CHAPTER 1

Noise and Frequency Stability

Frequency stability or instability is a very important parameter in both modern terrestrial and space communications, in high-performance computers, in GPS (global positional system), and many other digital systems. In this connection, even very small frequency or phase changes of steering frequency generators (exciting oscillators, clock generators, frequency synthesizers, amplifiers, etc.) are of fundamental importance. Since all physical processes are subject to some sort of uncertainties due to fluctuations of individual internal or external parameters, generally designated as noise, the investigation of the overall noise properties is of the highest importance for the analysis of frequency stability.

In practice, we encounter three fundamental types of noises that differ by the power in the time or frequency unit $S(f)$ (generally in the 1 Hz bandwidth), the latter being called the power spectral density (PSD—see Fig. 1.1).

There are three major types of noises:

1. **White noise** with a constant PSD: $S(f) \sim$ const.
2. **Flicker noise** or $1/f$ noise with the PSD $S(f) \sim 1/f^{\alpha}$, where the power α is very close to one.
3. **Random walk** or **Brownian motion** (often as the Wiener–Levy process) with $S(f) \sim f^{-2}$.

Frequency Stability. By Venceslav F. Kroupa

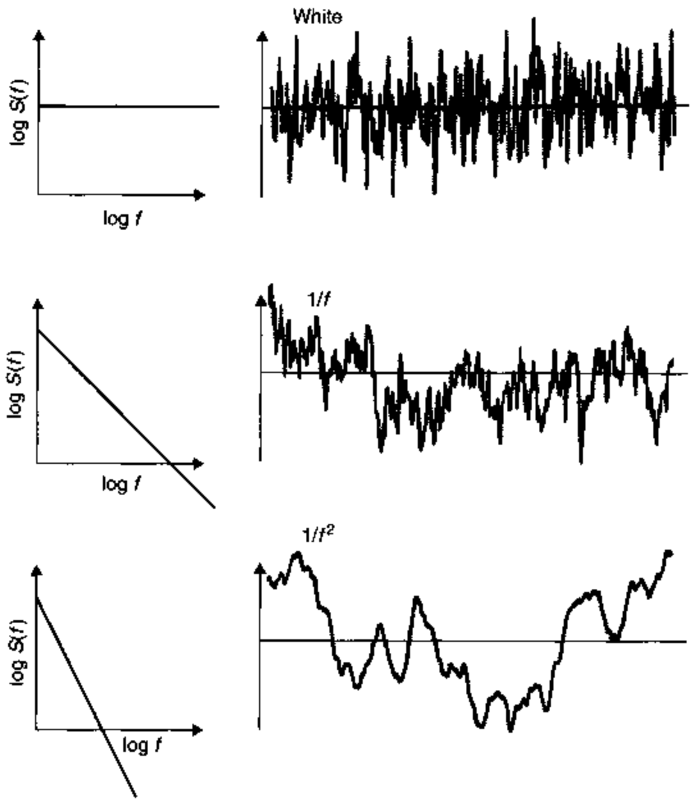


Fig. 1.1 Fundamental types of noises [1.1, 1.2]. (Copyright © IEEE. Reprinted with permission.)

The last two noise processes with their integrals [generating PSD $S(f)$ proportional to $\sim f^3$, $\sim f^4$, etc.] are often called *colored noises.*

1.1 WHITE NOISE

Typical representation of white noise consists of *black body radiation* or the thermal noise of resistors, or shot noise, in electronic devices.

1.1.1 Thermal Noise

In 1928, Johnson [1.3] and Nyquist [1.4] published a theory explaining the existence of thermal noise in conductors. It is caused by short current pulses generated by collisions of a large number of electrons. The result is such that a noiseless conductor is connected in series with a generator with a root mean square (rms) noise voltage, e_n (see Fig. 1.2)

$$e_n^2 = 4kTR\Delta f \qquad 1.1$$

where k is the Boltzmann constant, T the absolute temperature (see Table 1.1), R is the resistance of the conductor (Ω), and Δf is the frequency bandwidth (in Hz) used for the appreciation of the noise action.

After dividing (1.1) by the frequency bandwidth Δf, we arrive at the PSD in 1-Hz bandwidth, that is,

$$S_{e,n}(f) = 4kTR \qquad (\mathrm{V^2/Hz}) \qquad 1.2$$

Similarly, with the assistance of the Thevenin theorem we get the noise current, i_n, flowing into the resistance R or the conductivity $G = 1/R$, that is,

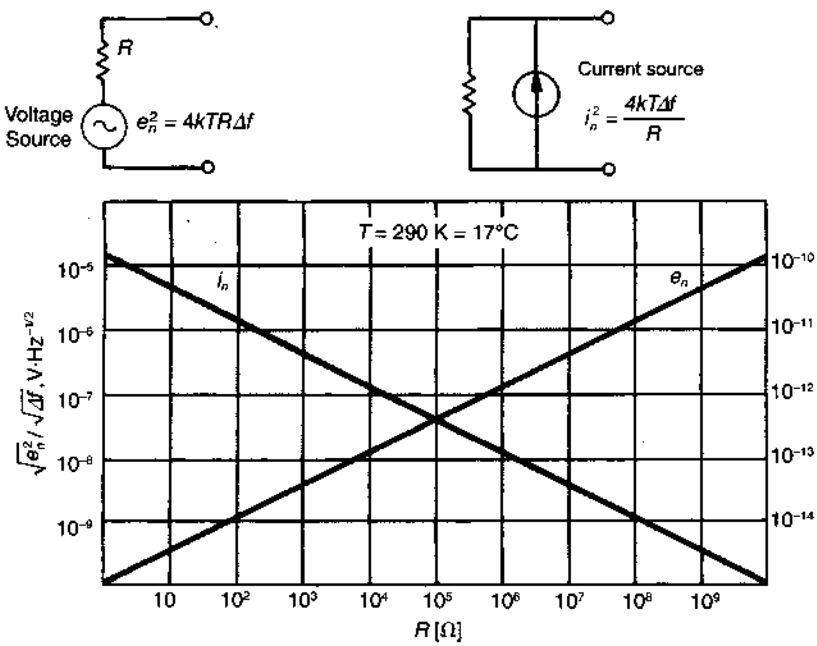


Fig. 1.2 Thermal noises of conductors [1.2]. (Copyright © IEEE. Reprinted with permission.)

Table 1.1 Several physical constants

Physical constants	Symbols	Numerical Values
Planck constant	h	6.625×10^{-35} (Js)
Boltzmann constant	k	1.380×10^{-23} (J/K)
Electron charge	q	1.6×10^{-19} (C)
Speed of light	c	299,792,458 (m/s)
Noise voltage	$4kTR$; $T = 296$ K, $R = 1\ \Omega$	$10^{-19.8}$ (V^2, rms)
Noise voltage	$4kTR$; $T = 296$ K, $R = 50\ \Omega$	$10^{-18.1}$ (V^2, rms)

$$S_{i,n}(f) = 4kTG \qquad (\mathrm{A^2/Hz}) \qquad 1.3$$

(See Fig. 1.2.) In instances with a general impedance or admittance, we introduce only the real parts into the above equations. Further, since both PSDs (1.2) and (1.3) are constant in a very large frequency bandwidth (with no filter at the output), we call this type of noise *white* (in accordance with optical physics). By considering the noise power in a frequency range $\Delta f = f_{\mathrm{high}} - f_{\mathrm{low}}$, we get

$$P_n = \int_{f_l}^{f_h} \frac{S_{e,n}(f)}{R} df = 4kT(f_{\mathrm{high}} - f_{\mathrm{low}}) \approx 4kTf_h \qquad (\mathrm{Ws}) \qquad 1.4$$

However, by increasing the upper bound f_{high} above all limits the noise power P_n would also increase above all limits. But this is not possible and the correct solution is provided by quantum mechanics, which changes noise PSD for extremely high frequencies into relation (1.5), where h is the Planck constant, $h = 6.625 \times 10^{-35}$ (Js).

EXAMPLE 1.1

Compute the thermal noise generated in the 1 Hz bandwidth in the $R = 50\text{-}\Omega$ resistor placed at room temperature

$$P_{n,f} = \frac{4hf}{e^{hf/kT} - 1} \qquad (\mathrm{Ws}) \qquad 1.5$$

$k = 1.38 \times 10^{-23}$ (J/K)
$T = 296$ (K)
$R = 50$ (Ω)
$\langle e_n^2 \rangle = 4 \times 1.38 \times 10^{-23} \times 296 \times 50 = 8.17 \times 10^{-19} = 10^{-18.1}$ (V^2)
or $e_n \approx 1.10^{-9}$ mV

1.1.2 Shot Noise

In all cases where the output current is composed of random arrivals of a large number of particles, we again witness fluctuations of the white noise type [1.5].

