Chapter 1

First-Order Differential Equations

1.1 MOTIVATION AND OVERVIEW

1.1.1 Introduction. Typically, phenomena in the natural sciences can be described, or "modeled," by equations involving derivatives of one or more unknown functions. Such equations are called **differential equations**.

To illustrate, consider the motion of a body of mass m that rests on an idealized frictionless table and is subjected to a force $F(t)$ where t is the time (Fig. 1). According to Newton's second law of motion, we have

$$m\frac{d^2x}{dt^2} = F(t), \tag{1}$$

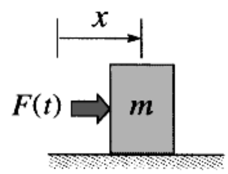


Figure 1. The motion of a mass on a frictionless table subjected to a force $F(t)$.

in which $x(t)$ is the mass's displacement. If we know the displacement history $x(t)$ and wish to determine the force $F(t)$ required to produce that displacement, the solution is simple: According to (1), merely differentiate the given $x(t)$ twice and multiply the result by m.

However, if we know the applied force $F(t)$ and wish to determine the displacement $x(t)$ that results, then we say that (1) is a "differential equation" governing the unknown function $x(t)$ because it involves derivatives of $x(t)$ with respect to t. Here, t is the independent variable and x is the dependent variable. The question is: *What function or functions $x(t)$, when differentiated twice with respect to t and then multiplied by m (which is a constant), give the prescribed function $F(t)$?*

To solve (1) for $x(t)$ we need to undo the differentiations; that is, we need to integrate (1) twice. To illustrate, suppose $F(t) = F_0$ is a constant, so

$$m\frac{d^2x}{dt^2} = F_0. \tag{2}$$

Integrating (2) once with respect to t gives

$$\int m\frac{d}{dt}\left(\frac{dx}{dt}\right) dt = \int F_0\, dt$$

From the calculus, $\int \frac{du}{dt}\, dt = \int du = u$ plus an arbitrary constant.

or

$$m\frac{dx}{dt} + C_1 = F_0 t + C_2 \tag{3}$$

in which C_1 and C_2 are the arbitrary constants of integration. Equivalently,

$$m\frac{dx}{dt} = F_0 t + A, \tag{4}$$

in which the combined constant $A = C_2 - C_1$ is arbitrary. Integrating again gives $mx = F_0 t^2/2 + At + B$, so

$$x(t) = \frac{1}{m}\left(\frac{F_0}{2}t^2 + At + B\right). \tag{5}$$

It is a good habit to express the functional dependence explicitly, as we did in (5) when we wrote $x(t)$ instead of just x.

We say that a function is a **solution** *of a given differential equation, on an interval of the independent variable, if its substitution into the equation reduces that equation to an identity everywhere on that interval.* If so, we say that the function **satisfies** the differential equation on that interval. Accordingly, (5) is a solution of (2) on the interval $-\infty < t < \infty$ because if we substitute it into (2) we obtain $F_0 = F_0$, which is true for all t.

Actually, (5) is a whole "family" of solutions because A and B are arbitrary. Each choice of A and B in (5) gives one member of that family. That may sound confusing, for weren't we expecting to find "the" solution, not a whole collection of solutions? What's missing is that we haven't specified "starting conditions," for how can we expect to fully determine the ensuing motion $x(t)$ if we don't specify how it starts, namely, the displacement and velocity at the starting time $t = 0$? If we specify those values, say $x(0) = x_0$ and $x'(0) = x_0'$ where x_0 and x_0' are prescribed numbers, then the problem becomes

$$m\frac{d^2x}{dt^2} = F_0, \qquad (0 < t < \infty) \tag{6a}$$

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = x_0', \tag{6b}$$

rather than consisting only of the differential equation (2). We seek a function or functions $x(t)$ that satisfy the differential equation $m\, d^2x/dt^2 = F_0$ on the interval $0 < t < \infty$ as well as the conditions $x(0) = x_0$ and $\frac{dx}{dt}(0) = x_0'$. We call (6b) **initial conditions**, and since the problem (6) includes one or more initial conditions we call it an **initial value problem** or **IVP**. Application of the initial conditions to the solution (5) gives

Initial value problem is often abbreviated as IVP.

$$x(0) = x_0 = \frac{1}{m}(0 + 0 + B) \qquad \text{[from (5)]}, \tag{7a}$$

$$\frac{dx}{dt}(0) = x_0' = \frac{1}{m}(0 + A) \qquad \text{[from (4)]}, \tag{7b}$$

so $A = mx_0'$ and $B = mx_0$, and we have the solution

$$x(t) = \frac{F_0}{2m}t^2 + x_0't + x_0 \tag{8}$$

of (6). Thus, from the differential equation (6a), which is a statement of Newton's second law, and the initial conditions (6b), we've been able to predict the displacement history $x(t)$ for all $t > 0$.

Whereas the differential equation (2), by itself, has the whole family of solutions given by (5), there is only one within that family that also satisfies the initial conditions (6b), the solution given by (8).

Unfortunately, most differential equations cannot be solved that readily, merely by undoing the derivatives by integration. For instance, suppose the mass is restrained by an ordinary coil spring that supplies a restoring force (i.e., in the direction opposite to the displacement) proportional to the displacement x, with constant of proportionality k (Fig. 2a). Then the total force on the mass when it is displaced to the right a distance x is $-kx + F(t)$, where the minus sign is because the kx force is in the negative x direction (Fig. 2b). Thus, now the differential equation governing the motion is

$$m\frac{d^2x}{dt^2} = -kx + F(t).$$

Finally, gathering all the unknown x terms on the left, as is customary, gives

$$m\frac{d^2x}{dt^2} + kx = F(t). \tag{9}$$

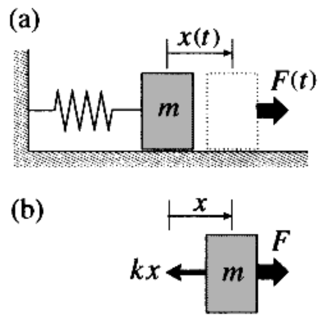


Figure 2. (a) The mass/spring system. (b) The forces on the mass. NOTE: The force kx exerted on the mass by the spring is proportional to the stretch in the spring, x, and the (empirically determined) constant of proportionality is k.

Let us try to solve (9) for $x(t)$ in the same way that we solved (2), by integrating twice with respect to t. One integration gives

$$m\frac{dx}{dt} + k\int x(t)\,dt = \int F(t)\,dt + A, \tag{10}$$

in which A is an arbitrary constant of integration. Since the function $F(t)$ is prescribed, the integral of $F(t)$ in (10) can be evaluated. However, *since the solution $x(t)$ is not yet known, the integral $\int x(t)\,dt$ cannot be evaluated, and we cannot proceed with our solution by repeated integration.*

Be sure to understand this point.

Thus, solving differential equations is, in general, not merely a matter of undoing the derivatives by integration. The theory and technique involved is considerable and will occupy us throughout this book. To develop that theory we will need to establish distinctions — definitions, some of which are given below.

1.1.2 Modeling. Besides *solving* the differential equations that arise in applications, we must *derive* them in the first place. Their derivation is called the *modeling* part of the analysis because it leads to the mathematical problem that is to be solved. To model the motion of the mass shown in Fig. 1, for instance, we defined the displacement variable x, identified the relevant logic as Newton's second law of motion, and arrived at the differential equation (1) that models the motion of the mass, subject to the approximations that the friction force exerted on the mass by the table and the force on it do to air resistance are negligible. The upshot is that mathematical models are not "off the shelf" items, they require thoughtful development.

1.1.3 The order of a differential equation. The **order** of a differential equation is the order of the highest derivative (of the unknown function or functions) in the equation. For instance, (9) is a second-order differential equation.

As additional examples,

$$\frac{dN}{dt} = r\left(1-\frac{1}{K}N\right)N - \frac{N^2}{1+N^2} \qquad (0 < t < \infty) \tag{11}$$

for $N(t)$ and

$$EI\frac{d^4y}{dx^4} = -w(x) \qquad (0 < x < L) \tag{12}$$

for $y(x)$ are of first and fourth order, respectively.

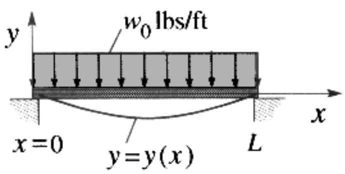


Figure 3. A beam subjected to a uniform load w lbs/ft; $y = y(x)$ is the deflection that results.

In (11) the independent variable is the time t, the dependent variable is the population N of *spruce budworms*, and r and K are known constants. The "population dynamics" of the spruce budworm has been the subject of research because budworms eat the foliage on balsam fir trees and a budworm outbreak can result in the defoliation and destruction of an entire forest.

Equation (12) governs the vertical deflection $y(x)$ of a beam of length L subjected to a prescribed load $w(x)$ lb/ft, and will be encountered in subsequent chapters. In Fig. 3 we've taken $w(x)$ to be a constant, w_0, so the total load is w_0L. Equation (12) is derived in a sophomore mechanical or civil engineering course on solid mechanics. In it, E and I are physical constants regarding the beam material and cross-sectional dimensions, respectively.

Equation (12) is similar to (2) in that it can be solved by repeated integration. To solve (2) we integrated (with respect to t) twice, and in doing so there arose two arbitrary "constants of integration." Similarly, to solve (12) we can integrate (with respect to x) four times, so there will be four arbitrary constants (Exercise 11).

These few examples hardly indicate the proliferation of differential equations that arise in applications — not just in engineering and physics, but in such diverse fields as biology, economics, psychology, chemistry, and agriculture. Since the applications are diverse, the independent and dependent variables differ from one application to another; for instance, in (2) the dependent variable is displacement and in (11) it is population. Often, though not necessarily, the independent variable will be a space coordinate x [as in (12)] or the time t [as in (1) and (11)]. As generic variables we will generally use x and y as the independent and dependent variables, respectively. With this notation, we can express our general nth-order differential equation for $y(x)$ as

We will often use x and y as generic independent and dependent variables, respectively.

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n}\right) = 0 \tag{13}$$

or, using the more compact prime notation for derivatives, $F(x, y, y', \dots, y^{(n)}) = 0$, in which $y'(x)$ means dy/dx, $y''(x)$ means d^2y/dx^2, and so on. In (12), for instance, $F(x, y, y', ..., y'''')$ is $EIy'''' + w(x)$, and in (11), in which the variables are t and N instead of the generic variables x and y, we can identify $F(t, N, N')$ as $N' - r(1-N/K)N - N^2/(1+N^2)$.

1.1.4 Linear and nonlinear equations. In studying curves in the x, y plane, analytic geometry, one begins with straight lines, defined by equations of the form $ax + by = c$. And in studying surfaces in x, y, z space one begins with planes, defined by equations of the form $ax + by + cz = d$. Such equations are **linear** because the variables occur as a linear combination.

A **linear combination** of quantities $x_1, \dots, x_n$ means a constant times x_1, plus a constant times $x_2, \dots$, plus a constant times x_n.

Likewise, to study differential equations it is best not to begin with the general case (13), but with *linear equations, ones in which the unknown function and its derivatives [namely,* $y, y', ..., y^{(n)}$ *] occur as a linear combination,*

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y(x) = f(x), \tag{14}$$

in which the **coefficients** $a_0(x), \dots, a_n(x)$ and the $f(x)$ on the right-hand side are prescribed functions of the independent variable x. An nth-order differential equation is **linear** if it is expressible in the form (14) and **nonlinear** if it is not. That is, (14) is a linear nth-order differential equation for $y(x)$ because it is in the form of a linear combination of y, y', ... , $y^{(n)}$ equaling some prescribed function of x.

To illustrate (14), (9) is a linear second-order equation [with $x(t)$ instead of $y(x)$] with $a_0(t) = m, a_1(t) = 0, a_2(t) = k$, and $f(t) = F(t)$, and (12) is a linear fourth-order equation with $a_0(x) = EI, a_1(x) = a_2(x) = a_3(x) = a_4(x) = 0$, and $f(x) = w(x)$. However, the first-order equation (11) is nonlinear; it cannot be put in the linear form $a_0(t)N' + a_1(t)N = f(t)$ because of the N^2 and $N^2/(1+N^2)$ terms, which we refer to as *nonlinear terms*.

Further, the linear differential equation (14) is **homogeneous** if $f(x)$ is zero and **nonhomogeneous** if $f(x)$ is not zero. For instance, (9) is nonhomogeneous because of the $F(t)$, and (12) is nonhomogeneous because of the $-w(x)$, but the linear second-order equation $y'' - e^x y' + 4y = 0$, for instance, is homogeneous because the right-hand side [after all of the y, y', and y'' terms are put on the left, as in (14)] is zero.

What physical or mathematical significance can we attach to the $f(x)$ term in (14)? In (1), for instance, $F(t)$ was an applied force that acted on the mass over the t interval of interest; in (12), $-w(x)$ was an applied force or load distribution that acted on the beam over the x interval of interest. Thus, it is common to call $f(x)$ in (14) a **forcing function** — even if it is not physically a force. For instance, in the linear differential equation governing the charge on a capacitor in an electrical circuit the forcing function will be seen in Section 1.3.5 to be an applied voltage, not an applied force, yet we will still call it a forcing function.

We call the right-hand side of (14) the "forcing function." Think of it as an "input," along with any initial conditions.

It is useful to think in terms of "inputs" and "outputs." If the linear equation (14) is augmented by initial conditions, for instance as (6a) was augmented by the initial conditions (6b), then both the forcing function $f(x)$ and the initial conditions are called **inputs**, and the response $y(x)$ to those inputs is the **output**. For instance, in the solution $x(t) = F_0t^2/2m + x_0't + x_0$ to the IVP (6), the term $F_0t^2/2m$ is the response to the forcing function F_0 in (6a), and the term $x_0't + x_0$ is the response to the initial conditions (6b). The idea is indicated schematically in Fig. 4.

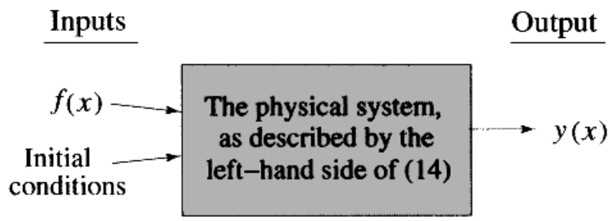


Figure 4. Schematic of the input/output nature of a linear initial value problem with differential equation (14).

1.1.5 Our plan. We will find that nonlinear differential equations are generally much more difficult than linear ones, and also that higher-order equations are more difficult than lower-order ones. Thus, *we will begin our study in Section 1.2 by considering differential equations that are both linear and of the lowest order — first order.*

To motivate our plan (which is typical, not unique to this text), think of one's early studies of algebra. Probably, it began with a single equation in one unknown, $ax = b$. From there, we proceeded in each of two different directions: higher-order algebraic equations in one unknown (quadratic, cubic, and so on), and also systems of linear equations in more than one unknown, such as the two equations $6x+y = 7$ and $2x - 8y = 5$ for x and y. The same is a good idea in differential equations. Following our study of first-order linear equations, in Chapter 1, we will proceed to higher-order linear equations in Chapters 2 and 3, and to **systems** of differential equations in Chapters 4 and 7.

We will develop three different approaches to solving and studying differential equations: **analytical**, **numerical**, and **qualitative**. Our derivation of the solution (8) of the problem (6) illustrates what we mean by *analytical*; that is, by carrying out a sequence of calculus-based steps we were able to end up with an expression for the unknown function. Most of our attention in this text is on analytical solution methods and the theory on which they are based.

Many differential equations, such as the budworm equation (11), are too difficult to solve analytically, but we can turn to a *numerical* method such as Euler's method. The idea, in numerical solution, is to give up on finding an expression for the solution $N(t)$ and to be content to numerically generate approximate values of $N(t)$ at a sequence of discrete t's, the spacing between them being called the *step size* of the calculation. To illustrate, let $r = K = 1$ in (11), let the initial condition be $N(0) = 3$, and let the step size be 0.2. The result of the Euler calculation is shown by the points in Fig. 5 along with the exact solution. Don't be concerned that the Euler-generated points are so inaccurate in this illustration, so far from the exact solution; one can increase the accuracy by reducing the step size.

Finally, by *qualitative* methods we mean methods that give information about solutions, without actually finding them analytically or numerically. One qualitative method that we will use is the "direction field," which we will use in Section 1.2.

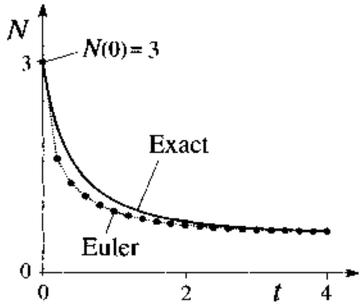


Figure 5. Solution of the budworm equation (11) for $r = K = 1$ and $N(0) = 3$. The dots are the approximate numerical solution (using Euler's method with a step size of 0.2) and the solid curve is the exact solution.

1.1.6 Direction field. If we can solve a given first-order equation $F(x, y, y') = 0$ for y', by algebra, we can express the equation in the form $y' = f(x, y)$, that is,

$$\frac{dy}{dx} = f(x, y), \tag{15}$$

which we take as our starting point.

To discuss the direction field of (15) we must first define the term "solution curve." A **solution curve** or **integral curve** of (15) is the graph of a solution $y(x)$ of that equation. Observe from (15) that at each point in the x, y plane at which $f(x, y)$ is defined, $f(x, y)$ gives the slope dy/dx of the solution curve through that point. For instance, for the differential equation

$$\frac{dy}{dx} = 4 - 3x - y \qquad (-\infty < x < \infty) \tag{16}$$

the slope of the solution curve through the point $(2, 1)$ is given by $f(2, 1) = 4 - 3(2) - 1 = -3$.

(16) is a linear first-order equation. Comparing it with (14) we see that n is 1, $a_0(x)$ is 1, $a_1(x)$ is 1, and $f(x)$ is $4-3x$.

In Fig. 6 we've plotted **the direction field** or **slope field** corresponding to (16), namely, a field of short line segments through a discrete set of points called a **grid**. Each line segment is called a *lineal element*, and the lineal element through any given grid point has the same slope as the solution curve through that point and is therefore a short tangent line to that solution curve. In computer graphics packages we can specify lines with or without arrowheads; we omitted arrowheads in Fig. 6.

In intuitive language, the direction field shows the overall "flow" of solution curves. Consider for instance the initial point $(0, -5)$ shown in Fig. 6 by the heavy dot; that is, consider the initial condition $y(0) = -5$ to be appended to (16). By following the direction field, we can sketch by hand the solution curve passing through that point. (Actually, we obtained that solution curve by computer, but we could just as well have sketched it by hand.) Four other solution curves are included as well.

You may wonder why we've shown the solution curve through $(0, -5)$ both to the right and to the left; if $(0, -5)$ is an "initial point," then shouldn't the solution through that point extend only to the right, over $0 < x < \infty$? If the independent variable is the time t, then the t interval of interest is usually to the right of the initial time. But in the present example the interval of interest of the independent variable x was stated in (16) to be $-\infty < x < \infty$. Hence, we extended the solution curve in Fig. 6 both to the right and to the left of the initial point.

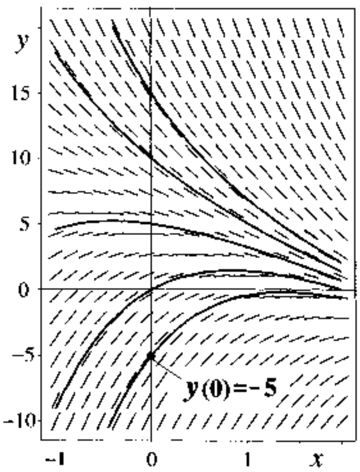


Figure 6. Direction field for $y' = 4 - 3x - y$, and representative solution curves.

Incidentally, (16) is linear [because it can be expressed in the form (14) as $dy/dx + y = 4 - 3x$], but (15) admits a direction field whether it is linear or nonlinear. In fact, direction fields are particularly valuable for nonlinear equations because those are more difficult, in general, and we may need all the help we can get to obtain information about their solutions.

1.1.7 Computer software. There are powerful computer software systems, such as *Maple*, *Mathematica*, and *MATLAB*, that can be used to implement much of the mathematics presented in this text — symbolically, numerically, and graphically. Though the reading is not tied to any particular software, it is anticipated that you will be using some such system as you go through this text. Thus, included among the exercises are some that call for the use of computer software, and the Student Solution Manual includes *Maple*, MATLAB, and *Mathematica* tutorials specifically for this text, chapter by chapter. Even if an exercise does not call for the use of software, and the answer is not given at the back of the book, you may be able to use computer software to solve the problem and check your work, and to plot your results if you wish.

Closure. We've introduced the idea of a differential equation and enough terminology to get us started. We defined the order of the equation as the order of the highest-order derivative in the equation, and we classified the equation as linear if it is expressible in the form (14), and nonlinear otherwise. We found that some differential equations, such as (2), can be solved merely by repeated integration, but in general that strategy does not work. However, whether or not the solution process proceeds by direct integrations, we can think of the arbitrary constants that will arise as "integration constants." The presence of these arbitrary constants makes it possible for the solution to satisfy initial conditions, such as the initial displacement and the initial velocity in (6b). Later, we will see that for differential equations of second order and higher it may be appropriate to specify conditions at *more than one point*. This case is illustrated in Exercise 11.

We've begun to classify different types of differential equations — for instance as linear or nonlinear, as homogeneous or nonhomogeneous, by order, and so on. Why do we do that? Because the most general differential equation is far too difficult for us to solve. Thus, we break the set of all possible differential equations into various categories and develop theory and solution strategies that are tailored to a given category. Historically, however, the early work on differential equations — by such great mathematicians as *Leonhard Euler* (1707–1783), *James Bernoulli* (1654–1705) and his brother *John* (1667–1748), *Joseph-Louis Lagrange* (1736–1813), *Alexis-Claude Clairaut* (1713–1765), and *Jean le Rond d'Alembert* (1717–1783) — generally involved attempts at solving specific equations rather than developing a general theory.

From the point of view of applications, we will find that in many cases diverse phenomena are modeled by the same differential equation. The remarkable conclusion is that *if one knows a lot about mechanical systems, for example, then one thereby knows a lot about electrical, biological, and social systems, for example, to* *whatever extent they are modeled by differential equations of the same form.* The significance of this fact can hardly be overstated as a justification for a careful study of the mathematical field of differential equations.

EXERCISES 1.1

NOTE: UNDERLINING OF AN EXERCISE NUMBER OR LETTER INDICATES THAT THAT EXERCISE IS INCLUDED AMONG THE ANSWERS TO THE SELECTED EXERCISES AT THE END OF THE TEXT.

1. Concepts of Order and Solution. State the order of each differential equation, and show whether or not the given functions are solutions of that equation.

(a) $y' = 3y$; $y_1(x) = e^{3x}$, $y_2(x) = 76e^{3x}$, $y_3(x) = e^{-3x}$

(b) $(y')^2 = 4y$; $y_1(x) = x^2$, $y_2(x) = 2x^2$, $y_3(x) = e^{-x}$

(c) $2yy' = 9\sin 2x$; $y_1(x) = \sin x$, $y_2(x) = 3\sin x$, $y_3(x) = e^x$

(d) $y'' - 9y = 0$; $y_1(x) = e^{3x} - e^x$, $y_2(x) = 3\sinh 3x$, $y_3(x) = 2e^{3x} - e^{-3x}$

(e) $(y')^2 - 4xy' + 4y = 0$; $y_1(x) = x^2 - x$, $y_2(x) = 2x - 1$

(f) $y'' + 9y = 0$; $y_1(x) = 4\sin 3x + 3\cos 3x$, $y_2(x) = 6\sin(3x + 2)$

(g) $y'' - y' - 2y = 6$; $y_1(x) = 5e^{2x} - 3$, $y_2(x) = -2e^{-x} - 3$

(h) $y''' - y'' = 6 - 6x$; $y_1(x) = 3e^x + x^3$

(i) $x^6 y''' = 6y^2$; $y_1(x) = x^3$, $y_2(x) = x^2$, $y_3(x) = 0$

(j) $y'' + y' = y^2 - 4$; $y_1(x) = x$, $y_2(x) = 1$, $y_3(x) = 2$

(k) $y' + 2xy = 1$; $y_1(x) = 4e^{-x^2}$, $y_2(x) = e^{-x^2}\left(\int_0^x e^{t^2}\,dt + A\right)$ for any value of A. HINT: For $y_2(x)$, recall the *fundamental theorem of the integral calculus*, that if $F(x) = \int_a^x f(t)\,dt$ and $f(t)$ is continuous on $a \le x \le b$, then $F'(x) = f(x)$ on $a \le x \le b$. [The reason we did not evaluate the integral in $y_2(x)$ is that it is too hard; it cannot be evaluated as a finite combination of elementary functions.]

(l) $y' - 4xy = x^2$; $y_1(x) = e^{2x^2}\int_1^x e^{-2t^2} t^2\,dt$
HINT: See the hint in part (k).

2. Including an Initial Condition; First-Order Equations. First, verify that the given function $y(x)$ is a solution of the given differential equation, for any value of A. Then, solve for A so that $y(x)$ satisfies the given initial condition.

(a) $y' + y = 1$; $y(x) = 1 + Ae^{-x}$; $y(0) = 3$

(b) $y' - y = x$; $y(x) = Ae^x - x - 1$; $y(2) = 5$

(c) $y' + 6y = 0$; $y(x) = Ae^{-6x}$; $y(4) = -1$

(d) $y' = 2xy^2$; $y(x) = -1/\left(x^2 + A\right)$; $y(0) = 5$

(e) $yy' = x$; $y(x) = \sqrt{x^2 + A}$; $y(1) = 10$

3. Second-Order Equations. First, verify that the given function is a solution of the given differential equation, for any constants A, B. Then, solve for A, B so that $y(x)$ satisfies the given initial conditions.

(a) $y'' + 4y = 8x^2$; $y(x) = 2x^2 - 1 + A\sin 2x + B\cos 2x$; $y(0) = 1$, $y'(0) = 0$

(b) $y'' - y = x^2$; $y(x) = -x^2 - 2 + A\sinh x + B\cosh x$; $y(0) = -2$, $y'(0) = 0$

(c) $y'' - 2y' + y = 0$; $y(x) = (A + Bx)e^x$; $y(0) = 0$, $y'(0) = 0$

(d) $y'' - y' = 0$; $y(x) = A + Be^x$; $y(0) = 1$, $y'(0) = 0$

(e) $y'' + 2y' = 4x$; $y(x) = A + Be^{-2x} + x^2 - x$; $y(0) = 0$, $y'(0) = 0$

4. Linear or Nonlinear? Classify each equation as linear or nonlinear:

(a) $y' + e^x y = 4$

(b) $yy' = x + y$

(c) $e^x y' = x - 2y$

(d) $y' - e^y = \sin x$

(e) $y'' + (\sin x)y = x^2$

(f) $y'' - y = e^x$

(g) $yy''' + 4y = 3x$

(h) $y''' = y$

(i) $\dfrac{y'' - y}{y' + y} = 4$

(j) $y''' + y^2 + 6y = x$

(k) $y'' = x^3 y'$

(l) $y''' + y''y' = 3x$

(m) $y'' - xy' = 3y + 4$

(n) $y''' = 4y$

5. Exponential Solutions. Each of the following is a homogeneous linear equation with constant coefficients [i.e., the coefficients $a_0(x), \ldots, a_n(x)$ in (14) are constants]. As we will see in Chapter 2, such equations necessarily admit solutions of exponential type, that is, of the form $y(x) = e^{rx}$ in which r is a constant. For the given equation, determine the value(s) of r such that $y(x) = e^{rx}$ is a solution. HINT: Put $y(x) = e^{rx}$ into the equation and determine any values of r such that the equation is satisfied, that is, reduced to an identity.

(a) $y' + 3y = 0$ (b) $2y' - y = 0$
(c) $y'' - 3y' + 2y = 0$ (d) $y'' - 2y' + y = 0$
(e) $y'' - 2y' - 3y = 0$ (f) $y'' + 5y' + 6y = 0$
(g) $y''' - y' = 0$ (h) $y''' - 2y'' - y' + 2y = 0$
(i) $y'''' - 6y'' + 5y = 0$ (j) $y'''' - 10y'' + 9y = 0$

6. Powers of x as Solutions. Unlike the equations in Exercise 5, the following equations admit solutions of the form $y(x) = x^r$, in which r is a constant. For the given equation determine the value(s) of r for which $y(x) = x^r$ is a solution.

(a) $xy' + y = 0$ (b) $xy' - y = 0$
(c) $xy'' + y' = 0$ (d) $xy'' - 4y' = 0$
(e) $x^2y'' + xy' - 9y = 0$ (f) $x^2y'' + xy' - y = 0$
(g) $x^2y'' + 3xy' - 2y = 0$ (h) $x^2y'' - 2y = 0$

7. Figure 6. Five representative solution curves are shown in Fig. 6. There is also one solution curve, not shown in the figure, that is a straight line. Find the equation of that straight-line solution. HINT: Seek a solution of (16) in the form $y(x) = mx + b$. Put that into (16) and see if you can find m and b such that the equation is satisfied. Does your result *look* correct — in terms of the direction field shown in the figure?

8. Straight-Line Solutions. First, read Exercise 7. For each given differential equation find any straight-line solutions, that is, of the form $y(x) = mx + b$. If there are none, state that.

(a) $y' + 2y = 2x - 1$ (b) $y' + 4y = 20$
(c) $y'' + y'^2 = 9$ (d) $y'' - 2y' + y = 0$
(e) $yy' + x = 0$ (f) $y' = y^2$
(g) $y' = y^2 - 4x^2 - 12x - 7$ (h) $yy' - y^2 = -x^2 + 3x - 2$
(i) $y' = y^2 - 4x^2 - 2$ (j) $y'' + y' + y = 3x$
(k) $y'' + y = x^2 + 7$ (l) $y'' - y' = 24x$
(m) A differential equation supplied by your instructor.

9. Grade This. Asked to solve the differential equation $\frac{dx}{dt} + x = 10t$, a student proposes this solution: By integrating with respect to t, obtain

$$x + xt = 5t^2 + A, \text{ so } x(t) = \frac{5t^2 + A}{1 + t}.$$

Is this correct? Explain.

10. No Solutions. (a) Show that the differential equation

$$\left|\frac{dy}{dx}\right| + |y| + 3 = 0 \tag{10.1}$$

has *no* solutions on any x interval. NOTE: This example shows that it is *possible* for a differential equation to have no solutions.
(b) Is (10.1) linear? Explain.

11. Deflection of a Loaded Beam; Boundary Conditions. Consider the beam shown in Fig. 3. Its deflection $y(x)$ is modeled by the fourth-order linear differential equation

$$EI\frac{d^4y}{dx^4} = -w_0. \tag{11.1}$$

(a) By repeated integration of (11.1), show that

$$y(x) = \frac{1}{EI}\left(-\frac{w_0}{24}x^4 + \frac{A}{6}x^3 + \frac{B}{2}x^2 + Cx + D\right). \tag{11.2}$$

(b) From Fig. 3 it is obvious that $y(0) = 0$ and $y(L) = 0$. Not so obvious (without some knowledge of Euler beam theory) is that $y''(0) = 0$ and $y''(L) = 0$ (because no moments are applied at the two ends). Use those four conditions to evaluate A, B, C, D in (11.2), and thus show that

$$y(x) = -\frac{w_0}{24EI}\left(x^4 - 2Lx^3 + L^3x\right). \tag{11.3}$$

NOTE: In this application the conditions are at two points, $x = 0$ and $x = L$, rather than one, so they are called **boundary conditions** rather than initial conditions, and the problem is a **boundary value problem** rather than an initial value problem.
(c) From (11.3), show that the largest deflection is $-5w_0L^4/384EI$.

1.2 LINEAR FIRST-ORDER EQUATIONS

We begin with the general *linear* first-order differential equation

$$a_0(x)\frac{dy}{dx} + a_1(x)y = f(x), \tag{1}$$

in which $a_0(x)$, $a_1(x)$, and $f(x)$ are prescribed. We assume $a_0(x)$ is nonzero on the x interval of interest, so we can divide (1) by $a_0(x)$ and obtain the simpler-looking version

$$\boxed{\frac{dy}{dx} + p(x)y = q(x),} \tag{2}$$

That is, $a_1(x)/a_0(x)$ is $p(x)$ and $f(x)/a_0(x)$ is $q(x)$.

which is the **standard form** of the linear first-order equation. It is assumed throughout this section that $p(x)$ and $q(x)$ are continuous on the x interval of interest. As noted in Section 1.1, we cannot solve (2) merely by integrating it because integration gives

$$y(x) + \int p(x)y(x)\,dx = \int q(x)\,dx + C, \tag{3}$$

and we don't yet know the $y(x)$ in the integrand of $\int p(x)y(x)\,dx$.

