_L(t) = V_L \cos(\omega t + \theta_L)$.

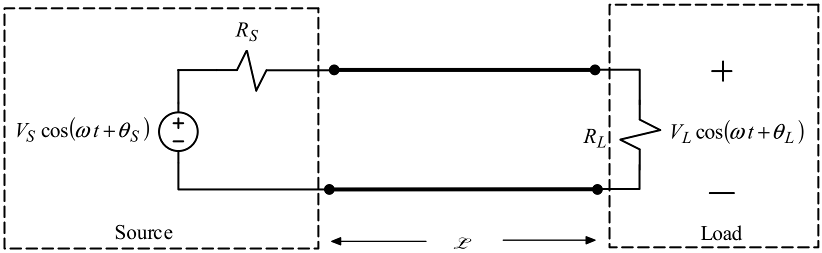


FIGURE 1.16. General source–load configuration.

The source and load are separated by a parallel pair of wires or a pair of lands of length $\mathscr{L}$. The lumped-circuit model *ignores* the two interconnect conductors of length $\mathscr{L}$. Analyzing this configuration as a lumped circuit gives (using voltage division and ignoring the interconnect conductors) the ratio of the source and load voltage magnitudes as

$$\frac{V_L}{V_S} = \frac{R_L}{R_S + R_L}$$

and the phase angles are identical: $\theta_S = \theta_L$. These, according to a lumped-circuit model of the line, remain the same for *all source frequencies*!

Consider the specific configuration shown in Fig. 1.17. The parameters are $R_S = 10\,\Omega$ and $R_L = 1000\,\Omega$ for a line of total length of $\mathscr{L} = 0.3$ m (or about 12 in). Ignoring the effects of the interconnect conductors gives $V_L/V_S = 0.99$, and the phases are related as $\theta_L - \theta_S = 0^{\circ}$. The *exact solution* is obtained by including the two interconnect conductors of length $\mathscr{L}$ as a distributed-parameter *transmission line*. The circuit analysis computer program, PSPICE, contains an exact transmission-line model of the interconnect conductors. Figure 1.18 shows the *exact* ratio of the voltage magnitude, V_L/V_S, and voltage angle, $\theta_L - \theta_S$, versus the frequency of the source as it is swept in frequency from 1 MHz to 1 GHz. Model the interconnect conductors as a distributed-parameter transmission line having a characteristic impedance of $Z_C = 50\,\Omega$ and a one-way delay of the interconnect line of

$$T_D = \frac{\mathscr{L} = 0.3 \text{ m}}{v_0 = 3 \times 10^8 \text{ m/s}} = 1 \text{ ns}$$

The entire configuration is analyzed using PSPICE. The AC mode of analysis in PSPICE is used to obtain the *frequency-domain* transfer function of this system. The PSPICE program is

```
EXAMPLE
VS 1 0 AC 1 0
RS 1 2 10
T 2 0 3 0 Z0=50 TD=1N
RL 3 0 1K
.AC DEC 50 1MEG 1G
.PRINT AC VM(1) VP(1) VM(3) VP(3)
.PROBE
.END
```

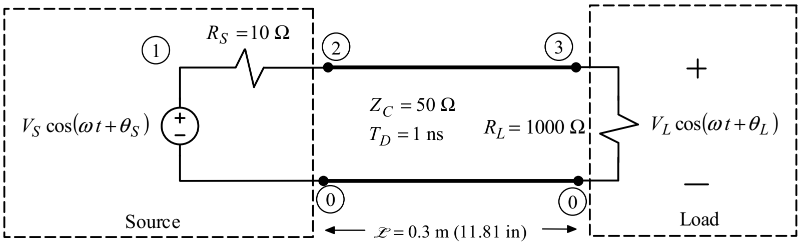


FIGURE 1.17. Specific example treating the connection lands as a transmission line.