By considering an idealized transition in Fig. 1.3, where electrons flow randomly from A to B and holes flow from B to A, in a negligible transit time, each particle arrival is connected with transport of a current pulse. Consequently, in a time unit τ (s) the number of n charges generates the current

$$i = \frac{q}{\tau} n \qquad 1.6$$

where q is the electron or hole charge, $q = 1.6 \times 10^{-19}$ (C). It was shown earlier [1.5] that the probability of the transition of the charge carriers was subjected to the Poisson distribution (see also Section 1.4.2.3):

$$p(n) = \frac{<n>^n}{n!} e^{-<n>} \qquad 1.7$$

where $<n>$ is the mean value of the number of carriers in the time unit. In such cases, the variance is equal to

$$\sigma^2(n) = <n> \qquad 1.8$$

By reverting to (1.6), we get for the mean current

$$I = \frac{q}{\tau} <n> \qquad 1.9$$

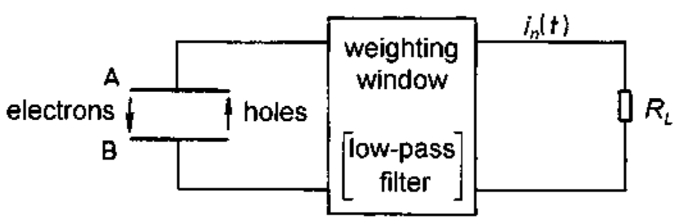


Fig. 1.3 Circuit model for the shot noise [1.2]. (Copyright © IEEE. Reprinted with permission.)

and for its variance value,

$$\sigma^2(I) = \frac{q^2}{\tau^2} < n > = \frac{q}{\tau} I \qquad 1.10$$

To arrive at the PSD, we use a bit heuristic approach with the assistance of the autocorrelation [cf. (1.94)]

$$S_{i,n} = 2\int_0^\infty \sigma^2 \cos(\omega t)dt \approx 2\int_0^\tau \sigma^2(i)\cos(\omega t)dt =$$
$$2\sigma^2(i)\frac{\sin \omega t}{\omega} /_0^\tau \approx 2\frac{q}{\tau} I \frac{\omega\tau}{\omega} = 2qI \qquad 1.11$$

EXAMPLE 1.2

Find the PSD $S_{i,n}(f)$ of the shot noise for the transistor current

$I = 1$ mA $\qquad S_{i,n} = 2 \times 1.6 \times 10^{-19} \times 10^{-3} = 3.2 \times 10^{-22}$

For PSD of other currents, see Fig. 1.4.

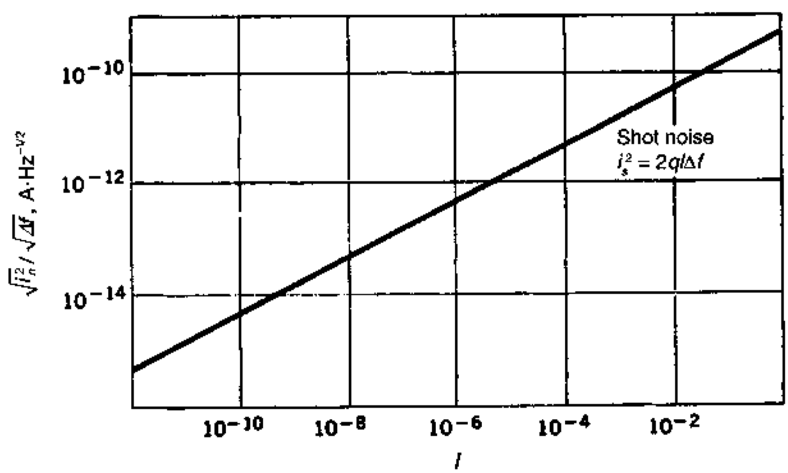


Fig. 1.4 Noise current through a semiconductor junction [1.2]. (Copyright © IEEE. Reprinted with permission.)

1.2 COLORED NOISES

Until now we have discussed PSD of noises generated by slow time-independent fluctuations, that is, with constant PSD over a large Fourier frequency range. However, with oscillators and other frequency generators, we encounter phase fluctuations with frequency-dependent PSDs proportional to $1/f$, $1/f^2$, or even to $1/f^3$, $1/f^4$, at very low Fourier frequencies that are often called *colored noises*.

In the mid-1920s, Johnson [1.3] found that at very low frequencies the shot noise in vacuum tubes did not follow white noise at low frequencies and he introduced for the additive noise the name flicker noise. This name is still used. Subsequent observations proved the $1/f$ law for a much larger set of physical phenomena on one hand and its validity at very low frequencies on the other hand. Some years later, Bernamont [1.6] suggested a law for its PSD:

$$S_n(f) \approx \frac{1}{f^{\alpha}} \qquad 1.12$$

where the power of α was in the vicinity of *one*. In electronic devices, the higher order noises are often generated by integration in the corresponding Fourier transform division by s (cf. Table 1.2). The only exception presents $1/f$ noise fluctuations encountered both in crystal resonators and oscillators, and in many other physical systems [1.7] (dispersions of cars on highways [1.8], frequency change around 50 or 60 Hz in power line systems [1.9], or even flooding in the Nile river valley [1.1]; the latter reference is based on the time dependence of generating fluctuations). Note that all are based on the time.

The problem of colored noises was investigated by many authors in the past from different points of approach and often with different results; particularly, with the ever-present $1/f$ noise. For example, Keshner [1.10] investigated noises with different slopes, $1/f^{\alpha}$, and arrived at a number of variables needed for generation of the desired colored noise (cf. Fig. 1.5). His finding for $1/f$ noise is one degree of freedom per decade.

1.2.1 Mathematical Models of $1/f^{\alpha}$ Processes

Characterization of the frequency stability of all types of generators, inclusive of phase-locked loops (PLLs), is important for applications;

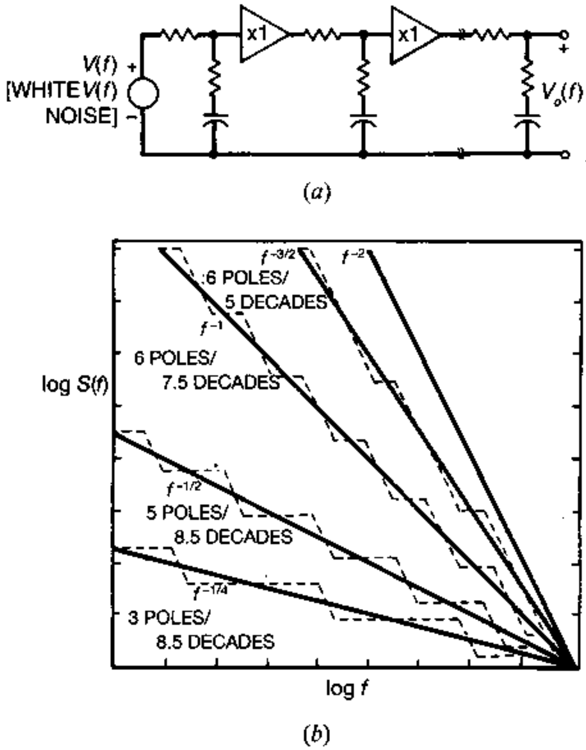


Fig. 1.5 (*a*) A linear system yielding $1/f$ noise (approximately) (each section has a one-state variable, which is the capacitor voltage). (*b*) A curve fit of the power spectral densities of an approximating linear system to obtain $1/f^{\alpha}$ (α = 0.25, 0.5, 1, 1.5, 2) [Adapted from 1.10.]

in the first place for their designers, and vice versa for users. In the mid-1960s, theoretical principles of the phase noise theory in frequency generators were established [1.11] and later a number of practical papers were published (e.g. [1.12, 1.13]). Here, we will briefly recall the corresponding theory.