1.2.1 The simplest case. When stuck, it is good to simplify the problem temporarily, to get started. In this case we might do that by letting $p(x)$ or $q(x)$ be zero. If we let $p(x) = 0$, so the differential equation is simply

$$\frac{dy}{dx} = q(x), \tag{4}$$

then the $\int p(x)y(x)\,dx$ term causing the trouble in (3) drops out and we successfully obtain the solution by integrating (4) and obtaining

$$y(x) = \int q(x)\,dx + A, \tag{5}$$

in which the integration constant A is arbitrary. The integral in (5) does exist (i.e., converge) because we're assuming that $p(x)$ and $q(x)$ are continuous. Reversing our steps, differentiation of (5) shows that (5) does satisfy the original differential equation (4), because $\frac{dy}{dx} = \frac{d}{dx}\left(\int q(x)\,dx + A\right) = q(x)$.

We call (5) a **general solution** of (4) because it contains *all* solutions of (4). Put differently, (4) implies (5), and (5) implies (4), as we've seen. In fact, (5) is a whole "family" of solutions, a **one-parameter family** in which the parameter is the arbitrary constant A. Each choice of A gives a member of that family, called a **particular solution** of (4). For instance, if $q(x) = 6e^{2x}$, then the general solution is given by (5) as $y(x) = 3e^{2x} + A$, the graph of which is shown, for several values of A, in Fig. 1.

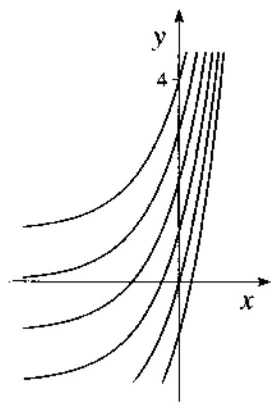


Figure 1. The solutions $y(x) = 3e^{2x} + A$ of the differential equation $\frac{dy}{dx} = 6e^{2x}$, for several values of A.

1.2.2 The homogeneous equation. Now consider the special case of (2) for which $q(x) = 0$ instead,

$$\frac{dy}{dx} + p(x)y = 0. \tag{6}$$

To solve (6), first divide both terms by y [which is permissible if $y(x) \neq 0$ on the x interval, which we tentatively assume], then integrate with respect to x:

$$\int \frac{1}{y}\frac{dy}{dx}\,dx + \int p(x)\,dx = 0, \tag{7a}$$

$$\int \frac{1}{y}\,dy + \int p(x)\,dx = 0, \tag{7b}$$

$$\ln|y| + \int p(x)\,dx = C, \tag{7c}$$

$$|y| = e^{-\int p(x)\,dx + C} = e^C e^{-\int p(x)\,dx}, \tag{7d}$$

and it follows from (7d) that

$$y(x) = \pm e^C e^{-\int p(x)\,dx}.$$

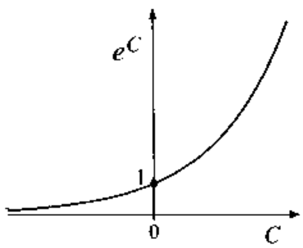


Figure 2. $0 < e^C < \infty$; e^C is not zero for any finite value of C.

The integration constant C is arbitrary so $-\infty < C < \infty$, and therefore $0 < e^C < \infty$ (Fig. 2). If we abbreviate $\pm e^C$ as A, then A is *any* number, positive or negative, but not zero because the exponential e^C is nonzero (Fig. 2). Thus, we can write $y(x)$ in the friendlier form

$$y(x) = A\,e^{-\int p(x)\,dx} \tag{8}$$

in which A is an arbitrary constant, positive or negative but not zero.

When we evaluate $\int p(x)\,dx$ in (8) we don't need to include an additive arbitrary integration constant; we already did in (7c).

Because we tentatively assumed that $y \neq 0$ in (7a), we must check the case $y = 0$ separately. In fact, we see that $y(x) = 0$ satisfies (6) because it reduces (6) to $0 + 0 = 0$. We can bring this additional solution under the umbrella of (8) if we now allow A to be zero. The upshot is that the general solution of (6) is given by (8) where A is an *arbitrary constant*: $-\infty < A < \infty$.

By (8) being the general solution of (6), we mean that it contains *all* solutions of (6). Each individual solution corresponds to a particular choice of the arbitrary constant A.

The preceding reasoning regarding the inclusion of the solution $y(x) = 0$ is similar to the reasoning involved in solving the algebraic equation $x^2 + 2x = 0$ for x. If we divide through by x, tentatively assuming that $x \neq 0$, then we obtain $x + 2 = 0$ and the root $x = -2$. Unless we then check the disallowed case $x = 0$, to see if it satisfies the equation $x^2 + 2x = 0$, we will have missed the root $x = 0$.

The key to our solution of (6) was dividing the equation by y because that step enabled us to end up [in (7b)] with one integral on y alone and one on x alone. The process of separating the x and y variables is called **separation of variables** and will be used again in Section 1.4 to solve certain *nonlinear* equations. Verification that (8) satisfies (6) is left for the exercises.

EXAMPLE 1. One to Remember Forever. If $p(x)$ is merely a constant in (6), then

$\int p(x)\,dx = px$, and (8) gives the general solution of

$$\frac{dy}{dx} + py = 0. \tag{9}$$

on $-\infty < x < \infty$, as

$$y(x) = Ae^{-px}, \tag{10}$$

Roughly put, this example is as important in the study and application of differential equations as is the straight line in the study of curves.

with A an arbitrary constant. Recall that the graphs of the solutions, $y = Ae^{-px}$ in this case, are called the solution curves or integral curves. These are plotted in Fig. 3 for several representative values of A, along with the direction field. Notice, in the figure, how the

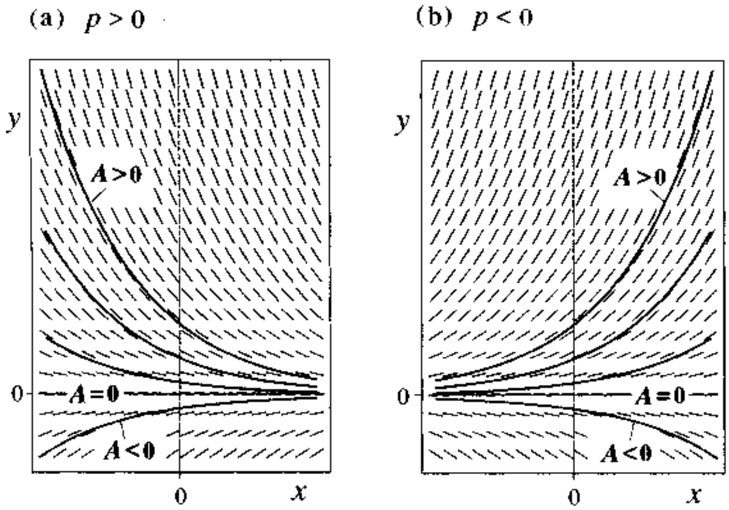


Figure 3. Representative solution curves $y(x) = Ae^{-px}$ for the equation $y' + py = 0$; direction field included.

solution curves follow the "flow" that is indicated by the direction field. ∎

EXAMPLE 2. Solve

$$y' + (\sin x)y = 0 \qquad (-\infty < x < \infty). \tag{11}$$

By comparing (11) with (6) we see that $p(x) = \sin x$. Then (8) gives

$$y(x) = Ae^{-\int \sin x\,dx} = Ae^{\cos x} \tag{12}$$

in which A is arbitrary.

Besides using the "off-the-shelf" formula (8), it is instructive to solve (11) by carrying out the separation of variables method that we used to derive (8) — as if stranded on a

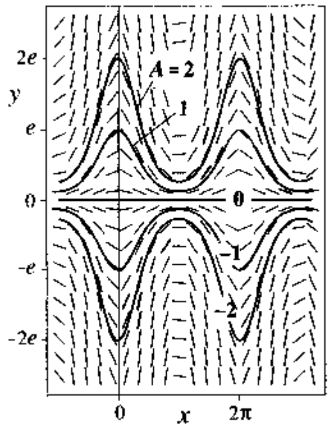


Figure 4. Representative solution curves $y(x) = Ae^{\cos x}$ for the equation $y' + (\sin x)y = 0$; direction field included.

desert island, our textbook having gone down with the ship:

$$\frac{dy}{y} + \sin x\,dx = 0, \tag{13a}$$

$$\int \frac{dy}{y} + \int \sin x\,dx = 0, \tag{13b}$$

$$\ln|y| - \cos x = C, \tag{13c}$$

$$|y| = e^{\cos x + C} = e^{\cos x}e^{C}, \tag{13d}$$

$$y(x) = \pm e^{C}e^{\cos x} = Ae^{\cos x}, \tag{13e}$$

which is the same result as we obtained in (12) by putting $p(x) = \sin x$ into (8). The solution curves are displayed for several values of A in Fig. 4. ∎

1.2.3 Solving the full equation by the integrating factor method. We're now prepared to solve the full equation

$$\boxed{\frac{dy}{dx} + p(x)y = q(x),} \tag{14}$$

including both $p(x)$ and $q(x)$. This time our separation of variables technique fails because when we try to separate variables by re-expressing (14) as

$$\frac{1}{y}\,dy + p(x)\,dx = \frac{q(x)}{y}\,dx$$

the term on the right-hand side spoils the separation because $q(x)/y$ is a function not only of x but also of y. Instead of separation of variables, we will use an "integrating factor" method invented by the great mathematician *Leonhard Euler* (1707–1783).[1] We first motivate Euler's idea with an example:

(Pronounced "oiler," not "yuler.")

EXAMPLE 3. Motivating Euler's Integrating Factor Method. We wish to solve the equation

$$\frac{dy}{dx} + \frac{1}{x}y = 12x^2 \qquad (0 < x < \infty) \tag{15}$$

for $y(x)$. Notice that if we multiply (15) through by x and obtain

$$xy' + y = 12x^3, \tag{16}$$

[1]Euler is among the greatest and most productive mathematicians of all time. He contributed to virtually every branch of mathematics and to the application of mathematics to the science of mechanics. During the last 17 years of his life he was totally blind but produced several books and some 400 research papers. He knew by heart the entire *Aeneid* by Virgil, and he knew the first six powers of the first 100 prime numbers. If the latter does not seem impressive, note that the 100th prime number is 541 and its sixth power is 25,071,688,922,457,241.

then the left-hand side is the derivative of the product $xy(x)$ because $[xy(x)]' = y(x) + xy'(x)$. Thus, (16) can be expressed as

$$\frac{d}{dx}(xy) = 12x^3 \tag{17}$$

which can now be solved by integration:

$$\int d(xy) = \int 12x^3\,dx, \tag{18a}$$

$$xy = 12\frac{x^4}{4} + C, \tag{18b}$$

where C is arbitrary. Thus, we obtain the general solution

$$y(x) = 3x^3 + \frac{C}{x} \tag{19}$$

of (15). We can readily verify that substitution of (19) into (15) produces an identity (namely, $12x^2 = 12x^2$) on the interval $0 < x < \infty$ specified in (15). ∎

The integrating factor method is similar to the familiar method of solving a quadratic equation $ax^2 + bx + c = 0$ by completing the square: We add a suitable number to both sides so that the left-hand side becomes a "perfect square;" then the equation can be solved by the inverse operation — by taking square roots. *Analogously, in the integrating factor method we multiply both sides of (14) by a suitable function so that the left-hand side becomes a "perfect derivative;" then the equation can be solved by the inverse operation — by integration.* In Example 3 the integrating factor was x; when we multiplied (15) by x the left-hand side became the derivative $(xy)'$. Then $(xy)' = 12x^3$ could be solved [in (18)] by integration.

The integrating factor method is similar to the method of solving a quadratic equation by completing the square.

To apply Euler's method to the general equation (14), multiply (14) by a (not yet known) **integrating factor** $\sigma(x)$:

$$\underline{\sigma y' + \sigma p y} = \sigma q. \tag{20}$$

Our aim is to determine $\sigma(x)$ so the left-hand side of (20) is the derivative of σy, namely,

$$\frac{d}{dx}(\sigma y) \quad \text{or, written out,} \quad \underline{\sigma y' + \sigma' y}. \tag{21}$$

In (15), we "noticed" that $\sigma(x) = x$ works, but in general we cannot expect to find $\sigma(x)$ by inspection.

To match the underlined terms in (20) and (21), we need merely choose $\sigma(x)$ so that $\sigma p = \sigma'$:

$$\sigma' = \sigma p. \tag{22}$$

But the latter, rewritten as

$$\sigma' - p(x)\sigma = 0, \tag{23}$$

is of the same form as the equation $y' + p(x)y = 0$ that we solved in Section 1.2.2 [if we change $y(x)$ to $\sigma(x)$ and $p(x)$ to $-p(x)$], so its solution is given by (8) as

$$\sigma(x) = Ae^{\int p(x)\,dx}. \tag{24}$$

We don't need the most general integrating factor, we simply need *an* integrating factor, so we can choose $A = 1$ without loss. Then

This is an integrating factor for (14).

$$\boxed{\sigma(x) = e^{\int p(x)\,dx}.} \tag{25}$$

With $\sigma(x)$ so chosen, (20) becomes

$$(\sigma y)' = \sigma q \quad \text{or} \quad \frac{d(\sigma y)}{dx} = \sigma q,$$

which can be integrated to give

$$\int d(\sigma y) = \int \sigma(x)q(x)\,dx, \tag{26a}$$

$$\sigma y = \int \sigma(x)q(x)\,dx + C, \tag{26b}$$

CAUTION: (27) is *not* the same as $y(x) = \frac{1}{\sigma(x)}\int \sigma(x)q(x)\,dx + C$. That is, don't merely "tack on" an integration constant at the end of the analysis; carry it along from the point at which it arises.

so a general solution of $y' + p(x)y = q(x)$ is

$$\boxed{y(x) = \frac{1}{\sigma(x)}\left(\int \sigma(x)q(x)\,dx + C\right),} \tag{27}$$

with the integrating factor $\sigma(x)$ given by (25).

The abbreviation IVP.

EXAMPLE 4. Solution by Integrating Factor Method. Solve the initial value problem (IVP for brevity)

$$\frac{dy}{dx} + 3y = 9x \qquad (-\infty < x < \infty), \tag{28a}$$

$$y(2) = 1. \tag{28b}$$

To solve, we could simply use the solution formula (27), or we could carry out the steps of the integrating factor method that led to (27). To use (27) "off the shelf," first compare (28a) with $y' + p(x)y = q(x)$ to identify $p(x)$ and $q(x)$: $p(x) = 3$ and $q(x) = 9x$. Then, (25) gives

$$\sigma(x) = e^{\int p(x)\,dx} = e^{\int 3\,dx} = e^{3x}, \tag{29}$$

and (27) gives a general solution of (28a) as

$\int xe^{ax}\,dx = (ax-1)\frac{e^{ax}}{a^2}$.

$$y(x) = e^{-3x}\left(\int e^{3x}9x\,dx + C\right) = 3x - 1 + Ce^{-3x}. \tag{30}$$

Finally, apply the initial condition (28b) to (30) to determine C:

$$y(2) = 1 = 6 - 1 + Ce^{-6},$$

so $C = -4e^6$. Hence, the solution of the IVP (28) is

$$y(x) = 3x - 1 - 4e^{-3(x-2)}, \tag{31}$$

which is plotted as the solid curve in Fig. 5.

Alternatively, let us solve (28) using not the solution formula (27), but the **integrating factor method**. First, multiply (28a) through by $\sigma(x)$:

$$\underline{\sigma y' + 3\sigma y} = 9\sigma x. \tag{32}$$

We want to choose σ so the left side of (32) is a "perfect derivative" $(\sigma y)'$ or, written out,

$$\underline{\sigma y' + \sigma' y}. \tag{33}$$

For the underlined terms in (32) and (33) to be identical we need merely match the coefficients 3σ and σ' of y. Thus,

$$\sigma' = 3\sigma, \tag{34}$$

which gives $\sigma(x) = e^{3x}$. Then (32) is in the desired form $(\sigma y)' = 9x\sigma$, which can be integrated to give $\sigma y = \int 9x\sigma\, dx + C$, or,

$$y(x) = e^{-3x}\left(\int 9xe^{3x}\, dx + C\right) = 3x - 1 + Ce^{-3x}, \tag{35}$$

which is the same result as that given in (30).

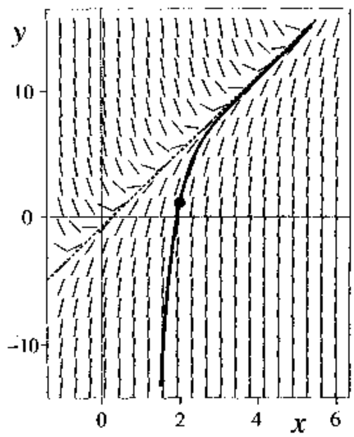


Figure 5. Graph of the solution (31) to the IVP (28), with the direction field. The heavy dot marks the initial point $y(2) = 1$.

COMMENT 1. Know and be comfortable with both approaches: memorizing and using (27) or, instead, using the integrating factor *method*.

COMMENT 2. We can see from (31) that the e^{-3x} term tends to zero as x increases, so every solution curve is asymptotic to the straight line $y = 3x - 1$. In fact, $y(x) = 3x - 1$ is itself a particular solution of (28a), corresponding to the choice $C = 0$ in (30), and is indicated in Fig. 5 by the dotted line. ▮

Can the integrating factor method fail? Perhaps for a given equation $y' + p(x)y = q(x)$ an integrating factor does not exist? No, $\sigma(x)$ is given by (25) and the only way that equation can fail to give $\sigma(x)$ is if the integral $\int p(x)\, dx$ does not exist. However, our assumption that $p(x)$ is continuous on the x interval of interest guarantees that the integral does exist.

1.2.4 Existence and uniqueness for the linear equation. A fundamental question in the theory of differential equations is whether a given differential equation for $y(x)$ *has* a solution through a given initial point $y(a) = b$ in the x, y plane and, if so, on what x interval it is valid. That is the question of **existence**. If a solution does exist, then the next question is that of **uniqueness**: Is that solution unique? That is, is there only one solution or is there more than one?

For linear initial value problems we have the following result.

THEOREM 1.2.1 *Existence and Uniqueness for Linear Initial Value Problems*
The linear initial value problem

$$y' + p(x)y = q(x); \quad y(a) = b \tag{36}$$

has a solution

$$y(x) = \frac{1}{\sigma(x)}\left(\int_a^x \sigma(s)q(s)\,ds + b\sigma(a)\right), \tag{37}$$

where $\sigma(x) = e^{\int p(x)\,dx}$ is an integrating factor of the differential equation in (36). That solution exists and is unique *at least* on the broadest open x interval, containing the initial point $x = a$, on which $p(x)$ and $q(x)$ are continuous.

In (37), s is just a dummy integration variable. Partial check of (37): Setting $x = a$ in (37) gives $y(a) = \frac{1}{\sigma(a)}[0 + b\sigma(a)] = b$, so (37) does satisfy the initial condition $y(a) = b$.

Unlike (27), (37) includes a definite integral instead of an indefinite integral, and C has been chosen so that the initial condition $y(a) = b$ is satisfied. We leave the derivation of (37) to the exercises, and turn to applications of the theorem.

EXAMPLE 5. Existence on $-\infty < x < \infty$. Consider the IVP (28) again, in the light of Theorem 1.2.1: $p(x) = 3$ and $q(x) = 9x$ are continuous for all x, so Theorem 1.2.1 guarantees that there exists a unique solution of (28) on $-\infty < x < \infty$. That solution was given by (31) and was plotted as the solid curve in Fig. 5. ▮

EXAMPLE 6. The Possibilities of Existence on a Limited Interval, and of No Solution. Consider the IVP

$$x\frac{dy}{dx} + y = 12x^3, \tag{38a}$$

$$y(1) = b \tag{38b}$$

First, identify $p(x)$ and $q(x)$ by getting (38a) into the standard form $y' + p(x)y = q(x)$.

We've left b unspecified so we can consider several different b's. Here, $p(x) = 1/x$, $q(x) = 12x^2$, and $a = 1$. Although $q(x)$ is continuous for all x, $p(x) = 1/x$ is discontinuous at $x = 0$, so Theorem 1.2.1 guarantees the existence and uniqueness of a solution to the IVP (38) at *least* on $0 < x < \infty$, because that is the broadest open x interval, containing the initial point $x = 1$, on which both $p(x)$ and $q(x)$ are continuous.

In fact, the general solution of (38a) was found in Example 3 to be

$$y(x) = 3x^3 + \frac{C}{x}, \tag{39}$$

and for the representative initial conditions $y(1) = 0$, $y(1) = 3$, and $y(1) = 5$ we obtain $C = -3, 0$, and 2, respectively. These solutions are plotted in Fig. 6, and we see that we can *think* of the vertical line $x = 0$ as a barrier or wall; if the initial point $(1, b)$ is above the curve $y = 3x^3$ the solution "climbs the wall" to $+\infty$ as $x \to 0$ and if the initial point is below $y = 3x^3$ the solution approaches $-\infty$ as $x \to 0$, because of the C/x term in (39).

There is just one solution, corresponding to $y(1) = 3$, that manages to cross the barrier, for then we obtain $C = 0$; then $y(x) = 3x^3$ and the C/x term that "blows up" at $x = 0$ is not present. Thus, through the initial point $y(1) = 3$ the unique solution $y(x) = 3x^3$ exists for *all* x, on $-\infty < x < \infty$. The presence of this exceptional solution does not violate the theorem because of the words "at least" in the last sentence of the theorem.

Thus far we've considered initial conditions at $x = 1$. Since $p(x) = 1/x$ and $q(x) = 12x^2$ are both continuous at $x = 1$, the existence of unique solutions through those initial points was guaranteed, and the only question concerned their "intervals of existence." Now consider initial points at $x = 0$, at which $p(x) = 1/x$ is discontinuous. That is, consider initial points on the y axis. Since $p(x)$ is not continuous in any neighborhood of $x = 0$, Theorem 1.2.1 simply gives no information. In fact, through the initial point $y(0) = 0$ (the origin) there is the unique solution $y(x) = 3x^3$, which exists on $-\infty < x < \infty$, as noted above. But, through every other point on the y axis there is *no solution* because (39) gives $y(0) = b = 0 + C/0$, which cannot be satisfied by any value of C. ∎

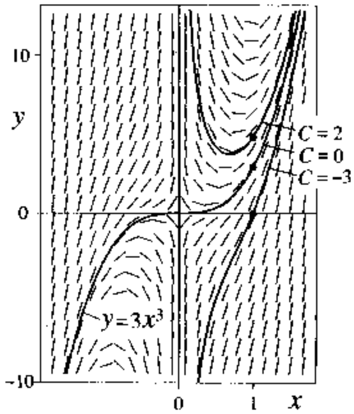


Figure 6. Representative solution curves $y(x) = 3x^3 + C/x$, with the direction field included.

Interval of existence.

The broadest interval on which a solution exists is called the **interval of existence** of that solution. For instance, in Example 6 consider the solution satisfying the initial condition $y(1) = 5$, its graph being the uppermost of the three shown in Fig. 6. Both $y(x) = 3x^3 + 2/x$ and $y' = 6x^2 - 2/x^2$ are undefined at $x = 0$, where they "blow up." Thus, the interval of existence of that solution is $0 < x < \infty$. In contrast, the initial condition $y(1) = 3$ gives $C = 0$ in (39), so the singular C/x term drops out and the solution $y(x) = 3x^3$ has, as its interval of existence, $-\infty < x < \infty$.

EXAMPLE 7. Occurrence of Nonuniqueness. The only case not illustrated in Examples 5 and 6 is that of nonuniqueness, so consider one more example,

$$x\frac{dy}{dx} = y, \tag{40a}$$

$$y(a) = b, \tag{40b}$$

so $p(x) = -1/x$ and $q(x) = 0$. Here, $p(x)$ is discontinuous at $x = 0$. The general solution of (40a) is found to be

$$y = Cx, \tag{41}$$

Use (8) and remember that $e^{\ln x} = x$.

and the initial condition (40b) gives $y(a) = Ca = b$. Now, if $a \neq 0$, the latter gives $C = b/a$ and we have the unique solution $y(x) = bx/a$ with interval of existence $-\infty < x < \infty$. [That interval happens to exceed the minimum interval of existence indicated by Theorem 1.2.1, which is $0 < x < \infty$ if $a > 0$ and $-\infty < x < 0$ if $a < 0$.]

However, consider the case $a = 0$ so the initial point lies on the y axis. If $b \neq 0$, then $(C)(0) = b$ has no solution for C and the IVP (40) has no solution. But if $b = 0$ (so the initial point is the origin), then $(C)(0) = 0$ is satisfied by *any* finite value of C, and (40) has the *nonunique* solution $y = Cx$ where C is an arbitrary finite value.

Summary: If the initial point is not on the y axis there is a unique solution, but if it is on the y axis [where $p(x) = -1/x$ is discontinuous] there are two cases: if it is not at the origin

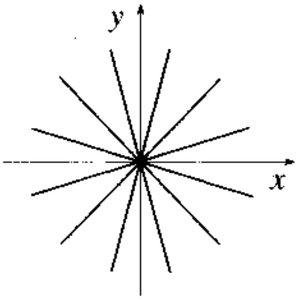


Figure 7. The solutions of (40).

there is no solution, and if it is at the origin there is a nonunique solution, namely, every line $y = Cx$ with finite slope C, as summarized in Fig 7. ∎

Closure. To study the general linear first-order equation $y' + p(x)y = q(x)$, we considered first the homogeneous case $y' + p(x)y = 0$, and used a separation of variables method to derive the general solution (8). For the nonhomogeneous case, separation of variables failed, but we were able to find a general solution by using an integrating factor. The result was the general solution (27), with the integrating factor $\sigma(x)$ given by (25).

Finally, we gave the fundamental existence and uniqueness theorem, Theorem 1.2.1, which states that a solution of the IVP (36) exists and is unique *at least* on the broadest open x interval, containing the initial point $x = a$, on which $p(x)$ and $q(x)$ are continuous; that solution is given by (37).

EXERCISES 1.2

CAUTION: The right-hand sides of equations (8) and (25) are similar, but have different signs in the exponents.

1. Verify, by direct substitution and with the help of chain differentiation, that
(a) (8) satisfies (i.e., is a solution of) $y' + p(x)y = 0$, for any value of A.
(b) (27) satisfies $y' + p(x)y = q(x)$, for any value of C.

2. Homogeneous Equations. Find the particular solution satisfying the initial condition $y(3) = 1$, and give its interval of existence.

(a) $y' = 6x^2y$ (b) $y' + 2(\sin x)y = 0$
(c) $y' - (\cos x)y = 0$ (d) $xy' - y = 0$
(e) $xy' + 3y = 0$ (f) $(\cos x)y' = (\sin x)y$
(g) $(\sin x)y' = (\cos x)y$ (h) $xy' + (1 + x)y = 0$
(i) $x^2y' - y = 0$ (j) $(2 + x)(6 - x)y' = 8y$
(k) $x(5 - x)y' = 5y$ (l) $(1 - x^2)y' - y = 0$
(m) $(2 + x)^2y' + 5y = 0$ (n) $(1 + x)y' - 2y = 0$
(o) $(1 + x)y' + 4y = 0$ (p) $(4 - x^2)y' - 2y = 0$

3. Nonhomogeneous Equations. Find the particular solution satisfying the initial condition $y(2) = 0$ and give its interval of existence.

(a) $y' - y = 3e^x$ (b) $y' + 4y = 8$
(c) $x^2y' + 3xy = 4$ (d) $xy' = 2y + 4x^3$
(e) $xy' + 2y = 10x^3$ (f) $y' - y = 8\sin x$
(g) $y' - 2x = -y - x$ (h) $2xe^xy' = 4 - 2e^xy$
(i) $xy' + y = \sin x + 2\cos x$ (j) $(9 - x^2)y' - 2xy = 10$
(k) $xy' = \sin x - y$ (l) $e^xy' + e^xy = 50$

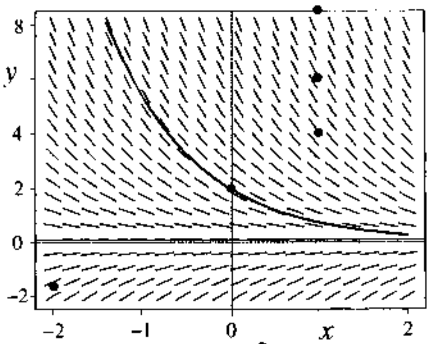
4. The following equations are not linear, so the methods of this section seem not to apply. However, in these examples you will find that if you interchange the independent and dependent variables and consider $x(y)$ instead of $y(x)$, then the result will be a linear equation for $x(y)$. To do that, merely replace the dy/dx by $1/(dx/dy)$ and put the equation into the standard linear form. Solve it for $x(y)$, subject to the given initial condition. If you can, then solve for $y(x)$ from that result, and give its interval of existence.

(a) $y' = y/(4y - x);\ y(2) = 1$
(b) $y' = y^2/(4y^3 - 2xy);\ y(1) = -1$
(c) $(2y - x)y' = y;\ y(0) = 1$
(d) $(x + 2e^{-y})y' = 1;\ y(1) = 0$

5. Computer; Example 4. Obtain a computer plot of the direction field of (28a) and the solutions satisfying the initial conditions $y(1) = -10$, $y(3) = -10$, $y(1) = 20$, $y(3) = 20$, and $y(0) = -1$, within the rectangle $-2 \le x \le 6$ and $-10 \le y \le 20$.

6. Direction Fields. The following are direction fields of first-order linear differential equations. In each case sketch by hand, on a photocopy of the figure, the solution curve through each of the four initial points (that are denoted by heavy dots). To illustrate, we have shown the solution curve through the initial point $y(0) = 2$ in (a), and through $y(1) = 2$ in (b).

