CHAPTER 1

Probability

The study of reliability engineering requires an understanding of the fundamentals of probability theory. In this chapter these fundamentals are described and illustrated by examples. They are applied in Sections 1.8 to 1.13 to the computation of the reliability of variously configured systems in terms of the reliability of the system's components and the way in which the components are arranged.

Probability is a numerical measure that expresses, as a number between 0 and 1, the degree of certainty that a specific outcome will occur when some random experiment is conducted. The term random experiment refers to any act whose outcome cannot be predicted. Coin and die tossing are examples. A probability of 0 is taken to mean that the outcome will never occur. A probability of 1.0 means that the outcome is certain to occur. The relative frequency interpretation is that the probability is the limit as the number of trials N grows large, of the ratio of the number of times that the outcome of interest occurs divided by the number of trials, that is,

$$p = \lim_{N \to \infty} \frac{n}{N} \tag{1.1}$$

where n denotes the number of times that the event in question occurs. As will be seen, it is sometimes possible to deduce p by making assumptions about the relative likelihood of all of the other events that could occur. Often this is not possible, however, and an experimental determination must be made. Since it is impossible to conduct an infinite number of trials, the probability determined from a finite value of N, however large, is considered an estimate of p and is distinguished from the unknown true value by an overstrike, most usually a caret, that is, $\hat{p}$.

Using the Weibull Distribution: Reliability, Modeling, and Inference, First Edition. John I. McCool.

1.1 SAMPLE SPACES AND EVENTS

The relationship among probabilities is generally discussed in the language of set theory. The set of outcomes that can possibly occur when the random experiment is conducted is termed the sample space. This set is often referred to by the symbol Ω. As an example, when a single die is tossed with the intent of observing the number of spots on the upward face, the sample space consists of the set of numbers from 1 to 6. This may be noted symbolically as $\Omega = \{1, 2, \ldots 6\}$. When a card is drawn from a bridge deck for the purpose of determining its suit, the sample space may be written: Ω = {diamond, heart, club, spade}. On the other hand, if the purpose of the experiment is to determine the value and suit of the card, the sample space will contain the 52 possible combinations of value and suit. The detail needed in a sample space description thus depends on the purpose of the experiment. When a coin is flipped and the upward face is identified, the sample space is Ω = {Head, Tail}. At a more practical level, when a commercial product is put into service and observed for a fixed amount of time such as a predefined mission time or a warranty period, and its functioning state is assessed at the end of that period, the sample space is Ω = {functioning, not functioning} or more succinctly, Ω = {S, F} for success and failure. This sample space could also be made more elaborate if it were necessary to distinguish among failure modes or to describe levels of partial failure.

Various outcomes of interest associated with the experiment are called *Events* and are subsets of the sample space. For example, in the die tossing experiment, if we agree that an event named A occurs when the number on the upward face of a tossed die is a 1 or a 6, then the corresponding subset is A = {1, 6}. The individual members of the sample space are known as elementary events. If the event B is defined by the phrase "an even number is tossed," then the set B is {2, 4, 6}. In the card example, an event C defined by "card suit is red" would define the subset C = {diamond, heart}. Notationally, the probability that some event "E" occurs is denoted P(E). Since the sample space comprises all of the possible elementary outcomes, one must have $P(\Omega) = 1.0$.

1.2 MUTUALLY EXCLUSIVE EVENTS

Two events are mutually exclusive if they do not have any elementary events in common. For example, in the die tossing case, the events A = {1, 2} and B = {3, 4} are mutually exclusive. If the event A occurred, it implies that the event B did not. On the other hand, the same event A and the event C = {2, 3, 4} are not mutually exclusive since, if the upward face turned out to be a 2, both A and C will have occurred. The elementary event "2" belongs to the intersection of sets A and C. The set formed by the intersection of sets A and C is written as $A \cap C$. The probability that the outcome will be a member of sets A and C is written as $P(A \cap C)$.

When events are mutually exclusive, the probabilities associated with the events are additive. One can then claim that the probability of the mutually exclusive sets A and B is the sum of $P(A)$ and $P(B)$.

In the notation of set theory, the set that contains the elements of both A and B is called the union of A and B and designated $A \cup B$. Thus, one may compute the probability that either of the mutually exclusive events A or B occurs as:

$$P(A \cup B) = P(A) + P(B) \tag{1.2}$$

The same result holds for three or more mutually exclusive events; the probability of the union is the sum of the probabilities of the individual events.

The *elementary* events of a sample space are mutually exclusive, so for the die example one must have:

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = P(\Omega) = 1.0. \tag{1.3}$$

Now reasoning from the uniformity of shape of the die and homogeneity of the die material, one might make a leap of faith and conclude that the probability of the elementary events must all be equal and so,

$$P(1) = P(2) = \ldots = P(6) = p.$$

If that is true then the sum in Equation 1.3 will equal $6p$, and, since $6p = 1$, $p = 1/6$. The same kind of reasoning with respect to coin tossing leads to the conclusion that the probability of a head is the same as the probability of a tail so that P(H) = P(T) = 1/2. Dice and coins whose outcomes are equally likely are said to be "fair." In the card selection experiment, if we assume that the card is randomly selected, by which we mean each of the 52 cards has an equal chance of being the one selected, then the probability of selecting a specific card is 1/52. Since there are 13 cards in each suit, the probability of the event "card is a diamond" is 13/52 = 1/4.