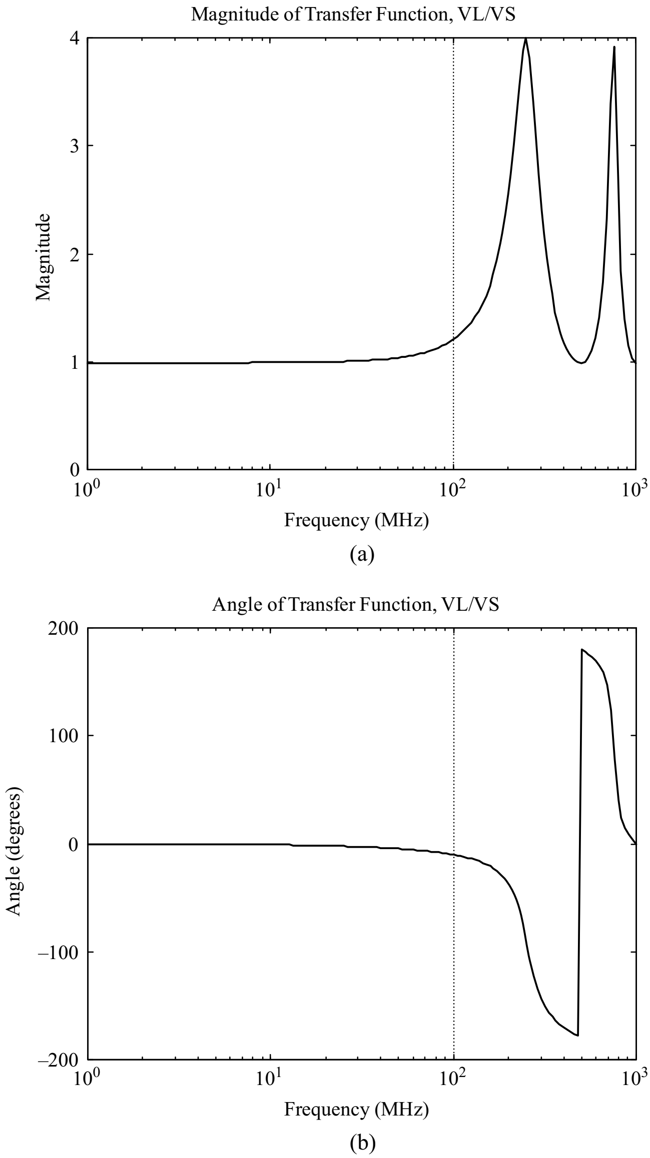


FIGURE 1.18. Frequency response of the line in Fig. 1.17.

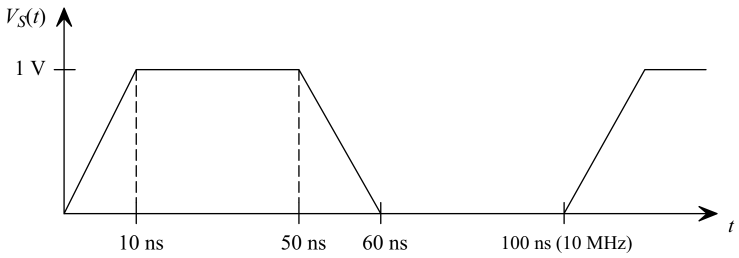


FIGURE 1.19. Source voltage.

Figure 1.18 shows that the magnitudes and angles of the *transfer function* voltages, V_L/V_S and $\theta_L - \theta_S$, begin to deviate rather drastically from the low-frequency lumped-circuit analysis result of $V_L/V_S = 0.99$ and $\theta_L - \theta_S = 0^\circ$ above about 100 MHz. The line is one-tenth of a wavelength (electrically short) at

$$f|_{\mathscr{L}=(1/10)\lambda} = \frac{1}{10T_D = 10\,\text{ns}} = 100\,\text{MHz}$$

(denoted by the vertical line at 100 MHz in both plots). This is evident in the plots in Fig. 1.18. Hence the interconnect line is electrically long above 100 MHz. The interconnect line is one wavelength at 1 GHz:

$$f|_{\mathscr{L}=\lambda} = \frac{1}{T_D = 1\,\text{ns}} = 1\,\text{GHz}$$

Observe that the magnitude plot in Fig. 1.18(a) shows two peaks of 250 MHz and 750 MHz where the interconnect line electrical length is $\lambda/4$ and $\frac{3}{4}\lambda$, respectively, and the magnitude of the transfer function increases to a level of 4. There are two minima at 500 MHz and 1 GHz,where the interconnect line electrical length is $\lambda/2$ and λ, respectively. Above 1 GHz (the last frequency plotted) the pattern replicates, which is a general property of transmission lines.

Finally, we investigate the *time-domain* response of the line where we drive the line with a clock signal of 10 MHz fundamental frequency (a period of 100 ns), an amplitude of 1 V, rise and fall times of 10 ns, and a 50% duty cycle as shown in Fig. 1.19. It is typical for the rise and fall times of digital waveforms to be chosen to be around 10% of the period T in order to give adequate "setup" and "hold" times. The exact *time-domain* load voltage waveform, $V_L(t)$, is obtained with the .TRAN module of PSPICE for the waveform in Fig. 1.19. The PSPICE program used is

```
EXAMPLE
VS 1 0 PULSE(0 1 0 10N 10N 40N 100N)
RS 1 2 10
T 2 0 3 0 Z0=50 TD=1N
RL 3 0 1K
.TRAN 0.1N 200N 0 0.1N
.PRINT TRAN V(1) V (3)
.PROBE
.END
```

Figure 1.20 shows a comparison of the load voltage waveform, $V_L(t)$, and the source voltage waveform, $V_S(t)$, for this source waveform over two cycles of the source. The source voltage and load voltage waveforms are virtually identical, and the interconnect line clearly has no substantial effect. From the frequency response of the waveform in Fig. 1.18, we see that the first 10 harmonics of this waveform (the bandwidth of the waveform is $\mathrm{BW} = 1/\tau_r = 100\,\mathrm{MHz}$)—10, 20, 30, 40, 50, 60, 70, 80, 90, 100 MHz—all fall below the frequency where the line ceases to be electrically short: 100 MHz. This is what we expect when the major harmonic components of the waveform (its BW) fall into the frequency range where the line is electrically short for all of them.