Solution of the noise problems is performed with the assistance of statistics by investigating correlations and by the transformation of the time domain processes into the (complex) frequency domain via the Laplace transform (cf. Appendix at the end of this chapter).

$$\mathcal{L}(f(t)) = \int_0^{\infty} f(t)e^{-st}dt \qquad 1.13$$

In instances where the lower bound of the above integral is $-\infty$ the Laplace transform changes into the Fourier transform, the corresponding pairs for important time functions encountered in practice are summarized in Table 1.2.

Reverting to the investigated time function, $f(t)$ in (1.13), the process can be represented as a power series:

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots; \qquad \lim_{n\to\infty}(a_n t^n) = \text{const} \qquad 1.14$$

By retaining only the first two terms, we arrive at the exponential approximation that represents a large set of actual situations of the time domain fluctuations,

$$f(t) \approx a_0 - a_1 + n(t) \approx a_o e^{(-a_1/a_o)t} = a_o e^{-at} \qquad 1.15$$

with the respective Fourier transform (cf. Table 1.2),

$$F(s) = \frac{a_o}{s+a} \qquad 1.16$$

After multiplication with the complex conjugate of $F(s)$, we arrive at the so-called Lorenzian PSD (cf. Section 1.5.1, Brownian Motion):

$$S(f) = \frac{a_o^2}{f^2 + a^2} \qquad 1.17$$

Table 1.2 Fourier and Laplace transform pairs for important time functions, encountered in practice

Type	$f(t)$	$F(s)$	Process
Unit step	$u(t)$	$1/s$ $[F(s)/s]$	Integration
Ramp	t	$1/s^2$	Aging
Differentiation	$df(t)/dt$	$sF(s)$	
Time delay	$f(t-\tau)$	$F(s)e^{-s\tau}$	
	$1/\sqrt{\pi t}$	$1/\sqrt{s}$	
	$2/\sqrt{t/\pi}$	$s^{-3/2}$	
	t^{k-1}	$\Gamma(k)/s^k$ $(k>0)$	$\Gamma(k) = (k-1)!$
	e^{-at}	$1/(a+s)$	Exponential decay
	te^{-at}	$1/(a+s)^2$	Exponential decay with aging
	$\sin(at)$	$a/(a^2+s^2)$	

1.2.2 1/f Noise (Flicker Noise)

To generate the PSD of the $1/f$ slope, so often observed in practice, we encounter a large number of approaches. McWorther [1.14] suggested the mathematical model (for the flicker noise generated in semiconductors) as a multistep process composed of single events (cf. Fig. 1.6a):

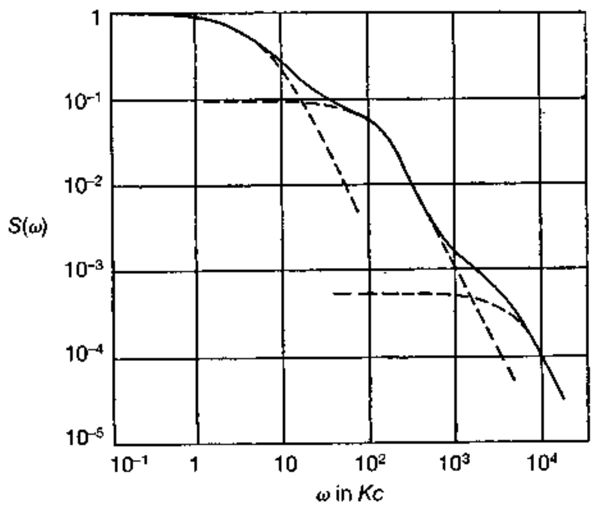
$$\sum_{\tau_i} \frac{\tau_i}{1+(2\pi f \tau_i)^2} \quad 1.18$$

By assuming that in the time domain we have a set of events of the type in equation (1.15), the PSD will retain the shape as in equation (1.17) as long as the time constants, a, do not change appreciably from one. The final amplitude of the PSD is then still a_o^2 at low Fourier frequencies. To arrive at the flicker noise behavior, we start with inspection of the PSD, $S(f)$, in (1.17) and find that in the neighborhood of the corner frequency, $2\pi f \approx 1/\tau$, its slope is approximately proportional to $1/f$. Evidently, by proportionally increasing the time constant and decreasing the amplitude in the corresponding series,

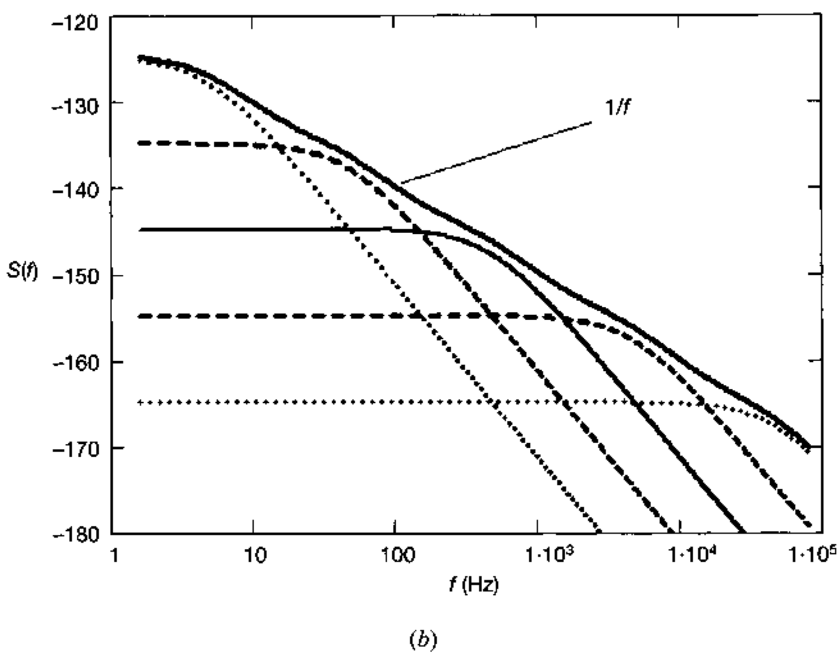
$$S(f) = \sum_{\tau_i} \frac{a_o^2 / \tau_i}{(2\pi f + 1/\tau_i)^2} \quad 1.19$$

The summation reveals a slope of $1/f$ (see the example in Fig. 1.6*b*), where we have chosen $\tau_i/\tau_{i+1} \approx 10$ and arrived at a nearly perfect slope of $1/f$. This finding is in a good agreement with a discussion by Keshner [1.10]. However, note a rather forceful, not random, condition on the amplitudes and time constants in the set of the Lorentzian noise characteristics (1.19) needed for the generation of the flicker noise system. The difficulty is that this is true for voltage or current fluctuations (e.g., [1.15, 1.16]), however, in instances of other physical quantities (transfer of power, flow of cars on a highway, etc.) the rms of (1.19) must be used. Effectively, we face a fractional integration discussed in connection with the $1/f$ fluctuations by Halford [1.17] or suggested by Radeka [1.18].

In another approach, let us again consider a flow (e.g., of power) with losses during defined time periods (cf. Fig. 1.7). In that case, we introduce a sampling process with the dissipated energy, P_{diss}, during



(*a*)



(*b*)

Fig. 1.6 (a) Flicker phase noise generated by a set of several $1/f^2$ noises. Their summation (the solid line) presents the ideal slope $1/f$ [1.14]. (*b*) The simulated slope $1/f$, with five $1/f^2$ noise characteristics providing the background set.

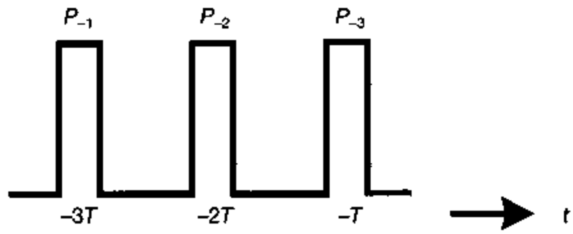


Fig. 1.7 Generation of the phase noise in the sampled form reduced to a set of pulses.

one sampling period T_o. In the next period, we encounter nearly the same energy losses, and so on. Generalization reveals a sampling process whose noise model in the z-transform is (cf. [1.19, 1.20])

$$P_{\text{noise}}(z) = \frac{1}{P_o}\left(P_{\text{diss},0} + P_{\text{diss},1}z^{-1} + P_{\text{diss},2}z^{-2} + P_{\text{diss},3}z^{-3} + \cdots\right) \approx \frac{P_{\text{diss}}}{P_o} \cdot \frac{1}{1-z^{-1}} \tag{1.20}$$

where P_o is the energy of the flux. To get the corresponding Fourier transform, we have to replace z^{-1} with e^{-sT_o} and multiply by the transfer function $H(s)$

$$H(s) = \frac{1-e^{-sT_o}}{s} \tag{1.21}$$

with the result

$$P_{\text{noise}}(s) = \frac{P_{\text{diss}}}{P_o} \cdot \frac{1}{s} \tag{1.22}$$

However, to get the PSD of the noise power we must apply on the complex product of $P_{\text{noise}} \times P^*_{\text{noise}}$ the rms operation and thus we arrive at the $1/f$ slope (i.e., at the slope of PSD 10 dB/dec in 1-Hz bandwidth):

$$S_{\text{noise}}(f) = \frac{P_{\text{diss}}}{P_o} \cdot \frac{1}{f} \tag{1.23}$$

Where $P_{diss,i}$ are losses in individual periods and P_o the overall power in the steady-state flow.