1.3 VENN DIAGRAMS

Event probabilities and their relationship are most commonly displayed by means of a Venn diagram named for the British philosopher and mathematician John Venn, who introduced the Venn diagram in 1881. In the Venn diagram a rectangle symbolically represents the set of outcomes constituting the sample space Ω; that is, it contains all of the elementary events. Other events, comprising subsets of the elementary outcomes, are shown as circles within the rectangle. The Venn diagram in Figure 1.1 shows a single event A.

The region outside of the circle representing the event contains all of the elementary outcomes not encompassed by A. The set outside of A, Ω-A, is

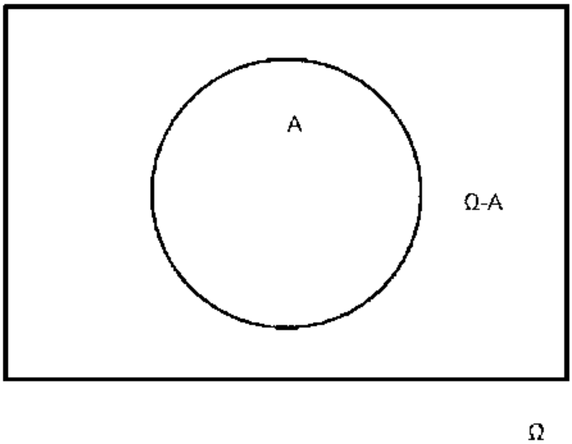

Figure 1.1 Venn diagram showing a single event A.

generally called "not-A" and is indicated by a bar overstrike $\bar{A}$. Since A and $\bar{A}$ are mutually exclusive and sum to the whole sample space, we have:

$$P(A) + P(\bar{A}) = P(\Omega) = 1.0. \tag{1.4}$$

Therefore, the probability of the event not-A may be found simply as:

$$P(\bar{A}) = 1 - P(A). \tag{1.5}$$

Thus, if A is the event that a bearing fails within the next 1000 hours, and $P(A) = 0.2$, the probability that it will survive is $1 - 0.2 = 0.8$. The *odds* of an event occurring is the ratio of the probability that the event occurs to the probability that it does not. The odds that the bearing survives are thus $0.8/0.2 = 4$ or 4 to 1.

Since mutually exclusive events have no elements in common, they appear as nonoverlapping circles on a Venn diagram as shown in Figure 1.2 for the two mutually exclusive events A and B:

1.4 UNIONS OF EVENTS AND JOINT PROBABILITY

The Venn diagram in Figure 1.3 shows two nonmutually exclusive events, A and B, depicted by overlapping circles. The region of overlap represents the set of elementary events shared by events A and B. The probability associated with the region of overlap is sometimes called the joint probability of the two events.

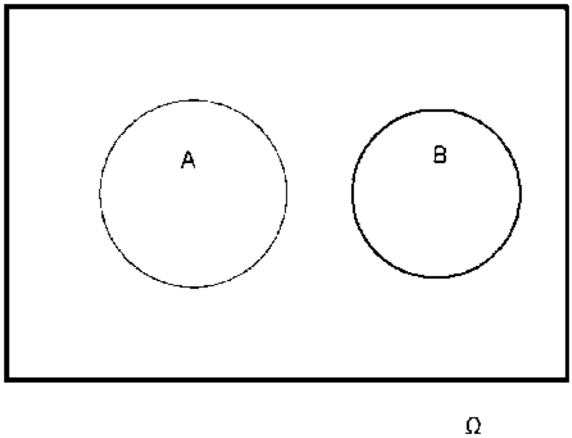

Figure 1.2 Venn diagram for mutually exclusive events A and B.

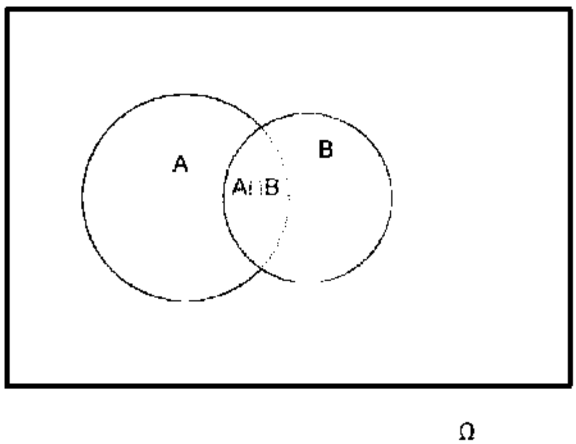

Figure 1.3 Venn diagram for overlapping events A and B.