(a)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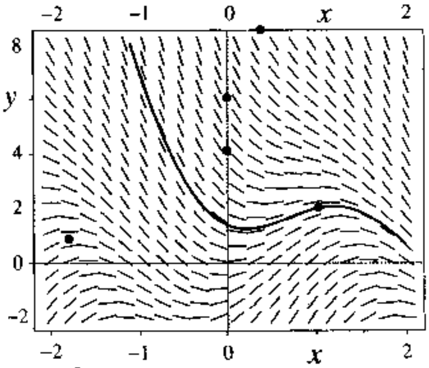
(b)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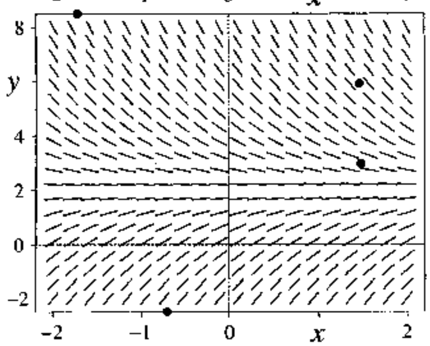
(c)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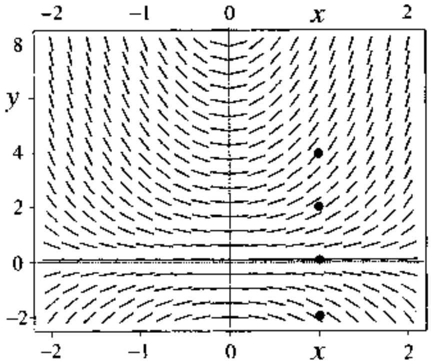
(d)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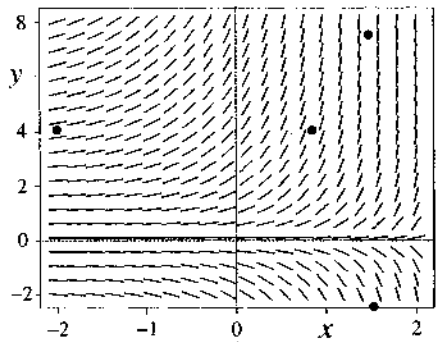
(e)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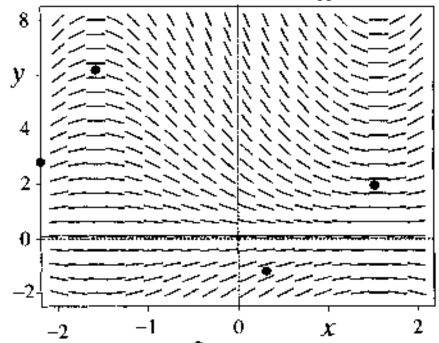
(f)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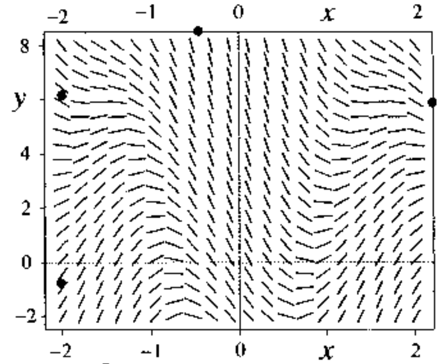
(g)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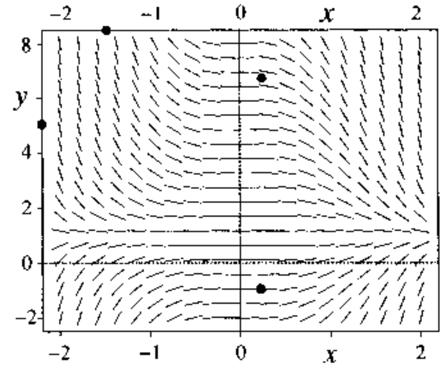
(h)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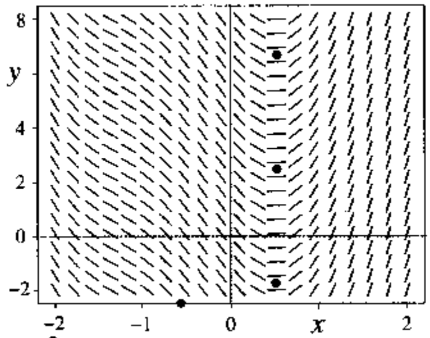
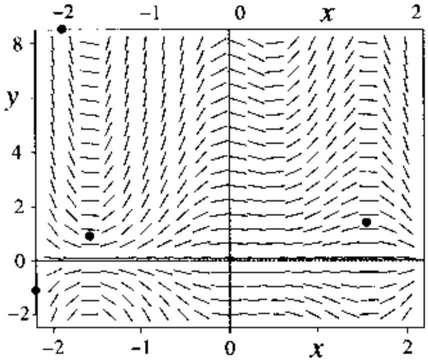
(i)



(j)



7. Matching. The differential equations whose direction fields are given in Exercise 6(a)–6(d) are these:

$$y' + y = 3\sin 2x, \tag{7.1}$$

$$y' + y = 2, \tag{7.2}$$

$$y' + y = 0, \tag{7.3}$$

$$y' = xy. \tag{7.4}$$

Match these four differential equations with the corresponding direction fields shown in 6(a)–6(d), and state your reasoning. HINT: Write the equation in the form $y' = f(x, y)$ and compare $f(x, y)$ with the directions shown in the figure. For example, (7.1) is $y' = 3\sin 2x - y$, so along the line $y = 0$, for instance, the slope $3\sin 2x$ should be oscillatory. Of the direction fields in 6(a)–(d), above, the only one with that property is (b), so we can match 6(b) with equation (7.1).

8. First, read Exercise 7. The differential equations whose direction fields are given in Exercise 6(e)–6(g) are these:

$$y' + y = -6\cos 2x \tag{8.1}$$

$$y' = e^x y, \tag{8.2}$$

$$y' + (\cos x)y = 0. \tag{8.3}$$

Match these differential equations with the corresponding direction fields shown in 6(e)–6(g), and state your reasoning.

9. First, read Exercise 7. The differential equations whose direction fields are given in Exercise 6(h)–6(j) are these:

$$y' + x^2(y-1) = 0, \tag{9.1}$$

$$y' + (1-2x)(\sin 2x)y = 0, \tag{9.2}$$

$$y' = 4(x - \cos 2x). \tag{9.3}$$

Match these differential equations with the corresponding direction fields shown in 6(h)–6(j), and state your reasoning.

10. Straight-Line Solutions. Straight-line solutions of $y' + p(x)y = q(x)$ are striking because of their simple form; for instance, in Example 4 there was one, and in Example 6 there were none. For the given differential equation, find all straight-line solutions, if any. HINT: You can find the general solution and then look within that family of solutions for any that are of the form $y = mx + b$, but it is more direct to seek solutions specifically in that form. This idea, of seeking solutions of a certain form, is prominent in the study of differential equations.

(a) $y' + 3xy = 6x^2 + 15x + 2$
(b) $y' + 3xy = 12x^2 + 15x + 2$
(c) $y' + e^x y = (1-3x)e^x - 3$
(d) $xy' + 2y = 15x - 4$
(e) $xy' = x^2 + y$
(f) $(x-1)y' - y = -3$
(g) $(x+3)y' = y + 1$
(h) $e^x y' + y = x + e^x - 2$

11. Form of General Solution. Observe that the form of the general solution (27) is $y(x) = F(x) + CG(x)$, in which the constant C is arbitrary. Show that $F(x)$ is a particular solution [i.e., of the full equation $y' + p(x)y = q(x)$] and that $G(x)$ is a homogeneous solution [i.e., of the "homogenized" version $y' + p(x)y = 0$]. HINT: Substitute $y(x) = F(x) + CG(x)$ into $y' + p(x)y = q(x)$ and use the fact that C is arbitrary.

12. Working Backwards. If possible, find an equation (or equations) $y' + p(x)y = q(x)$ that has the following functions among its solutions.

(a) $y_1(x) = 1,\ y_2(x) = x$
(b) $y_1(x) = e^x,\ y_2(x) = 5e^x$
(c) $y_1(x) = e^x,\ y_2(x) = e^{-x}$
(d) $y_1(x) = 0,\ y_2(x) = e^x,\ y_3(x) = 6e^x$
(e) $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$
(f) $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = 2x - 1$

13. Interval of Existence. (a) Make up any differential equation $y' + p(x)y = q(x)$ and initial condition that give a unique solution on $-1 < x < 1$ but not on any larger interval; give

that solution. Show your steps and reasoning.
(b) Make up another one.

14. Suppose an equation $y' + p(x)y = q(x)$ has solutions $y_1(x)$ and $y_2(x)$, the graphs of which cross at $x = a$. What can we infer, from that crossing, about the behavior of $p(x)$ and $q(x)$?

15. Change of Variables and the Bernoulli Equation. Sometimes it is possible to convert a nonlinear equation to a linear one (which is desirable because we know how to solve linear first-order equations). This idea will be developed in Section 1.8; but since you may not cover that section, we introduce the topic here as an exercise. The equation

$$\boxed{y' + p(x)y = q(x)y^n,} \tag{15.1}$$

in which n is a constant (not necessarily an integer), is called **Bernoulli's equation**, after the Swiss mathematician *James Bernoulli*. James (1654–1705), his brother John (1667–1748), and John's son Daniel (1700–1782) are the best known of the eight members of the Bernoulli family who were prominent mathematicians and scientists.

(a) Give the general solution of (15.1) for the special cases $n = 0$ and $n = 1$, in which case (15.1) is linear.
(b) If n is neither 0 nor 1, then (15.1) is *nonlinear* because of the y^n term. Nevertheless, show that by transforming the dependent variable from $y(x)$ to $v(x)$ according to

$$\boxed{v = y^{1-n}} \tag{15.2}$$

(for $n \neq 0, 1$), (15.1) can be converted to the equation

$$v' + (1-n)p(x)v = (1-n)q(x), \tag{15.3}$$

which is *linear* and which can be solved by the methods developed in this section. This method of solution was discovered by *Gottfried Wilhelm Leibniz* (1646–1716) in 1696.

16. Use the method suggested in Exercise 15(b) to solve each of the following. Give the interval of existence. HINT: To solve, identify n, $p(x)$, and $q(x)$, then use (15.3).

(a) $y' + y = -3e^x y^2; \quad y(0) = 1$
(b) $y' + 2y = -12e^{3x}y^{3/2} \quad (y > 0); \quad y(0) = 1$
(c) $(1 + x)y' + 2y = 2\sqrt{y} \quad (y > 0); \quad y(3) = 4$
(d) $xy' - 2y = 5x^3y^2; \quad y(1) = 4$
(e) $3y' + y = x/\sqrt{y} \quad (y > 0); \quad y(3) = 1$

17. (a)–(p) For the corresponding part of Exercise 2, what minimum interval of existence and uniqueness is predicted by Theorem 1.2.1 for the initial condition $y(0.7) = 2$?

18. (a)–(l) For the corresponding part of Exercise 3, what minimum interval of existence and uniqueness is predicted by Theorem 1.2.1 for the initial condition $y(-2) = 5$?

19. Proof of Existence Part of Theorem 1.2.1. To prove existence it suffices to put forward a solution, and (37) is indeed a solution. Thus, to prove the existence part of the theorem you need merely verify that (37) satisfies the differential equation and the boundary condition:

(a) Verify that (37) satisfies the differential equation in (36). HINT: Since $p(x)$ is continuous, (25) shows that $\sigma(x)$ is also continuous and nonzero. Also, $\frac{d}{dx}\int_a^x \sigma(s)q(s)\,ds = \sigma(x)p(x)$ because $\sigma(x)$ and $q(x)$ are continuous.
(b) Verify that (37) also satisfies the initial condition $y(a) = b$.
(c) We wrote (37) without derivation. Derive it. HINT: Instead of using indefinite integrals when you integrate $(\sigma y)' = \sigma q$, use definite integrals, from a to x.

20. Proof of Uniqueness Part of Theorem 1.2.1. To prove that a problem has a unique solution, the standard approach is to consider any two solutions and to show that their difference must be identically zero, so the two solutions must be identical and hence the solution must be unique. Accordingly, suppose $y_1(x)$ and $y_2(x)$ satisfy (36), in which $p(x)$ and $q(x)$ satisfy the continuity condition stated in the theorem. Then

$$y_1' + p(x)y_1 = q(x); \quad y_1(a) = b, \tag{20.1}$$

$$y_2' + p(x)y_2 = q(x); \quad y_2(a) = b. \tag{20.2}$$

Denote the difference $y_1(x) - y_2(x)$ as $u(x)$.

(a) By subtracting (20.2) from (20.1), show that $u(x)$ satisfies the "homogenized" problem

$$u' + p(x)u = 0; \quad u(a) = 0. \tag{20.3}$$

[We say (20.3) is homogeneous because both the forcing function on the right-hand side of the differential equation is zero and the initial condition is zero as well; there are no "inputs."]

(b) Solve (20.3) and show that its only solution is $u(x) = 0$. It follows that $y_1(x) - y_2(x) = 0$ so $y_1(x) = y_2(x)$. Hence, the solution of (36) is unique. HINT: Use an integrating factor $\sigma(x) = e^{\int_a^x p(s)\,ds}$.

21. Alternative Solution Method: Variation of Parameters. We derived a general solution of the linear first-order equation

$$y' + p(x)y = q(x) \tag{21.1}$$

by the integrating factor method. An alternative method of solution is as follows. First, recall that the *homogeneous* equation $y' + p(x)y = 0$ has the general solution

$$y(x) = Ae^{-\int p(x)\,dx}, \tag{21.2}$$

where A is an arbitrary constant. To solve the *nonhomogeneous* equation (21.1), seek $y(x)$ in the form

$$y(x) = A(x)e^{-\int p(x)\,dx}; \tag{21.3}$$

that is, let the constant A in the homogeneous solution (21.2) vary. (The motivation behind this step is not obvious, but we will see that it works.) Substitute (21.3) into (21.1) and show, after canceling two terms, that you obtain

$$A'(x) = e^{\int p(x)\,dx}q(x) \tag{21.4}$$

so

$$A(x) = \int e^{\int p(x)\,dx}q(x)\,dx + C \tag{21.5}$$

and

$$y(x) = e^{-\int p\,dx}\left(\int e^{\int p\,dx}q\,dx + C\right), \tag{21.6}$$

which agrees with the solution (27) obtained earlier by using an integrating factor. This method is called **variation of parameters** because the key is in letting the parameter A vary.

1.3 APPLICATIONS OF LINEAR FIRST-ORDER EQUATIONS

Having solved the first-order linear differential equation, we now give representative physical applications — to population dynamics, radioactive decay, mixing problems, and electrical circuits, with additional applications in the exercises.

1.3.1 Population dynamics; exponential model. We want to model the population dynamics of a certain species, such as bass in a lake or the malaria parasite introduced into the host's bloodstream. That is, we want to develop a mathematical problem that governs the variation of the population $N(t)$ of that species with the time t.

That is, the birth and death rates β and κ are *per capita*.

Let β be the birth rate (births per individual per unit time) and δ the death rate, with β and δ assumed to be known constants over the time of interest. Then, for any time interval Δt,

$$N(t + \Delta t) = N(t) + \beta N(t)\Delta t - \delta N(t)\Delta t, \tag{1}$$

Equations (1) and (2) hold for *any* time interval Δt, so it is permissible to let $\Delta t \to 0$ in (2), which step gives the differential equation (3).

or,

$$\frac{N(t + \Delta t) - N(t)}{\Delta t} = (\beta - \delta)N(t). \tag{2}$$

Equation (1) is simply bookkeeping: The number of individuals at time $t + \Delta t$ equals the number that we start with at time t plus the number that are born minus the number that die over the Δt time interval. If we let $\Delta t \to 0$ in (2), and denote the *net* birth/death rate $\beta - \delta$, called the **growth rate**, as κ, we obtain[1]

$$\boxed{N' = \kappa N,} \tag{3}$$

which is a linear first-order homogeneous differential equation for $N(t)$, homogeneous because it is $N' - \kappa N = 0$.

The latter is of the same form as $y' + py = 0$, studied in Section 1.2, with N in place of y, t in place of x, and $p = -\kappa$, so its general solution is

$$N(t) = Ae^{\kappa t}, \tag{4}$$

with A an arbitrary constant. Alternative to obtaining (4) by using the memorized solution formula, let us solve (3) by separation of variables, as review:

$$\int \frac{dN}{N} = \int \kappa\, dt, \quad \ln N = \kappa t + C, \quad N(t) = e^{\kappa t + C} = e^{C} e^{\kappa t} = Ae^{\kappa t}, \tag{5}$$

$\ln N$ rather than $\ln|N|$ in (5) because the population $N(t)$ cannot be negative, so there is no need for absolute values.

as in (4). If we have an initial condition

$$N(0) = N_0, \tag{6}$$

then $N(0) = N_0 = Ae^{(\kappa)(0)} = A$ so $A = N_0$, and (4) becomes

$$\boxed{N(t) = N_0 e^{\kappa t},} \tag{7}$$

which we've plotted in Fig. 1 for several values of κ.

COMMENT 1. Equation (3) is often called the **Malthus model** after the British economist *Thomas Malthus* (1766–1834), who observed that many biological populations change at a rate that is proportional to their population. It is also known as the **exponential model** because of the exponential form of its solution.

COMMENT 2. If the growth rate κ is negative, then (7) predicts an exponential decrease to zero as $t \to \infty$, which seems reasonable (although when N becomes small enough our approximation of N as a continuous and differentiable function of t comes into question). But if κ is positive, then (7) predicts exponential growth, with $N(t)$ tending to infinity as $t \to \infty$. Such sustained growth is not reasonable because if N becomes sufficiently large then other factors will no doubt come into play, such as insufficient food, factors that have not been accounted for in our

[1] Strictly speaking, $N(t)$ is integer-valued since one cannot have a population of 28.37, say. Its graph develops in a stepwise manner so $N(t)$ is a discontinuous function of t. Hence, it is not differentiable and the $N'(t)$ in (3) does not exist. However, if N is sufficiently large *so that the steps are sufficiently small compared to* N, then we can regard $N(t)$ as a continuous function of t.

Recall that $\kappa < 0$ if the death rate exceeds the birth rate, and $\kappa > 0$ if the birth rate exceeds the death rate.

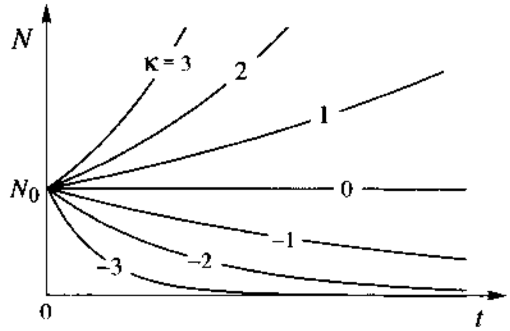


Figure 1. Exponential growth and decay given by (7); $N(t) = N_0 e^{\kappa t}$.

The logistic equation (8) is also called the **Verhulst equation** after the Belgian mathematician *P. F. Verhulst* (1804–1849).

simple model. Specifically, we expect κ not really to be a constant but to be a function of N, decreasing as N increases. As a first approximation of such behavior, suppose κ varies linearly with N: $\kappa = a - bN$, where a and b are positive constants. Then (3) is replaced by

$$\frac{dN}{dt} = (a - bN)N, \tag{8}$$

which is well known as the **logistic equation**. However, the logistic equation is *nonlinear* because of the N^2 term, and will be studied later, in Section 1.6.[1]

1.3.2 Radioactive decay; carbon dating. Another classical application of linear first-order equations involves radioactive decay and carbon dating.

Radioactive materials, such as carbon-14, plutonium-241, radium-226, and thorium-234, are observed to decay at a rate that is proportional to the amount of radioactive material present. Thus, the number of nuclei disintegrating per unit time will be proportional to the number of nuclei present, so

$$\frac{dN}{dt} = -kN, \tag{9}$$

in which $N(t)$ is the number of atoms of the radioactive element at time t, and the positive constant k is the *decay rate*, which we assume is known. However, it is inconvenient to work with N since one cannot count the number of atoms in a given batch of material. Thus, multiply both sides of (9) by the atomic mass (mass per atom). Since the atomic mass times $N(t)$ is the mass $m(t)$ of the radioactive material, (9) gives

$$\boxed{\frac{dm}{dt} = -km} \tag{10}$$

[1] Verhulst studied human population but did not have sufficient census data to test the accuracy of his model. Later researchers turned to species with much shorter life spans, such as *Drosophila melanogaster* (fruit fly), which could be accurately monitored in the laboratory over many generations, and they did obtain good agreement using Verhulst's logistic model.

for $m(t)$, which is more readily measured than $N(t)$. Solving (10) gives

$$m(t) = m_0 e^{-kt}, \tag{11}$$

where $m(0) = m_0$ is the initial amount of the radioactive mass (Fig. 2). This result agrees well with experiment.

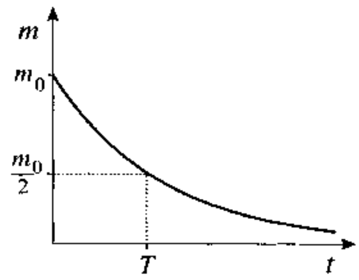


Figure 2. The exponential decay $m(t) = m_0e^{-kt}$ and the half-life T.

The decay rate k determines the **half-life** T of the material, the time required for any initial amount of mass m_0 to be reduced by half, to $m_0/2$. It is more common and more convenient to work with T than k, so we will eliminate k from (11), in favor of T, as follows: When t is T, in (11), $m(t)$ is $m_0/2$, so $m_0/2 = m_0e^{-kT}$, which gives $\boldsymbol{k = (\ln 2)/T}$. If we put the latter into (11), $m(t)$ can be re-expressed in terms of the half-life T as

$$m(t) = m_0 2^{-t/T}. \tag{12}$$

For instance, at $t = 0, T, 2T, 3T, 4T, \ldots$, (12) gives $m(t) = m_0, m_0/2, m_0/4, m_0/8$, and so on.

The steps leading from (11) to (12) involve the properties of the exponential and logarithmic functions, which are among the review formulas on the inside cover of this book. We leave those steps for the exercises.

Radioactivity has had an important archaeological application in connection with **dating**. The idea behind any dating technique is to identify a physical process that proceeds at a known rate. If we measure the state of the system now, and we know its state at the initial time, then from these two quantities together with the known rate of the process we can infer how much time has elapsed; the mathematics enables us to "travel back in time as easily as a wanderer walks up a frozen river."[1]

For instance, consider carbon dating, developed by the American chemist *Willard Libby* in the 1950's. Cosmic rays consisting of high-velocity nuclei penetrate the earth's lower atmosphere. Collisions of these nuclei with atmospheric gases produce free neutrons. These collide with nitrogen, changing it to carbon-14, which is radioactive and which decays to nitrogen-14 with a half-life of around 5,570 years. Thus, some of the carbon dioxide (CO_2) in the atmosphere contains this radioactive C-14. Plants take in both radioactive and nonradioactive CO_2, and humans and animals inhale both and eat the plants. Consequently, the plants and animals living today contain both the nonradioactive C-12 and, to a much lesser extent, its radioactive isotope C-14, in a ratio that is the same from one plant or animal to another. When a plant or animal dies its C-12 remains fixed but its C-14 decreases with time by radioactive decay. The resulting "shortage" of C-14 at any given time is a measure of how long ago the plant or animal died.

Radioactive carbon, C-14, is called radiocarbon because it decays radioactively.

For instance, suppose we wish to carbon date a given sample of wood, that is, to determine how long ago it died. To do so we make two assumptions: First, assume that the ratio of radioactive to nonradioactive carbon (C-14 to C-12) in living material at the time the tree died was the same as it is in living material today. Second, assume that the rate of radioactive decay of C-14 has been constant over that period of time. Subject to these assumptions (which cannot be verified because they are historical in nature), here is how the method works. Measure the mass of C-14 present in the sample now, which is the $m(t)$ on the left-hand side of (12), and assume that the initial mass of C-14 (when the tree died), m_0, is the same

The first assumption establishes the initial condition, the second establishes the differential equation.

[1]Ivar Ekeland, ***Mathematics and the Unexpected*** (Chicago: University of Chicago Press, 1988).

as the mass of C-14 in a sample of the same weight that is alive today. Knowing $m(t)$, m_0, and the half-life T, solve (12) for t, which is the time that has elapsed since the tree died.

EXAMPLE 1. Carbon Dating a Sample of Petrified Wood.
Consider a petrified wood sample that we wish to date. Since C-14 emits approximately 15 beta particles (i.e., high-speed electrons) per minute per gram, we can determine how many grams of C-14 are contained in the sample by measuring the rate of beta particle emission. Suppose we find that the sample contains 0.002 grams of C-14, whereas if it were alive today it would, based upon its weight, contain around 0.0045 grams. Assuming it contained 0.0045 grams of C-14 when it died, then that mass of C-14 will have decayed, over the subsequent time t, to 0.002 grams. Then (12) gives

$$0.002 = (0.0045)\, 2^{-t/5570},$$

and, solving for t, we determine the sample to be around $t = 6,520$ years old. ∎

1.3.3 Mixing problems; a one-compartment model. Consider a mixing tank, as in a chemical plant, with a constant inflow of Q gallons per minute and an equal outflow (Fig. 3). The inflow is at a constant concentration c_i (pounds per gallon) of a particular solute such as salt, and the tank is stirred so the concentration $c(t)$ is uniform throughout the tank; t is the time. Hence, the outflow is at concentration $c(t)$. Let v be the liquid volume within the tank, in gallons; v is constant because the inflow and outflow rates are equal. We want to determine the solute concentration $c(t)$.

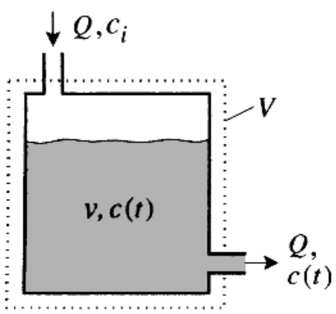


Figure 3. Mixing tank.

To derive a differential equation for $c(t)$, carry out a mass balance for the "control volume" V (dashed lines in the figure):

$$\text{Rate of increase of mass of solute within } V = \text{Rate in} - \text{Rate out}, \tag{13}$$

$$\frac{d}{dt}[c(t)v] = Qc_i - Qc(t) \tag{14}$$

or, since v is constant,

$$\boxed{\frac{dc}{dt} + \frac{Q}{v}c = \frac{c_i Q}{v},} \tag{15}$$

The units of each term in (14) are lb/min. For instance, the first term on the right is $(Q\frac{\text{gal}}{\text{min}})(c_i\frac{\text{lb}}{\text{gal}})$, or $Qc_i\frac{\text{lb}}{\text{min}}$.

which is a first-order linear differential equation for $c(t)$.

The tank in Fig. 3 could, literally, be a mixing tank in a chemical plant, but in some applications the figure may be only schematic. For instance, in biological applications it is common to represent the interacting parts of the biological system as one or more interconnected **compartments**, with inflows, outflows, and exchanges between compartments.[1] One compartment could be an organ such as the liver; another could be all the blood in the circulatory system. The system represented in Fig. 3 is an example of a one-compartment system; the compartment is the tank.

EXAMPLE 2. Mixing Tank; Approach to Steady-State Operation. Let the initial concentration in the tank be $c(0) = 0$, so

$$\frac{dc}{dt} + \frac{Q}{v}c = \frac{c_i Q}{v}; \qquad c(0) = 0. \tag{16}$$

The integrating factor is $\sigma(t) = e^{\int p(t)\,dt} = e^{\int (Q/v)\,dt} = e^{Qt/v}$ and the general solution of the differential equation in (16) is

$$\begin{aligned} c(t) &= e^{-Qt/v}\left(\int e^{Qt/v}\frac{c_i Q}{v}\,dt + C\right) \\ &= c_i + Ce^{-Qt/v}. \end{aligned} \tag{17}$$

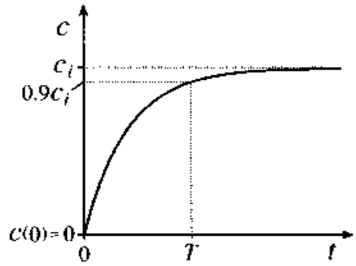


Figure 4. Exponential approach of $c(t)$ to its steady-state value c_i.

Finally, $c(0) = 0 = c_i + C$ gives $C = -c_i$, so

$$\boxed{c(t) = c_i\left(1 - e^{-Qt/v}\right).} \tag{18}$$

Since $e^{-Qt/v} \to 0$ as $t \to \infty$, it follows from (18) that $c(t) \to c_i$ as $t \to \infty$, as we might have expected since the inflow is maintained at that concentration. This asymptotic behavior is seen in Fig. 4.

The time T that it takes for $c(t)$ to reach $0.9c_i$, say, can be found from (18):

$$c(T) = 0.9c_i = c_i\left(1 - e^{-QT/v}\right),$$

which gives

$$T = (\ln 10)\frac{v}{Q}. \tag{19}$$

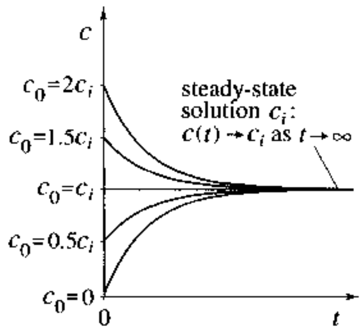


Figure 5. Varying the initial condition c_0, with Q fixed.

Thus far we've taken $c(0) = 0$, but now suppose it is not necessarily zero. Let $c(0) = c_0$, with $c_0 \geq 0$ because a negative concentration $c_0 < 0$ is impossible. Then (17) gives $c(0) = c_0 = c_i + C$ so $C = c_0 - c_i$, and in place of (18) we have

$$c(t) = \underbrace{(c_0 - c_i)e^{-Qt/v}}_{\text{transient}} + \underbrace{c_i.}_{\text{steady state}} \tag{20}$$

As $t \to \infty$, the exponential term in (20) tends to zero and $c(t) \to c_i$. Thus we call the $(c_0 - c_i)e^{-Qt/v}$ term the **transient** part of the solution, and we call the c_i term the **steady-state** solution. Graphs of $c(t)$ in Fig. 5 show the approach to steady state for several different initial conditions. The bottom curve corresponds to the one in Fig. 4. ∎

[1]For a discussion of compartmental analysis in biology see L. Edelstein-Keshet, *Mathematical Models in Biology* (New York: Random House, 1988) or John A. Jacquez, *Compartmental Analysis in Biology and Medicine*, 3rd ed (Ann Arbor, MI: BioMedware, 1996).

The $\equiv$ in (21) means *equal to by definition*. That is, let $(Q/v)(c_i - c)$ be called "$f(c)$."

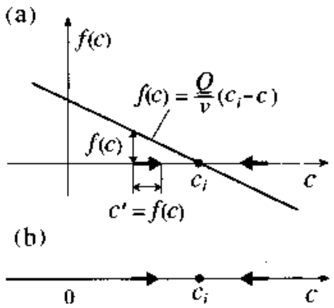


Figure 6. The "flow" along the phase line, implied by (21). The phase line is shown in (b).

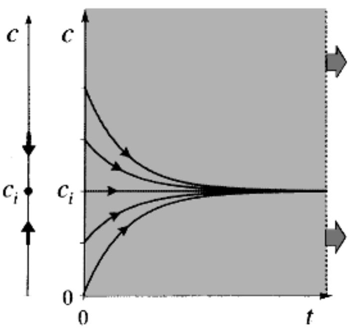


Figure 7. Connection between the phase line flow and the solution curves in the t, c plane.

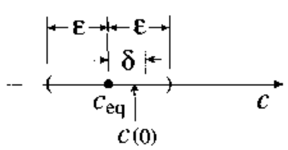


Figure 8. Stability of an equilibrium point c_{eq} on the phase line.

1.3.4 The phase line, equilibrium points, and stability. Another graphical idea is useful. If we write the differential equation in (16) as

$$\frac{dc}{dt} = \frac{Q}{v}(c_i - c) \equiv f(c), \tag{21}$$

we see that the equation is **autonomous**, which means that the right-hand side contains no explicit t dependence, for it is of the form $f(c)$ rather than $f(c,t)$.[1] It is informative to plot $f(c)$ versus c, as we've done in Fig. 6a. Since c is a function of t we can imagine each point on the c axis as moving, with time, along that axis, with its velocity equal to the value of $f(c)$ at that point [because $c' = f(c)$], to the right if $f(c) > 0$ and to the left if $f(c) < 0$. The point c_i is not moving because $f(c_i) = 0$, points to the left of c_i are moving rightward, and points to the left of c_i are moving leftward, as indicated by the two arrows in Fig. 6a.

Thus, we can think of the movement of points along the c axis as a one-dimensional "flow," and we call the line along which that flow takes place the **phase line**. From that point of view the steady-state solution $c = c_i$ in Fig. 5 corresponds to an **equilibrium point** of the flow along the phase line (Fig. 6b) because the flow velocity dc/dt is zero there.

We could show many arrows on the phase line, rather than just the two in Fig. 6b, and could even scale them according to their magnitude, but we will keep phase line displays simple and just show any equilibrium points (with heavy dots) and single arrows to indicate flow directions.

To see the connection between Fig. 5 and Fig. 6b, we've shown them together in Fig. 7, with the phase line arranged vertically at the left, and we've included arrows on the solution curves in the c, t plot — in the direction of increasing time. To see how the flow on the phase line is related to the flow in the c, t plane, imagine the c, t plot as resulting if (on our imaginary computer screen) we click on the phase line and drag it to the right, in time, as suggested by the two large arrows at the right. Conversely, if we drag that dotted line back to the left, then the c, t graphs get "squashed," and all we're left with is the flow along the phase line, shown at the left of the figure.

Along with the concept of equilibrium comes the concept of **stability**.

For instance, the equilibrium of a marble on a hilltop is "unstable" and the equilibrium of a marble in a valley is "stable." To define the stability of an equilibrium point on the phase line, let c_{eq} be an equilibrium point on the phase line of $\frac{dc}{dt} = f(c)$; that is, $f(c_{eq}) = 0$. We say that c_{eq} is **stable** if points that start out close to it remain close to it, and **unstable** if it is not stable.[2]

---

[1] If any of Q, c_i, v were functions of time, then $\frac{dc}{dt} = \frac{Q(t)}{v(t)}[c_i(t) - c] = f(c,t)$ would not be autonomous, it would be *nonautonomous*.

[2] Let us make that intuitively stated definition precise: an equilibrium point c_{eq} is **stable** if, for any $\epsilon > 0$ (i.e., no matter how small), there corresponds a $\delta > 0$ such that $c(t)$ remains closer to c_{eq} than ϵ for all $t > 0$ if $c(0)$ is closer to c_{eq} than δ (Fig. 8). That is, if $|c(0) - c_{eq}| < \delta$ then $|c(t) - c_{eq}| < \epsilon$ for all $t > 0$. If c_{eq} is not stable, it is **unstable**.