In this connection, we recall the paper by Kasdin and Walter [1.19] who suggested the sampling process for generation of the flicker noise. By assuming the memory system shown schematically in Fig. 1.8 and the corresponding z-transform, they assumed both $X(z)$ and $Y(z)$ to be energy during one sampling period. After very complicated computations, they arrived at the desired $1/f$ slope and found the approach acceptable from the stochastic point of view but had to apply the fractional integration (i.e., to arrive at the PSD, application of the rms operation on the respective transfer function).

Finally, we have to mention Hooge's formula presented 1969 [1.21], since it was intensively studied, for relation of the power spectral density of current or resistance fluctuations:

$$\frac{S_I(f)}{I^2} = \frac{S_{\Delta R}(f)}{R^2} = \text{const}\,\frac{1}{f} \qquad 1.24$$

(see Section 1.5.2.2.)

1.2.3 $1/f^2$, $1/f^3$, and $1/f^4$ Noises

The PSD of the first type of noises, also designated as the so-called random walk, is generated by the randomly distributed pulses of the type (1.15). With the assistance of (1.17) we get for the system of n pulses the PSD:

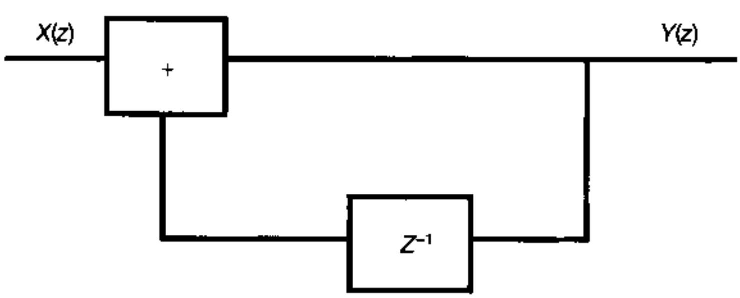


Fig. 1.8 Basic block diagram of the z-transform approach for generation of $1/f$ noise.

$$S(f) = \sum_n \frac{a_o^2}{f^2 + a_n^2} \qquad 1.25$$

Note that the above PSD changes into a pure $1/f^2$ spectrum for high Fourier frequencies ($f > a_n$). To this class of noise generators, the stochastic Wiener–Levy process (Sec. 1.5.2) may be enclosed. Another origin of the $1/f^2$ noise is integration of white noise (e.g., in oscillators) and due to the Laplace transform the process as realized by dividing by s (cf. Table 1.2).

EXAMPLE 1.3

In oscillators, the thermal white noise originating in the maintaining electronics generates frequency fluctuations with PSD, as in (1.2):

$$S_\varphi(f) \approx \frac{4kTR}{V_o^2} = \frac{2kT}{P_o} \qquad 1.26$$

However, the oscillating condition requires that the phase around the loop is equal to $2\pi n$ ($n = 1, 2, \ldots$). But this condition is connected with integration (i.e., division by s in the Fourier transform), which changes the white phase noise inside the resonance range into the random walk with the PSD (cf. Chapter 2, Sec. 2.1.3):

$$S_\varphi(f) = \frac{2kT}{P_o f^2} \qquad 1.27$$

Similarly, the higher order noises ($1/f$ and $1/f^2$) in the oscillator maintaining electronics generate the phase noise with PSD inversely proportional to f^{-3} or f^{-4} due to the integration process.

1.3 SMALL AND BAND LIMITED PERTURBATIONS OF SINUSOIDAL SIGNALS

Till now, we have considered single-frequency generators, that is, oscillators with rather small and continuous amplitude and phase perturbations. However, in frequency synthesizers, particularly *direct digital synthesizers* (DDS), we encounter many spurious signals. In the following sections, we will investigate some of their properties [1.2, 1.22].

1.3.1 Superposition of One Large and a Set of Small Signals

In actual frequency synthesizers, PLL systems not excluded, we always encounter many generally small spurious signals accompanying the carrier. The composite signal may be written as

$$v(t) = \sum_{n=1}^{N} V_n \cos(\omega_n t + \phi_n) \tag{1.28}$$

After introducing

$$\omega_n t + \phi_n = \omega_1 + (\omega_n - \omega_1)t + \phi_n = \omega_1 t + \Phi_n(t) \tag{1.29}$$

and after putting

$$\alpha_n = \frac{V_n}{V_1} \tag{1.30}$$

we get

$$\begin{aligned} v(t) &= V_1 \sum_{n=1}^{N} \alpha_n \cos[\omega_1 t + \Phi_n(t)] = \\ & V_1 \cos(\omega_1 t) \sum_{n=1}^{N} \alpha_n \cos[\Phi_n(t)] - V_1 \sin(\omega_1 t) \sum_{n=1}^{N} \alpha_n \sin[\Phi_n(t)] \end{aligned} \tag{1.31}$$

or

$$v(t) = V_1 \alpha(t) \cos[\omega_1 t + \Phi(t)] \tag{1.32}$$

where $\alpha(t)$ is the normalized instantaneous amplitude

$$\alpha^2(t) = \left[\sum_{n=1}^{N} \alpha_n \cos \Phi_n(t) \right]^2 + \left[\sum_{n=1}^{N} \alpha_n \sin \Phi_n(t) \right]^2 \tag{1.33}$$

and $\Phi(t)$ is the instantaneous phase departure

$$\Phi(t) = \arctan \frac{\sum_{n=1}^{N} \alpha_n \sin \Phi_n(t)}{\sum_{n=1}^{N} \alpha_n \cos \Phi_n(t)} \qquad 1.34$$

Without any loss of generality, it is possible to choose the time scale in such a way that

$$\phi_1 = \Phi_1(t) = 0 \qquad 1.35$$

If only small perturbations are assumed one may put

$$\alpha_1 \gg \alpha_n; \ (n = 2,3,\ldots,N) \ \ (\text{note } \alpha_1 = 1) \qquad 1.36$$

With this situation, the spurious amplitude is

$$\alpha(t) \approx 1 + \sum_{n=1}^{N} \alpha_n \cos \Phi_n(t) \qquad 1.37$$

and the spurious phase is

$$\Phi(t) = \arctan \sum_{n=1}^{N} \alpha_n \sin \Phi_n(t) \approx \sum_{n=1}^{N} \alpha_n \sin \Phi_n(t) \qquad 1.38$$

EXAMPLE 1.4

For the superposition of one strong signal $V_1 \cos(\omega_1 t)$ and one weak signal $V_2 \cos(\omega_2 t + \varphi_2)$ (i.e., $V_1 \gg V_2$) we get

$$v(t) \approx V_1 \left[1 + \frac{V_2}{V_1} \cos(\Omega t + \phi_2)\right] \cos\left[\omega_1 t + \frac{V_2}{V_1} \sin(\Omega t + \phi_2)\right] \quad 1.39$$

Evidently, in the first approximation we face a simultaneous amplitude and phase modulation of the stronger signal at the rate of difference frequency, $\Omega = |\omega_2 - \omega_1|$, with the modulation indexes V_2/V_1. In the case of its larger value, higher order terms must be added.