In this case, computing the probability of the occurrence of event A or B or both as the sum of P(A) and P(B) will add the probability of the shared events twice. The correct formula is obtained by subtracting the probability of the intersection from the sum of the probabilities to correct for the double inclusion:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B). \tag{1.6}$$

As an example, consider again the toss of a single die with the assumption that the elementary events are equally likely and thus each have a probability of occurrence of p = 1/6. Define the events A = {1, 2, 3, 4} and B = {3, 4, 5}. Then P(A) = 4/6, P(B) = 3/6 and since the set $(A \cap B) = \{3, 4\}$, it follows that $P(A \cap B) = 2/6$. The probability that the event A or B occurs may now be written:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{6} + \frac{3}{6} - \frac{2}{6} = 5/6.$$

The formula above applies even to mutually exclusive events when it is recalled that for mutually exclusive events,

$$P(A \cap B) = 0.$$

Similar reasoning leads to the following expression for the union of three events:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \quad (1.7)$$

The expression consists of the sum of all the individual event probabilities minus the joint probabilities of all pairings of the events plus the probability of the intersection of all three events. The generalization to more events is similar. For four events one would sum the individual probabilities, subtract all the probabilities of pairs, add the probability of all triples, and finally subtract the probability of the four-way intersection. This calculation is sometimes referred to as the inclusion–exclusion principle since successive groupings of additional element intersections are added or subtracted in sequence until terminating with the inclusion of the intersection of every event under consideration.

1.5 CONDITIONAL PROBABILITY

We know that our assessments of probabilities change as new information becomes available. Consider the event that a randomly selected automobile survives a trip from coast to coast with no major mechanical problems. Whatever the probability of this event may be, we know it will be different (smaller) if we are told that the automobile is 20 years old. This modification of probabilities upon the receipt of additional information can be accommodated within the set theory framework discussed here. Suppose that in the situation above involving overlapping events A and B, we were given the information that event A had indeed occurred. The question is, having learned this, what then is our revised assessment of the probability of the event B? The probability of B conditional on A having occurred is written P(B|A). It is read as "the probability of B given A." Clearly, had the specified events been mutually exclusive instead of overlapping, the knowledge that A occurred would eliminate the possibility of B occurring and so P(B|A) = 0. In general, knowing that A occurred changes the set of elementary events at issue from those in the set Ω to those in the set A. The set A has become the new sample space. Within that new sample space, the points corresponding to the occurrence of B are those contained within the intersection $A \cap B$. The probability of B given A is now the proportion of P(A) occupied by the intersection probability $P(A \cap B)$. Thus:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}. \tag{1.8}$$

Similarly, P(A|B) is given by:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}. \tag{1.9}$$

The numerator is common to these two expressions and therefore by cross multiplication we see that:

$$P(A \cap B) = P(B)P(A \mid B) = P(A)P(B \mid A). \tag{1.10}$$

One application of this formula is in sampling from finite lots. If a lot of size 25 contains five defects, and two items are drawn randomly from the lot, what is the probability that both are defective? Let A be the event that the first item sampled is defective, and let B be the event that the second item is also defective. Then since every one of the 25 items is equally likely to be selected, P(A) = 5/25. Given that A occurred, the lot now contains 24 items of which four are defective, so P(B|A) = 4/24. The probability that both are defective is then calculated as:

$$\text{Prob(both defective)} = P(A \cap B) = P(A)P(B \mid A) = \frac{5}{25} \cdot \frac{4}{24} = \frac{1}{30}.$$

A similar problem occurs in determining the probability of picking two cards from a deck and finding them both to be diamonds. The result would be (13/52) (12/51) = 0.0588.

When three events are involved, the probability of their intersection could be written as

$$P(A \cap B \cap C) = P(A)P(B \mid A)P(C \mid A \cap B). \tag{1.11}$$

This applies to any ordering of the events A, B, and C. For four or more events the probability of the intersection may be expressed analogously.

1.6 INDEPENDENCE

Two events A and B are said to be *independent* if the conditional probability P(A|B) is equal to P(A). What this says in essence is that knowing that B occurred provides no basis for reassessing the probability that A will occur. The events are unconnected in any way. An example might be if someone tosses a fair coin and the occurrence of a head is termed event A, and perhaps someone else in another country, throws a die and the event B is associated with 1, 2, or 3 spots appearing on the upward face. Knowing that the event B occurred, P(A|B) remains P(A) = 1/2. When two events are independent, the probability of their intersection becomes the product of their individual probabilities:

$$P(A \cap B) = P(B) \cdot P(A \mid B) = P(B) \cdot P(A). \tag{1.12}$$

This result holds for any number of independent events. The probability of their joint occurrence is the product of the individual event probabilities. Reconsider the previous example of drawing a sample of size 2 from a lot of 25 items containing five defective items but now assume that each item is replaced after it is drawn. In this case the proportion defective remains constant at 5/25 = 0.2 from draw to draw and the probability of two defects is $0.2^2 = 0.04$. This result would apply approximately if sampling was done without replacement and the lot were very large so that the proportion defective remained essentially constant as successive items are drawn.

When events A and B are independent the probability of either A or B occurring reduces to:

$$P(A \cup B) = P(A) + P(B) - P(A)P(B). \tag{1.13}$$

Example

A system comprises two components and can function as long as at least one of the components functions. Such a system is referred to as a parallel system and will be discussed further in a later section. Let A be the event that component 1 survives a specified life and let B be the event that component 2 survives that life. If P(A) = 0.8 and P(B) = 0.9, then assuming the events are independent:

$$P[\textit{at least one survives}] = 0.8 + 0.9 - 0.8 \cdot 0.9 = 0.98.$$

Another useful approach is to compute the probability that both fail. The complement of this event is that at least one survives. The failure probabilities are 0.2 and 0.1 so the system survival probability is:

$$P[\textit{at least one survives}] = 1 - (0.2)(0.1) = 1 - .02 = 0.98$$

When the system has more than two components in parallel this latter approach has the advantage of simplicity over the method of inclusion and exclusion shown earlier.