We can see that the equilibrium point c_i in Fig. 6b is stable because the flow approaches c_i from both sides, so if we start close to c_i then we remain close to it for all $t > 0$. Actually, the stability of an equilibrium point c_{eq} does not require $c(t)$ to *approach* c_{eq} as $t \to \infty$, but only to remain close to it. If it does approach c_{eq}, that is, if $c(t) \to c_{eq}$ as $t \to \infty$, then the equilibrium point is not only stable, it is **asymptotically stable.** In Fig. 7, c_i is a steady-state solution because that point is an asymptotically stable equilibrium point of the phase line.

Realize that we obtained the graph in Fig. 6a, and hence the phase line in Fig. 6b, merely by plotting the right-hand side of the differential equation $c' = (Q/v)(c_i - c)$ versus c; *we did not need to solve the differential equation.*

There is good news and bad news regarding the phase line: The bad news is that it contains less information than the traditional plots of c versus t. Of *course*: To get the phase line we merely plotted the right-hand side of the differential equation versus c, we did not solve it, so it makes sense that the detailed time history, contained in the c, t plot, is not available from the phase line. But the good news is that the phase line is readily obtained and contains key information. The key information in this example is the equilibrium point c_i and its stability. After all, from the c, t plot we see that after some time goes by, the line $c(t) = c_i$ is where all the solution curves "end up," for it is approached as $t \to \infty$. Furthermore, from the phase line, at the left in Fig. 7, we could even sketch the solution curves in the c, t plane, if only qualitatively, without actually solving the differential equation.

Thus, the phase line concept is more *qualitative* than quantitative. It is not so impressive for linear equations because linear equations can be solved analytically anyhow, but nonlinear equations are much more difficult in general, and in that case we will need to rely more heavily on other approaches — qualitative ones such as direction fields and the phase line, and quantitative ones involving numerical solution by computer. In any case, remember that the phase line method applies only if the differential equation is autonomous.

1.3.5 Electrical circuits. Consider electrical circuits consisting of closed wire loops and a number of circuit elements such as resistors, inductors, capacitors, and voltage sources such as batteries.

An electric current is a flow of charges: The *current* through a given control surface, such as the cross section of a wire, is the charge per unit time crossing that surface. Each electron carries a negative charge of 1.6×10^{-19} *coulomb*, and each proton carries an equal positive charge. Current is measured in *amperes*, one ampere being a flow of one coulomb per second. A current is counted as positive in a given direction if it is the flow of positive charge in that direction. While, in general, currents can involve the flow of positive or negative charges, the flow, typically, is of negative charges, free electrons. Thus, when one speaks of a current of one ampere in a given direction in an electrical circuit, one really means the flow of one coulomb per second of negative charges (electrons) in the opposite direction.

Just as heat flows due to a temperature difference, from one point to another, an electric current flows due to a difference in electric potential, or *voltage*, measured in *volts*. Thus, we will need to know the relationship between the voltage difference across a given circuit element and the corresponding current flow through it. The circuit elements considered here are resistors, inductors, and capacitors.

For a **resistor**, the voltage drop $e(t)$, where t is the time, is proportional to the current $i(t)$ through it:

$$e(t) = Ri(t), \tag{22}$$

where the constant of proportionality R is called the *resistance* and is measured in *ohms*; (22) is called **Ohm's law**. By a resistor we mean an electrical device, often made of carbon, that offers a specified resistance — such as 100 ohms, 500 ohms, and so on. The standard symbolic representation of a resistor is shown in Fig. 9.

For an **inductor**, the voltage drop is proportional to the time rate of change of current through it:

$$e(t) = L\frac{di(t)}{dt}, \tag{23}$$

in which the constant of proportionality L is called the **inductance** and is measured in *henrys*. Physically, most inductors are coils of wire, hence the symbolic representation in Fig. 9.

For a **capacitor**, the voltage drop is proportional to the charge $Q(t)$ on the capacitor:

$$e(t) = \frac{1}{C}Q(t), \tag{24}$$

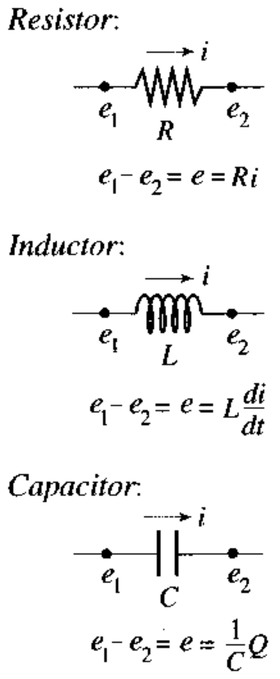


Figure 9. Circuit elements.

where C is called the **capacitance** and is measured in *farads*. Physically, a capacitor consists of two plates separated by a gap across which no current flows, and $Q(t)$ is the charge on one plate relative to the other. Though no current flows across the gap, there will be a current $i(t)$ that flows through the (closed) circuit that links the two plates and is equal to the time rate of change of charge on the capacitor:

$$i(t) = \frac{dQ(t)}{dt}. \tag{25}$$

Equations (22)–(24) give the behavior of the respective circuit elements, but we also need to know the physics of the circuit itself, which consists of two laws named after the German physicist *Gustav Robert Kirchhoff* (1824–1887):

Kirchhoff's current law states that the sum of the currents approaching any point P of a circuit equals the sum of the currents leaving that point. The latter is a *conservation law*, namely, that electric charge is conserved; it is neither created nor destroyed at P. To illustrate, consider the portion of a circuit shown in Fig. 10a. Application of Kirchhoff's current law to point P, say, gives

$$i_1 = i_2 + i_3.$$

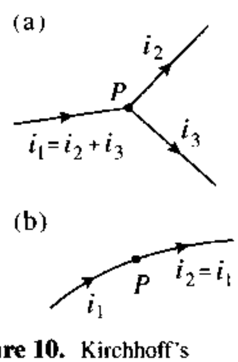


Figure 10. Kirchhoff's current law.

Kirchhoff's voltage law states that the algebraic sum of the voltage drops around any loop of a circuit is zero. Since voltage is an electric potential (i.e., electric potential energy), the statement that the potential drops around any loop of a circuit is zero is also a conservation law, this time the conservation of *energy*.

To apply these ideas, consider the circuit shown in Fig. 11, consisting of a single loop containing a resistor, an inductor, a capacitor, a voltage source (such as a battery or generator), and the necessary wiring. Take the current $i(t)$ to be positive clockwise; if it flows counterclockwise, its numerical value will be negative. In this case, Kirchhoff's current law simply says that the current i is a constant from point to point within the circuit and therefore varies only with time. That is, the current law states that at *every* point P in the circuit the currents i_1 and i_2 (Fig. 10b) are the same, namely, $i(t)$. Next, Kirchhoff's voltage law gives

$$(e_a - e_d) + (e_b - e_a) + (e_c - e_b) + (e_d - e_c) = 0, \tag{26}$$

which, canceling terms, is simply an algebraic identity. If we use (22)–(24), (26) gives

$$e(t) - Ri - L\frac{di}{dt} - \frac{1}{C}Q(t) = 0. \tag{27}$$

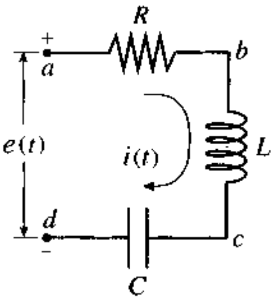


Figure 11. RLC circuit.

If we differentiate (27) with respect to t and use (25) to eliminate $Q(t)$ in favor of $i(t)$, we obtain

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{de(t)}{dt}, \tag{28}$$

If the voltage source is a battery, then $e(t)$ is a constant. More generally, a generator can be a time-varying voltage source.

which is a linear second-order differential equation for $i(t)$, in which $e(t)$ is known — prescribed. Alternatively, we could use $Q(t)$ instead of $i(t)$ as our dependent variable. In that case we again use (25) in (27), but this time to eliminate the $i(t)$'s in favor of $Q(t)$, and we obtain the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = e(t) \tag{29}$$

for $Q(t)$. Either way, we have a linear second-order differential equation.

In this chapter our interest is in *first*-order equations, but we do obtain first-order equations in the following two special cases.

EXAMPLE 3. ***RC* Circuit.**
If $L = 0$ (i.e., if we remove the inductor from the circuit in Fig. 11, as shown in Fig. 12a, then (28) reduces to the linear first-order equation

$$R\frac{di}{dt} + \frac{1}{C}i = \frac{de(t)}{dt} \tag{30}$$

for $i(t)$. ∎

EXAMPLE 4. RL Circuit.
If, instead of removing the inductor from the circuit shown in Fig. 11, we remove the capacitor (Fig. 12b), then (27) gives the first-order equation[1]

$$\boxed{L\frac{di}{dt} + Ri = e(t)} \tag{31}$$

for $i(t)$. ∎

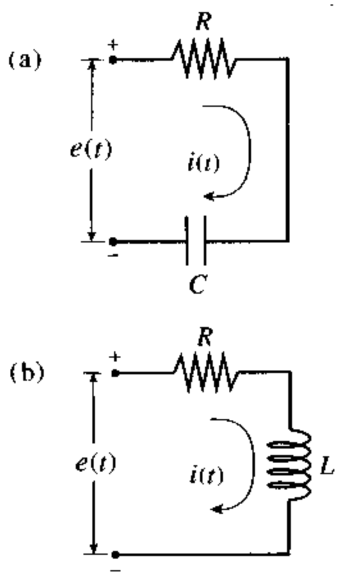


Figure 12. Two special cases of the RLC circuit shown in Fig. 11; RC and RL circuits.

Although the RC and RL circuits are different, their governing equations are of the same form — first-order linear equations with constant coefficients on the left-hand side. Thus, it will suffice to consider just one of the two circuits in Fig. 12, for instance the RL circuit in Fig. 12a, modeled by (31). Hence, we've highlighted (31). Dividing by L to put the equation into the standard form $i' + p(t)i = q(t)$, and appending an initial condition, consider the IVP

$$i' + \frac{R}{L}i = \frac{1}{L}e(t); \quad i(0) = i_0. \tag{32}$$

Identifying $p(t)$ as R/L and $q(t)$ as $e(t)/L$, the results in Section 1.2 give a general solution of the differential equation as

$$i(t) = e^{-Rt/L}\left(\int e^{Rt/L}\,\frac{e(t)}{L}\,dt + A\right), \tag{33}$$

in which A can be found by applying the initial condition $i(0) = i_0$. In Examples 5–7 we will specify several typical $e(t)$'s and complete the solution.

EXAMPLE 5. RL Circuit with Constant Applied Voltage.
Suppose the applied voltage is a constant, $e(t) = \text{constant} = E_0$. Then (33), together with the initial condition, gives

$$i(t) = \underbrace{\left(i_0 - \frac{E_0}{R}\right)e^{-Rt/L}}_{\text{transient}} + \underbrace{\frac{E_0}{R}}_{\text{steady state}}, \tag{34}$$

and representative solution curves are plotted in Fig. 13.

---

[1]It would be natural to expect that removing the capacitor is equivalent to setting $C = 0$, yet in that case the capacitor term in (27) becomes infinite rather than zero. Rather, to remove the capacitor, move its plates together until they touch. The capacitance C is *inversely* proportional to the gap dimension, so as the gap diminishes to zero $C \to \infty$ and the capacitor term in (27) does indeed drop out because in that limit the $1/C$ factor becomes zero.

Does Fig. 13 look familiar? It should, for the IVP (32) [with $e(t)$ = constant = E_0] and its solution (34) are identical to the IVP (16) and its solution (20), respectively, with the correspondences

$$c(t) \leftrightarrow i(t), \quad c_0 \leftrightarrow i_0, \quad c_1 \leftrightarrow \frac{E}{R}, \quad \frac{Q}{v} \leftrightarrow \frac{R}{L}. \tag{35}$$

As $t \to \infty$, the exponential term in (34) tends to zero and $i(t)$ tends to the steady-state value E_0/R. Outside of name changes, the only difference between Figs. 5 and 13 is that in Fig. 13 we've added the case $i_0 = -0.5E/R$ because whereas the concentration c_0 cannot be negative, the current i_0 *can* be negative. ∎

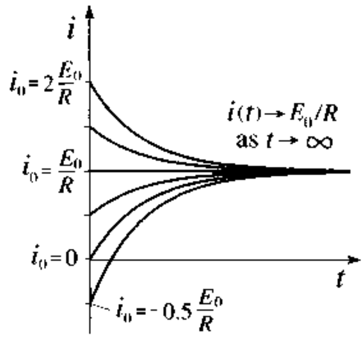


Figure 13. Response $i(t)$ for the case $e(t)$ = constant = E_0, for six different i_0's. It is convenient to express the i_0's as multiples of the steady-state value E_0/R, which is a natural reference value.

The correspondence just noted between the mixing tank and the RL circuit is important, for if two different systems are modeled by the same IVP, to within name changes, then their solutions are identical to within those name changes. Such systems are called **analogs** of each other.

EXAMPLE 6. *RL* Circuit with Sinusoidal Applied Voltage.

Now let the applied voltage be oscillatory instead, for instance $e(t) = E_0 \sin \omega t$, with amplitude E_0 and frequency ω. With this expression for $e(t)$ we can evaluate the integral in (33) and then apply the initial condition to evaluate A. Doing so, we obtain

$$i(t) = \underbrace{\left(i_0 + \frac{E_0 \omega L}{R^2 + (\omega L)^2}\right) e^{-Rt/L}}_{\text{transient}} + \underbrace{\frac{E_0 R}{R^2 + (\omega L)^2}\left(\sin \omega t - \frac{\omega L}{R}\cos \omega t\right)}_{\text{steady state}}. \tag{36}$$

By "steady state" we don't necessarily mean that the dependent variable is constant; in this example it is a "steady oscillation."

Once again we have a transient response, transient in that it tends to zero as $t \to \infty$ because of the negative exponential function, plus a steady-state response, namely, that which is left after the transients have died out. Note *that steady state does not necessarily mean constant*. In Example 5 the applied voltage was constant and the steady state was, likewise, a constant or "steady" current, but in this example the applied voltage is oscillatory and the steady state is oscillatory as well. The response is plotted, for representative values of various parameters, in Fig. 14; the transient and steady-state parts are shown

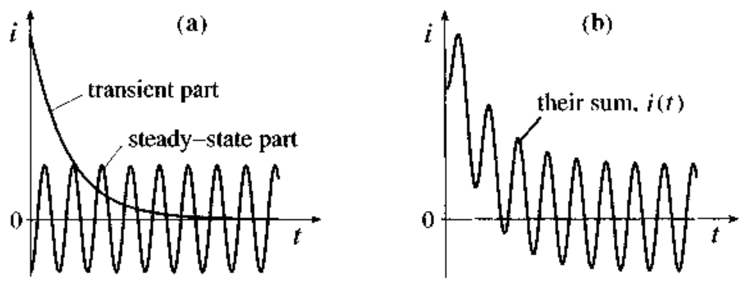


Figure 14. The response (36), using the representative values $E_0 = 1$, $R = 1$, $L = 4$, $\omega = 2$, and $i_0 = 0.3$.

in (a) and the total response $i(t)$, their sum, is shown in (b). For the choice of parameters used to generate Fig. 14, we see that the transient part of $i(t)$ has practically died out after five or six cycles. ∎

Closure. We studied representative applications of linear first-order equations, relying on solution techniques and formulas that were derived in Section 1.2, but two new ideas did arise.

First, in some problems the solution $x(t)$, say, approaches a steady state that is a constant or a steady oscillation as $t \to \infty$. In that case it is convenient to express

$$\boxed{x(t) = x_{\text{tr}}(t) + x_{\text{ss}}(t)} \tag{37}$$

where the transient part $x_{\text{tr}}(t)$ tends to zero as $t \to \infty$ and the steady-state part $x_{\text{ss}}(t)$ is a constant or a steady oscillation. The steady-state part is of particular interest because that is, after all, what we end up with after the transient part has died out.

Second, we introduced the idea of a one-dimensional phase line for autonomous systems, namely, systems of the form

$$\boxed{\frac{dx}{dt} = f(x).} \tag{38}$$

The idea is to plot the derivative x' versus x; that is, $f(x)$ versus x. Where $f(x) > 0$ the flow on the x axis phase line is rightward, where $f(x) < 0$ it is leftward, and where $f(x) = 0$ there is an equilibrium point. This qualitative flow diagram helps us to sketch the solution curves, without solving the differential equation.

EXERCISES 1.3

1. If a population governed by the exponential model has 4,500 members after five years and 6,230 after 10 years, what is its growth rate? Its initial population?

2. If a population governed by the exponential model has 500 members after two years and 460 after five years, what is its growth rate? Its initial population?

3. The world population is increasing at approximately 1.3% per year. If its growth rate remains constant, how many years will it take for its population to double? To triple?

4. If a population governed by the exponential model doubles after m days, after how many days will it have tripled?

5. A certain population is initially 1,000, grows to 1,200 after 10 years, and to 1,400 after another five years. Do you think it might be well described by the exponential model (3)? Explain.

6. ***E. Coli* Cultures.** The bacterium *Escherichia coli*, which inhabits the human intestine, multiplies by cell division. Since it is capable of rapid growth and can be grown in the laboratory it is a useful subject for experiments on population dynamics. It can be grown in culture and the population can be estimated indirectly, by measuring the turbidity of the culture through its scattering of incident light. The population $N(t)$ of a colony of *E. coli* cells can be modeled by the Malthus equation $N' = kN$. Suppose a colony of *E. coli* is grown in a culture having a growth rate $k = 0.2$ per hour. (From

$N' = kN$ we can see that the units of k are 1/time.) At the end of 5 hours the culture conditions are modified (e.g. by increasing the nutrient concentration in the medium) so that the new growth rate is $k = 0.5$ per hour. If the initial population is $N(0) = 500$, evaluate N at $t = 20$.

7. Allowing for Migration. Thus far we've used $N' = kN$ to model the population dynamics of a single species. Implicit in that equation is that the system is *closed*; that is, its borders are closed to influx or efflux of that species due, for instance, to migration. How would you modify that equation to account for a known migration rate $r(t)$ (individuals per unit time, counted as positive if it is immigration and negative if it is emigration)?

8. Radioactive Decay. We claimed that if we put $k = (\ln 2)/T$ into (11) we obtain (12). Fill in the missing steps.

9. (a) A seashell contains 90% as much C-14 as a living shell of the same size. How old is it? NOTE: The half-life of C-14 is $T = 5,570$ years.
(b) How many years did it take for its C-14 content to diminish from its initial value to 99% of that value?

10. Suppose 10 grams of some radioactive substance reduces to 8 grams in 60 years.
(a) How many more years until 2 grams are left?
(b) What is its half-life?

11. If 20% of a radioactive substance disappears in 70 days, what is its half-life?

12. Suppose an element X decays radioactively to an element Y with a half-life T_{xy}, that Y in turn decays to an element Z with a half-life T_{yz}, and that Z is not radioactive.

(a) Let $x(t), y(t), z(t)$ denote the masses of X, Y, Z, respectively, in a given sample. Write down a set of three differential equations for $x(t), y(t), z(t)$. NOTE: Recall that the rate constant k in (10) is expressible in terms of the half-life T as $k = (\ln 2)/T$.
(b) By adding the three differential equations, show that $x(t) + y(t) + z(t)$ is a constant, and explain why that result makes sense.
(c) Let the initial conditions be $x(0) = 100$ g, $y(0) = 50$ g, $z(0) = 20$ g, and let $T_{xy} = 50$ yr, and $T_{yz} = 200$ yr. Solve the three IVPs for $x(t), y(t), z(t)$.

13. A radioactive substance having a mass m_1 at time t_1 decays to a mass m_2 at time t_2. Use that information to solve for its half-life T in terms of m_1, m_2, t_1, t_2.

14. Mixing Tank. In Example 2, let $v = 500$ gal. For $c(t)$ to diminish to 98% of c_i in one hour, what flow rate Q (gal/min) is required?

15. For the mixing tank in Fig. 3, let the initial concentration in the tank be $c(0) = 0$. Beginning at time T the inflow concentration is changed from c_i to zero.

(a) Solve for $c(t)$, both for $t < T$ and for $t > T$. HINT: Break the problem into two parts: for $t < T$ solve $c' + (Q/v)c = c_iQ/v$; $c(0) = 0$, and for $t > T$ solve $c' + (Q/v)c = 0$ subject to an initial condition that $c(T)$ that is the final value (i.e., at $t = T$) from the first solution (i.e., on $t < T$).
(b) Give a labeled hand sketch of the graph of $c(t)$.

16. For the mixing tank shown in Fig. 3, let $c(0) = 0$. Beginning at time T the flow rate Q is increased to $10Q$. Solve for $c(t)$, both for $t < T$ and for $t > T$. Is $c(t)$ continuous at $t = T$? How about $c'(t)$? Explain. HINT: See the hint in Exercise 15. The idea in this problem is the same.

17. Runoff Into Your Pond. Your garden pond is 300 ft^2, with an average depth of 3 ft. It rains hard for one hour, during which time the pond receives runoff from your neighbor at a rate $Q = 20$ ft^3/hr, with a concentration of a weed killer, Di-Bolic, equal to 0.01 lb/ft^3. The pond volume remains constant because there is an overflow pipe. Considering the concentration of Di-Bolic in the pond to be spatially uniform (hence, a function only of the time t), calculate its value $c(t)$ at the end of the hour if $c(0) = 0$.

18. Inflow and Outflow Rates Unequal. If, for the mixing tank shown in Fig. 3, the inflow is $Q_i = 5$ gal/min and the outflow is $Q_o = 12$ gal/min, then the liquid volume v in the tank is *not constant, so (15) does not apply.* But in place of (15) you can use (14), which becomes

$$[v(t)c(t)]' = Q_i c_i - Q_o c(t), \tag{18.1}$$

because it holds even if v is a function of t.

(a) Let the liquid volume $v(t)$ be 1,000 gal at $t = 0$, so $v(t) = 1,000 + (5-12)t = 1,000 - 7t$, let $c_i = 2$ lb/gal, and let $c(0) = 0$. Write down the IVP for $c(t)$ and solve for $c(t)$.
(b) What is $c(t)$ at the instant when the last bit of liquid is draining from the tank?
(c) You should have found, in (b), that $c(t)$ tends to the incoming concentration $c_i = 2$ as the last bit of liquid is draining from the tank. Is that result is a coincidence? Explain.

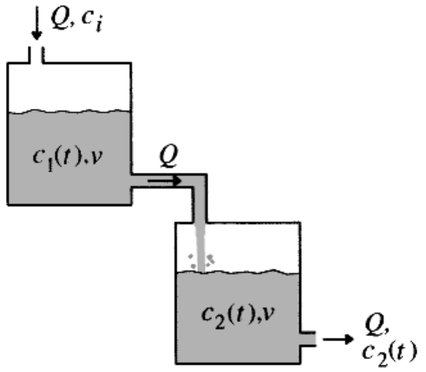
19. A tank initially contains 100 gal of fresh water. Brine containing 0.5 lb/gal of salt flows in at the rate of 8 gal/min and brine at concentration $c(t)$ flows out at the rate of 5 gal/min.

(a) Solve for the concentration $c(t)$ in the tank. HINT: The volume v is not a constant, so equation (15) does not apply; use (18.1) instead, in Exercise 18.
(b) How long will it take for there to be 40 lb of salt in the tank?

If a difficult algebraic equation arises, use computer software to solve it.

20. For the mixing tank shown in Fig. 3, let the initial concentration in the tank be $c(0) = c_0$. Beginning at time T the inflow is shut off, while the outflow rate Q is maintained. Solve for $c(t)$ for $t < T$ and for $t > T$. HINT: See the hint for Exercise 15. The idea here is the same. Also, for $t > T$ the liquid volume v is not a constant, so use (14) instead of (15).

21. Mixing Tanks in Series. Consider two tanks in series, as shown below, with an inflow of Q gal/min of solution containing c_i lb/gal of solute and an equal outflow rate Q. Let the liquid volume in each be v gal, and let $c_1(0) = c_2(0) = 0$.



(a) Use a mass balance for each tank to derive the IVPs

$$c_1'(t) + \frac{Q}{v}c_1(t) = \frac{Q}{v}c_i; \qquad c_1(0) = 0, \qquad (21.1a)$$

$$c_2'(t) + \frac{Q}{v}c_2(t) - \frac{Q}{v}c_1(t) = 0; \quad c_2(0) = 0. \qquad (21.1b)$$

(b) Solve (21.1a,b) and show that

$$c_2(t) = c_i\Big[1 - \big(1 + \frac{Qt}{v}\big)e^{-Qt/v}\Big]. \qquad (21.2)$$

NOTE: Actually, (21.1) is a *system* of coupled differential equations for c_1 and c_2, and systems are not studied until Chapter 4. However, although both c_1 and c_2 are present in (21.1b), only c_1 is present in (21.1a), so you can solve (21.1a) for c_1. Then, put that result into (21.1b) and solve the latter for c_2.
(c) From (21.2), show that $c_2(t) \to c_i$ as $t \to \infty$; that is, show that the steady-state outflow concentration equals the inflow concentration c_i. Further, show that $c_2(t) \to c_i$ and $c_1(t) \to c_i$ as $t \to \infty$, directly from (21.1). HINT: By the definition of steady state, set $c_1'(t)$ and $c_2'(t)$ to zero in the differential equations, and solve the resulting algebraic equations for the steady-state concentrations.

22. Computer. This exercise is to provide experience with some of the differential equation software.

(a) Use computer software to obtain an analytic solution of the IVP

$$100i' + 500i = 25(1+t)/(2+t); \quad i(0) = 0. \qquad (22.1)$$

(You will find that the solution is messy, and involves a nonelementary function denoted as Ei and called the *exponential integral function.*) Also, use computer software to generate the graph of $i(t)$ on $0 \le t \le 25$. Finally, obtain a computer tabulation of the solution values at $t = 0, 5, 10$.
(b) Same as (a), for the IVP

$$(2+t)i' + i = -50\sin t/(4+t); \quad i(0) = 2. \qquad (22.2)$$

(For the analytical solution you will again run into nonelementary functions, this time the *cosine integral* and the *sine integral* functions Ci and Si, respectively.)

23. Phase Line. Develop the phase line, as we did in Fig. 6, identify any equilibrium points, and state whether each is stable or unstable. Then, use that phase line to develop a hand sketch of the solutions corresponding to a handful of representative initial conditions, as we did in Fig. 7 (without the shaded area and large dragging arrows, of course).

(a) $x' = x^2 - x$ (b) $x' = x + x^2$ (c) $x' = x^3 - x$
(d) $x' = (x-1)^2$ (e) $x' = \sin x$
(f) an equation supplied by your instructor

24. Light Extinction; Lambert's Law. Consider window glass subjected to light rays normal to its surface, and let x be a coordinate normal to that surface with $x = 0$ at the incident face. It is found that the light intensity I in the glass is not constant, but decreases with the penetration distance x, as light is "absorbed" by the glass. According to *Lambert's law*, the fractional loss in intensity between x and $x + dx$, $-dI/I$ (with the minus sign included because dI is negative), is proportional to dx: $-dI/I = k\,dx$, where k is a positive constant. Thus, $I(x)$ satisfies the differential equation

$$\boxed{\frac{dI}{dx} = -kI.} \qquad (24.1)$$

(a) If 80% of the light penetrates a 1-inch thick slab of this glass, how thin must the glass be to let 95% penetrate?
(b) If 50% of the light penetrates five inches, how far does 25% penetrate? How far does only 1% penetrate?

25. Modeling Mothballs and "Mothcylinders." (a) A spherical mothball evaporates with time. (For a mothball that is not a defect; it is *supposed to* evaporate.) Does it completely disappear in finite time or only as $t \to \infty$? NOTE: You will need to model the evaporation process in some simple and reasonable way so as to derive a differential equation for the radius $r(t)$. Solve it for $r(t)$, assuming an initial radius $r(0) = r_0$.

(b) Suppose that instead of being spherical the mothball is a circular cylinder of radius $r(t)$, with an initial radius $r(0) = r_0$ and an initial length L that is much larger than r_0. Again, model this problem so as to obtain a differential equation for $r(t)$. Solve for $r(t)$ in terms of r_0. Does this type mothball (really, "mothcylinder") evaporate in finite time? HINT: Use the fact that the initial length is much larger than the initial radius to help you to obtain an *approximate* differential equation for $r(t)$. Indicate the approximations that you adopt in obtaining your differential equation. For instance, do you need to take into account evaporation at the two ends, or only along the lateral surface?

26. Compound Interest. If a sum of money S earns interest at a rate k per unit time, compounded continuously, then in time dt we have $dS/S = kdt$, so $S(t)$ satisfies

$$\frac{dS}{dt} = kS. \tag{26.1}$$

Thus, if $S(0) = S_0$, then

$$S(t) = S_0 e^{kt}. \tag{26.2}$$

If, instead, interest is compounded yearly, then after t years

$$S(t) = S_0(1+k)^t. \tag{26.3}$$

Finally, if it is compounded n times per year, then

$$S(t) = S_0\left(1 + \frac{k}{n}\right)^{nt}. \tag{26.4}$$

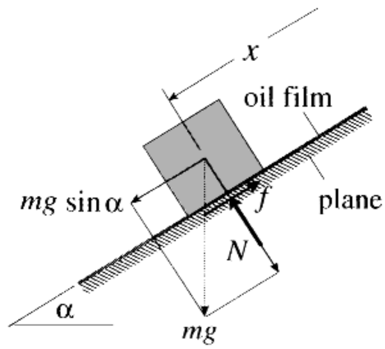
(a) Show that if we let $n \to \infty$ in (26.4), then we do obtain the continuous compounding result (26.2). HINT: Recall, from the calculus, that $\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$.

(b) Let $k = 0.05$ (i.e., 5% interest) and compare $S(t)/S_0$ after 1 year [i.e., at $t = 1$] if interest is compounded yearly, monthly, weekly, daily, and continuously.

27. Mass Sliding Down a Lubricated Plane. A block of mass m slides down a plane that is at an angle α with respect to the horizontal, under the action of gravity and friction, air resistance being negligible. Applying Newton's second law to the motion in the tangential and normal directions gives

$$mx'' = -f + mg\sin\alpha, \tag{27.1a}$$

$$0 = N - mg\cos\alpha, \tag{27.1b}$$



respectively. The left-hand side of (27.1b) is zero because the acceleration normal to the plane is zero. If the plane and block are lubricated then the friction force is, to a good approximation, proportional to the block's velocity $x'(t)$, so $f \approx cx'$ where c is a constant, and (27.1a) gives the second-order differential equation

$$mx'' + cx' = mg\sin\alpha \tag{27.2}$$

for the motion $x(t)$. Assume that $x(0) = 0$ and $x'(0) = 0$.

(a) It's true that (27.2) is a second-order equation whereas this chapter is about first-order equations, but you can integrate it once with respect to t to obtain a first-order equation. Do that, then solve that equation for $x(t)$ and show that

$$x(t) = \frac{m^2 g\sin\alpha}{c^2}\left(\frac{ct}{m} + e^{-ct/m} - 1\right). \tag{27.3}$$

(b) The constant of proportionality c in $f = cx'$ is $\mu A/h$ where A is the area of the bottom of the block, h is the normal distance between the bottom of the block and the plane (i.e., the oil film thickness), and μ is the viscosity of the oil. Suppose, in an experiment, $m = 0.5$ slugs, $A = 0.6$ ft^2, $\alpha = 30°$, $h = 0.003$ ft, and that when $t = 5$ sec we measure x to be 114 ft. Also, $g = 32.2$ ft/sec^2. Use that data in (27.3) to solve for the viscosity of the oil, μ, by computer if necessary. NOTE: That is, think of this as an experiment aimed at the determination of the viscosity of a given lubricating oil.