1.3.2 Narrow Bandwidth Noise

In instances where the noise power is concentrated in a relatively narrow band around the frequency ω_1, the noise voltage can be expressed as

$$e(t) = e_c(t)\cos\omega_1 t - e_s(t)\sin\omega_1 t \qquad 1.40$$

where the slowly varying time functions $e_c(t)$ and $e_s(t)$ are statistically independent. Note that (1.40) resembles (1.31). Consequently, the product of the mean values is zero if $<e(t)>$ is zero, that is,

$$< e_c(t)e_s(t) > = < e_c(t) >< e_s(t) > = 0 \qquad 1.41$$

and also

$$< e_c(t) > = < e_s(t) > = 0 \text{ and } < e^2(t) > = < e_c^2(t) > = < e_s^2(t) > \qquad 1.42$$

1.4 STATISTICAL APPROACH

In the previous sections, frequency stability was discussed from the point of view of common noises. However, the problem is much more complicated since actual situations may contain both continuous and sampled systems, both random and discrete fluctuations, and even other processes solved with the advantage of statistical approaches whose basic properties are discussed briefly in the following sections.

1.4.1 Probability

When inspecting physical, biological, economical, and many other processes, we find either a deterministic model or start from experimental observations and guess the details. One of the tools we use is the appreciation of the outcome with the assistance of probability theory, which is the ratio of the positive outcomes of an experiment n_p to all trials n_m:

$$P = \frac{n_p}{n_m} \qquad 1.43$$

The corresponding theory was well established in the past (e.g., [1.22]). Here, we recall the three axioms, namely, that the result is always a positive number between zero and one, that the probability of mutually independent events is equal to the sum of individual probabilities, and that the probability of the whole set of events is equal to 1 (cf. Fig. 1.9)

$$P(e_i < x) \geq 0; \; P(x = S) = 1$$
$$P(A_x + B_x) = P(A_x) + P(B_x) - P(A_x)P(B_x) \qquad 1.44$$

In addition, conditional probability of mutually independent events is equal to their product

$$P(A_x \mid B_x) = P(A_x) \cdot P(B_x) \qquad 1.45$$

Note that all operations are performed on sets subjected to the Boole summations and multiplications.

1.4.2 Random Variables, Distribution Function, Density of Probability

Let us assume an experiment E with events e_i identified by real or complex numbers, $\xi(e_i)$, which will be designated as *random variables*. Next, we define the probability of a set of events meeting condition

$$\xi(e_i) \leq x \quad P(\xi \leq x) = F_\xi(x) \qquad 1.46$$

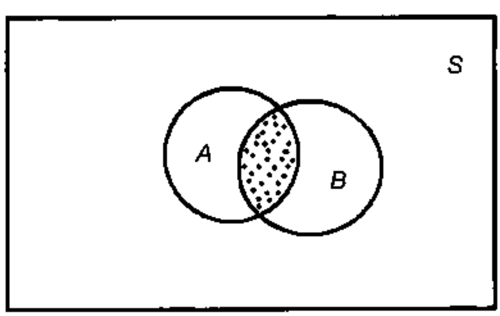


Fig. 1.9 Idealization of the probability space $S = 1$ of all the events e_n. A represents the set of the e_{A_i} events and B the set of the e_{B_i} events.

where $F_\xi(x)$ defines the *distribution of the probability,* which is a monotonic nondeclining function from 0 to 1. A typical behavior is presented in Fig. 1.10 for both continuous and discrete variables.

By reverting to the continuous variable, we define its derivative and designate it as the *probability density function* or simply *probability density f(x)* or *p(x)*,

$$f(x) = \frac{\delta F(x)}{\delta x} \qquad 1.47$$

if this derivative exists. Further, the probability for x between a and b is

$$P(a \leq x \leq b) = F(b) - F(a) \qquad 1.48$$

The mean or the expected value (the moment of the first order) is

$$E(x) = \mu(x) = \int_{-\infty}^{\infty} x f(x) dx \quad \text{or} \quad E(x) = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad 1.49$$

Generally, the mean values of the nth order of random variables are designated as the nth-order moments:

$$m_{n\xi} = <\xi^n> = \int_{-\infty}^{\infty} x^n f(x) dx \qquad 1.50$$

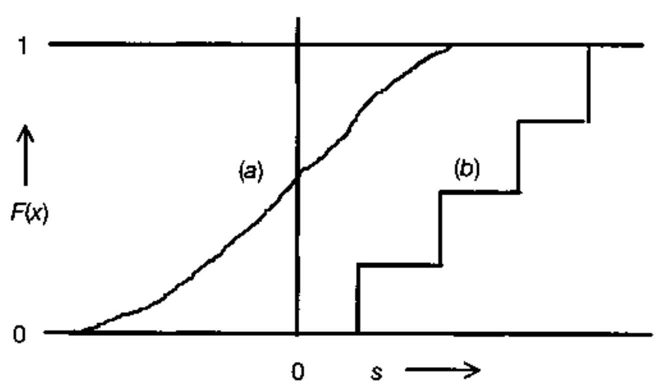


Fig. 1.10 Probability distribution function: (*a*) a continuous variable and (*b*) a discrete variable.

Central moments are important:

$$\mu_n = < (\xi - m_{n,\xi})^n > = \int_{-\infty}^{\infty} (x - m_{n,\xi})^n f(x) dx \qquad 1.51$$

The central moment of the second order defines the *variance,* σ^2, or distribution and the corresponding rms, σ, the so-called dispersion:

$$\mu_2 = \sigma^2 = m_{2\xi} - m_{1\xi}^2 \qquad 1.52$$

1.4.2.1 The Uniform Distribution

The simplest *probability density function f(x)* is assumed to be constant between a and b on the x axis and zero otherwise (cf. Fig. 1.11):

$$f(x) = \frac{1}{b-a} \quad \text{and} \quad 0 \ \text{elswere} \qquad 1.53$$

The corresponding mean and variance are

$$\mu = \frac{b-a}{2} \qquad \sigma^2 = \frac{1}{12}(b-a)^2 \qquad 1.54$$

1.4.2.2 Binomial Distribution

We face a discrete distribution and the task is to compute the probability of the event ξ in n trials. Let p be the probability of the positive

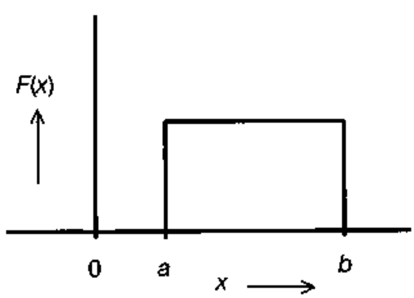


Fig. 1.11 The probability density function $f(x)$ of the uniform distribution.

outcome and $q = 1 - p$ the probability of the opposite. The probability of the occurrence of the event ξ occurring only once in a sequence of trials is

$$P_n(\xi)' = p \cdot q^{n-1} \qquad 1.55$$

In instances where the sequence is unimportant, the probability is

$$P_n(\xi) = n \cdot p \cdot q^{n-1} \qquad 1.56$$

Finally, if the event ξ should repeat k *times* the probability is

$$P_{n,k}(\xi) = \binom{n}{k} p^k q^{n-k} \qquad 1.57$$

The first and second moments are computed in Section 1.4.3.1:

$$\mu = pn \qquad \sigma^2 = pn \qquad 1.58$$

1.4.2.3 Poisson Distribution

For a very large number of trials, $n \to \infty$, and, $p \to 0$, the binomial distribution passes into the Poisson distribution [1.22 p. 72] from (1.57):

$$\begin{aligned} p_{n,k}(\xi) = \binom{n}{k} p^k q^{n-k} \approx \frac{n!}{k!(n-k)!} p^k q^n \approx \frac{(np)^k}{k!}(1 - pn) \approx \\ \frac{(np)^k}{k!} e^{\ pn} = \frac{\lambda^k}{k!} e^{-\lambda} \end{aligned} \qquad 1.59$$

The mean and variance are the same as above, namely, equal to $\lambda = pn$ (cf. shot noise).