The terms independence and mutual exclusivity are sometimes confused. Both carry a connotation of "having nothing to do with each other." However, mutually exclusive events are not independent. In fact they are strongly dependent since $P(A \cap B) = 0$ and not $P(A)P(B)$ as required for independence.

1.7 PARTITIONS AND THE LAW OF TOTAL PROBABILITY

When a number of events are mutually exclusive and collectively contain all the elementary events, they are said to form a partition of the sample space. An example would be the three events $A = \{1, 2\}$, $B = \{3, 4, 5\}$, and $C = \{6\}$. The probability of their union is thus $P(\Omega) = 1.0$. The Venn diagram fails us in representing a partition since circles cannot exhaust the area of a rectangle. Partitions are therefore ordinarily visualized as an irregular division of a rectangle without regard to shape or size as shown in Figure 1.4.

Alternate language to describe a partition is to say that the events are disjoint (no overlap) and exhaustive (they embody all the elementary events). When an event, say D, intersects with a set of events that form a partition, the probability of that event may be expressed as the sum of the intersections of D with the events forming the partition. The Venn diagram in Figure 1.5 shows three events, A, B, C, that form a partition. Superimposed is an event D that intersects each of the partitioning events.

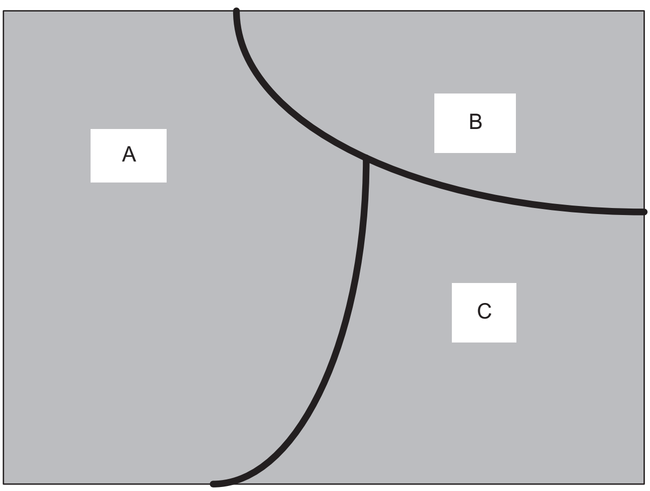

Figure 1.4 Mutually exclusive events A, B, and C forming a partition of the sample space.

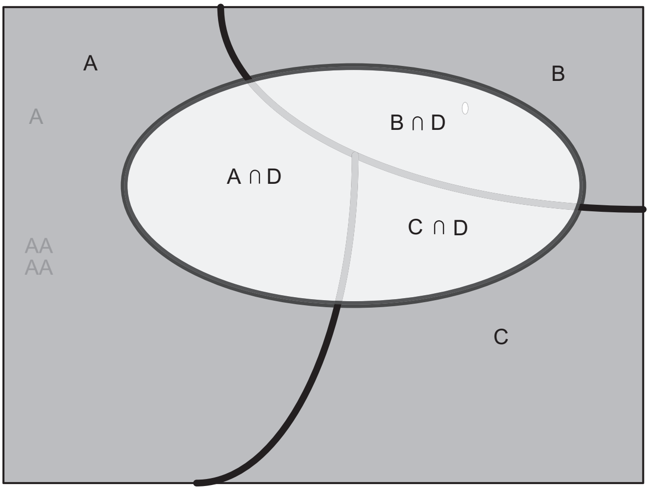

Figure 1.5 An event D superimposed on a partition.

The probability of the event D can be expressed as the sum of the probabilities of the intersections of D with A, B, and C:

$$P(D) = P(A \cap D) + P(B \cap D) + P(C \cap D). \tag{1.14}$$

Using the expression for the joint probability in terms of the probability of D conditioned on each of the other three events, this becomes:

$$P(D) = P(A)P(D \mid A) + P(B)P(D \mid B) + P(C)P(D \mid C). \tag{1.15}$$

This formula is commonly called the Law of Total Probability. It is frequently the only practical way of computing the probability of certain events of interest. One example of its usefulness is in computing overall product quality in terms of the relative amount of product contributed by different suppliers and the associated quality performance of those suppliers.

Example

A company has three suppliers, designated A, B, and C. The relative amounts of a certain product purchased from each of the suppliers are 50%, 35%, and 15%, respectively. The proportion defective produced by each supplier are 1%, 2% and 3%, respectively. If the company selects a product at random from its inventory the probability that it will have been supplied by supplier A is 0.5 and the probability that it is defective given that it was produced by supplier A is 0.01. Let A, B, and C denote the event that a randomly selected part drawn randomly from the company's inventory was provided by suppliers A, B, and C, we have:

$$P(A) = 0.5, P(B) = 0.35 \text{ and } P(C) = 0.15.$$

The event that an item randomly drawn from inventory is defective is denoted as event D. The following conditional probabilities apply:

$$P(D \mid A) = 0.01, P(D \mid B) = 0.02 \text{ and } P(D \mid C) = 0.03.$$

P(D) then represents the overall proportion defective and may be computed from the Law of Total Probability.

$$P(D) = 0.5 \times 0.01 + 0.35 \times 0.02 + 0.15 \times 0.03 = 0.0165.$$

The company's inventory is thus 1.65% defective.