28. Sliding With Dry Friction. In the preceding exercise the block/plane interface was lubricated and the friction force f in (27.1a) was of the form $f = cx'$, proportional to the velocity x'. Suppose instead that the interface is dry (not lubricated). Then (27.1a,b) still hold, but in that case (if α is large enough for slipping to be initiated in the first place) the friction force f is proportional to the normal force N: $f = \mu N$ where the constant of proportionality μ is the coefficient of sliding friction. With $f = \mu N$ in (27.1a), and N given by (27.1b), solve for $x(t)$ subject to the conditions $x(0) = 0$ and $x'(0) = 0$, and show that

$$x(t) = \frac{g}{2}(\sin\alpha - \mu\cos\alpha)t^2. \tag{28.1}$$

29. The R, L Circuit in Fig. 12b. The R, L circuit in Fig. 12b was modeled by (32) and its solution was given by (33). Let $R = 2$ ohms, $L = 10$ henrys, and $i_0 = 0$ amperes.
(a) Let $e(t) = 5\sin t$ volts for $0 < t < 6\pi$ and 0 for $t > 6\pi$ seconds. Solve for $i(t)$, both for $0 < t < 6\pi$ and for $t > 6\pi$, either by hand or by computer. HINT: First, for $0 < t < 6\pi$ solve $i' + 0.2i = (0.1)(5)\sin t$ with $i(0) = 0$, and call the solution $i_1(t)$. Then, for $6\pi < t < \infty$ solve $i' + 0.2i = 0$ with initial condition $i(6\pi) = i_1(6\pi)$.
(b) Obtain a computer plot of the solution obtained in part (a). HINT: How can we plot the two parts of the solution together? Denote the solutions on $0 < t < 6\pi$ and $6\pi < t < \infty$ as $i_1(t)$ and $i_2(t)$, respectively. Then a single expression valid on $0 < t < \infty$ is

$$i(t) = i_1(t) + H(t-6\pi)[i_2(t) - i_1(t)] \tag{29.1}$$

in which $H(t)$ is the **Heaviside function** which is defined as 0 for $t < 0$ and 1 for $t > 0$. Thus, $H(t - 6\pi)$ is 0 for $t < 6\pi$ and 1 for $t > 6\pi$, so the right-hand side of (29.1) is $i_1(t)$ for $t < 6\pi$ and $i_2(t)$ for $t > 6\pi$. In *Maple*, for instance, $H(t)$ is entered as Heaviside(t).
(c) Now let $e(t) = 5t$ volts for $0 < t < 10$ and 10 for $t > 10$. Solve for $i(t)$, both for $0 < t < 10$ and for $t > 10$, by hand or by computer.
(d) Obtain a computer plot of the solution obtained in part (c).

30. Newton's Law of Cooling. Newton's law of cooling states that a body that is hotter than its environment will cool at a rate proportional to the temperature difference between the body and its environment, so that the temperature $u(t)$ of the body is modeled by the differential equation

$$\boxed{\frac{du}{dt} = k(U - u),} \tag{30.1}$$

in which U is the temperature of the environment (assumed here to be a constant), t is the time, and k is a positive constant of proportionality. NOTE: Parts (b), (c), (d), below, are *independent problems* that apply the results of part (a) to different situations. [Note also that if $u > U$ then (30.1) does indeed model the *cooling* of the body by Newton's law of cooling. But, (30.1) holds for "Newton heating" as well, that is, if $U > u$. In that case the right-hand side of (30.1) is positive, so $u(t)$ is an increasing function of t, as the body is being heated by the environment.]

(a) Derive the general solution of (30.1),

$$u(t) = U + Ae^{-kt}. \tag{30.2}$$

(b) A cup of coffee in a room that is at 70° F is initially at 200° F. After 10 minutes it has cooled to 180° F. How long will it take to cool to 100° F? What will its temperature be three hours after it was poured?

(c) Yoshiko takes a cup of tea, initially at 200°F, outdoors at noon. By 12:06 pm it has cooled to 188° and by 12:12 pm it has cooled to 177°. By what time will it have cooled to 130°, assuming that the ambient temperature remains constant over that time period?

(d) An interesting application of (30.1) and its solution (30.2) is in connection with estimating the time of death in a homicide. For instance, suppose a body is discovered at a time T after death and its temperature is measured to be 90°F. We wish to determine T. Suppose the ambient temperature is $U = 70$°F and assume that the temperature of the body at the time of death was $u_0 = 98.6$°F. If we put this information into (30.2) we can solve for T, provided that we know k, but we don't. Proceeding indirectly, we can infer the value of k by taking one more temperature reading. Thus, suppose we wait an hour and again measure the temperature of the body, and find that $u(T+1) = 87$° F. [$u(T+1)$ is u at time $T+1$, not u *times* $T+1$.] Use this information to solve for T (in hours).

ADDITIONAL EXERCISES

31. Newton Heating and Cooling of a House. First, read the introduction to Exercise 30. Let $u(t)$ in (30.1) be the temperature inside a house that is subjected to a time-varying outside temperature $U(t) = 70 - 15\cos(\pi t/12)$ degrees Fahrenheit, where t is in hours and $t = 0$ corresponds to midnight. [If unclear about our choice of $U(t)$, sketch its graph and see that it is a reasonable choice of a daily temperature fluctuation, from a low of 55° at midnight to a high of 85° at noon.] Suppose that neither heating nor cooling are being used inside the house. Then (30.1) applies, with $U(t)$ as given above. Let the initial condition be $u(0) = 50$° although, looking ahead, this value will not affect the steady-state temperature fluctuation, which is our chief interest. Then we have the IVP

$$\begin{aligned} \frac{du}{dt} &= k\,[U(t) - u(t)] \\ &= k\Big[70 - 15\cos\frac{\pi t}{12} - u(t)\Big]; \;\; u(0) = 50. \end{aligned} \tag{31.1}$$

To understand the physical significance of k, consider the extreme cases in which $k \to \infty$ and $k \to 0$. As $k \to \infty$ it follows from (31.1) that $U(t) - u(t) \to 0$. Since in that case $u(t) = U(t)$ we see that $k = \infty$ corresponds to there being no insulation at all in the walls of the house. At the other extreme, in which $k \to 0$, it follows from (31.1) that $du/dt = 0$ so $u(t) = \text{constant} = u(0)$, which indicates that the house is "infinitely" insulated so there is no heat exchange at all with the outside. Here, let $k = 0.05$.

(a) Solve (31.1) by computer (with $k = 0.05$). You should find that

$$u(t) = -19.45e^{-0.05t} + 70 - 0.5275\cos 0.2618t - 2.764\sin 0.2618t. \tag{31.2}$$

As $t \to \infty$ the exponential term tends to zero and leaves a steady oscillation which we call the *steady-state solution*,

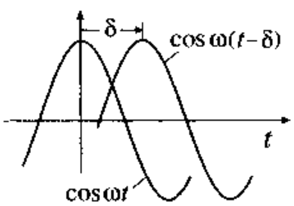
$$u_s(t) = 70 - (0.5275\cos 0.2618t + 2.764\sin 0.2618t). \tag{31.3}$$

(b) Obtain computer plots of the outside temperature $U(t)$ and the inside temperature $u(t)$, over a long enough time for the transient part of the response to die out.

(c) **Time Lag.** Verify that (31.3) can be re-expressed in the form

$$u_s(t) = 70 - 2.814\cos[0.2618(t - 5.280)]. \tag{31.4}$$

Our purpose in converting (31.3) to the form (31.4) is that in the latter form we can more readily compare it with the outside temperature $U(t) = 70 - 15\cos 0.2618t$. Doing so, we can see two effects. First, the presence of insulation causes a reduction in the amplitude of the temperature variation, from 15° outside to the more comfortable value of around 2.8° inside. Second, we see that there is a **time lag** of 5.28 hours in the response (see the figure, below), so although the outside



temperature peaks at noon, the temperature inside does not peak until after 5pm. Do these two results agree with your computer plots obtained in part (a)? HINT: To verify (31.4) use the trigonometric identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

(d) Now consider the more general case in which $U(t) = U_0 + a\cos(\pi t/12)$, for any values of the average outdoor temperature U_0, the amplitude a, the constant k, and the initial temperature $u(0)$. Solve the differential equation

$$\frac{du}{dt} = k\left[U_0 + a\cos\frac{\pi t}{12} - u(t)\right] \tag{31.5}$$

and discuss the effect on the amplitude and time lag of the steady-state response in the limits as $k \to \infty$ (no insulation) and as $k \to 0$ (perfect insulation).

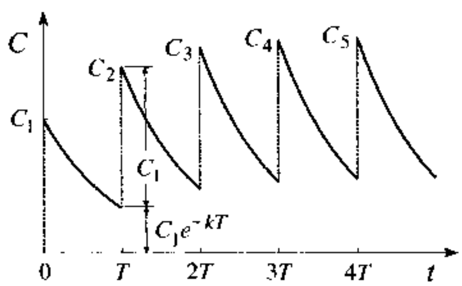
32. Drug Delivery in Pharmacology. Suppose we take a dose of a certain drug, either orally or intravenously. As the blood circulates, the drug will disperse and its concentration will tend to become uniform throughout the circulatory system. That will probably happen so quickly (particularly if the dose is administered intravenously), compared to the time T between doses (such as 24 hours), that we can idealize the situation and regard the concentration $C(t)$ (the mass of the drug per unit blood volume) as rising instantaneously to C_1 (which is the dosage divided by the total blood volume), and then diminishing relatively slowly as the drug passes through the the walls of the circulatory system into the muscles and organs of the body. Clinical studies show that $C(t)$ will diminish with time, approximately, according to the differential equation

$$\boxed{\frac{dC}{dt} = -kC,} \tag{32.1}$$

in which k is a positive experimentally known constant. The solution to (32.1), subject to the initial condition $C(0) = C_1$ is

$$C(t) = C_1 e^{-kt}. \tag{32.2}$$

At time T the concentration has fallen to $C_1 e^{-kT}$, so when we administer another dose the concentration jumps up "instantaneously" from that value by an additional C_1, to a new peak given by $C_2 = C_1 e^{-kT} + C_1$, as shown in the figure (in which all the vertical rises are of the same magnitude, C_1). The phar-



macology problem that we want to solve is to determine the correct dosage (which is the concentration C_1 times the patient's total blood volume v) and the time interval T between doses. The constraints are that the drug is helpful if its concentration is above some known value $C_{\min}$ and harmful if it exceeds some known value $C_{\max}$.

(a) Show that the successive peaks are

$$C_2 = C_1 + C_1 e^{-kT},$$
$$C_3 = C_1 + C_2 e^{-kT} = C_1\left(1 + e^{-kT} + e^{-2kT}\right),$$

and so on, so

$$C_n = C_1\left(1 + e^{-kT} + e^{-2kT} + \cdots + e^{-nkT}\right). \qquad (32.3)$$

(b) We can see that $C_1, C_2, \ldots$ is an increasing sequence because C_2 is C_1 plus the positive quantity $C_1 e^{-kT}$, C_3 is C_2 plus the positive quantity $C_2 e^{-kT}$, and so on. If that increasing sequence diverges to infinity then $C_{\max}$ will be exceeded, so we are concerned with whether or not it converges and, if it does, to what value. To see if it does, recall that the geometric series $1 + x + x^2 + \cdots$ converges to $1/(1-x)$ if $|x| < 1$, and hence show that

$$\lim_{n\to\infty} C_n = C_1/\left(1 - e^{-kT}\right) \equiv C_\infty. \qquad (32.4)$$

That is, as $t \to \infty$ the sequence of peaks C_n converges, and $C(t)$ tends to a steady-state oscillation, with peak values C_∞ given by (32.4).

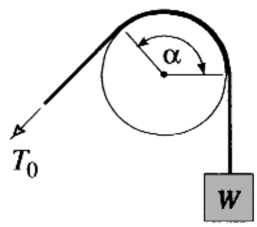
(c) Following such a peak, the concentration diminishes to a minimum value that occurs immediately before the next dose. Show that that minimum value is $C_\infty e^{-kT}$.

(d) Now that we understand the time history of $C(t)$, we can design the drug protocol: Set $C_\infty = C_{\max}$ and $C_\infty e^{-kT} = C_{\min}$. Thus, solve for the time T between doses and the dosage D, say, in terms of the known values $C_{\max}$, $C_{\min}$, the blood volume v, and the empirical constant k.

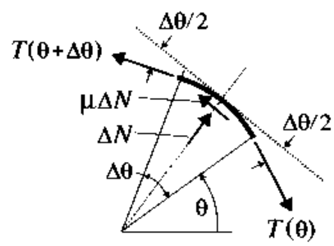
(e) If we forget to take a pill, we're tempted to take two the next time, but the instructions tell us not to do that, but to return to taking one pill every T hours. Explain the reasoning behind those instructions.

NOTE: Observe how important the figure was in facilitating this analysis. It is difficult to imagine successfully analyzing this problem without it. More generally, be aware of the importance of supporting your work with suitable sketches.

33. Belt Friction. We know from experience that if a belt (or rope) is wrapped around a cylinder such as a tree trunk, then a large force on one end of the belt can be supported, without the belt slipping, by a relatively small force on the other end of the belt, thanks to the friction between the cylinder and the belt. For instance, consider a flexible belt hanging over a fixed horizontal circular cylinder, as shown in the figure.



A weight force W is applied at one end and the problem is to find the tension T_0 at the other end, as a function of the wrap angle α, that will keep the belt from slipping. Consider a typical infinitesimal element of the belt and the forces that act upon it, as shown below. The force exerted on the element by the cylinder can be broken into a radial component



or normal force ΔN, say, and a tangential component due to the friction between the cylinder and the belt. It is known from physics that the friction force that can be sustained, without the belt slipping, is proportional to the normal force. The constant of proportionality is the coefficient of static friction μ, which is an experimentally known constant that depends upon the two materials (the belt and the cylinder). Thus, the tangential friction force is $\mu\Delta N$, as shown in the figure. Assume that the belt is sufficiently light (compared to W) for us to neglect its weight. Thus, we have not included a weight force on the belt element in the figure. For static equilibrium the net force on the element must be zero, so the net tangential force and the net radial force must each be zero:

$$\Sigma F_{\text{tang}} = T(\theta+\Delta\theta)\cos\frac{\Delta\theta}{2} - T(\theta)\cos\frac{\Delta\theta}{2} + \mu\Delta N = 0, \qquad (33.1a)$$

$$\Sigma F_{\text{radial}} = \Delta N - T(\theta+\Delta\theta)\sin\frac{\Delta\theta}{2} - T(\theta)\sin\frac{\Delta\theta}{2} = 0. \qquad (33.1b)$$

(a) We're not interested in the normal force distribution, so eliminate ΔN between (33.1a) and (33.1b) by algebra. In the resulting equation let $\Delta\theta \to 0$ and thus show that

$$\boxed{\frac{dT}{d\theta} = -\mu T,} \tag{33.2}$$

which is a differential equation for the tension T in the belt as a function of angular position θ.

(b) From (33.2) show that the tension T_0 in the first figure, needed to support the weight without slipping, is

$$T_0 = We^{-\mu\alpha}, \tag{33.3}$$

which is our final result.

COMMENTS: (i) Thus, the force T_0 that is needed decreases *exponentially* with the wrapping angle α. For instance, suppose that $\mu = 0.4$, corresponding to leather on metal. If $\alpha = \pi$ then $T_0 = We^{-(0.4)\pi} = 0.285W$, and if $\alpha = 5\pi$ (so the belt is wrapped around the cylinder two and one half times), then $T_0 = We^{-(0.4)5\pi} = 0.0019W$. For instance, if $W = 1000$ lbs, then the force T_0 needed to support it is only around 2 lb.

(ii) Our derivation of (33.2) is typical of the method used in engineering courses and textbooks on mechanics: Isolate a typical infinitesimal element of the system, show the forces acting on it, write down the governing physics (Newton's second law of motion in this case), and then let the infinitesimal increment ($\Delta\theta$ in this case) tend to zero.

(iii) Instead of setting the radial and tangential force components equal to zero, as we did in (33.1), we could have set the horizontal and vertical force components equal to zero, but the former was more convenient.

(iv) The foregoing derivation of (33.2) is typical of the derivation of the differential equations governing the variety of phenomena encountered in undergraduate engineering curricula: Isolate a typical arbitrarily small element; indicate the forces, fluxes, etc.; write down the governing physical principle(s); and take the limit as the spatial or temporal increment tends to zero. In this case the physical law was Newton's second law of motion which, for static equilibrium, amounts to the sum of the forces being zero.

34. Differential equations of the form $y' + py = 0$, in which p is a constant, have arisen, in this section, in modeling a wide range of applications. List as many as you can find, in both the text and the exercises. For instance, one would be (33.2) in Exercise 33, for the tension in a belt. Refer to texts on application areas, such as bioengineering, if you wish.

1.4 NONLINEAR FIRST-ORDER EQUATIONS THAT ARE SEPARABLE

Having thus far studied only the linear equation

$$\frac{dy}{dx} + p(x)y = q(x), \tag{1}$$

we now consider the general equation

$$F(x, y, y') = 0,$$

and assume that we can solve it by algebra for y' and thus express it in the **standard form**

$$\boxed{\frac{dy}{dx} = f(x, y).} \tag{2}$$

Standard form.

Of course, (2) includes the linear equation (1) as a special case, but we've already studied that case, so here we focus on the case where (2) is *nonlinear*.

For the linear equation (1), we were successful in deriving an explicit formula for its general solution, in terms of $p(x)$ and $q(x)$. If an initial condition was prescribed, then the solution thus obtained was unique, and a minimum interval of existence could be determined for it in advance, by examining $p(x)$ and $q(x)$. For the nonlinear equation (2) we are not so fortunate. It is not possible to obtain an explicit solution formula in terms of $f(x, y)$, and the issues of existence, uniqueness, and interval of existence are more subtle.

Therefore, we consider only some special cases of (2), for which solution methods are available. The most prominent is the case where (2) is **separable**, which means that $f(x, y)$ can be factored as a function of x times a function of y:

$$\boxed{\frac{dy}{dx} = X(x)Y(y).} \tag{3}$$

For instance, $y' = xe^{x+2y}$ is separable because it can be written as $y' = (xe^x)(e^{2y})$, but $y' = 3x + y$ is not, because $3x + y$ cannot be written as a function of x times a function of y.

Actually, the linear homogeneous equation $y' + p(x)y = 0$ that we studied in Section 1.2.2 was separable because it can be expressed as $y' = X(x)Y(y)$ with $X(x) = -p(x)$ and $Y(y) = y$. In that case we solved by *separation of variables*: We divided both sides by y, multiplied both sides by dx, and integrated:

Or $X(x) = p(x)$ and $Y(y) = -y$, of course.

$$\int \frac{dy}{y} = -\int p(x)\,dx. \tag{4}$$

We can use that same **separation of variables** method to solve (3), whether it is linear or not: Divide both sides by $Y(y)$ [tentatively assuming that $Y(y) \neq 0$ because division by zero is not permissible], multiply both sides by dx, and integrate:

Thanks to the separable form of (3), there are no x's in the y integral and no y's in the x integral.

$$\boxed{\int \frac{dy}{Y(y)} = \int X(x)\,dx.} \tag{5}$$

Then evaluate the integrals in (5), if we can, including the usual additive arbitrary constant of integration.

EXAMPLE 1. Solution by Separation of Variables. Solve

$$\frac{dy}{dx} = 2(x-1)e^{-y}. \tag{6}$$

First, identify (6) as separable, with $X(x) = 2(x-1)$ and $Y(y) = e^{-y}$. In this case $Y(y) \neq 0$ for all y so we can divide both sides of (6) by e^{-y} (or, equivalently, multiply by e^y), multiply by dx, and integrate. Thus,

$$\int e^y\,dy = 2\int (x-1)\,dx, \tag{7}$$

so

$$e^y = x^2 - 2x + C, \tag{8}$$

in which C is an arbitrary constant. Solving (8) for y gives

$$y(x) = \ln(x^2 - 2x + C). \tag{9}$$

The latter is plotted in Fig. 1 for the representative values $C = 1, 3, 5, 7$, together with the direction field.

Now consider applying initial conditions, say $y(4) = 5$ and $y(0) = 0$, in turn.

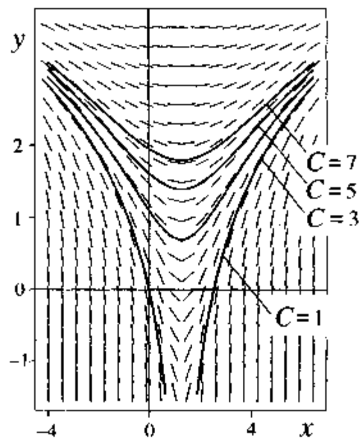


Figure 1. Representative members of the family of solutions (9), and the direction field.

$\boldsymbol{y(4) = 5}$: Applying this condition to (9) gives $5 = \ln(16 - 8 + C)$ so $C = e^5 - 8$. Hence,

$$\begin{aligned} y(x) &= \ln(x^2 - 2x + e^5 - 8) \\ &= \ln\left[(x-1)^2 + e^5 - 9\right], \end{aligned} \tag{10}$$

the graph of which is plotted in Fig. 2.

What is its interval of existence? It's true that the logarithm function tends to $-\infty$ as its argument tends to zero (Fig. 3), but the argument $(x-1)^2 + e^5 - 9$ is positive for all x, its smallest value being $e^5 - 9$. Thus, the right-hand side of (10) is defined for all x, and the interval of existence of (10) is $-\infty < x < \infty$.

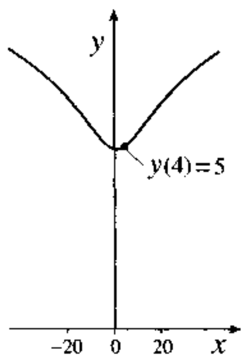


Figure 2. Particular solution (10), satisfying $y(4) = 5$, with domain of existence $-\infty < x < \infty$.

$\boldsymbol{y(0) = 0}$: In this case, (9) gives $y(0) = 0 = \ln C$, so $C = 1$ and

$$y(x) = \ln(x^2 - 2x + 1) = \ln(x-1)^2 = 2\ln|x-1|, \tag{11}$$

the graph of which is given in Fig. 1, for $C = 1$, and is displayed by itself in Fig. 4. This time the logarithm does "blow up," namely, at $x = 1$. That is, $2\ln|x-1| \to -\infty$ as $x \to 1$, so we can think of the graph as consisting of two branches, one to the left of $x = 1$ (labeled L in Fig. 4) and one to the right of $x = 1$ (labeled R). The solution through $(0, 0)$ can be extended arbitrarily far to the left, along L, but it cannot be extended to the right up to or beyond $x = 1$ because the solution (11) tends to $-\infty$ as $x \to 1$ from the left, and becomes undefined at $x = 1$. Thus, the solution through the initial point $(0, 0)$ consists only of the left-hand branch L, and its domain of existence is $-\infty < x < 1$. The right-hand branch R is to be discarded, as we've suggested in Fig. 4 by using a dotted line for its graph.

COMMENT 1. To solve for C we applied the initial condition $y(4) = 5$ to the solution (9), but it would have been slightly simpler to apply the initial condition to (8).

COMMENT 2. A potential error is to omit the constant C in (8) and then include it in (9), writing $y(x) = \ln(x^2 - 2x) + C$ instead of (9). That is incorrect, and the two are not equivalent; C is an integration constant so it must be inserted immediately upon doing the integrations, in (8). ∎

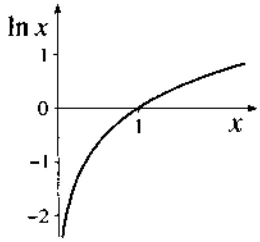


Figure 3. Recall: $\ln x$ tends to $-\infty$ as $x \to 0$ and to $+\infty$ as $x \to \infty$.

EXAMPLE 2. Solve

$$\frac{dy}{dx} = -y^2. \tag{12}$$

It might appear that the right-hand side of (12) is not of the form $X(x)Y(y)$ because we see no x's, but it is; we can take $X(x) = -1$ and $Y(y) = y^2$. Now proceed. If $y \neq 0$, we

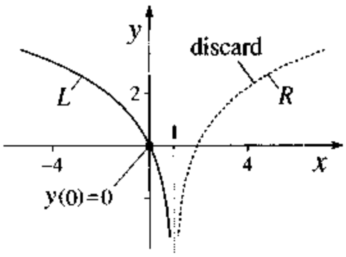


Figure 4. Particular solution (11) corresponding to the initial condition $y(0)=0$. These curves are also in Fig. 1, for $C=1$.

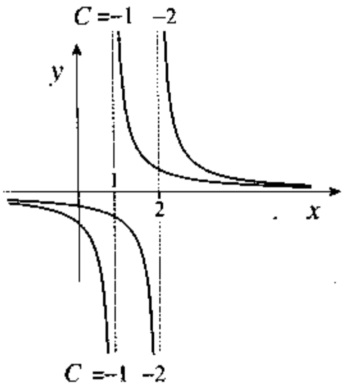


Figure 5. The graph of (13) for representative values of C; note the infinite jump discontinuities.

can divide (12) by y^2, separate the variables, and obtain

$$\int \frac{dy}{y^2} = -\int dx.$$

Evaluating the integrals and solving for y gives

$$y(x) = \frac{1}{x+C}, \tag{13}$$

with C arbitrary ($-\infty < C < \infty$). The solution (13) has an infinite jump discontinuity at $x=-C$ and is plotted in Fig. 5 for representative values of C.

Having tentatively assumed that $y \neq 0$ when we divided (12) by y^2, we must consider that case separately. In fact, we see that

$$y(x) = 0 \tag{14}$$

satisfies (12) because it reduces (12) to the identity $0=0$. The solution (14) is not contained in (13) by any finite choice of C, so it is an additional solution, in addition to (13). In summary, the solutions of (12) consist of the set of functions (13), for all values of C in $-\infty < C < \infty$, together with the additional solution $y(x)=0$.

Now consider appending representative initial conditions to (12): $y(0)=1$, $y(2)=-3$, and $y(1)=0$, in turn.

$\boldsymbol{y(0)=1}$**:** Applying this condition to (13) gives $C=1$, so $y(x)=1/(x+1)$, which is displayed in Fig. 6a. We can see that the solution "blows up" at $x=-1$; $y \to +\infty$ as $x \to -1$ from the right. Surely (12) is not satisfied by $y(x)=1/(x+1)$ *at* $x=-1$ because neither the y' on the left nor the y on the right of (12) is defined at $x=-1$. Thus, keep the right-hand branch in Fig. 6a, discard the left-hand branch, and conclude that the interval of existence of the solution $y(x)=1/(x+1)$ through the initial point $y(0)=1$ is $-1 < x < \infty$.

$\boldsymbol{y(2)=-3}$**:** Applying this condition to (13) gives $C=-7/3$, so $y(x)=1/(x-7/3)$, which is plotted in Fig. 6b. This time keep the left-hand branch, discard the right-hand branch, and conclude that the interval of existence of the solution $y(x)=1/(x-7/3)$ is $-\infty < x < 7/3$.

$\mathbf{y(1)=0}$: Applying this condition to (13) gives $y(1)=0=1/(1+C)$, but the latter cannot be solved for C. However, the additional solution $y(x)=0$, given by (14), satisfies this initial condition and the graph of that solution is shown in Fig. 6c. This solution exists on $-\infty < x < \infty$.

COMMENT. The factorization $X(x)=-1$ and $Y(y)=y^2$ is unique only to within an inconsequential scale factor. For instance, we could have taken $X(x)=1$ and $Y(y)=-y^2$, or $X(x)=378$ and $Y(y)=-y^2/378$, and so on. ∎

In summary, the separation of variables process is this: *Identify the factors $X(x)$ and $Y(y)$ (if the equation is separable), divide both sides by $Y(y)$ under the tentative assumption that $Y(y) \neq 0$, multiply both sides by dx, and integrate.*

The case where $Y(y) = 0$ has one or more roots for y must be treated separately. If y_0 is any root of $Y(y) = 0$, then $y(x) = y_0$ is a solution of $y' = X(x)Y(y)$ because setting $y(x) = y_0$ reduces the differential equation to the identity $d(y_0)/dx = X(x)Y(y_0)$, namely, $0 = X(x)(0)$. Thus, first assume that $Y(y) \neq 0$ and obtain the solution family obtained from (5). Then solve $Y(y) = 0$. If there are any (real) roots $y_1, \ldots, y_k$, then besides the solution family (5) there are the additional constant solutions $y(x) = y_1, \ldots, y(x) = y_k$. In Example 1 there were no such additional solutions, and in Example 2 there was one, namely, $y(x) = 0$.

There is no analog of these "additional solutions" for the *linear* differential equation $y' + p(x)y = q(x)$. In that case we were able to obtain the *general solution* with confidence that it contained *all* solutions. For nonlinear equations, however, being certain that we have the set of all solutions is a more subtle matter. *To avoid calling a solution a "general solution" without proving that it does contain all solutions, we will not use the term general solution for nonlinear differential equations.* Thus, in Example 2, for instance, we did not call (13) a general solution; we said that we found the "family of solutions" (13) plus the "additional solution" (14).

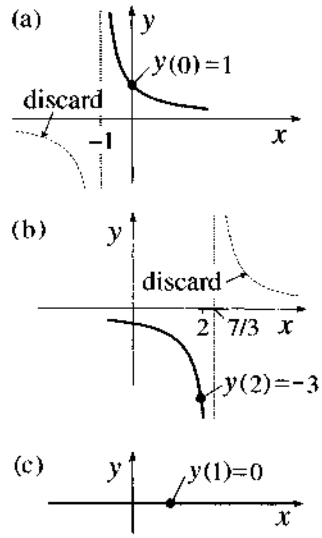


Figure 6. Particular solutions of (12) for $y(0) = 1$, $y(2) = -3$, and $y(1) = 0$.

EXAMPLE 3. Implicit Solution. Solve the IVP

$$\frac{dy}{dx} = \frac{(\sin x - 3x^2)(y-3)}{y-2}; \qquad y(0) = 5. \tag{15}$$

Separating variables and integrating gives

$$\int \frac{y-2}{y-3}\,dy = \int (\sin x - 3x^2)\,dx, \tag{16}$$

$$y - 3 + \ln|y-3| = -\cos x - x^3 + C. \tag{17}$$

To integrate, let $y - 3 = z$ and get $\int (1 + \frac{1}{z})\,dz$.

Unfortunately, we cannot solve (17) for y. Nevertheless, we can apply the initial condition to (17) to evaluate C: $2 + \ln 2 = -1 + C$ gives $C = 3 + \ln 2$, so the solution of (15) is given by

$$y - 3 + \ln|y-3| = -\cos x - x^3 + 3 + \ln 2. \tag{18}$$

Since we are not able to solve the latter for y, we accept it as it is, a **relation** on x and y that defines $y(x)$ only implicitly rather than explicitly. Thus, we say that the solution (18) is in **implicit form** rather than explicit form.

In spite of its implicit form, (18) can be used to obtain a computer plot of y versus x. The result is shown in Fig. 7.

COMMENT. In this example, $Y(y) = (y-3)/(y-2) = 0$ has the root $y = 3$ so, in addition to the family of solutions given implicitly by (18), $y(x) = 3$ is a solution as well. That additional solution turns out not to be relevant in that it does not satisfy the initial condition $y(0) = 5$, but if the initial condition were $y(0) = 3$ instead, then the solution would be that additional solution $y(x) = 3$. By the way, the solution shown in Fig. 7 is *not* identically 3 beyond $x \approx 2$; it is simply extremely close to 3. For instance, $y(2.5) \approx 3.0000147$. ∎

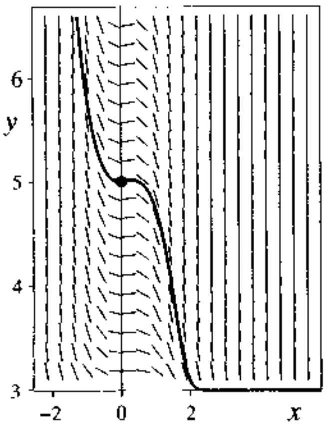


Figure 7. The solution (18) of the IVP (15). The dot shows the initial condition $y(0) = 5$.

Closure. We began our study of nonlinear equations in this section by considering the important special case in which the equation is separable, namely, of the form $y' = X(x)Y(y)$. Assuming first that $Y(y) \neq 0$, separation of variables gives

$$\int \frac{dy}{Y(y)} = \int X(x)\,dx, \tag{19}$$

but the latter *may* give $y(x)$ only in implicit rather than explicit form, as occurred in Example 3. In contrast, the general linear equation $y' + p(x)y = q(x)$ can always be solved in explicit form, and its explicit solution was given in Section 1.2.

If $Y(y) = 0$ has (real) roots $y_1, \dots, y_k$, then besides the family of solutions obtained from (19) there are additional constant solutions $y(x) = y_1, \dots, y(x) = y_k$. When we are done we must go back and recover them, and include them as "additional solutions."

If that point is unclear, it may help to consider a simple algebraic analogy, such as the equation $x^2 - 5x = 2x$ or, $x(x-5) = 2x$. If we cancel x's we obtain $x - 5 = 2$ and hence $x = 7$. However, canceling the two x's (i.e., dividing both sides of the equation by x) is permissible only if $x \neq 0$, so after obtaining $x = 7$ we must return to the original equation to check the case $x = 0$. Indeed, $x = 0$ *is* a solution, so we must augment the solution set to $x = 7, 0$.

In this section we've emphasized the separation of variables solution method for equations $y' = f(x, y)$ that are separable. In the next section we continue to consider the general case $y' = f(x, y)$, and give a fundamental existence/uniqueness theorem.