1.4.2.4 Gaussian Distribution

Another limiting process of the binomial distribution for large n results in the *Gaussian* or *normal distribution*. Computation of the probability density is a cumbersome one. The asymptotic solution is based on introduction of a new variable, $k = np + x$, and application of the Stirling approximation for factorials, and after approximating powers of binomials close to one,

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1+\frac{1}{12n}\right) \qquad (1+\alpha)^\beta \approx e^{\beta lg(1+\alpha)} \approx e^{\beta\alpha(1-\alpha/2)} \qquad 1.60$$

After introduction of these approximations into (1.59), we finally arrive at the *density function:*

$$p_{n,k} \approx \frac{(pn)^{(pn+x)}}{(pn+x)^{(pn+x)} e^{-(pn+x)}\sqrt{2\pi(pn+x)}} e^{-pn}$$

$$\approx \frac{1}{\left(1+\frac{x}{pn}\right)^{(pn+x)} e^{-x}\sqrt{2\pi(pn+x)}} \qquad 1.61$$

$$\approx \frac{1}{e^{[(pn+x)\frac{x}{pn}\left(1-\frac{1}{2pn}\right)-x]} * \sqrt{2\pi(pn+x)}} \approx \frac{e^{\frac{x^2}{2pn}}}{\sqrt{2\pi pn}} = \frac{e^{-\frac{(x-x_o)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} = f(x)$$

and after integration we arrive at the Gaussian *distribution function F(x):*

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{(x-x_o)^2/(2\sigma^2)}dx \quad x_o = \mu \qquad 1.62$$

where σ^2 is the variance, σ is the dispersion, and μ is the mean value of the process. Note that this distribution is also designated as the *normal distribution* and is, by far, the most important. It is often postulated in concrete situations when solving actual probability problems. Numerical values of $F(x)$ are published in tables or on-spot computed. The normalized distribution function $F(x)$ for $\sigma = 1$ and $\mu = 0$ is depicted in Fig. 1.12. Note that there are other distributions; however, we feel that those mentioned here are sufficient for information needed in this book. The sample distributions χ^2 is discussed in Chapter 5, Sec. 5.2.4.

EXAMPLE 1.5

Determination of the density function of $f_y\ (y) = g(x)$, where x is randomly distributed in the interval $(-\pi,\pi)$. Papoulis [1.22] pos-

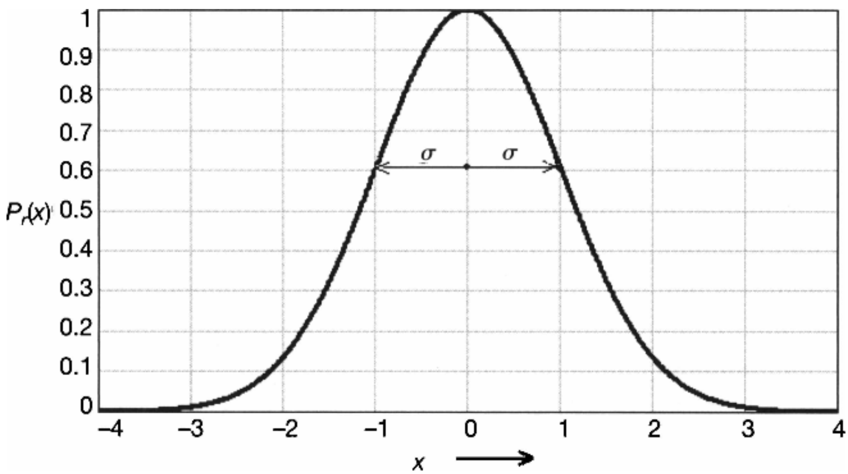


Fig. 1.12 The normal distribution for $\sigma = 1$ and $\mu = 0$.

tulated the fundamental theorem: If x_1, x_2, ..., x_n, ... are all real roots of $f_y(y) = g(x)$, then the density function is

$$f_y(y) = \frac{f_x(x_1)}{g'(x_1)} + \cdots + \frac{f_x(x_n)}{g'(x_n)} + \cdots + \qquad g'(x) = \frac{dg(x)}{dx} \qquad 1.63$$

For an important case (cf. Fig. 5.16*b*),

$$y = \sin(x+\theta) \qquad f_y(y) = \frac{2}{2\pi\sqrt{1-y^2}} \qquad 1.64$$

1.4.3 Characteristic Functions

The Fourier transform of the probability density $f_\xi(x)$ is the so-called characteristic function of the random variable ξ:

$$\Phi_\xi(u) = \langle e^{ju\xi} \rangle = \int_{-\infty}^{\infty} f_\xi(x) e^{jux} dx \qquad 1.65$$

We have seen that the introduction of the moments simplified some conclusions and certain computations encountered in the probability applications. Similarly, adoption of the characteristic function and its logarithm simplifies some statements about moments. Expansion of the e-funtion in (1.65) in series reveals

$$\Phi_\xi(u) = \int_{-\infty}^{\infty} f_\xi(x)\left[1 + jux + \cdots + \frac{(jux)^k}{k!} + \cdots +\right]dx =$$
$$1 + ju<\xi> - \frac{u^2<\xi^2>}{2} - \cdots + \frac{(ju)^k}{k!}<\xi^k> + \cdots \quad 1.66$$

Note that the characteristic function of the random variable ξ is easily evaluated from the knowledge of moments, the first and second order often suffice.

1.4.3.1 Characteristic Function of the Binomial Distribution

For a large n, the integration in (1.65) can be approximated with the summation of the binomial series:

$$\Phi_\xi(u) \approx \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{juk} = (pe^{ju} + q)^n \quad 1.67$$

The first derivation reveals the mean value (the first moment):

$$m_{1,\xi} = -jn(pe^{ju} + q)^{n-1}(pe^{ju}j)/_{u=0} = np = <\xi> \quad 1.68$$

Similarly, the second moment and the variance are:

$$m_{2,\xi} = <\xi^2> = np(np+1) \qquad \sigma^2 = <\xi^2> - <\xi>^2 = np \quad 1.69$$

1.4.3.2 Characteristic Function of the Gaussian Distribution

After introducing the probability density function of the Gaussian distribution $f(x)$ (1.62) into the characteristic function definition (1.65), we have

$$\Phi(u) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-x_o)^2/(2\sigma^2)} e^{jxu} dx \approx$$
$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-x_o)^2/(2\sigma^2)} \cos(jxu) dx \qquad 1.70$$

For the evaluation, we used the property that $f(x)$ is an even function, that is, $f(-x) = f(x)$, and that it is concentrated around $x = 0$. With the assistance of the Dwight formula 861.20 [1.23] we arrive at

$$\Phi(u) = \frac{1}{2} e^{-u^2\sigma^2/2 + jux_o} \qquad 1.71$$

1.4.3.3 Characteristic Function of the Sum of Distributions

The Fourier transform of the probability density simplifies some solutions, for example, of sums of random variables:

$$\eta = a \cdot \xi + b$$
$$\Phi_\eta(u) = \langle e^{ju\eta} \rangle = \langle e^{ju(a\xi+b)} \rangle = e^{jub}\Phi_\xi(au) \qquad 1.72$$

or generally for

$$\zeta = \xi_1 + \xi_2 + \cdots + \xi_n$$
$$\Phi_\zeta(u) = \Phi_{\xi,1}(u)\Phi_{\xi,2}(u)\ldots\Phi_{\xi,n}(u) \qquad 1.73$$

Note, that limitation of the characteristic function to the two first moments reveals

$$\Phi_\zeta(u) \approx \left(1 + \sum_i^n ju\xi_i\right) - \frac{u^2}{2}\sum_i^n \langle \xi_i^2 \rangle + \text{higher order terms}) \qquad 1.74$$

from which it follows that the mean value is the sum of the individual mean values and the variance is the sum of individual variances. Now, let us calculate the distribution for n-Gaussian probability distributions:

$$F(x) = \frac{1}{2\pi} \int_{-\infty} e^{-(u^2/2)(\sigma_1^2+\sigma_2^2+\ldots+\sigma_n^2)+ju(x_{o1}+x_{o2}+\ldots+x_{on})} du =$$

$$\frac{1}{\sigma\sqrt{2\ \pi}} e^{-(x-x_o)^2/(2\sigma)^2} \quad \left[x_O = \sum_i x_{Oi} \quad \sigma^2 = \sum_i \sigma_{Oi}^2 \right] \qquad 1.75$$

After comparing (1.74) with (1.75), we conclude that the sum of partial distributions, irrespective of the type, finally results in the Gaussian or normal probability distribution. The process is often designated as the central limit theorem.

1.4.4 Stochastic Processes

In instances where the evaluated system is accompanied with a lot of disturbing signals subjected, in addition, to time fluctuations, it is labeled as a stochastic process encountered in different fields of engineering. The situation is generally so complicated that it is difficult to solve in the closed form; the proper situation is with frequency stability of oscillators, frequency synthesizers, communications channels, and so on. Now, let us assume an experiment formed by a set of e_i events with assigned time functions $x_i(t)$; in such a case, we face a class forming a stochastic process $\boldsymbol{x(t)}$, where time t may have any value, either continuous or discrete. In instances where the time is fixed, $t = t_i$, then $x_i(t)$ is a random variable of the event e_i.