One creative use of the Law of Total Probability is in the analysis of the randomized response questionnaire (cf. Warner 1965). This questionnaire is aimed at determining the proportion of people who have participated in an activity, such as tax evasion, that they might be loathe to admit if directly asked. Instead two questions are posed, Q1 and Q2. The participant randomly chooses to answer Q1 or Q2 based on a random mechanism such as flipping a coin. Let us say that if the coin is a head, they answer Q1 and otherwise Q2. If the coin is fair, P(Q1) = 1/2 and P(Q2) = 1/2. Now Question Q1 is chosen so that the fraction of affirmative responses is known. For example:

Q1: Is the last digit of your social security number even? Yes/No. The probability of a Yes answer given that Q1 is answered is therefore P(Y|Q1) = 0.5.

Question Q2 is the focus of actual interest and could be something like:

Q2: Have you ever cheated on your taxes? Yes/No.

From the respondent's viewpoint, a Yes answer is not incriminating since it would not be apparent whether that answer was given in response to Q1 or to Q2.

The overall probability of a Yes response may be written as:

$$P(Yes) = P(Y \mid Q1)P(Q1) + P(Y \mid Q2)P(Q2) = (1/2)(1/2) + 1/2(Y \mid Q2).$$

When the survey results are received the proportion of Yes answers are determined and used as an estimate of P(Yes) in the equation above. For example, suppose that out of 1000 people surveyed, 300 answered Yes. P(Yes) may therefore be estimated as 300/1000 = 0.30.

Substituting this estimate gives:

$$0.30 = 0.25 + 0.5P[Y \mid Q2]$$

So that P[Y| Q2] may be estimated as: (0.30–0.25)/0.5 = 0.10.

1.8 RELIABILITY

One source of probability problems that arise in reliability theory is the computation of the reliability of systems in terms of the reliability of the components comprising the system. These problems use the very same principles as discussed above and are only a context change from the familiar dice, cards, and coins problems typically used to illustrate the laws of probability. We use the term reliability in the narrow sense defined as "the probability that an item will perform a required function under stated conditions for a stated period of time." This definition coincides with what Rausand and Høyland (2004) call survival probability. They use a much more encompassing definition of reliability in compliance with ISO 840 and of which survival probability is only one measure.

Reliability relationships between systems and their components are readily communicated by means of a reliability block diagram. Reliability block diagrams are analogous to circuit diagrams used by electrical engineers. The reliability block diagram in Figure 1.6 identifies a type of system known as a series system. It has the appearance of a series circuit.

1.9 SERIES SYSTEMS

In Figure 1.6, R_i represents the probability that the i-th component ($i = 1\ldots4$) functions for whatever time and conditions are at issue. A series circuit functions if there is an unbroken path through the components that form the system. In the same sense, a series system functions if every one of the components displayed also functions. The reliability of the system is the probability of the intersection of the events that correspond to the functioning of each component:

$$R_{system} = Prob[1\ functions \cap 2\ functions \cap 3\ functions \cap 4\ functions]. \quad (1.16)$$

If the components are assumed to be independent in their functioning, then,

$$R_{system} = R_1 \cdot R_2 \cdot R_3 \cdot R_4 = \prod_{i=1}^{4} R_i. \quad (1.17)$$

It is readily seen that the reliability of a series system is always lower than the reliability of the least reliable component. Suppose that R_3 were lower than

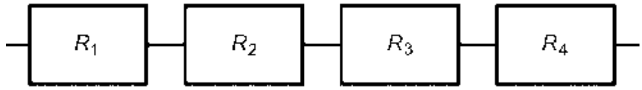

Figure 1.6 A reliability block diagram.

the others, that is, suppose component 3 is the least reliable of the four components in the system. Since R_3 is being multiplied by the product $R_1R_2R_4$, which is necessarily less than or equal to 1.0, the system reliability cannot exceed R_3.

As an example of a series system calculation, if $R_1 = R_2 = 0.9$, $R_3 = 0.8$, and $R_4 = 0.95$, the system reliability is $(0.9)^2(0.8)(0.95) = 0.6156$.

It is clear that a series system comprising a large number of relatively reliable components may nevertheless be quite unreliable. For example, a series system with 10 components each having $R = 0.95$ has a system reliability of only $(0.95)^{10} = 0.599$. One way of improving the system reliability is to provide duplicates of some of the components such that the system will function if any one of these duplicates functions. The practice of designing with duplicates is called redundancy and gives rise to design problems involving optimum tradeoffs of complexity, weight, cost, and reliability.

1.10 PARALLEL SYSTEMS

The reliability of a system that functions as long as at least one of its two components functions may be computed using the rule for the union of two events where the two events are (i) component 1 functions and (ii) component 2 functions. Assuming independence the probability that both function is the product of the probabilities that each do. Thus, the probability that component 1 or component 2 or both function is:

$$R_{system} = R_1 + R_2 - R_1 \cdot R_2. \tag{1.18}$$

Systems of this type are known as parallel systems since there are as many parallel paths through the reliability block diagram as there are components.

The reliability block diagram in Figure 1.7 shows a parallel system having four components. The direct approach shown above for computing system reliability gets more complicated in this case requiring the use of the inclusion–exclusion principle. A simpler but less direct approach is based on the recognition that a parallel system fails only when all of the n components fail.