EXERCISES 1.4

1. Solution by Separation of Variables. Solve the given IVP by separation of variables. Sketch the graph of the solution(s) or, if you prefer, use computer software to obtain both the graph of the solution(s) and also the direction field; indicate initial conditions by heavy dots. Determine the interval(s) of existence of the solution(s). If more than one initial condition is given, consider each, in turn.

(a) $y' - 3x^2 e^{-y} = 0$; $y(0) = 0$
(b) $xyy' = 2$; $y(1) = 2$
(c) $y' + 4y^3 = 0$; $y(-1) = -1$
(d) $y' - 4y^3 = 0$; $y(1) = 1$
(e) $y' = y^2 + 1$; $y(0) = 1$
(f) $y' = (\sin x)y$; $y(1) = 0$, $y(1) = 1$
(g) $y' = (y+1)^2$; $y(0) = -3$, $y(0) = -1$, $y(0) = 3$
(h) $y' = 4y^2$ $y(1) = -1$, $y(1) = 0$, $y(1) = 1$
(i) $y' + e^{y-x} = 0$; $y(0) = 0$
(j) $y' = e^{y-x}$; $y(0) = 0$
(k) $2xy' = y$; $y(3) = -1$
(l) $y' = e^{x-y}$; $y(-1) = 0$
(m) $[\tan^2(y-4) + 1]y' = 1$; $y(3) = 4$
(n) $2(1+y)y' = 1$; $y(3) = -2$
(o) $y = \ln y'$; $y(-2) = 0$

2. Implicit Solutions. The problems in Exercise 1 led to explicit solutions. The following lead to solutions in implicit form — although you *may* be able to solve for y and thus convert your solution to explicit form. Solve the IVP and determine its interval of existence, for each initial condition that is given. NOTE: Implicit solutions are more challenging regarding determination of the interval of existence, and plotting and examining their graphs may be particularly helpful.

(a) $(\cos y)y' + e^x = 0$; $y(0) = 0$ and $y(0) = 2\pi$.
(b) $(1 + e^y)y' = 1$; $y(0) = 0$. Give asymptotic expressions for $y(x)$ as $x \to \pm\infty$.
(c) $x(1 + y)y' = -y$; $y(1) = 2$
(d) $(1 + e^y)y' = 1 + \sin x$; $y(0) = 0$
(e) $(1 + e^y)y' = e^{x+y}$; $y(2) = 0$. Show that $y(x) \sim 1.874$ as $x \to -\infty$ and that $y(x) \sim e^x$ as $x \to +\infty$.
(f) $(2 - \cos y)y' = 2 - \cos x$; $y(0) = 0$.
(g) $(3y^4 - 1)y' = 0.2$; $y(0) = 0$
(h) $(30y^4 - 1)y' = x$; $y(0) = 0$
(i) $(2y + 0.06y^5)y' + 8x = 0$; $y(0) = -1$
(j) $x(1 - y)y' = (x - 1)y$; $y(1) = 0.1$

3. By separation of variables, solve the IVP

$$y' = y(y - 2) \tag{3.1}$$

subject to the initial conditions $y(0) = -1, 0, 1, 2, 2.4$, in turn. Sketch the graphs of the solutions, or use computer software to obtain both the graphs of the solutions and also the direction field; indicate initial conditions by heavy dots. Determine the interval of existence of each solution. HINT: To integrate $\int dy/[y(y - 2)]$, subject to the condition that $y \neq 0$ and $y \neq 2$ [which cases correspond to additional solutions], use partial fractions and obtain

$$y(x) = 2/(1 - Ae^{2x}). \tag{3.2}$$

4. Consider the equation

$$xy' = y \ln y \quad (y > 0). \tag{4.1}$$

(a) Derive the solution family $y(x) = e^{Cx}$ of (4.1), in which the constant C is arbitrary.
(b) Sketch (or plot) the solutions on $-\infty < x < \infty$ for representative values of C.
(c) Show that *one and only one* of those solutions satisfies $y(a) = b$ for any $a \neq 0$ and for any $b > 0$, that *none* of those solutions satisfies $y(0) = b$ if $b \neq 1$, and that *infinitely many* solutions in the family $y(x) = e^{Cx}$ satisfy the initial condition $y(0) = 1$.

5. Find the solutions, if any, of $x(1 + y)y' + y = 0$ subject to the initial conditions $y(5) = 0.4$, and $y(5) = -0.2$, in turn. What are their intervals of existence?

6. Relative Rates of Growth. It is important to understand relative orders of magnitude. For instance, if two populations $N_1(t)$ and $N_2(t)$ are given by $N_1(t) = 100t^{50}$ and $N_2(t) = 100e^{0.0001t}$, it is evident that both tend to infinity as $t \to \infty$, but which one grows more quickly? [Of course, whether the independent variable is temporal (t), spatial (x), or whatever, doesn't matter in this discussion.] Let us compare three common types of growth (as $t \to \infty$):

logarithmic growth,	$\ln t$;
algebraic growth,	$t^\alpha \ (\alpha > 0)$
exponential growth,	$e^{\beta t} \ (\beta > 0)$

(a) Show that *algebraic growth dominates logarithmic growth*, as $t \to \infty$. Namely, show that $t^\alpha / \ln t \to \infty$ as $t \to \infty$, for any $\alpha > 0$ no matter how small. Since t^α dominates $\ln t$ for *any* α, even $\alpha = 10^{-12}$, say, logarithmic growth is extremely weak in comparison with algebraic growth.
(b) Show that *exponential growth dominates algebraic growth*, as $t \to \infty$, no matter howsmall β is and no matter how large α is. Thus, algebraic growth is extremely weak in comparison with exponential growth. For instance $N_2(t)$, given above, dominates $N_1(t)$ as $t \to \infty$; i.e., $N_2(t)/N_1(t) \to \infty$ as $t \to \infty$.

7. Relative Rates of Decay. Analogous to Exercise 6, compare these types of decay (as $t \to \infty$):

algebraic decay,	$t^{-\alpha} \ (\alpha > 0)$;
exponential decay,	$e^{-\beta t} \ (\beta > 0)$.

Show that *exponential decay dominates algebraic decay*, as $t \to \infty$, namely, that $e^{-\beta t}/t^{-\alpha} \to 0$ as $t \to \infty$, no matter how small β is and no matter how large α is.

8. Algebraic, Exponential, and Explosive Growth. We saw in Section 1.3 that the population model

$$\frac{dN}{dt} = \kappa N \qquad (\kappa > 0) \tag{8.1}$$

gives the exponential growth $N(t) = Ae^{\kappa t}$, so $N \to \infty$ as $t \to \infty$ (if $A > 0$). More generally, consider the model

$$\boxed{\frac{dN}{dt} = \kappa N^p, \qquad (\kappa > 0)} \tag{8.2}$$

in which p is a positive constant. Our purpose in this exercise is to examine how the rate of growth of $N(t)$ varies with the exponent p in (8.2).

(a) Solve (8.2) and show that if $0 < p < 1$, then the solution exhibits **algebraic growth** [i.e., $N(t) \sim at^b$ as $t \to \infty$, where a and b are positive constants that depend upon p].
(b) Show that as $p \to 0$ the exponent b tends to unity, and as $p \to 1$ the exponent b tends to infinity. (Of course, when $p = 1$ we have **exponential growth**, as mentioned above, so we can

think, crudely, of exponential growth as a limiting case of algebraic growth, in the limit as the exponent b becomes infinite. Thus, exponential growth is powerful indeed.)
(c) If p is increased beyond 1 then we expect the growth to be even more spectacular. Show that if $p > 1$ then the solution exhibits **explosive growth**, explosive in the sense that not only does $N \to \infty$, but it does so in *finite* time, namely as $t \to T$ where

$$T = \frac{1}{\kappa(p-1)N_0^{p-1}} \tag{8.3}$$

and N_0 denotes the initial value $N(0)$. Observe that not only does the growth become explosive when p is increased beyond 1, but that the time T until "blowup" decreases as p increases and tends to 0 as $p \to \infty$.

9. Exponential Decay Versus Explosive Growth. We know that $N' + N = 0$ gives exponential decay and [Exercise 8(c)] that $N' = N^2$ gives explosive growth, as $t \to \infty$. If we combine both forms and write

$$N' + N = N^2; \quad N(0) = N_0, \tag{9.1}$$

which one wins? That is, does $N(t)$ exhibit exponential decay or explosive growth as $t \to \infty$, or perhaps a different behavior altogether? [Think of N as population, so N_0 and $N(t)$ are nonnegative.]

1.5 EXISTENCE AND UNIQUENESS

1.5.1 An existence and uniqueness theorem. Recall from Theorem 1.2.1 that if $p(x)$ and $q(x)$ are continuous at a, then the *linear* equation $y' + p(x)y = q(x)$ admits a solution through an initial point $y(a) = b$ that exists at least on the broadest open x interval containing $x = a$, on which p and q are continuous, and is unique. What can be said about existence and uniqueness for the initial value problem

$$y' = f(x, y); \quad y(a) = b$$

if the latter is *nonlinear*, that is, if $f(x, y)$ is not a function of x times y plus a function of x? We have the following theorem.[1]

Theorem 1.5.1 applies whether (1) is nonlinear *or* linear, but we already have the stronger Theorem 1.2.1 for the linear case. Thus, our interest in Theorem 1.5.1 is for the nonlinear case.

THEOREM 1.5.1 *Existence and Uniqueness for Initial Value Problems*
If f and $\partial f/\partial y$ are continuous functions of x and y in an open disk D about the initial point (a, b) (Fig. 1), then the initial value problem

$$\boxed{\frac{dy}{dx} = f(x, y); \quad y(a) = b} \tag{1}$$

has a unique solution at *least* on the open x interval $x_1 < x < x_2$, where x_1 and x_2 denote the x locations of the points at which the solution curve intersects the

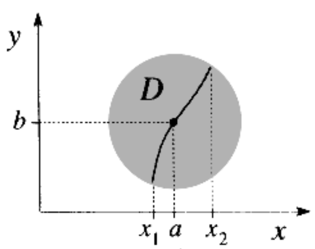


Figure 1. The disk D in Theorem 1.5.1.

[1]For further discussion of this fundamental theorem and its proof, at about the same level as this text, see Section 2.11 of W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 6th ed. (NY: John Wiley, 1997). We also recommend J. Polking, A. Boggess, and D. Arnold, "Differential Equations with Boundary Value Problems," 2nd ed. [Upper Saddle River, NJ: Pearson, 2005]. See Theorems 7.6 and 7.16 and the related discussion.

circular boundary of D.

Whereas Theorem 1.2.1 gave a minimum interval of existence and uniqueness for the linear equation [namely, the largest open x interval containing $x = a$ on which both $p(x)$ and $q(x)$ are continuous], Theorem 1.5.1 ensures existence and uniqueness for the general equation $y' = f(x, y)$ on an interval $x_1 < x < x_2$, but it does not tell us what x_1 and x_2 are. In fact, the interval $x_1 < x < x_2$ can be arbitrarily small if the solution curve through (a, b) happens to be steep. Thus, Theorem 1.5.1 is a *local* result in the sense that it tells us that under the stipulated conditions there does exist a unique solution in *some* neighborhood of the initial point $x = a$, but it does not tell us the size of that neighborhood.

Of course, Theorem 1.5.1 applies also if the differential equation in (1) happens to be linear, but there is little point in using it for linear equations because, as we've mentioned, Theorem 1.2.1 for linear equations is much more informative.

1.5.2 Illustrating the theorem. We will illustrate Theorem 1.5.1 with two examples before discussing the significance of existence and uniqueness in a physical application in the next subsection.

EXAMPLE 1. Consider the IVP

$$4y\frac{dy}{dx} = -x; \quad y(3) = 1. \tag{2}$$

To identify $f(x, y)$, first put the differential equation into standard form by dividing both sides by $4y$.

To apply Theorem 1.5.1 to (2), observe that both $f(x, y) = -x/(4y)$ and $\partial f/\partial y = x/(4y^2)$ are continuous everywhere in the plane except on the line $y = 0$ (the x axis). Since the initial point $y(3) = 1$ is not on that line, it follows from the theorem that the IVP (2) does have a solution, a unique solution, passing through the largest disk D centered at (3,1), throughout which both f and $\partial f/\partial y$ are continuous, namely the shaded disk of unit radius shown in Fig. 2. The size of that disk is limited by the presence of the line $y = 0$ along which the theorem's continuity conditions are not met.

In fact, the differential equation in (2) is separable and gives the solution in implicit form as the ellipse $x^2 + 4y^2 = C$. The initial condition $y(3) = 1$ then gives $C = 13$, so

$$y(x) = \pm\frac{1}{2}\sqrt{13 - x^2}. \tag{3}$$

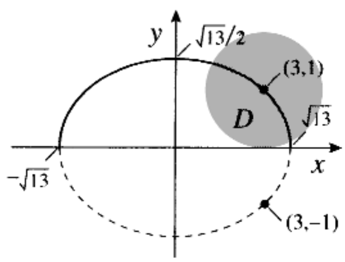


Figure 2. The solution curves through $(3, 1)$ and $(3, -1)$, shown as solid and dotted, respectively. The largest possible disk D at (3,1) is shown as shaded. Its radius is 1.

Choose the plus sign in (3) because it gives the upper half of the ellipse, which passes through the initial point (3,1), whereas the minus sign gives the lower half of the ellipse, which does not pass through the initial point. The graph of the solution is the solid curve in Fig. 2.

Theorem 1.5.1 assures us of the existence of a unique solution in some interval about $x = 3$, but it does not give a minimum size of that interval. However, in this example we did not need a prediction of the interval of existence because we were able to *solve* (2); we can now simply examine its solution, the graph of which is the upper half of the ellipse in Fig. 2. We can see from the figure that the solution exists on $-\sqrt{13} < x < \sqrt{13}$; the endpoints $x = \pm\sqrt{13}$ are not included because the slope y' is undefined (infinite) at those points.

COMMENT 1. Similarly, through $(3, -1)$ there is the unique solution

$$y = -\frac{1}{2}\sqrt{13 - x^2}$$

on $-\sqrt{13} < x < \sqrt{13}$, corresponding to the lower branch of the ellipse (the dotted curve in Fig. 2).

COMMENT 2. If the initial point is *on* the x axis then the theorem does not guarantee the existence or uniqueness of a solution. It simply gives *no information* because then the continuity conditions on f and $\partial f/\partial y$ are not satisfied in any open disk D centered at the initial point. In fact, if the initial point is on the x axis then there is *no solution* through that point because the slope of the ellipse passing through that point is infinite there; that is, y' is undefined, so the differential equation in (2) cannot be satisfied there or in any x interval containing that point.[1] ∎

In Example 1, Theorem 1.5.1 assured us that there exists a unique solution through the given initial point, although it did not guarantee that that solution would exist at *least* on "such and such" an x interval. However, we were able to solve (2), and hence to determine the interval of existence simply by examining the solution. In other cases the IVP may be too difficult for us to solve, and our interest is in determining some guaranteed interval of existence, even in the absence of having the solution in hand. In the next example, we will use Theorem 1.5.1 to see what we can do about determining an interval of existence.

EXAMPLE 2. Consider the IVP

$$y' = y^2; \quad y(0) = 1. \tag{4}$$

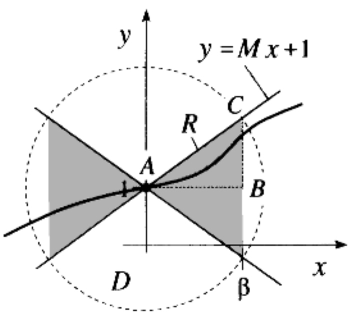


Figure 3. Interval of existence and uniqueness in Example 2.

Here, $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere in the x, y plane, so Theorem 1.5.1 assures us that there is a unique solution of the IVP (4) — in *some* interval about the initial point.

What we can learn (without peeking at the solution, as we did in Example 1) about the interval of existence and uniqueness of that solution? [Actually, (4) can be solved readily by separation of variables, but let us see what we can determine even in the absence of having the solution in hand to examine.]

Since the continuity conditions are satisfied throughout the plane, we can make the disk D any size we like. Begin by drawing the disk D, of radius R, about the initial point (0,1), as in Fig. 3. Everywhere in D, $|y'| = |y^2| < (R + 1)^2$ because the maximum y is at the top of the disk, where $y = R + 1$. Thus, the absolute magnitude of the slope of

[1] In Section 1.1.6 we stated that an **integral curve** is simply the graph of a solution. Actually, it can be the union of such graphs. For instance, the entire ellipse in Fig. 2 is called an integral curve of $4yy' = -x$, even though it is not the graph of a single solution curve but, rather, the union of the upper and lower solution curves.

the solution curve through (0,1) is less than $(R+1)^2$, which we will denote as M, so the solution curve must fall within the shaded "bow tie" region. After all, for the solution curve to break out of the bow tie its slope would have to exceed M at the point of break out, and that cannot happen because $|y'| < M$ everywhere in D.

Hence, the interval of existence and uniqueness is *at least* $-\beta < x < \beta$. To determine β, write the Pythagorean theorem for the right triangle ABC: $AB^2 + BC^2 = R^2$ or, $\beta^2 + (M\beta)^2 = R^2$, which can be solved for β as

$$\beta = \frac{R}{\sqrt{1+M^2}} = \frac{R}{\sqrt{1+(1+R)^4}}. \tag{5}$$

For instance, $R = 1$ gives $\beta = 0.2425$. Since we can choose R as large or small as we like, we might as well choose R so as to maximize the right hand side of (5). To do that, set $d\beta/dR = 0$ and obtain $R^4 + 2R^3 - 2R - 2 = 0$ which (using computer software) gives $R = 1.1069$; putting that into (5) then gives $\beta = 0.2031$. Thus, we have shown that the interval of existence and uniqueness is *at least* $-0.2031 < x < 0.2031$.

In fact, (4) is readily solved, its solution being

$$y(x) = \frac{1}{1-x}, \tag{6}$$

the graph of which is given in Fig. 4. Thus, the actual interval of existence and uniqueness is $-\infty < x < 1$, so the interval $-0.2031 < x < 0.2031$ is correct, but falls well short of capturing the full interval of existence. ∎

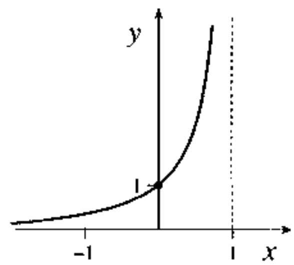


Figure 4. The exact solution (6) of the IVP (4).

1.5.3 Application to free fall; physical significance of nonuniqueness.

It is important to give a physical application as well, so the impression is not left that the questions of existence and uniqueness are only of theoretical interest. Such an application can be found even in the simple problem of a body of mass m that is dropped from rest at time $t = 0$. Let the mass's downward displacement from the point of release be $x(t)$ (Fig. 5). Neglecting air resistance, Newton's second law gives $mx'' = mg$, so we have the IVP

$$x'' = g, \qquad 0 \le t < \infty, \tag{7a}$$

$$x(0) = 0, \quad x'(0) = 0. \tag{7b}$$

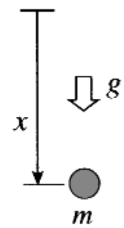


Figure 5. Free fall under the influence of gravity, neglecting air resistance.

We can integrate (7a) twice with respect to t and use the initial conditions in (7b) to evaluate the two constants of integration. Doing so gives the solution

$$x(t) = \frac{1}{2}gt^2 \tag{8}$$

that is probably familiar from a first course in physics. The graph of (8) is the parabola shown in Fig. 6.

However, it will be instructive to work not with Newton's second law but with an "energy equation." To derive an energy equation, multiply Newton's law $mx'' = mg$ not by dt but by dx:[2]

[2]Multiplying the terms in Newton's law by dx will lead to an energy equation because dx is distance, force times distance is work (mg in Newton's law is the force), and work is manifested as energy.

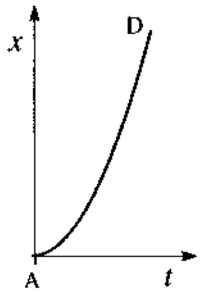


Figure 6. The solution $x(t) = gt^2/2$ of (6).

We say that (9e) is a "first integral" of (7a). In place of the second-order equation (7a) we now have the first-order equation (10a). Our starting point is now the IVP (10), not (7).

$$mx''\,dx = mg\,dx, \tag{9a}$$

$$m\frac{dx'}{dt}\,dx = mg\,dx, \tag{9b}$$

$$mdx'\,\frac{dx}{dt} = mg\,dx, \tag{9c}$$

$$\int mx'\,dx' = \int mg\,dx, \tag{9d}$$

$$\frac{1}{2}mx'^2 = mgx + A \tag{9e}$$

or $\frac{1}{2}mx'^2 + (-mgx) = A$. The latter is a statement of conservation of energy: The kinetic energy $mx'^2/2$ plus the (gravitational) potential energy $-mgx$ is a constant. Putting $t = 0$ in (9e) gives $0 = 0 + A$ so $A = 0$, and it follows from (9e) that $x' = \sqrt{2gx}$. The latter is a first-order differential equation, so append the single initial condition $x(0) = 0$. Then we have the IVP

$$\frac{dx}{dt} = \sqrt{2g}\,x^{1/2}, \qquad 0 \le t < \infty, \tag{10a}$$

$$x(0) = 0. \tag{10b}$$

Our interest here is in considering the IVP (10) in the light of Theorem 1.5.1. Solve (10a) by separation of variables. If $x \neq 0$ we can divide both sides by $x^{1/2}$, multiply by dt, integrate, and obtain

$$x(t) = \frac{1}{4}\left(\sqrt{2g}\,t + C\right)^2. \tag{11}$$

Then the initial condition $x(0) = 0$ gives $C = 0$ so

$$x(t) = \frac{1}{2}gt^2, \tag{12}$$

which is the same as (8). However, recall from Section 1.4 that a separable equation $y' = X(x)Y(y)$ can have solutions $y(x) = constant$ coming from any roots of $Y(y) = 0$, in addition to the family of solutions obtained by separation of variables. In the present case (where the variables are t, x instead of x, y) the root $x = 0$ of $x^{1/2} = 0$ gives the solution $x = 0$ [i.e., $x(t) = 0$] of (10a), and that solution is additional since it is not contained in (11) by any choice of the constant C. That solution also satisfies the initial condition (10b), so besides the solution (12) of (10) (which corresponds to AD in Fig. 7) we also have the solution $x(t) = 0$ (which corresponds to the positive t axis in Fig. 7). Thus, the solution of (10) is nonunique.

With that result in mind, turn to Theorem 1.5.1 to see what it can tell us about the existence and uniqueness of solutions of the IVP (10). Realizing that x and y in the theorem correspond here to t and x, respectively, observe that $\partial f/\partial x = \partial(\sqrt{2g}\,x^{1/2})/\partial x = \sqrt{2g}/(2\sqrt{x}\,)$ is not continuous in any neighborhood of the initial point $(0,0)$ in the t, x plane because it "blows up" to infinity at $x = 0$, that

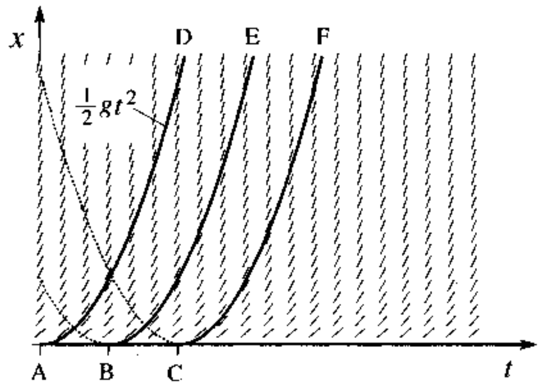


Figure 7. Solutions of (10).

is, all along the t axis. Thus, the conditions of the theorem are not met, so the theorem simply gives no information for the IVP (10).[1]

COMMENT 1. The additional solution $x(t) = 0$ of (10) amounts to the mass levitating: it does not fall! While the latter is a legitimate solution of (10) and does not violate the conservation of energy expressed by (10a), it *does* violate Newton's second law (7a) because if we put $x(t) = 0$ into (7a) we obtain the contradiction $0 = g$. Thus, the levitation solution can be discarded, finally, as nonphysical.

This is a general situation: *energy formulations may lead to solutions that are unacceptable.* Still, there is a nagging question: If we derived (10a) from Newton's law, in equations (9a) through (9e), then how did this nonphysical solution get its foot in the door? It entered in (9a), for if $x(t)$ = constant then dx is zero, so when we multiplied both sides of $mx'' = mg$ by dx, the resulting equation $mx''dx = mgdx$ does not imply that $mx'' = g$, because the dx's are zero.

COMMENT 2. Actually, (10) admits not only the two solutions $x(t) = gt^2/2$ (the curve AD in Fig. 7) and $x(t) = 0$, but an infinite number of other solutions as well. For instance, the segment AB of the t axis (in Fig. 7) followed by the half-parabola BE is also a solution curve, as is AC followed by CF, and so on. That is, the energy equation (10a) and initial condition (10b) are both satisfied if the mass levitates for a while, and *then* falls.

Closure. Theorem 1.5.1 gives sufficient conditions for the existence of a unique solution to the IVP (1). It is less informative than the corresponding Theorem 1.2.1 for the linear case: Theorem 1.2.1 gave a formula for the solution and a minimium x interval on which that solution exists and is unique. Theorem 1.5.1 assures existence but does not give the solution, and whereas it assures existence and uniqueness on "some" x interval, it does not indicate how broad that interval will be.

[1]The lack of satisfaction of the conditions of the theorem does *not* imply that (10) does not have a unique solution, because the theorem says "if," not "if and only if." That is, its conditions are sufficient, not necessary and sufficient.

EXERCISES 1.5

1. Application of the Theorems. First, solve the given IVP. If there is no solution state that. Is the solution unique? If possible, give the interval of existence of each solution. Then, show that your findings are consistent with the existence and uniqueness theorem; if the equation is nonlinear use Theorem 1.5.1, and if it is linear use the more informative Theorem 1.2.1. HINT: In difficult cases a computer plot of the direction field may help.

(a) $y' = 2xy;\quad y(0) = 2$ (b) $y' = x + y;\quad y(0) = 3$
(c) $yy' = x;\quad y(0) = 2$ (d) $yy' = x;\quad y(0) = 0$
(e) $yy' = x;\quad y(0) = -2$ (f) $xy' + y = 0;\quad y(1) = 1$
(g) $xy' + y = 0;\quad y(0) = 5$ (h) $xy' + y = 0;\quad y(2) = 0$

(i) $xy' + 2y = 0;\quad y(-1) = -4$
(j) $xy' - 2y = 0;\quad y(0) = 0$
(k) $2yy' = 1;\quad y(3) = -1$
(l) $x^2 y' + y^2 = 0;\quad y(-2) = -1$
(m) $y' = \tan y;\quad y(0) = -3$
(n) $y' = 6y^{1/3};\quad y(0) = 0\ (x \geq 0)$
(o) $y' = 6y^{1/3};\quad y(0) = 1$

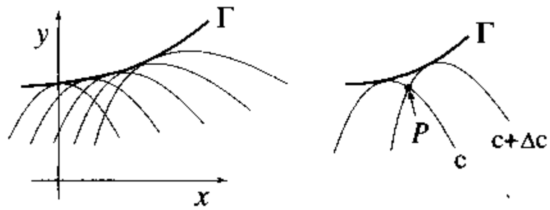
2. Estimating a Minimum Interval of Existence and Uniqueness. For the nonlinear IVP (4) we used the "bow tie" idea to estimate the interval of existence and uniqueness; we showed that it is *at least* $-0.2031 < x < 0.2031$. For the problems in this exercise, follow that same idea to obtain a formula for $\beta(R)$ analogous to that in (5).

(a) For the IVP $y' = 1 + y^2$ with $y(0) = 0$, obtain $\beta = R/\sqrt{R^4 + 2R^2 + 2}$ and show that the maximum β is 0.45509, so that existence and uniqueness is assured *at least* on $-0.45509 < x < 0.45509$. Further, solve the IVP for $y(x)$ and show that the actual interval of existence and uniqueness is $-\pi/2 < x < \pi/2$.
(b) For the IVP $yy' = x$ with $y(0) = -2$, obtain $\beta = R(2-R)/\sqrt{2R^2 - 2R + 4}$. You need not maximize the latter, but show that it is at least 0.5, so that existence and uniqueness is assured *at least* on $-0.5 < x < 0.5$. Further, solve the IVP for $y(x)$ and show that the actual interval of existence and uniqueness is $-\infty < x < \infty$.
(c) For the IVP $yy' = x$ with $y(2) = 3$, obtain $\beta = R(3-R)/\sqrt{2R^2 - 2R + 13}$. You need not maximize the latter, but show that it is at least 0.5, so that existence and uniqueness is assured *at least* on $1.5 < x < 2.5$. Further, solve the IVP for $y(x)$ and show that the actual interval of existence and uniqueness is $-\infty < x < \infty$.
(d) Given the IVP $y' = y^2/x^2$ with $y(1) = 0.5$, we simply want to be assured that a unique solution exists on $|x - 1| < 0.1$. Show that that is the case. Further, solve the IVP for $y(x)$ and determine its actual interval of existence and uniqueness.

3. Envelopes. In this exercise we introduce the geometric concept of the "envelope" of a one-parameter family of curves in a plane, and in subsequent exercises we will show what envelopes have to do with first-order nonlinear differential equations. Consider a one-parameter family of curves

$$g(x, y, c) = 0, \tag{3.1}$$

in which c is the parameter. For instance, $x^2 + y^2 - c^2 = 0$ is the family of concentric circles centered at the origin, each one corresponding to a different value of the parameter c. Such a family of curves may, but need not, have an envelope, such as the curve Γ in the left-hand figure. (A curve Γ is an **envelope**



of a family of curves if every member of the family is tangent to Γ and if Γ is tangent, at each of its points, to some member of the family.) If we are given $g(x, y, c)$, how can we *find* any such envelopes? The coordinates x, y of point P (in the right-hand figure) must satisfy both $g(x, y, c) = 0$ and $g(x, y, c + \Delta c) = 0$ or, equivalently,

$$g(x, y, c) = 0 \tag{3.2}$$

and

$$\frac{g(x, y, c + \Delta c) - g(x, y, c)}{\Delta c} = 0. \tag{3.3}$$

Equation (3.3) is valid for Δc arbitrarily small, so it must hold in the limit as $\Delta c \to 0$, in which limit P approaches Γ. Thus (3.2) and (3.3) become

$$\boxed{g(x, y, c) = 0, \quad \frac{\partial g}{\partial c}(x, y, c) = 0.} \tag{3.4a,b}$$

Eliminating c between (3.4a) and (3.4b) gives the desired equation of Γ, if the family does indeed have an envelope. To illustrate, consider the family of circles $(x-c)^2+y^2=9$. Equations (3.4a) and (3.4b) give $g=(x-c)^2+y^2-9=0$ and $\partial g/\partial c=-2(x-c)=0$, and eliminating c between these gives the two straight line envelopes $y=+3$ and $y=-3$ which, from a sketch of the family of circles, is seen to be correct. On the other hand, consider the family of parallel lines $y=x+c$, which has no envelope. (To see that, sketch the lines for several different c's.) In this case (3.4a) and (3.4b) give $g=y-x-c=0$ and $\partial g/\partial c=-1=0$. These cannot be satisfied (because $-1=0$ cannot be satisifed), so the family $y=x+c$ has no envelopes.

The problem: In each case use (3.4a) and (3.4b) to determine all envelopes, if any, of the given family of curves, and illustrate with a labeled sketch (or computer plot).

(a) $y=cx+1/c$ (b) $(x-c)^2+y^2=c^2/2$

(c) $y=(x-c)^3$ (d) $y=(x-c)^2+x$

4. Envelope Solutions of Differential Equations. Let the differential equation

$$y'=f(x,y) \tag{4.1}$$

have a one-parameter family of solutions

$$g(x,y,c)=0, \tag{4.2}$$

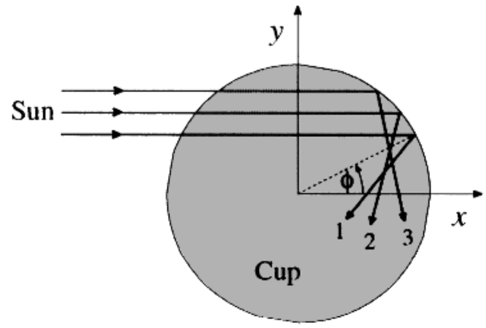
where c is a constant of integration, and suppose that the family of solution curves (4.2) has an envelope Γ (as in the figure in Exercise 3). At each point on Γ the values of x,y and the slope y' are such that (4.1) is satisfied, so Γ *itself is a solution curve*. That solution is not contained in (4.2) because Γ is not itself a member of the family (4.2).