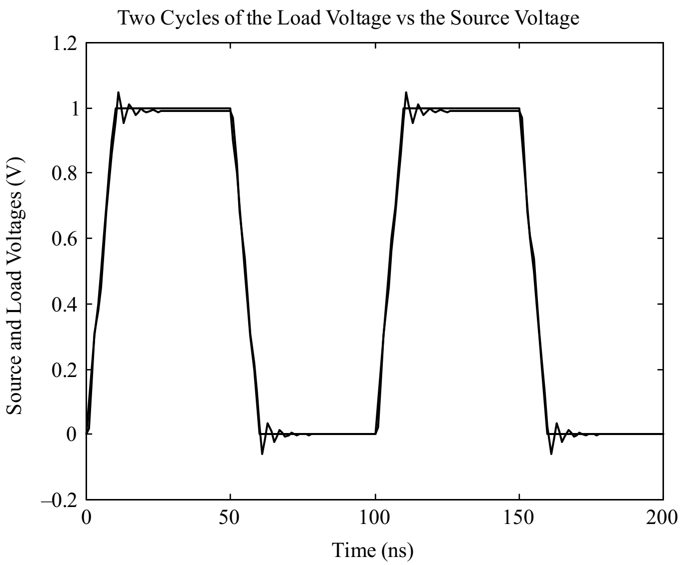


FIGURE 1.20. Comparison of the source and load waveforms for a 1-V, 10-MHz waveform with rise and fall times of $\tau_r = \tau_f = 10\,\mathrm{ns}$ and a 50% duty cycle (see Fig. 1.19).

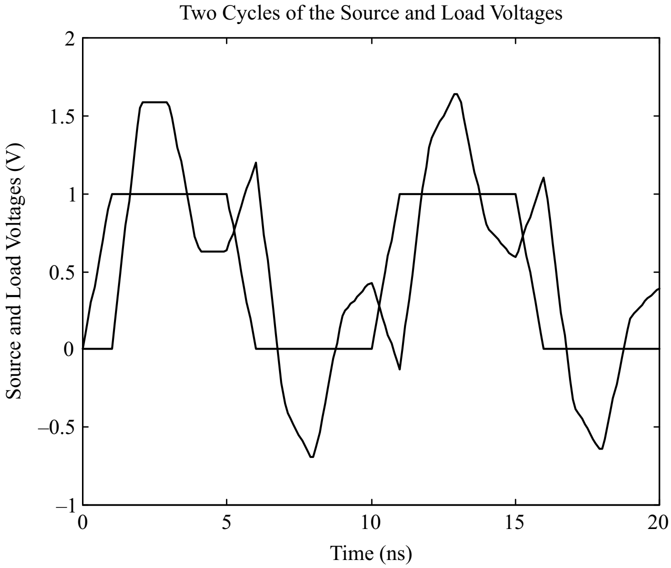


FIGURE 1.21. Comparison of the source and load waveforms for a 1-V, 100-MHz waveform with rise and fall times of $\tau_r = \tau_f = 1$ ns and a 50% duty cycle.

Figure 1.21 shows the same comparison when the source parameters are changed to a 100-MHz fundamental frequency (a period of 10 ns) having an amplitude of 1 V, rise and fall times of 1 ns, and a 50% duty cycle. The PSPICE program is changed slightly just by changing the parameters of the "PULSE" function and the ".TRAN" lines to

```
EXAMPLE
VS 1 0 PULSE(0 1 0 1N 1N 4N 10N)
RS 1 2 10
T 2 0 3 0 Z0=50 TD=1N
RL 3 0 1K
.TRAN 0.01N 20N 0 0.01N
.PRINT TRAN V(1) V (3)
.PROBE
.END
```

From Fig. 1.18, this waveform contains the first 10 harmonics that constitute the major components in its bandwidth (BW $= 1/\tau_r = 1$ GHz): 100 MHz, 200 MHz, 300 MHz, 400 MHz, 500 MHz, 600 MHz, 700 MHz, 800 MHz, 900 MHz, and 1 GHz. The line length is $\lambda/10$ at its fundamental frequency, 100 MHz, and 1λ at its tenth harmonic of 1 GHz. Observe that the load voltage waveform bears no resemblance to the source waveform. From the frequency response of the system in Figure 1.18, we see that all of these harmonics fall in the frequency range where the interconnect line is *electrically long* ($>$100 MHz), so this is expected.

This has shown that as the frequencies of the sources increase to the point where the interconnect lines connecting the source and the load become *electrically long*, the standard lumped-circuit models are no longer valid and give erroneous answers. The requirement to model electrically long interconnects requires that we master transmission-line modeling.