Assuming independence, the probability that the system functions is most readily computed as 1-Prob[system fails to function]:

$$R_{system} = 1 - \prod_{i=1}^{n}(1 - R_i). \tag{1.19}$$

For $n = 2$, this results in:

$$R_{system} = 1 - (1 - R_1)(1 - R_2) = R_1 + R_2 - R_1 \cdot R_2. \tag{1.20}$$

In agreement with the direct method given in Equation 1.18.

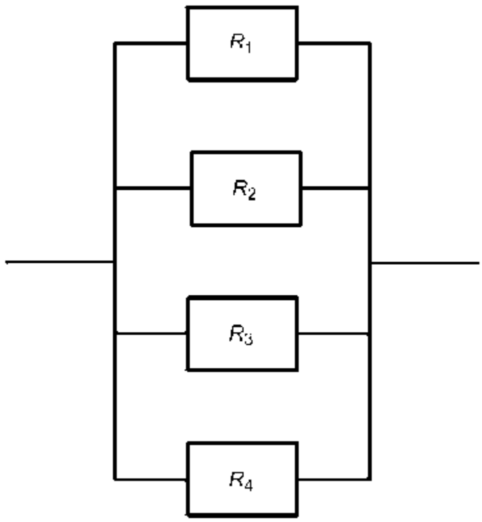

Figure 1.7 A parallel system reliability diagram.

For the component reliabilities considered in the series system depicted in Figure 1.1, letting $R_1 = R_2 = 0.90$, $R_3 = 0.8$, and $R_4 = 0.95$, the reliability of the corresponding parallel system is:

$$R_{system} = 1 - (0.10 \times 0.10 \times 0.20 \times 0.05) = 0.9999.$$

Note that in the parallel case the system reliability is greater than the reliability of the best component. This is generally true. Without loss of generality let R_1 denote the component having the greatest reliability. Subtract R_1 from both sides of Equation 1.19,

$$R_{system} - R_1 = 1 - R_1 - (1 - R_1) \cdot \prod_{i=2}^{n} (1 - R_i). \tag{1.21}$$

Factoring out $(1 - R_1)$, this becomes,

$$R_{system} - R_1 = (1 - R_1) \cdot \left\{ 1 - \prod_{i=2}^{n} (1 - R_i) \right\}. \tag{1.22}$$

Since the values of R_i are all less than or equal to 1.0, the two bracketed terms on the right-hand side are positive and hence $R_{system} \geq R_1$.

We may conclude that the reliability of any system composed of a given set of components is always greater than or equal to the reliability of the series combination and less than or equal to the reliability of the parallel combination of those components.

1.11 COMPLEX SYSTEMS

Systems consisting of combinations of parallel and series arrangements of components can be resolved into a purely parallel or a purely series system. The system depicted in Figure 1.8 is an example:

Replace the series elements on each path by a module whose reliability is equal to that of the series combinations. Multiplying the reliabilities of the three series branches results in the equivalent system of three modules in parallel shown in Figure 1.9.

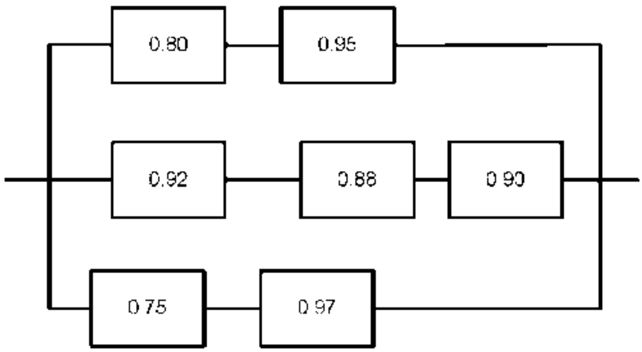

Figure 1.8 Combined series and parallel system reliability block diagram.

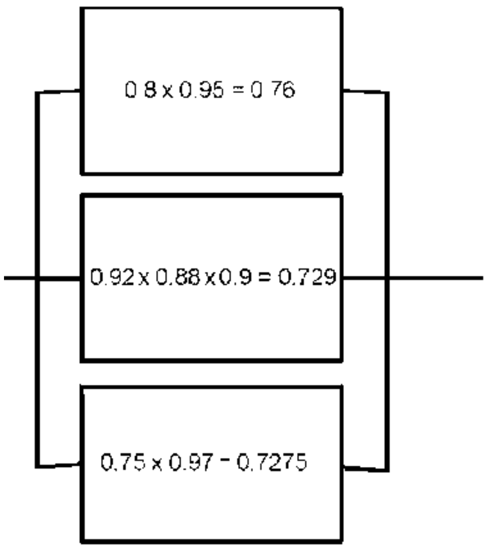

Figure 1.9 System resolved into an equivalent parallel system.

The system reliability may now be computed from the rule for parallel systems:

$$R_{system} = 1-(1-.76)(1-0.729)(1-.7275) = 0.984.$$

1.12 CROSSLINKED SYSTEMS

The system reliability for more complex systems involving crosslinking can sometimes be found by exploiting the Law of Total Probability. Before examining an example let us consider a simple parallel structure with two components having reliabilities R_1 and R_2. Let us consider two mutually exclusive situations: component 2 functions and component 2 does not function. Figure 1.10 shows the original parallel system and what it becomes in the two mutually exclusive circumstances that (i) component 2 functions and (ii) component 2 fails.