The problem: Show that the levitation solution $x(t)=0$ in Section 1.5.3 is such an envelope solution.

NOTE: The concept of the envelope of a family of curves is of interest not only in connection with solutions of differential equations but also in optics and acoustics. The next exercise illustrates its relevance in optics.

ADDITIONAL EXERCISES

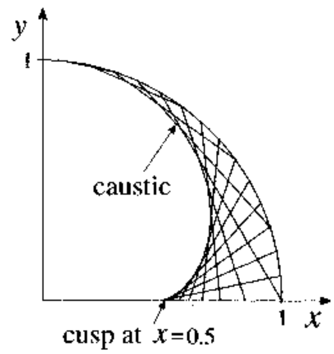
5. Application of Envelopes to Caustics in a Coffee Cup. In this exercise we apply the geometric idea of envelopes (Exercise 3) — not to differential equations but, for fun, to the reflected light pattern in a coffee cup. In the morning, when the sun is low, the sunlight striking the inside lip of a coffee cup is reflected by the inside of the lip. Continuing its slightly downward trajectory it strikes the coffee surface and is reflected to our eye, revealing a bright geometric pattern called a **caustic**. The latter is the envelope of the light rays reflected off the lip, such as those labeled 1, 2, and 3 in the following figure:



(a) Let the inner radius of the cup be 1 inch and assume that the angle of reflection off the lip equals the angle of incidence. Show that the equation of the typical reflected ray is

$$y=\big(\tan 2\phi\big)x+(\sin\phi-\tan 2\phi\cos\phi). \tag{5.1}$$

(b) Using computer graphics, plot enough of the lines defined by (5.1) to reveal the shape of the caustic, as we have in the second figure. It suffices to plot them on $0\le y\le 1$



rather than on $-1\le y\le 1$ because surely the caustic will be symmetric about the x axis.

(c) Show that the caustic has a cusp at $x=0.5$, that is, that the slope of the caustic tends to zero as $y\to 0$.

NOTE: The envelope of light rays amounts to their mutual reinforcement to the extent that the caustic becomes visible.

6. Iterative Solution; Picard's Method. Suppose we try to solve the IVP

$$y'=f(x,y);\quad y(a)=b \tag{6.1}$$

by integrating the differential equation from the initial point a to an arbitrary point x, obtaining

$$y(x)-y(a)=\int_a^x f(s,y(s))\,ds$$

in which s is a dummy integration variable, or, imposing the initial condition $y(a) = b$,

$$y(x) = b + \int_a^x f(s, y(s))\, ds. \tag{6.2}$$

Unfortunately, the latter is not the solution of (6.1) because the unknown y appears inside the integral. Rather, it is an **integral equation** for y because the unknown y appears under the integral sign. Thus, all we've accomplished is the conversion of the differential equation IVP (6.1) to an integral equation. (Integral equations will not be studied in this text except insofar as they occur here and in the chapter on the Laplace transform.) Consider the solution of (6.2) by "iteration." That is, since the complication in (6.2) is the presence of y inside the integral, let us approximate that y (inside the integral) and integrate. Let $y_0(x) = b$ be a first approximation of the desired solution $y(x)$; the latter probably doesn't satisfy the differential equation in (6.1) but at least it satisfies the initial condition $y(a) = b$. Because of this approximation on the right, the $y(x)$ on the left will likewise not (in general) satisfy (6.1) exactly, but only approximately. Hopefully, it will be an improvement over the initial approximation $y_0(x) = b$. If we denote that new approximation as $y_1(x)$ then

$$y_1(x) = b + \int_a^x f(s, y_0(s))\, ds. \tag{6.3}$$

We can repeat the process and use the function $y_1(x)$ computed from (6.3) as, hopefully, a better approximation of the y inside the integrand, obtaining

$$y_2(x) = b + \int_a^x f(s, y_1(s))\, ds,$$

and so on. That is, beginning with $y_0(x) = b$ we can use the iterative formula

$$\boxed{y_{n+1}(x) = b + \int_a^x f(s, y_n(s))\, ds} \tag{6.4}$$

with $n = 0, 1, 2, \ldots$ in turn, to generate a sequence of iterates $y_0(x), y_1(x), y_2(x)$, and so on. Hopefully, if (6.1) has a unique solution then the $y_n(x)$ sequence thus generated will converge to that solution. This iterative method is due to the French mathematician *Emile Picard* (1856–1941) and is known as **Picard's method**. If we assume that $f(x, y)$ satisfies the conditions of Theorem 1.5.1 then it can be shown that that sequence does converge to the exact solution $y(x)$,

$$\lim_{n\to\infty} y_n(x) = y(x), \tag{6.5}$$

on some open interval containing the initial point a; in fact, Picard iteration is a traditional method of proof of the existence part of Theorem 1.5.1. See, for instance, Section 2.11 of W. Boyce and R. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 6th ed. (NY: John Wiley, 1997). Our purpose here is not to attempt that proof but only to explore the idea of iterative solution and to provide guidance through some exploratory examples.

The problem: For the example

$$y' = -y; \quad y(0) = 1, \tag{6.6}$$

beginning with $y_0(x) = y(0) = 1$, use (6.4) to generate the first several iterates:

$$y_1(x) = 1 - x, \tag{6.7a}$$

$$y_2(x) = 1 - x + \tfrac{1}{2}x^2, \tag{6.7b}$$

$$y_3(x) = 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3, \tag{6.7c}$$

$$y_4(x) = 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4. \tag{6.7d}$$

NOTE: Since the Taylor expansion of the exact solution is

$$y(x) = e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \cdots \quad (-\infty < x < \infty) \tag{6.8}$$

it appears that the Picard sequence is indeed converging to the exact solution.

7. Another Example of Picard's Method. Consider the IVP

$$y' = 2xy^2; \quad y(0) = 1. \tag{7.1}$$

(a) Derive, by separation of variables, the solution $y(x) = 1/(1 - x^2)$, which exists on $-1 < x < 1$.
(b) Now use the Picard method given above, beginning with $y_0(x) = 1$, to generate $y_n(x)$ for $n = 1, 2$, and 3. You can do this by hand, or using computer software.
(c) Plot those iterates, together with the exact solution, on the interval of existence $-1 < x < 1$.

8. One More. Consider the IVP

$$y' = e^x y; \quad y(0) = 1. \tag{8.1}$$

(a) Derive the solution $y(x) = \exp(e^x - 1)$, which exists on $-\infty < x < \infty$.

(b) Use the Picard method given above, beginning with $y_0(x) = 1$, to generate $y_n(x)$ for $n = 1, 2$, and 3.

(c) Plot those iterates, together with the exact solution, on $-3 < x < 3$, for instance.

1.6 APPLICATIONS OF NONLINEAR FIRST-ORDER EQUATIONS

In this section we consider the logistic model of population dynamics as a representative application of nonlinear first-order differential equations, and we give a variety of other applications in the exercises. Use of the phase line, from Section 1.3, will continue to be prominent, and we will introduce one new idea: linearized stability analysis.

1.6.1 The logistic model of population dynamics. In Section 1.3.1 we studied the simple exponential population model

$$\frac{dN}{dt} = \kappa N; \qquad N(0) = N_0, \tag{1}$$

with solution

$$N(t) = N_0 e^{\kappa t}. \tag{2}$$

We noted that the exponential model is not necessarily realistic for long time intervals if the net birth/death rate κ is positive, because in that case (2) indicates unbounded growth. As a more realistic model we suggested the logistic equation

$$\boxed{\frac{dN}{dt} = (a - bN)N,} \tag{3}$$

The well-known **logistic equation**.

which we wrote down in Section 1.3.1 but did not solve. In (3), a and b are known positive constants and $N(t)$ is the population, such as the number of bass in a lake. [Alternatively, we could take $N(t)$ to be the total mass, the *biomass*, of bass in the lake, or some other measure of the population.]

We can solve (3) by separation of variables:

$$\int \frac{dN}{(a-bN)N} = \int dt \qquad (\text{if } N \neq 0 \text{ and } N \neq a/b),$$

$$\int \left(-\frac{1}{a}\frac{1}{N-a/b} + \frac{1}{a}\frac{1}{N} \right) dN = t + C \qquad (-\infty < C < \infty),$$

We've expanded the $1/[(a-bN)N]$ in partial fractions.

$$-\ln|N - a/b| + \ln|N| = at + aC,$$

$$\left| \frac{N}{N-a/b} \right| = e^{at+aC} = e^{aC}e^{at},$$

$$\frac{N}{N-a/b} = \pm e^{aC}e^{at} \equiv Ae^{at} \tag{4}$$

or, solving (4) for N,

$$N(t) = \frac{aAe^{at}}{1+bAe^{at}}. \tag{5}$$

Since $-\infty < C < \infty$, the constant A can be any value other than zero (because $A = \pm e^{aC}$, and e^{aC} is not zero for any finite value of C).

Besides the solution family (5), $(a-bN)N = 0$ gives the additional solutions $N(t) = 0$ and $N(t) = a/b$. The former can be included in (5) if we allow the arbitrary-but-nonzero constant A to be zero, because setting $A = 0$ in (5) gives $N(t) = 0$. But $N(t) = a/b$ cannot be obtained from (5) by any finite choice of A so it is an additional solution of (3), in addition to (5).

If we apply an initial condition $N(0) = N_0$ to (5), we can solve for A and obtain $A = N_0/(a-bN_0)$. Then, after some algebra, (5) becomes

To solve for A it is simpler to apply $N(0) = N_0$ to (4) than to (5); try it both ways and see.

$$N(t) = \frac{a}{b}\frac{N_0}{N_0 + (\frac{a}{b} - N_0)e^{-at}}. \tag{6}$$

At this point we could use (6) to plot N versus t, for representative values of a, b, and N_0. Instead, put the solution (6) aside and return to the differential equation (3), to see what we can learn using a more qualitative approach. We see that (3) is autonomous, of the form

Autonomous differential equations and the phase line were discussed in Section 1.3.4.

$$\frac{dN}{dt} = (a-bN)N \equiv f(N), \tag{7}$$

so consider the phase line. Accordingly, we've plotted $N' = f(N) = (a-bN)N$ versus N in Fig. 1, from which we find equilibrium points at $N = 0$ and at $N = a/b$. (Of *course* $N = 0$ is an equilibrium point, because if we begin with no fish, we will never have any fish.)

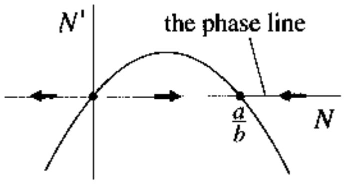


Figure 1. $N' = (a-bN)N$ and the resulting N-axis phase line. The two heavy dots denote equilibrium points.

If we take the phase line in Fig. 1 and place it vertically at the left of the N axis in a Cartesian t, N plane, as in Fig. 2, we can infer the qualititative shape of the solution curves in the t, N plane directly from the phase line flow. For instance, the dot at $N = a/b$ on the phase line indicates an equilibrium point there, so the solution curve springing from $N(0) = a/b$ in the t, N plane is simply a horizontal line. The downward flow on the phase line above a/b and the upward flow below a/b imply that the solution curves in the t, N plane approach the equilibrium solution $N(t) = a/b$ from above and below, respectively. And the dot at $N = 0$ gives the constant equilibrium solution $N(t) = 0$.

To sketch the solution curves in Fig.2 it would help to find the inflection points, if any. Like the phase line, that information can be obtained directly from the differential equation (3). Inflection points are points at which N'' vanishes and changes sign, so differentiate (3) and set $N'' = 0$:

Following the second equality, replace each N' by $(a-bN)N$ and simplify.

$$\begin{aligned} N'' &= \frac{d}{dt}[(a-bN)N] \\ &= -bN'N + (a-bN)N' \\ &= (a-2bN)(a-bN)N = 0, \end{aligned} \tag{8}$$

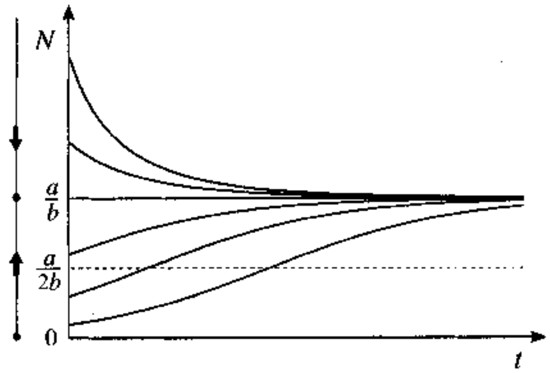


Figure 2. Using the phase line at the left to sketch solution curves for (3). Solution curves below the t axis are omitted because $N \geq 0$..

which gives the three roots $N = a/2b$, a/b, and 0. Of these, we can discard the last two, which are simply the horizontal-line equilibrium solutions $N(t) = a/b$ and $N(t) = 0$. Consider the first root, $a/2b$. The $a - 2bN$ factor in (8) changes sign at $a/2b$ but the $(a-bN)N$ part does not (since it is positive for $0 < N < a/b$). Thus, N'' both vanishes and changes sign at $a/2b$, so $N = a/2b$ is an inflection point. That is, all along the horizontal line $N = a/2b$ (dotted in Fig. 2) the solution curves have inflection points. That information enables us to complete our sketch of the solution curves in Fig. 2.[1]

Merely from the phase line, at the left in Fig. 2, we can see that the equilibrium points $N = 0$ and $N = a/b$ are unstable and stable, respectively, as can also be seen from the solution curves in the t, N plane. $N = a/b$ is an important quantity, the **environmental carrying capacity**, the population that can be supported by the environment.

The logistic equation (3) is often written, instead, as $N' = r(1 - \frac{N}{K})N$, in which r is called the intrinsic growth rate and K is the environmental carrying capacity.

1.6.2 Stability of equilibrium points and linearized stability analysis. Let us review and extend the definitions of equilibrium points and stability given in Section 1.3.4. Recall that N_{eq} is an **equilibrium point** of $N'(t) = f(N)$ if $f(N_{\text{eq}}) = 0$. The equilibrium point is **stable** if $N(t)$ can be kept arbitrarily close to N_{eq} for all $t \geq 0$ by taking it to be sufficiently close initially (at $t = 0$);[2] otherwise it is **unstable**. For the logistic model (3), we can see from Fig. 2 that the equilibrium points $N_{\text{eq}} = 0$ and $N_{\text{eq}} = a/b$ are unstable and stable, respectively.[3]

[1] We didn't really sketch Fig. 2 by hand; we plotted computer generated solutions, but we *could* have sketched it from the information that we've discussed.

[2] That is, corresponding to each number $\epsilon > 0$ (i.e., no matter how small) there exists a number $\delta > 0$ such that $|N(t) - N_{\text{eq}}| < \epsilon$ for all $t \geq 0$ if $|N(0) - N_{\text{eq}}| < \delta$.

[3] The former is unstable because we *cannot* keep $|N(t) - N_{\text{eq}}| = |N(t) - 0| = N(t) < \epsilon$ for all $t \geq 0$, where ϵ is arbitrarily small, no matter how close $N(0)$ is to $N_{\text{eq}} = 0$, for no matter how close the initial point is to the t axis, in Fig. 2, the solution curve moves upward, tending to the asymptote a/b as $t \to \infty$. And the equilibrium point $N_{\text{eq}} = a/b$ is stable because we *can* keep $|N(t) - N_{\text{eq}}| = |N(t) - a/b| < \epsilon$ for all $t \geq 0$, where ϵ is arbitrarily small, simply by starting out closer to a/b than ϵ.

Further, we classify a stable equilibrium point as **asymptotically stable** if $N(t)$ not only remains arbitrarily close to N_{eq} for all $t \geq 0$ but if it actually *tends to* N_{eq} as $t \to \infty$, that is, if $N(t) \to N_{eq}$ as $t \to \infty$. From Fig. 2 it seems evident that not only is $N_{eq} = a/b$ stable, but that it is asymptotically stable.

Proceeding one step further, we introduce the idea of "linearized stability analysis." The idea is simple. Suppose N_{eq} is an equilibrium point of an autonomous equation

$$\boxed{\frac{dN}{dt} = f(N),} \tag{9}$$

and that we wish to examine its stability. Since the stability concept used here is a "local" one, why not simplify the function $f(N)$ in (9) by approximating it in the neighborhood of the point N_{eq}? Specifically, expand $f(N)$ in a Taylor series about N_{eq} and cut off after the linear (i.e., the first-degree) term:

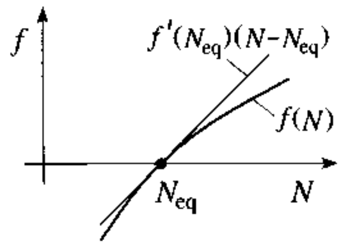


Figure 3. Local approximation of $f(N)$ in neighborhood of the equilibrium point N_{eq}.

$$\begin{aligned} f(N) &= f(N_{eq}) + f'(N_{eq})(N - N_{eq}) + \frac{1}{2!} f''(N_{eq})(N - N_{eq})^2 + \cdots \\ &\approx f'(N_{eq})(N - N_{eq}), \end{aligned} \tag{10}$$

in which we've also dropped the leading term $f(N_{eq})$ because $f(N_{eq}) = 0$ by the definition of equilibrium point; see Fig. 3.[1]

The approximation (10) reduces (9) to the simple *linear* equation

$$\boxed{\frac{dN}{dt} = f'(N_{eq})(N - N_{eq}),} \tag{11}$$

which is called the **linearized** version of (9). If we define the **deviation** from the equilibrium point as $\delta(t) \equiv N(t) - N_{eq}$, then $\delta'(t) = N'(t)$ and (11) becomes

$\delta(t) \equiv N(t) - N_{eq}$.

$$\frac{d\delta}{dt} = f'(N_{eq})\delta, \tag{12}$$

with solution

$$\delta(t) = \delta_0 e^{f'(N_{eq})t}. \tag{13}$$

Everything hinges on the sign of the number $f'(N_{eq})$ in the exponent: If

$$\boxed{f'(N_{eq}) < 0,} \tag{14a}$$

then (13) shows that $\delta(t) \to 0$ as $t \to \infty$ so the equilibrium point N_{eq} is evidently *asymptotically stable*, and if

$$\boxed{f'(N_{eq}) > 0,} \tag{14b}$$

[1] We've assumed that $f'(N_{eq}) \neq 0$ so that the approximation (10) does capture the first *nonvanishing* term of the series.

then $\delta(t)$ grows, instead, so the equilibrium point N_{eq} is evidently *unstable*. If $f'(N_{eq}) = 0$, the criterion gives no information. In that case we must go back to (10) and proceed farther into the Taylor series, to keep the first *non*vanishing term. This point is pursued in the exercises.

We said "evidently," above, because although it is reasonable to expect the *original* differential equation $dN/dt = f(N)$ to have the same behavior, near N_{eq}, as its linearized version, we have not proved that it does. However, the expected result is true, and we state it as a theorem:

THEOREM 1.6.1 *Stability Criterion for $dN/dt = f(N)$*
Let N_{eq} be an equilibrium point of $dN/dt = f(N)$, where $f(N)$ is differentiable at N_{eq}.
(a) If $f'(N_{eq}) < 0$, then N_{eq} is *asymptotically stable*.
(b) If $f'(N_{eq}) > 0$, then N_{eq} is *unstable*.

The criterion simply echos what we have already seen from our phase line pictures such as Fig. 1, namely, that if the slope f' at N_{eq} is negative then the flow is toward N_{eq} and the latter is stable, and if the slope there is positive then the flow is away from N_{eq} and the latter is unstable.

Realize that we can find the equilibrium points [by solving $f(N) = 0$ for N] and can then determine their stability [by determining the sign of f' at each equilibrium point] *without ever solving the differential equation (8) — even without plotting $f'(N)$ versus N and obtaining the phase line*! Indeed, (9) might be too difficult to solve, or its solution might be obtainable but intractably messy, or we might not be *interested* in the solution, but only in the equilibrium points and their stability.

EXAMPLE 1. Application of the Stability Criterion (14). To illustrate, we will apply the linearization procedure to the differential equation

$$\frac{dx}{dt} = \frac{1 - x^2}{1 + x^2}. \tag{15}$$

If we set $f(x) = (1-x^2)/(1+x^2) = 0$ to find the equilibrium points we obtain $x_{eq} = \pm 1$. To determine the stability of $x_{eq} = +1$, expand f in a Taylor series about that point and linearize,

$$\begin{aligned}\frac{dx}{dt} = \frac{1-x^2}{1+x^2} &= -(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{4}(x-1)^4 + \cdots \\ &\approx -(x-1)\end{aligned} \tag{16}$$

so, with $\delta = x - x_{eq} = x - 1$, (16) gives

$$\delta' = -\delta, \tag{17}$$

This example is just a made-up differential equation, not a population problem, so in place of $N(t)$ we revert to our generic $x(t)$ notation.

Recall that the Taylor series of f about $x = 1$ is $f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots$.

$$\delta(t) = \delta_0 e^{-t} \tag{18}$$

and because of the *negative* exponential in (18), $x_{eq} = +1$ is stable. Actually, we didn't need to carry out the solution of (17), or even the Taylor expansion in (16); we could simply have evaluated $f'(1)$ and examined its sign. Since $f'(1) = -1 < 0$, we could have concluded from Theorem 1.6.1 that $x_{eq} = +1$ is asymptotically stable.

For $x_{eq} = -1$ we will take the shortcut: We find that $f'(-1) = +1 > 0$, so $x_{eq} = -1$ is unstable. ∎

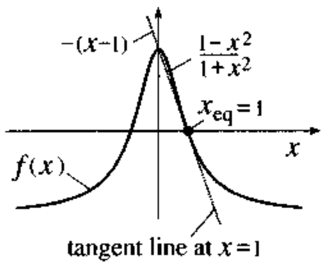


Figure 4. The linearization (16) is a tangent-line approximation of the function $f(x) = (1-x^2)/(1+x^2)$ at $x_{eq} = 1$.

To understand the linearization idea, think geometrically. Specifically, observe that the linearization of $f(x)$ about x_{eq} amounts to replacing the nonlinear function $f(x)$ in $x' = f(x)$ by its tangent-line approximation at that point. For instance, the approximation of $(1-x^2)/(1+x^2)$ by $-(x-1)$ in (16) amounts to the tangent-line approximation shown in Fig. 4. Just as the tangent line faithfully approximates $f(x)$ in the neighborhood of x_{eq}, so does the flow corresponding to the linearized differential equation faithfully approximate the flow corresponding to the original nonlinear differential equation in the neighborhood of x_{eq}.

Closure. In this section we used already-developed solution techniques and qualitative phase line methods to study representative problems involving separable first-order differential equations. The only new mathematical idea was that of linearization about an equilibrium point, which we used in Section 1.6.2 to determine the stability of equilibrium solutions of nonlinear autonomous differential equations.

EXERCISES 1.6

1. Incorporating Harvesting. Let $N(t)$ denote the fish population in a commercial fish pond. If we harvest fish at a rate h fish per unit time, we must modify the logistic equation (3) as

$$\boxed{N' = (a-bN)N - h.} \tag{1.1}$$

To maximize profits, we want to make h as large as possible, but if we make it too large then we will drive the fish population to zero and be out of business. Thus, our interest is not so much in solving (1.1) and obtaining traditional plots of $N(t)$ versus t, for instance, but in determining the maximum sustainable harvesting rate h. *The problem*: Determine that rate. NOTE: The phase line contains all the information that is needed. Sketch the phase line [i.e., the graph of $N' = (a-bN)N - h$ versus N] for $h = 0$, and again for $h > 0$, and see the effect of h on the flow along the phase line. This exercise illustrates the simplicity and value of the phase line — not to replace standard solution methods, but to complement them.

2. Incorporating a Threshold Population. Field studies indicate that if the population of a certain fish in a lake falls below a critical level, say P, then it will decline to zero (i.e., to extinction). Thus, to successfully stock the lake with that species one must supply enough fish so that the initial population is more than P fish. To incorporate this behavior, it is proposed that we modify the logistic equation (3) to the form

$$N' = -\kappa(P-N)(Q-N)N, \tag{2.1}$$

in which κ, P, and Q are positive constants and $Q > P$. Does the form of (2.1) seem reasonable? HINT: Consider the phase line.

3. Gompertz Growth Model. Let $W(t)$ be the weight of an organism as a function of the time t. One model of the growth of the organism is given by the **Gompertz** equation

$$W' = rW \ln \frac{K}{W} \tag{3.1}$$

or, equivalently, $W' = r(\ln K - \ln W)W$, in which r and K are positive constants.

(a) Solve (3.1) by separation of variables and show that

$$W(t) = Ke^{Ae^{-rt}}, \tag{3.2}$$

in which A is an arbitrary constant. HINT: To evaluate the integral that arises, the substitution $u = \ln W$ will help.
(b) Verify, by substitution, that (3.2) satisfies (3.1).
(c) Show that if the initial condition is $W(0) = W_0 > 0$, then (3.2) gives

$$W(t) = K\left(\frac{W_0}{K}\right)^{(e^{-rt})}, \tag{3.3}$$

that is, W_0/K to the e^{-rt} power, times K.
(d) Proceeding qualitatively instead, we can use the phase line because (3.1) is autonomous. Give labeled sketches analogous to Figs. 1 and 2 [which were for the logistic equation (3)]; consider only $W > 0$. Include any equilibrium points, sketch representative solution curves (as in Fig. 2), and show that the solution curves have inflection points at $W = 0.3679K$ (analogous to the line $N = a/2b$ in Fig. 2). Finally, classify each equilibrium point as stable or unstable.

4. Qualitative Analysis. Consider the autonomous equation $x' = f(x)$ on $0 \le t < \infty$ and for $-\infty < x < \infty$, where $f(x)$ is given below. Determine the equilibrium points, if any, sketch the graph of x' versus x and the phase line, and classify each equilibrium point as stable or unstable.

(a) $f(x) = e^x - 10$ (b) $f(x) = (x-2)^3$
(c) $f(x) = x^4 - 5x^2 + 4$ (d) $f(x) = x^3 + 8$
(e) $f(x) = x^4 - 1$ (f) $f(x) = 3x - \sin x$
(g) $f(x)$ supplied by your instructor

5. (a)–(g) Applying Theorem 1.6.1. For the $f(x)$ given in the corresponding part of Exercise 4, use Theorem 1.6.1 to determine the stability or instability of each equilibrium point. If the theorem gives no information, state that.

6. Speed of Approach to Equilibrium. For both equations

$$x' = -x \quad \text{and} \quad x' = -x^3, \tag{6.1a,b}$$

$x = 0$ is a stable equilibrium point. How does the *speed* of approach to the equilibrium point $x = 0$ compare, for (6.1a) and (6.1b)? Explain.

7. One-Compartment Biological Systems. Consider a one-compartment biological system, such as a single cell, or an organ such as a kidney, and consider a particular chemical with concentration $c(t)$ within it. If the difference between the concentration $c(t)$ inside the compartment and the concentration c_0 outside of it is sufficiently small, the transport of the chemical across the boundary of the compartment can be modeled as being proportional to the difference $c(t) - c_0$:

$$c'(t) = -k(c - c_0), \tag{7.1}$$

in which k is an empirically determined positive constant of proportionality. For larger concentration differences, a better model is probably the **Michaelis–Menten** equation

$$\boxed{c'(t) = -\frac{a(c-c_0)}{b + (c - c_0)},} \tag{7.2}$$

which contains two empirically determined positive parameters a and b. The right-hand side of (7.2) is designed so that for small concentration differences $b + (c - c_0) \sim b$ and (7.2) reduces to

$$c'(t) \sim -\frac{a}{b}(c - c_0), \tag{7.3}$$

which is of the form of (7.1), but for large concentration differences $b + (c - c_0) \sim (c - c_0)$ and (7.1) reduces to

$$c'(t) \sim -a. \tag{7.4}$$

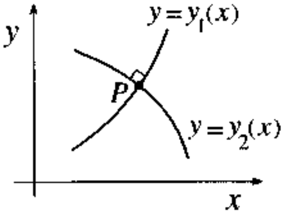
That is, the membrane cannot accomodate an arbitrarily large flow rate (any more than one can consume pizza at an arbitrarily large rate); the flow rate levels off as the concentration difference approaches infinity. NOTE: The right-hand side of the Michaelis–Menten equation is not derived, it is *designed* to be simple and to exhibit the two limiting behaviors indicated in (7.3) and (7.4).

(a) Determine any equilibrium points of (7.2) and their stability.
(b) If the initial concentration is a prescribed value $c(0)$, solve (7.2) and obtain the following solution, in implicit form,

$$b \ln\left|\frac{c(t) - c_0}{c(0) - c_0}\right| + c(t) = at + c(0). \tag{7.5}$$

(c) With $a = b = 10$ and $c_0 = 2$, say, obtain a computer plot of $c(t)$ versus t for each of the three initial conditions $c(0) = 1$, $c(0) = 2$, and $c(0) = 3$.

8. Orthogonal Families of Plane Curves. In a variety of applications, one is interested in two coplanar families of curves that intersect each other at right angles. Such families of curves are said to be **orthogonal**. For instance, the families of all concentric circles centered at the origin of an x, y plane and of all straight lines through the origin are orthogonal. (These are the constant-r and constant-θ curves of a polar coordinate system.) Consider two representative curves, one from each of the two families, defined by $y_1(x)$ and $y_2(x)$, and suppose they cross at P, as in the figure. With the help of a labeled



sketch, show that their slopes at P are negative inverses of each other:

$$\boxed{y_1'(x) = -\frac{1}{y_2'(x)}.} \tag{8.1}$$

9. Exercise 8, Continued. Suppose one family is comprised of the solutions of a given differential equation

$$y' = f(x, y), \tag{9.1}$$

and that we want to find the corresponding family of orthogonal curves. According to (8.1), to do so we must solve the differential equation

$$y' = -\frac{1}{f(x, y)}. \tag{9.2}$$

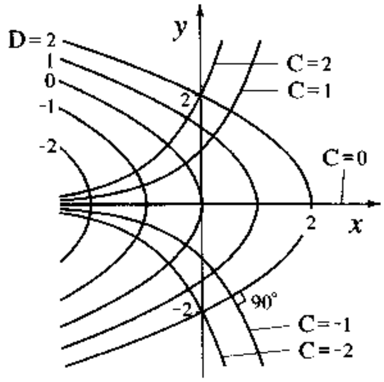
To illustrate, suppose we are given the differential equation

$$y' = y. \tag{9.3}$$

The family of solutions of (9.3) is the set of exponentials $y = Ce^x$, where C is an arbitrary constant. Then, to find the corresponding orthogonal curves form the negative inverse of the slope $y' = y$ given in (9.3) and solve

$$y' = -1/y. \tag{9.4}$$

That step gives the family of curves $y = \sqrt{2}\sqrt{D - x}$, in which D is an arbitrary constant. Representative members of the two orthogonal families are shown in the figure. Do the same for



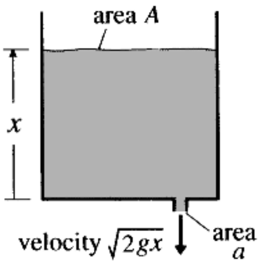
each given differential equation: find the two families of curves and give a hand sketch or computer plot of representative members of each family.

(a) $y' = -4y$ (b) $y' = 2y/x$
(c) $y' = -y/x$ (d) $y' = -x/y$

10. Exercises 8 and 9, Continued. Instead of being given the differential equation of one of the families, as we were in Exercise 9, in this exercise we give the family itself and ask you to find a second family, that is orthogonal to the one that is given. HINT: Work backwards and find a differential equation (9.1) for which the given family is the solution. Then proceed as in Exercise 9.

(a) $y = Ce^{2x}$ (b) $y = 1/(x + C)$ (c) $y = Cx^3$

11. The Draining of a Tank; Torricelli's Law. A tank, of uniform cross sectional area A, has a leak at the bottom due to a hole of cross-sectional area a, so the liquid depth x will diminish with time. We wish to predict the time T it will take the tank to empty if the initial liquid depth is x_0. According to



Torricelli's law, the efflux velocity from the hole is $\sqrt{2gx}$, which is the same as the velocity that would result from free fall, from rest, through the vertical distance x. (Actually, it will be $\beta\sqrt{2gx}$ for some positive constant $\beta < 1$, due to frictional losses, but we will neglect such effects and take $\beta = 1$.)