1.4.4.1 Distribution Functions and Probability Density

Similarly, as with the time-independent system, the probability distribution function is

$$F((x_1 \ldots x_n; t_1 \ldots t_n)) = P[x(t_1) \le x_1 \ldots x(t_n) \le x_n] \qquad 1.76$$

with the probability density

$$f(x_1 \ldots x_n; t_1 \ldots t_n) = \frac{\delta^n F(x_1 \ldots x_n; t_1 \ldots t_n t)}{\delta x_1 \ldots \delta x_n} \qquad 1.77$$

It is evident that the above equations are not suitable for solving stochastic processes. The solution provides either an analytical description, where the parameter is the random variable as, for example,

$$y(t) = A\sin(\omega_d t + \xi) \qquad 1.78$$

or we must content ourselves with partial information about the investigated stochastic process, such as the knowledge of the moments. Generally, the mean or the expected value which, however, remains a function of time is

$$m_{1x}(t) = \langle x(t) \rangle = \int_{-\infty}^{\infty} x f(x,t)dx \qquad 1.79$$

Similarly, the autocorrelation is the moment of the second order of the random variables $x(t_1)$ and $x(t_2)$, that is,

$$R(t_1,t_2) = \langle x(t_1)x(t_2) \geq \int_{-\infty}^{\infty} x_1 x_2 f(x_1,x_2;t_1,t_2)dx_1 dx_2 \qquad 1.80$$

Furthermore, the autocovariance is

$$C(t_1,t_2) = \langle [x(t_1) - \eta(t_1)][x(t_2) - \eta(t_2)] \rangle \qquad 1.81$$

and the variance of the random variable (r.v.) $x(t)$ is given by

$$\sigma^2_{x(t)} = C(t,t) = R(t,t) - \eta^2(t,t) \qquad 1.82$$

Autocorrelation or, eventually, autocovariance characterizes the statistical relationship of both random variables $x(t_1)$ and $x(t_2)$ for any times t_1 and t_2.

1.4.4.2 Stationary Stochastic Processes

Very important are stochastic processes that are invariant with respect to shift on the time axis and are designated as stationary in the strict sense. In this case, the probability density for any time shift δ is

$$f(x_1 \ldots x_n; t_1 + \delta \ldots t_n + \delta) = f(x_1 \ldots x_n; t_1 \ldots t_n) \qquad 1.83$$

When choosing $\delta = -t_1$ we find out that the *probability density* is a *constant,* and consequently the mean or expected value is also a constant

$$< x(t) > = E[x(t)] = m_{x,1} = \text{const} \quad 1.84$$

However, the second-order moment is a function of the time difference:

$$< [x(t_1)x(t_2)] > = < [x(t)x(t+\tau)] > = R_{xx}(\tau) \quad 1.85$$

In instances where $x(t)$ is real, the autocorrelation is also real and in addition is an even function:

$$R_{xx}(\tau) = R_{xx}(-\tau) \quad 1.86$$

and the autocovariance is

$$C_{xx}(\tau) = R_{xx}(\tau) - m_{x,1}^2 \quad 1.87$$

Furthermore, it follows that autocorrelation of a sum

$$z(t) = x(t) + y(t) \quad 1.88$$

may be expressed as a sum of autocorrelations:

$$R_{zz} = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) \quad 1.89$$

and for any time shift τ, the probability density is constant. However, the autocorrelation of a product

$$w(t) = x(t)y(t) \quad 1.90$$

generally cannot be expressed as a function of the second-order moments. Only in instances where both time processes are independent is the autocorrelation equal to the product of partial autocorrelations

$$R_{ww}(\tau) = R_{xx}(\tau) \cdot R_{yy}(\tau) < R_{ww}(0) \quad 1.91$$

Finally, we conclude with the fact that usually we do not know all the information needed for stationary processes (cf. 1.83). In such cases, we must be content with the statement that the process is stationary in the wide sense or weakly stationary.

1.4.4.3 Random Walk

The random walk is the sampling process taking equal steps either in the positive or negative sense (direction). By taking advantage of the central limit theorem, we may assume that the corresponding probability density of individual steps is

$$p(x) = xe^{-x^2/\sigma_i^2} \tag{1.92}$$

The corresponding variance after n steps is (Dwight [1.23], Eq. 860.12)

$$\sigma^2 = \sum_{i=1}^{n} \int_{-\infty}^{\infty} x\, e^{-x^2/\sigma_i^2} dx = n\sigma_i^2 \tag{1.93}$$

1.4.5 Ergodicity

Ergodicity deals with problems of determining the statistics of the process $\boldsymbol{x(t)}$: The process is ergodic in the most general form if all its statistics can be determined from a single function $x(t, \xi)$ of the process, or the process is ergodic if the time averages equal ensemble averages. The various criteria for ergodicity are discussed in detail by Papoulis in [1.22].

1.5 POWER SPECTRA OF STOCHASTIC PROCESSES

Power spectra or spectral density of the process $\boldsymbol{x(t)}$ is the Fourier transformation of its autocorrelation:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau \tag{1.94}$$

with the inversion formula

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} d\omega \tag{1.95}$$

and with the results for real processes

$$S(-\omega) = S(\omega)$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\cos(\omega\tau)d\tau \qquad R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos(\omega\tau)d\omega \tag{1.96}$$

Table 1.3 Shows the correspondence between a process $\boldsymbol{x(t)}$, its autocorrelation $R(\tau)$, and the power spectrum $S(\omega)$.

1.5.1 Brownian Motion

The random movement of particles immersed in liquids is referred to as Brownian motion. The first observations (1827) were provided with mechanical particles, however, later studies proved a more general process. Let us start with the velocity of a free particle in a viscous medium. With the assistance of the laws of motion, we arrive at the following differential equation:

$$m\frac{dv(t)}{dt} + bv(t) = B(t) \equiv m \cdot n(t) \tag{1.97}$$

where m is its mass, b is the friction force proportional to the velocity $v(t)$, and $B(t)$ represents the collision force [1.22]. In cases where the observation time is long, one may assume that $v(t)$, $B(t)$, and $n(t)$ are

Table 1.3 Correspondence between a process $x(t)$, its autocorrelation $(R)t$, and power spectrum $S(\omega)$ [1.22]

$x(t)$	$R(\tau)$	$S(\omega)$
$ax(t)$	$\lvert a\rvert^2 R(\tau)$	$\lvert a\rvert^2 S(\omega)$
$\frac{dx(t)}{dt}$	$-\frac{d^2R(\tau)}{d\tau^2}$	$\omega^2 S(\omega)$
$\frac{d^n x(t)}{dt^n}$	$(-1)^n\frac{d^{2n}R(\tau)}{d\tau^{2n}}$	$\omega^{2n}S(\omega)$
$x(t)e^{\pm j\omega_0 t}$	$R(\tau)e^{\pm j\omega_0\tau}$	$S(\omega \mp \omega_0)$
	$R(\tau)\cos\omega_0\tau$	$\frac{1}{2}[S(\omega+\omega_0)+S(\omega-\omega_0)]$

stochastic processes but $n(t)$ is normal white noise with a zero mean and spectrum $S_n\,(f) = a$. Now let us assume a long observation time, that is,

$$t \gg \frac{m}{b} = \frac{1}{\beta} \tag{1.98}$$

one may consider $v(t)$ as a stationary process and (1.97) as stochastic. With the rules for the derivation of the spectra of stochastic processes (cf. Table 1.3) we get

$$\omega^2 S_v(\omega) + \beta^2 S_v(\omega) = S_n(\omega) \tag{1.99}$$

Consequently, we arrive at the Lorezian spectrum

$$S_v(f) = \frac{a}{f^2 + (\beta / 2\pi)^2} \approx \frac{a}{f^2} \qquad (\omega \gg \beta) \tag{1.100}$$

1.5.2 Fractional Integration (Wiener–Levy Process)

Let us consider the situation where the output events are random, with nearly equal changes in one or the opposite direction, with the nearly constant variances $(\Delta e_i)^2$ in each period or time span. By taking into account the central limit theorem, the variance of the expected change, after n steps, will be

$$< (\Delta e)^2 > \approx nT_o < (\Delta e_{\text{one step}})^2 > \tag{1.101}$$

The situation is explained in the following example:

EXAMPLE 1.6

One hundred years ago, K. Pearson and Lord Rayleigh [1.24] presented the following random-walk problem: A man (presumably very drunk) takes steps of equal length m from a starting point O, one after the other in successively random directions. Where will he likely be after n steps? If n is large, the probability that he is at a distance r and $r + dr$ from the starting point is

$$P(r)dr = \frac{2r}{nm^2} e^{-r^2/(nm^2)} dr \tag{1.102}$$

His average distance is equal to the distribution σ, that is,

$$r_{av} = \int_0^{\infty} rP(r)dr = \frac{\sqrt{\pi}}{2}\sqrt{nm} \qquad 1.103$$

[cf. Eq. (1.93)] with the assistance of Dwight formula 860.12. Since each step takes some time, n is a measure of time, and so the distance will increase with the square root of time (see Fig. 1.13).