Applying the Law of Total Probability, we have in words:
Prob{system functions} = Prob{system functions|Component 2 functions} × Prob{component 2 functions} + Prob{system functions |component 2 does not function} × Prob(component 2 does not function).

Now, given that component 2 functions, the reliability of component 1 is irrelevant and the system reliability is 1.0. Given that component 2 does not function, the system reliability is simply R_1. Thus:

$$R_{system} = 1 \cdot R_2 + R_1 \cdot (1-R_2) = R_1 + R_2 - R_1 R_2.$$

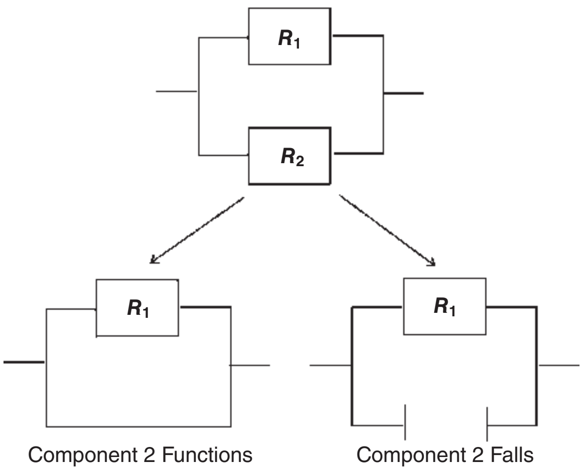

Figure 1.10 Decomposition of a parallel system based on the status of component 2.

We see that the resultant expression is in agreement with the expression previously found for a two-component parallel system. This method of analysis, often called the decomposition method, is always valid, but generally not used for systems that consist of simple combinations of series and parallel subsystems. The power of the decomposition method arises in the analysis of so-called crosslinked systems which cannot be handled by the direct approach used to analyze the system shown in Figure 1.10. Figure 1.11A is the reliability block diagram for such a crosslinked system.

The component labeled 3 causes this block diagram to differ from a pure series/parallel combination. Therefore, component 3 will be chosen as the pivot element in using the decomposition method. Let R^+ denote the system reliability when component 3 functions. The reduced system in this case becomes a series combination of two parallel components as shown in Figure 1.11B.

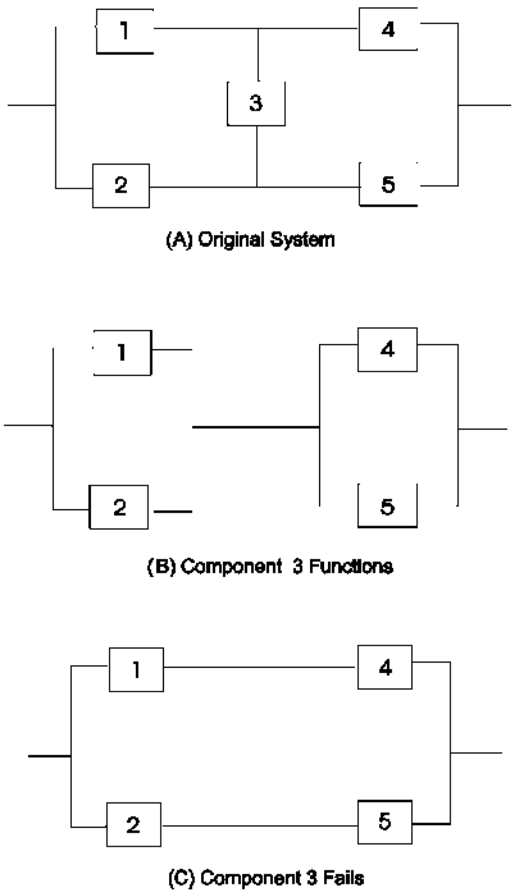

Figure 1.11 Decomposition of a complex system based on status of component 3.

The reliability R^+ is then the product of the reliabilities of the two parallel modules:

$$R^+ = [1-(1-R_1)(1-R_2)]\cdot[1-(1-R_4)(1-R_5)]. \tag{1.23}$$

When component 3 is in the failed state the reduced system is the parallel combination of two series modules as shown in Figure 1.11C.

The system reliability with component 3 failed is denoted R^- and may be expressed as:

$$R^- = 1-(1-R_1R_4)(1-R_2R_5). \tag{1.24}$$

Using the Law of Total Probability, the system reliability is then expressible as:

$$R_{system} = R_3R^+ + (1-R_3)R^-. \tag{1.25}$$

For example, suppose every component had a reliability of 0.9. In that case:

$$R^+ = [1-(0.01)^2][1-(0.01)^2] = 0.9998 \text{ and}$$
$$R^- = 1-[1-(0.9)^2][1-(0.9)^2] = 0.9639.$$

The system reliability is then:

$$R_{system} = 0.9 * 0.9998 + 0.1 * 0.9639 = 0.9962.$$

Another type of system which is somewhere between a series and a parallel system is known as a *k*/*n* system. The *k*/*n* system functions if k ($\leq n$) or more of its components function. An example might be a system containing eight pumps of which at least five must function for the system to perform satisfactorily. A series system could be regarded as the special case of an *n*/*n* system. A parallel system on the other hand is a 1/*n* system. The *k*/*n* is system is sometimes represented by a reliability block diagram with n parallel paths each showing k of the elements. For example with a 2/3 system there are 3 parallel paths. One shows the elements 1 and 2, another, the elements 1 and 3 and the third shows the elements 2 and 3. This might be a useful way to convey the situation but it can't be analyzed in the same manner as an ordinary parallel system since each element appears on 2 paths and thus the paths are not independent.

Let us assume that component i has reliability R_i for $i = 1, 2$, and 3. Define the following events:

A: (components 1 and 2 function),
B: (components 1 and 3 function), and
C: (components 2 and 3 function).

Using the inclusion–exclusion principle, the system reliability is:

$$R_{system} = P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \quad (1.26)$$

Now,

$$P(A) = R_1 R_2.$$

Likewise,

$$P(B) = R_1 R_3 \text{ and } P(C) = R_2 R_3.$$

The paired terms and the triple term are all equal to the product $R_1R_2R_3$. The final result is therefore:

$$R_{system} = R_1R_2 + R_1R_3 + R_2R_3 - 2R_1R_2R_3. \quad (1.27)$$

This calculation grows quite tedious for larger values of k and n.

For the case where all components have the same reliability the system reliability may easily be computed using the binomial distribution as shown in Section 2.5 of Chapter 2.

1.13 RELIABILITY IMPORTANCE

It is of interest to assess the relative impact that each component has on the reliability of the system in which it is employed, as a basis for allocating effort and resources aimed at improving system reliability. A measure of a component's reliability importance due to Birnbaum (1969) is the partial derivative of the system reliability with respect to the reliability of the component under consideration. For example, the system reliability for the series system shown in Figure 1.6 is:

$$R_s = R_1R_2R_3R_4. \quad (1.28)$$

The importance of component 1 is,

$$I = \frac{\partial R_s}{\partial R_1} = R_2R_3R_4 \quad (1.29)$$

and similarly for the other components. Suppose the component reliabilities were 0.95, 0.98, 0.9, and 0.85, respectively, for R_1 to R_4. The computed importance for each component is shown in the table below:

Component	Reliability	Importance
1	0.95	$R_2R_3R_4 = 0.7056$
2	0.98	$R_1R_3R_4 = 0.6840$
3	0.90	$R_1R_2R_4 = 0.7448$
4	0.80	$R_1R_2R_3 = 0.8379$

We see that the most important component, the one most deserving of attention in an attempt to improve system reliability is component 4, the least reliable component.

An alternate way of computing the importance of a component comes from the decomposition method. Suppose we seek the importance of component *i* in some system. We know that the system reliability can be expressed as:

$$R_s = R^+ R_i + R^- (1 - R_i).$$

Differentiating with respect to R_i shows that the importance of component *I* may be computed as:

$$I = R^+ - R^-. \tag{1.30}$$

Thus, the importance is the difference in the system reliabilities computed when component *i* functions and when it does not. Referring to the crosslinked Figure 1.11 and the associated computations, the importance of component 3 is the difference:

$$I = 0.9998 - 0.9639 = 0.0359.$$

There is an extensive literature on system reliability, and many other methods, approximations, and software are available for systems with large numbers of components. The book by Rausand and Høyland (2004) contains a good exposition of other computational methods and is a good guide to the published literature on systems reliability calculations. Another good source is the recent text by Modarres et al. (2010).

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EXERCISES

1. Employees at a particular plant were classified according to gender and political party affiliation. The results follow:

	Political Affiliation		
Gender	Democrat	Republican	Independent
Male	40	50	5
Female	18	8	4

If an employee is chosen at random, find the probability that the employee is:

a. Male

b. Republican

c. A female Democrat

d. Republican given that she is a female

e. Male given that he is a Republican

2. Three components, a, b, and c, have reliabilities 0.9, 0.95, and 0.99, respectively. One of these components is required for a certain system to function. Which of the following two options results in a higher system reliability?

a. Create two modules with a, b, and c in series. The system then consists of a parallel arrangement of two of these modules. This is called high-level redundancy.

b. The system consists of a parallel combination of two components of type a in series with similar parallel combinations of b and c. This is called low-level redundancy.

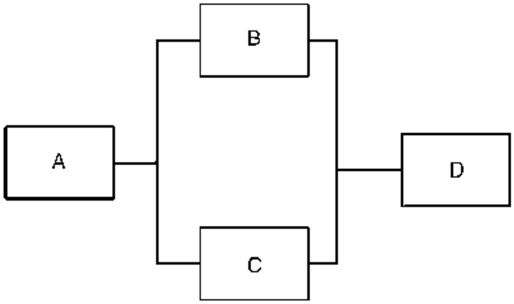
c. If in the low-level redundancy arrangement it were possible to add a third component of either type a or b or c, which would you choose? Why? Show work.

3. In the reliability diagram below, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

a. Decomposition using B as the keystone element.

b. The reduction method.

c. Compute the importance of each component if $R_A = 0.8$, $R_B = 0.9$, $R_C = 0.95$, and $R_D = 0.98$.



4. A message center has three incoming lines designated A, B, and C which handle 40%, 35%, and 25% of the traffic, respectively. The probability of a message over 100 characters in length is 5% on line A, 15% on line B, and 20% on line C. Compute the probability that a message, randomly selected at the message center, exceeds 100 characters in length.