1.5.2.1 Power Spectra with Fractional Integration Proportional to $\sqrt{t}$

Reverting to relation (1.102) and considering the time dependence of the final σ, we may also suppose Δe_i to be a function of time without any appreciable error. In the first approximation, we propose for its time dependence

$$\Delta e(t) \approx \sigma(\Delta e)\sqrt{t} \qquad 1.104$$

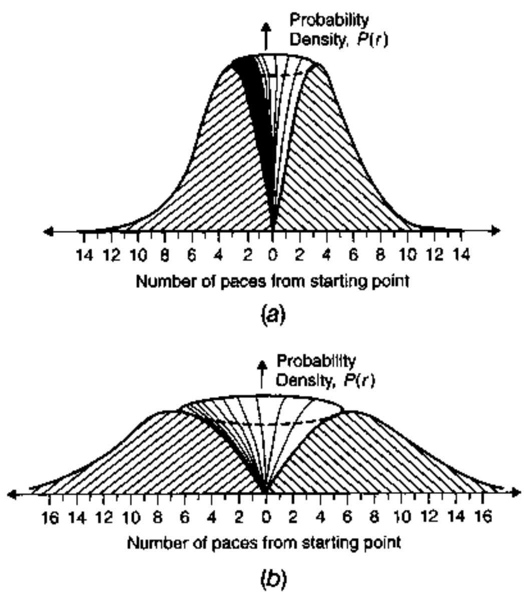


Fig. 1.13 Probability surface of a random walk (position of the drunken man): (*a*) after 18 random paces, and (*b*) after 72 random paces) [1.24].

With the assistance of the Table 1.2, we get for the Fourier transform

$$F(s)_{\Delta e} = s^{-3/2}\sigma(\Delta_e) \qquad 1.105$$

and for the corresponding PSD

$$S_{\Delta e}(f) = \frac{1}{f^3}\sigma^2(\Delta e) \qquad 1.106$$

1.5.2.2 Power Spectra with Fractional Integration Proportional to $1/\sqrt{t}$

In Section 1.2, we have showed that the origin of the flicker fluctuations in physical systems is based on the loss of energy [1.20]. Here we recall, once more, the problem in a much more general way, where the resonator (oscillator) system is supplied from an ideal voltage source V with current I. However, during its passage through the system some energy is lost; let it be E_{diss} during one time segment. With the assistance of the corresponding power, the loss is equal to

$$E_{\text{diss}} = P_{\text{dis}} T_o \qquad 1.107$$

Since the dissipated energy in each period (or time span) is rather constant, the effective power decreases and after n periods it may be equal to

$$P_{\text{diss},n} \approx < i^2_{\text{noise}} R > \approx \frac{nP_{\text{diss},n}T_o - (n-1)P_{\text{diss},n-1}T_o}{nT_o} = \frac{1}{t}P_{\text{diss}}T_o \qquad 1.108$$

which is inversely proportional to the elapsed time. Hence, the noise current is also a function of time and without any appreciable error, in the first approximation, we propose for its time dependence

$$i_{\text{noise}}(t) \approx \sqrt{\frac{1}{t}\cdot\frac{P_{\text{diss}}T_o}{R}} \qquad 1.109$$

whose Fourier transform is (Table 1.2)

$$I_{\text{noise}}(s) \approx \sqrt{\frac{\pi}{s}\cdot\frac{P_{\text{diss}}T_o}{R_{\text{diss}}}} \qquad 1.110$$

and the corresponding PSD with respect to the current I^2 (i.e., the phase noise) is

$$S_\phi(f) = \frac{\pi}{f} \cdot \frac{P_{\text{diss}} T_o}{I^2 R} \approx \frac{\pi}{f} \cdot \frac{P_{\text{diss}} T_o}{P_o} \approx \frac{1}{f} \cdot \frac{\pi}{f_o Q} = a_R \frac{1}{f} \qquad 1.111$$

where P_o is the effective power in the system and $P_o/P_{\text{diss}} = Q$ is the device quality factor. The last expression in (1.111) is valid for the quartz crystal resonators or oscillators (cf. Chapter 2). Since the process is much more general, here we recall an earlier example shown in Fig. 1.14, presenting the PSD $S_R(f)/R^2$ for an India ink resistor in accordance with (1.109). The validity extends for more than 10 decades with $\alpha \approx 1.21$ [1.1].

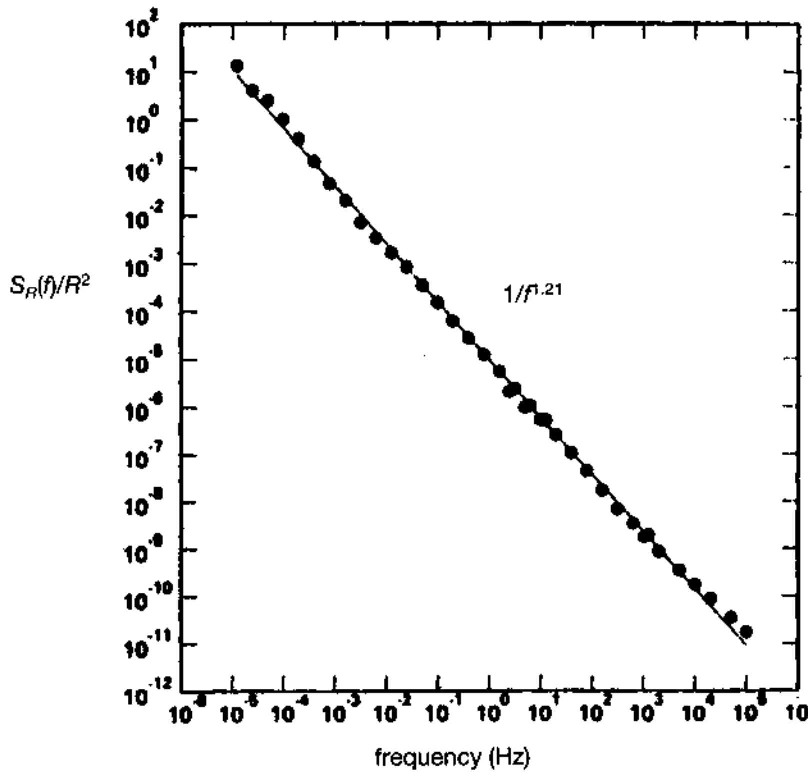


Fig. 1.14 Relative resistance fluctuation spectrum, $S_R(f)/R^2$, for India ink resistor.

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APPENDIX

Throughout the entire book we refer to the instantaneous phase or frequency,

$$\omega(t) \approx \frac{d}{dt}\left(\omega_o t + \varphi(t)\right) = \left(\omega_o + \dot{\varphi}(t)\right) \qquad \text{A.1}$$

and to their power spectral densities (PSD) $S_\varphi(f)$ or the normalized fractional frequency PSD $S_\varphi(f) = (f/f_o)^2 S_\varphi(f)$ (i.e., to the one-sided PSPs). Actually, however, we deal with the double-sided PSD:

$$S(\omega) \approx \frac{P_s^2}{2}[S_\varphi(\omega + \omega_o) + S_\varphi(\omega - \omega_o)] \qquad \text{A.2}$$

where P_s is the carrier power. The approximation is valid except for $\pm\,\omega_o$ and the surrounding narrow bands containing all the high modulation index, low frequency side bands. In the case of the validity of (A.2), we define the two-sided PSD $\mathcal{L}(\omega)$ as

$$\mathcal{L}(\omega) \approx S_\varphi(\omega + \omega_o) \approx S_\varphi(\omega - \omega_o) \qquad \text{A.3}$$