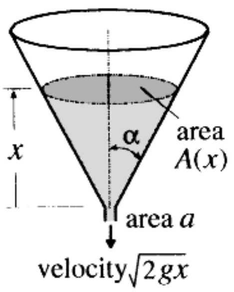
(a) Derive the IVP

$$x' = -\frac{a}{A}\sqrt{2gx}; \quad x(0) = x_0. \tag{11.1}$$

(b) Solve (11.1) and show that the tank empties not as $t \to \infty$, but in the finite time

$$T = \frac{A}{a}\sqrt{\frac{2x_0}{g}}. \tag{11.2}$$

12. A Conical Tank. First, read the introduction to Exercise 11, through Equation (11.1). Suppose the tank is not of uniform cross section but is conical, as shown below and, as in Exercise 11, suppose it has a hole of area a at the bottom.



(a) Recalling that the volume of a cone of base radius r and altitude h is $\pi r^2 h/3$, show that the IVP for the depth $x(t)$ is

$$x' = -\left(\frac{a\sqrt{2g}}{\pi \tan^2 \alpha}\right)x^{-3/2}; \quad x(0) = x_0. \tag{12.1}$$

(b) Solve (12.1) and show that the draining time is

$$T = \frac{2\pi \tan^2 \alpha}{5a\sqrt{2g}} x_0^{5/2}. \tag{12.2}$$

13. Streamline Pattern. Let

$$\frac{dx}{dt} = (1-y)x, \tag{13.1a}$$

$$\frac{dy}{dt} = (x-1)y \tag{13.1b}$$

be the x and y velocity components of a certain fluid flow in the first quadrant of the x, y plane. We wish to find the streamlines, that is, the paths of the fluid particles, and these are given by the integral curves of

$$\frac{dy}{dx} = \frac{(x-1)y}{(1-y)x}, \tag{13.2}$$

obtained by dividing (13.1b) by (13.1a).

(a) Derive the implicit solution

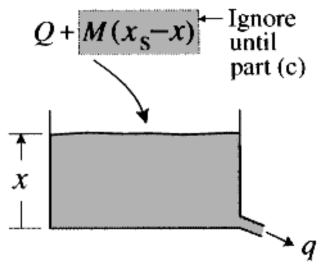
$$xy\, e^{-(x+y)} = C \tag{13.3}$$

of (13.2), and verify by differentiating (13.3), that it does satisfy (13.2).

(b) To see the streamline pattern, obtain computer-generated streamlines through the points (1,0), (1,0.1), (1,0.3), (1,0.5), (1,0.5), (1,0.9), and (0,1), and add flow direction arrows. Plot on the square $0 \le x \le 3$, $0 \le y \le 3$.

ADDITIONAL EXERCISES

14. Liquid Level Feedback Control. Liquid flows into a tank of horizontal cross sectional area A ft^2 at a constant rate Q ft^3/sec and leaves at the rate $q = \alpha\sqrt{x}$ ft^3/sec, where $x(t)$ is the liquid depth, t is the time, and α is an empirically known constant. Torricelli's law gives the exit velocity as $\sqrt{2gx}$,



and when we multiply that by the exit area we get a flow rate of the form $\kappa\sqrt{x}$, in which κ is a known positive constant. Equating the rate of increase of liquid volume in the tank to the rate in minus the rate out gives the differential equation

$$Ax' = Q - \kappa\sqrt{x} \tag{14.1}$$

for $x(t)$, with an equilibrium or steady-state x_s found from $0 = Q - \kappa\sqrt{x_s}$ as

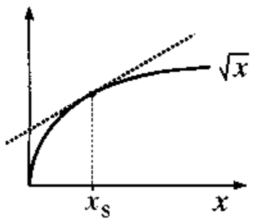
$$x_s = (Q/\kappa)^2. \tag{14.2}$$

The goal, in the operation of this "chemical plant," is to maintain $x(t)$ at its equilibrium value x_s, and to return it quickly to that value following any "disturbance" of x from its desired value x_s.

(a) From the phase line, show that x_s is a stable equilibrium point of (14.1).

(b) **Linearization.** If x is close to x_s, then it seems justified to linearize the nonlinear equation (14.1) about x_s.

Thus, expand the nonlinear $\sqrt{x}$ term in (14.1) in a Taylor series, about x_s, and cut it off after the first-order term, which amounts to using the tangent-line approximation illustrated in the next figure. Show that (14.1) is thus simplified to the



linearized equation

$$Ax' = -\beta(x - x_s) \quad (\beta = \frac{\kappa}{2\sqrt{x_s}}), \tag{14.3}$$

and, taking $x(0) = x_0 \neq x_s$, derive the solution of (14.3) as

$$x(t) = x_s + (x_0 - x_s)e^{-\beta t/A}. \tag{14.4}$$

(c) **Feedback Control.** Although (14.4) shows that $x(t) \to x_s$ as $t \to \infty$, that approach may be too slow for successful plant operation if β/A is small. To speed up the return to x_s suppose, using suitable equipment, that we continuously monitor x, compare its measured value with x_s to determine the instantaneous error

$$e(t) = x_s - x(t), \tag{14.5}$$

and augment the inflow Q by an amount proportional to that error, $M(x_s - x)$, as indicated in the first figure. Accordingly, re-write (14.3) as

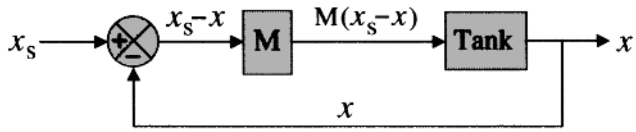
$$Ax' + \beta(x - x_s) = M(x_s - x), \tag{14.6}$$

show that $e(t)$ satisfies a linear differential equation with initial condition $e(0) = x_s - x_0$, with solution

$$e(t) = e(0)e^{-(\beta+M)t/A}. \tag{14.7}$$

COMMENT 1. The upshot is that β/A is enhanced to $(\beta + M)/A$, so disturbances from equilibrium die out exponentially faster with the "feedback" than without it.

COMMENT 2. This is an example of a **feedback control** system, because the error is fed back to the input as is indicated schematically in the next figure. Since the feedback $Me(t)$ is proportional to $e(t)$ it is an example of **proportional control**,



the amplification M being the **gain**. Also used are **derivative control** [proportional to $e'(t)$], **integral control** [proportional to $\int_0^t e(t)\,dt$], and combinations of the three. The human body is a whole hierarchy of control systems that control body temperature, heart rate, respiration, and so on. Control theory is normally taught, at the undergraduate and graduate levels, in engineering and bioengineering departments.

15. Free Fall and Terminal Velocity; Drag Proportional to Velocity Squared. The equation of motion of a body of mass m falling vertically in a fluid (such as air or water) follows from Newton's second law as

$$m\frac{dv}{dt} = mg - B - D, \tag{15.1}$$

in which $v(t)$ is the velocity (so dv/dt is the acceleration), B is the "buoyant force," and D is the "drag force" exerted on the body by the fluid. The buoyant force, by Archimedes' principle, is constant and equal to the weight of the fluid displaced by the body. The drag force is more complicated. For definiteness, suppose the body is spherical. It is shown, in a course on fluid mechanics, that if the *Reynolds number* parameter $\text{Re} = \rho v d/\mu$ is *small*, in which $d = 2r$ is the diameter of the sphere, and ρ and μ are the mass density and viscosity of the fluid, respectively, then the drag force D in (15.1) is proportional to the velocity v. In that case (15.1) is linear. Here, we consider instead the case of *large* Reynolds number, in the range

$$10^3 < \text{Re} < 10^5. \tag{15.2}$$

If (15.2) is satisfied then the drag force D on the sphere is approximately

$$D \approx 0.23\pi r^2 \rho v^2. \tag{15.3}$$

This time the quadratic dependence of D on v expressed by (15.3) results in the differential equation (15.1) for v being *nonlinear*. Take $g = 32.2$ ft/sec^2, and let the mass density of the body be 5 slugs/ft^3 (typical of stone). Let the fluid be water, with $\rho = 1.94$ slugs/ft^3 and $\mu = 2.36 \times 10^{-5}$ slugs/ft sec (at 60° F). With these values (15.1) becomes

$$6.67\pi r^3 \frac{dv}{dt} = 215\pi r^3 - 83.3\pi r^3 - (0.23)(1.94)\pi r^2 v^2,$$

or

$$\frac{dv}{dt} = 19.7 - \frac{0.067}{r}v^2. \tag{15.4}$$

(a) Using a phase line approach to (15.4), show that there is a steady-state value

$$v_s = 17.1\sqrt{r}\text{ ft/sec,} \tag{15.5}$$

called the *terminal velocity*, and that the latter is stable. [Note from (15.5) that large spheres, of a given mass density, fall faster than smaller ones.]

(b) Also, solve (15.4) with $v(0) = 0$ and show that

$$v(t) = 17.1\sqrt{r}\tanh(1.15t/\sqrt{r}). \tag{15.6}$$

Sketch the graph of $v(t)$, labeling any key values.

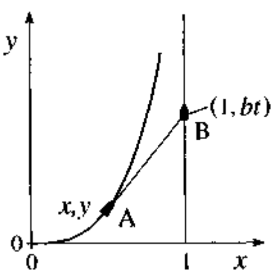
(c) Find the time, in terms of r, that it takes for the body to attain 90% of its terminal velocity.

(d) Remember that the approximate formula (15.3) for the drag force is accurate only for large Reynolds numbers in the interval defined by (15.2). In terms of the size of the sphere, (15.2) implies that our analysis is valid only if the diameter d of the sphere falls within certain limits. Show that with v given by (15.5), with $d = 2r$, and with the values of ρ and μ given above, the inequality (15.2) requires that $0.0050\text{ ft} < r < 0.108\text{ ft}$ or

$$0.06\text{ in} < r < 1.3\text{ in}. \tag{15.6}$$

For a stone of radius 0.06 inches (0.005 ft), (15.5) gives the terminal velocity as 1.21 ft/sec, and for a stone of radius 1.3 inches (0.108 ft) the terminal velocity is 5.62 ft/sec.

16. Curve of Pursuit. The following is a classical problem of pursuit — for instance, of one ship by another. Denote the pursued ship by point B and the pursuing ship by point A in the figure. Suppose B is at $(x, y) = (1, 0)$ at time $t = 0$ and moves



in the positive y direction with a constant speed b, and that A begins at the origin at $t = 0$, steers a course that is always directed at B, and moves with a constant speed a that is greater than b.

(a) Show, from the figure, that

$$\frac{dy}{dx} = \frac{bt - y}{1 - x}. \tag{16.1}$$

(b) Think about the variables used in (16.1): It is natural to think of x and y as functions of the time t, but in (16.1) the dy/dx implies that we are instead regarding x and y as independent and dependent variables, respectively. Fine, but then the t in (16.1) is not welcome. To eliminate it, differentiate (16.1) with respect to x. That step gives rise to a dt/dx term, and to obtain an expression for dt/dx use the formula

$$\frac{ds}{dt} = a = \frac{ds}{dx}\frac{dx}{dt} \tag{16.2}$$

in which s is the arclength along the curve of pursuit, from the origin to A. Show, from (16.2), that $dt/dx = \sqrt{1 + y'^2}/a$, and that (16.1) becomes

$$(1 - x)y'' = \frac{b}{a}\sqrt{1 + y'^2}. \tag{16.3}$$

(c) The latter is a *second*-order equation, and we haven't studied second-order equations yet, but the substitution $u(x) = y'(x)$ reduces it to a first-order equation for $u(x)$. Do that, solve for $u(x)$ by separation of variables, and obtain the solution (in implicit form)

$$u + \sqrt{u^2 + 1} = C(1 - x)^{-r}, \tag{16.4}$$

where $r = b/a < 1$ and C is an arbitrary constant. Applying the initial condition $u(0) = y'(0) = 0$, evaluate C.

(d) Solve (16.4) by algebra for u, replace u by $y'(x)$, and integrate again to show that

$$y(x) = -\frac{1}{2}\frac{(1 - x)^{1-r}}{1 - r} + \frac{1}{2}\frac{(1 - x)^{1+r}}{1 + r} + D, \tag{16.5}$$

where D is an arbitrary constant. Finally, apply the initial condition $y(0) = 0$ to solve for D, and thus show that the curve of pursuit is given by

$$y(x) = \frac{1}{2}\left[\frac{(1 - x)^{1+r}}{1 + r} - \frac{(1 - x)^{1-r}}{1 - r}\right] + \frac{r}{1 - r^2}. \tag{16.6}$$

(e) Determine the location of B when it is caught by A, and show that capture occurs at the time $T = a/(a^2 - b^2)$. Sketch the curve of pursuit up to the time of capture, with suitable labeling and with key features clearly rendered.

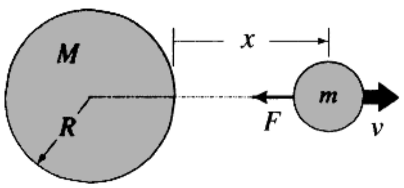
(f) For A to catch B, is it really an optimal strategy for it to always steer so as to be aiming at B? What would be the *optimal* pursuit path, optimal in the sense of overtaking B in minimal time, and what would be the time and place when that occurs? NOTE: This is the strategy used by a (good) baseball player in catching a "fly ball."

17. Projectile Dynamics and Escape Velocity. Consider a classical problem in Newtonian mechanics, the motion of a projectile subject to a gravitational force field. **Newton's law of gravitation** states that the force of attraction F exerted by one point mass M on another point mass m is

$$F = G\frac{Mm}{d^2}, \tag{17.1}$$

where d is the distance between them and $G = 6.67 \times 10^{-8}\,\mathrm{cm}^3/\mathrm{gm\,sec}^2$ is called the universal gravitational constant; (17.1) is called an **inverse-square law** because the force F varies as the inverse square of the distance d. (By M and m being point masses, we mean that their sizes are negligible compared to the distance between them; even an elephant could be a "point mass.")

Consider the linear motion of a projectile of mass m launched from the surface of the earth, as sketched in the figure, where M and R are the mass and radius of the earth respectively, and where any air resistance is neglected.



From Newton's second law of motion and his law of gravitation (17.1), it follows that the equation of motion of the projectile is

$$m\frac{d^2x}{dt^2} = -G\frac{Mm}{(x+R)^2}, \tag{17.2}$$

the minus sign because the force is in the negative x direction, and $x + R$ being the distance between mass centers. (It is not at all obvious, but can be proved, and indeed *was* proved by Newton and published in his *Principia Mathematica*, that the force of attraction of a spherical homogeneous mass M at any point outside that mass is the same as if the entire mass M were compressed to a point, at the center of that sphere.)
Here is the problem:

(a) Now, (17.2) is of second order and we have not yet studied equations of second order, but we can reduce it to a first-order equation as follows: If we integrate both sides with respect to t, the left side is simple; it merely gives $m\,dx/dt$, but the right-hand integral cannot be evaluated because the $x(t)$ in the integrand is not yet known. If we integrate both sides with respect to x instead, then the right-hand integral can be evaluated, but what can we do with the integral on the left, $\int mx''\,dx$? Proceeding as in Equation (9) of Section 1.5, show that the result of integrating (17.2) with respect to x is

$$\frac{1}{2}mx'^2 = \frac{GMm}{x+R} + C, \tag{17.3}$$

in which C is the arbitrary constant of integration. NOTE: Physically, if we express the latter in the form

$$\underbrace{\frac{1}{2}mx'^2}_{\text{KE}} + \underbrace{\left(-\frac{GMm}{x+R}\right)}_{\text{PE}} = C \tag{17.4}$$

we can understand it as a statement of **conservation of energy**, for it says the kinetic energy plus the gravitational potential energy is a constant over the course of the motion. If we solve (17.3) for $x'(t)$ by taking square roots of both sides, the resulting differential equation for $x(t)$ will be separable, but the x integration is difficult and leads to a messy solution in implicit form. Thus, let us forego the solution for $x(t)$ and see what we can learn, in the remainder of this exercise, directly from (17.3).

(b) Denote the velocity $x'(t)$ as $v(t)$ and let the launch velocity be $v(0) = V$. Applying that initial condition, solve (17.3) for C, and show that

$$v = \sqrt{V^2 - \frac{2GM}{R}\frac{x}{x+R}}. \tag{17.5}$$

It is possible to eliminate the universal gravitational constant G in favor of the more familiar constant g, the gravitational acceleration g at the earth's surface, by noting that when $x = 0$ the right-hand side of (17.2) must be the weight force $-mg$. Thus, show that $G = R^2g/M$ so (17.5) becomes

$$v = \sqrt{V^2 - 2gR\frac{x}{x+R}}. \tag{17.6}$$

(c) Show, from (17.6), that if V is less than a certain critical velocity, the **escape velocity** V_e, then the projectile reaches a maximum distance $x_{\max}$ from the earth and then returns to the earth, but if $V > V_e$, then the projectile escapes and does not return. Show that

$$x_{\max} = \frac{V^2 R}{2gR - V^2} \quad \text{and} \quad V_e = \sqrt{2gR}. \tag{17.7}$$

(d) Sketch the graph of v versus x for two representative launch velocities V, one smaller than V_e and one greater than V_e, and label any key values.

(e) Calculate V_e in km/sec and in miles/hr, using $R = 6378$ km = 3960 mi, and $g = 9.81$ m/sec^2 = 32.2 ft/sec^2.

HISTORICAL NOTE: Newton inferred (17.1) from **Kepler's laws** of planetary motion, which were, in turn, inferred empirically by Kepler from the voluminous measurements recorded by the Danish astronomer *Tycho Brahe* (1546–1601). Usually, in applications, one knows the force exerted on a mass and determines the motion by twice integrating Newton's second law of motion. In deriving (17.1), however, Newton worked "backwards:" The motion of the planets was described by Kepler's laws, and Newton used those laws to infer the force needed to cause that motion. It turned out to be an inverse-square force directed toward the sun. Newton then proposed the bold generalization that (17.1) holds not only between each planet and the sun, but between *any* two bodies in the universe; hence the name **universal law of gravitation**. Imagine how the idea of a force *acting at a distance*, rather than through physical contact, must have been incredible when first proposed. In fact, such great scientists and mathematicians as Huygens, Leibniz, and John Bernoulli called Newton's idea of gravitation absurd and revolting! But, Newton stood upon the results of his mathematics, in inferring the concept of gravitation, even in the face of such distinguished opposition. Remarkably, **Coulomb's law** subsequently stated an inverse-square type of attraction or repulsion between two electric charges. But, although the forms of the two laws are identical, the magnitudes of the forces are staggeringly different. Specifically, the electrical force of repulsion between two electrons is, independent of the distance of separation, 4.17×10^{42} times stronger than their gravitational attraction due to their mass!

1.7 EXACT EQUATIONS AND EQUATIONS THAT CAN BE MADE EXACT

Thus far we've developed solution techniques for first-order differential equations that are linear or separable. In this section we consider another important case, equations that are "exact." The method that we develop will be a version of the integrating factor method used in Section 1.2 to solve the linear equation $y' + p(x)y = q(x)$.

1.7.1 Exact differential equations. To motivate the idea of exact equations, consider

$$\frac{dy}{dx} = \frac{\sin y}{2y - x\cos y} \tag{1}$$

or, in differential form,

$$\sin y\,dx + (x\cos y - 2y)\,dy = 0. \tag{2}$$

The left-hand side is the differential of $F(x,y) = x\sin y - y^2$ because, by the chain rule,

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \sin y\,dx + (x\cos y - 2y)\,dy, \tag{3}$$

so (2) is simply $dF = 0$, which is readily integrated and gives F = constant. Thus,

$$F(x,y) = x\sin y - y^2 = C, \tag{4}$$

with C an arbitrary constant. Equation (4) is the solution to (1), in implicit form.

Here we are regarding x as independent variable and y as dependent variable.

To generalize the method outlined above, consider the differential equation

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}, \tag{5}$$

in which the minus sign is included merely so that when we re-express (5) in the differential form

$$\boxed{M(x,y)dx + N(x,y)dy = 0} \tag{6}$$

we end up with a plus sign in (6). For (1), for instance, we see by comparing (2) and (6) that $M = \sin y$ and $N = x\cos y - 2y$.

Here we change our viewpoint temporarily and regard both x and y as independent variables.

Before proceeding, notice that in equation (5) y is regarded as a function of x, as is implied by the presence of the derivative dy/dx; x is the independent variable and y is the dependent variable. But upon re-expressing (5) in the form (6) we've changed our viewpoint, and now consider x and y as having the same status: now, both are independent variables.

We've seen that integration of (6) is simple if $Mdx + Ndy$ happens to be the differential of some function $F(x,y)$, for if there is a function $F(x,y)$ such that

$$dF(x,y) = M(x,y)dx + N(x,y)dy, \tag{7}$$

then (6) is simply

$$dF(x,y) = 0, \tag{8}$$

which gives the solution

$$F(x,y) = C \tag{9}$$

of (6), with C an arbitrary constant.

Given $M(x,y)$ and $N(x,y)$, which we can identify when we write the given differential equation in the differential form (6), suppose there does exist an $F(x,y)$ such that $Mdx + Ndy = dF$. If so, we call $Mdx + Ndy$ an **exact differential**, and we call (6) an **exact differential equation**.

Two questions arise: *Given a first-order differential equation, expressed in the differential format (6), does such an $F(x,y)$ exist* and, *if so, how do we find it?* The first is answered by the following theorem.

THEOREM 1.7.1 *Test for Exactness*
Let $M(x,y)$, $N(x,y)$, $\partial M/\partial y$, and $\partial N/\partial x$ be continuous within a rectangle R in the x,y plane. Then $Mdx + Ndy$ is an exact differential in R if and only if

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \tag{10}$$

everywhere in R.

Partial Proof: Suppose $M dx + N dy$ is exact, so there is a function F such that $dF = M dx + N dy$. Then, by the chain rule,

$$M = \frac{\partial F}{\partial x} \tag{11a}$$

and

$$N = \frac{\partial F}{\partial y}. \tag{11b}$$

Differentiating (11a) partially with respect to y, and (11b) partially with respect to x, gives

$$M_y = F_{xy}, \tag{12a}$$

and

$$N_x = F_{yx}. \tag{12b}$$

In (12a) and below, we use subscripts for partial derivatives, for compactness. For example, $M_y = \frac{\partial M}{\partial y}$ and $F_{xy} = (F_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$.

Since M, N, M_y, and N_x have been assumed continuous in R, it follows from (11) and (12) that F_x, F_y, F_{xy}, and F_{yx} are too, so $F_{xy} = F_{yx}$.[1] Then it follows from (12) that $M_y = N_x$, which is equation (10). Because of the "if and only if" wording in the theorem, we must also prove the reverse, that the truth of (10) implies the existence of F, but we will omit that part.[2] ■

Assuming that the conditions of the theorem are met, so we are assured that such an F exists, how do we *find* it? From (11a) and (11b). We will illustrate the procedure by revisiting our introductory example.

EXAMPLE 1. Solving an Exact Equation. Consider equation (1) again, in differential form,

$$\sin y\, dx + (x \cos y - 2y)dy = 0. \tag{13}$$

Compare (13) with (6) and identify $M = \sin y$ and $N = x \cos y - 2y$. Clearly, M, N, M_y, and N_x are continuous in the whole plane, so turn to the exactness condition (10):

---

[1] Recall that the partial derivative notation F_{xy} means $(F_x)_y$: differentiate first with respect to x and then with respect to y. It is shown in the calculus that a sufficient condition for $F_{xy} = F_{yx}$ is that F_x, F_y, F_{xy}, and F_{yx} are all continuous within the region in question. This is typically true in applications.

[2] See, for example, William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 6th ed. (NY: Wiley, 1997), page 85. In applications, of course, the *existence* of F follows when we actually *find* F, as in our Example 1.

$M_y = \cos y$, and $N_x = \cos y$, so (10) is satisfied, and it follows from Theorem 1.7.1 that there does exist an $F(x, y)$ such that the left-hand side of (13) is dF. *To find F use (11)*:

$$\frac{\partial F}{\partial x} = \sin y, \tag{14a}$$
$$\frac{\partial F}{\partial y} = x \cos y - 2y. \tag{14b}$$

The "partial integration" notation $\int (\ ) \partial x$ is not standard. We use it here to remind us that any y's in the integrand are to be treated as constants.

Integrating (14a) partially, with respect to x, gives

$$F(x, y) = \int \sin y \, \partial x = x \sin y + A(y), \tag{15}$$

The $\sin y$ integrand was treated as a constant in the integration because we performed a "partial integration" on x, holding y fixed [just as y was held fixed in computing $\partial F/\partial x$ in (14a)]. The constant of integration A must therefore be allowed to depend on y since y was held fixed and was therefore constant. As a check, taking a partial derivative of (15) with respect to x does recover (14a). Next, (14b) will determine $A(y)$: Putting (15) into (14b) gives $\frac{\partial}{\partial y}[x \sin y + A(y)] = x \cos y - 2y$, or,

$$x \cos y + A'(y) = x \cos y - 2y, \tag{16}$$

in which $A'(y)$ denotes dA/dy. Canceling terms gives $A'(y) = -2y$, so

$$A(y) = -\int 2y \, dy = -y^2 + B. \tag{17}$$

[The integration in (17) was not a "partial integration;" it was an ordinary integration on y because $A'(y)$ in $A'(y) = -2y$ was an ordinary derivative.] Putting (17) into (15) gives

$$F(x, y) = x \sin y - y^2 + B = \text{constant}. \tag{18}$$

Finally, absorb B into the constant, and call the resulting constant C. Thus, we have the solution

$$x \sin y - y^2 = C \tag{19}$$

of (13), in implicit form.

How can the method *fail*? Be sure to understand this point.

COMMENT 1. It would be natural to wonder how this method can *fail*. After all, even if $M_y \neq N_x$ can't we integrate (11) to find F? The clue, in this example, is in (16). For suppose (16) were $2x \cos y + A'(y) = x \cos y - 2y$, for instance, instead. Then the $x \cos y$ terms would not cancel, as they did in (16), and we would have $A'(y) = -x \cos y - 2y$, which is impossible because it expresses a relationship between x and y, whereas x and y are independent variables. Put differently, $A'(y)$ is a function of y only, so it cannot depend on x. Thus, the cancelation of the $x \cos y$ terms in (16) was crucial and was not an accident, but was a consequence of M and N satisfying the exactness condition (10).

COMMENT 2. We integrated (14a) and then (14b), but the order doesn't matter. ∎

1.7.2 Making an equation exact; integrating factors. If M and N fail to satisfy (10), so the equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{20}$$

is *not* exact, we can try to find a function $\sigma(x,y)$ so that if we multiply (20) by that function, then the new equation,

$$\sigma(x,y)M(x,y)dx + \sigma(x,y)N(x,y)dy = 0, \tag{21}$$

is exact.

Here, σM is our new "M" and σN is our new "N." We are seeking a function $\sigma(x,y)$ so that the exactness condition

$$\boxed{\frac{\partial}{\partial y}(\sigma M) = \frac{\partial}{\partial x}(\sigma N)} \tag{22}$$

is satisfied for (21). If we can find a function $\sigma(x,y)$ satisfying (22), we call it an **integrating factor** of (20) because then the left-hand side of (21) is dF, the differential of some function $F(x,y)$. Then (21) is simply $dF = 0$, which gives the solution of (20) as $F(x,y) =$ constant.

How can we find σ? It is any (nontrivial) solution of (22), that is, of

By trivial solution we mean $\sigma(x,y) = 0$.

$$\sigma_y M + \sigma M_y = \sigma_x N + \sigma N_x, \tag{23}$$

in which subscripts denote partial derivatives. Since (23) contains partial derivatives of $\sigma(x,y)$ it is not an ordinary differential equation but a *partial differential equation* for σ. Partial differential equations are beyond the scope of this text, so we have made dubious headway: To solve the original ordinary differential equation on $y(x)$ we now need to solve the partial differential equation (23) for $\sigma(x,y)$!

However, perhaps an integrating factor σ can be found that is a function of x alone, $\sigma(x)$. In that case $\sigma_y = 0$ and (23) reduces to the *ordinary* differential equation

$$\sigma M_y = \frac{d\sigma}{dx}N + \sigma N_x$$

or

$$\frac{d\sigma}{dx} = \left(\frac{M_y - N_x}{N}\right)\sigma. \tag{24}$$

This idea succeeds if and only if the $(M_y - N_x)/N$ in (24) is a function of x only, for if it contained any y dependence, then (24) would amount to a contradiction: a function of x equaling a function of x *and* y, where x and y are independent variables. Thus, if

$$\boxed{\frac{M_y - N_x}{N} = \text{function of } x \text{ alone},} \tag{25}$$

For $\sigma(x)$.

If $M_y = N_x$, then (26) gives $\sigma(x) = 1$. After all, if $M_y = N_x$ then the equation was exact in the first place.

then (24) is separable and gives

$$\sigma(x) = e^{\int \frac{M_y - N_x}{N}\,dx}. \tag{26}$$

Actually, the general solution of (24) for $\sigma(x)$ is an arbitrary constant times the right-hand side of (26), but the constant can be taken to be 1 without loss since all we need is *an* integrating factor.

If $(M_y - N_x)/N$ is *not* a function of x alone, then an integrating factor $\sigma(x)$ does not exist, but perhaps we can find σ as a function of y alone, $\sigma(y)$. In that case, (23) reduces to

$$\frac{d\sigma}{dy}M + \sigma M_y = \sigma N_x$$

or

$$\frac{d\sigma}{dy} = -\left(\frac{M_y - N_x}{M}\right)\sigma.$$

This time, if

For $\sigma(y)$.

$$\frac{M_y - N_x}{M} = \text{function of } y \text{ alone}, \tag{27}$$

then

$$\sigma(y) = e^{-\int \frac{M_y - N_x}{M}\,dy}. \tag{28}$$

EXAMPLE 2. Finding and Using an Integrating Factor. Consider the equation $y' = 2xe^y/(e^y - 4)$, or

$$2xe^y\,dx + (4 - e^y)\,dy = 0. \tag{29}$$

Then $M(x,y) = 2xe^y$ and $N(x,y) = 4 - e^y$, so (10) is not satisfied and (29) is not exact. If we seek an integrating factor that is a function of x alone, we find that

$$\frac{M_y - N_x}{N} = \frac{2xe^y - 0}{4 - e^y} \neq \text{function of } x \text{ alone}, \tag{30}$$

so $\sigma(x)$ is not possible. Seeking instead an integrating factor that is a function of y alone,

$$\frac{M_y - N_x}{M} = \frac{2xe^y - 0}{2xe^y} = 1 = \text{function of } y \text{ alone}, \tag{31}$$

so $\sigma(y)$ *is* possible, and

$$\sigma(y) = e^{-\int \frac{M_y - N_x}{M}\,dy} = e^{-\int 1\,dy} = e^{-y}. \tag{32}$$

Thus, multiply (29) through by $\sigma(y) = e^{-y}$ and obtain

$$2x\,dx + \left(4e^{-y} - 1\right)dy = 0, \tag{33}$$

which *is* exact. Then, (33) gives

$$\frac{\partial F}{\partial x} = 2x \tag{34a}$$

and

$$\frac{\partial F}{\partial y} = 4e^{-y} - 1, \tag{34b}$$

and (34a) gives

$$F(x, y) = \int 2x\, \partial x = x^2 + A(y). \tag{35}$$

Next, put the right-hand side of (35) into the left-hand side of (34b):

$$\frac{\partial}{\partial y}\left[x^2 + A(y)\right] = 4e^{-y} - 1$$

or

$$A'(y) = 4e^{-y} - 1. \tag{36}$$

Thus,

$$A(y) = -4e^{-y} - y + B$$

and

$$F(x, y) = x^2 + A(y) = x^2 - 4e^{-y} - y + B = \text{constant}$$

or

$$x^2 = y + 4e^{-y} + C, \tag{37}$$

with C an arbitrary constant; (37) is the desired solution of (29), in implicit form.

Actually, the steps (34)–(37) are overkill, for if we have the form $f(x)\,dx + g(y)\,dy = 0$, as we do in (33), we can simply integrate. Doing so gives $x^2 - 4e^{-y} - y = C$, which is the same result as (37).

COMMENT. Suppose we impose an initial condition $y(2) = 0$. Then (37) becomes $4 = 0 + 4 + C$, so $C = 0$. Thus, in implicit form, we have the particular solution

$$x^2 = y + 4e^{-y}, \tag{38}$$

which is plotted in Fig.1. The curve consists of two branches, an upper branch AB and a lower branch AC. The initial point is on the lower branch so discard the upper branch AB and keep the lower branch AC. Point A can be determined from the fact that the slope is infinite there. By setting the denominator in the differential equation $y' = 2xe^y/(e^y - 4)$ equal to zero we obtain $e^y = 4$, so A is at $y = \ln 4$ and, as follows from (38), at $x = \sqrt{\ln 4 + 1}$. Thus, the interval of existence of the solution satisfying the initial condition $y(2) = 0$ is $\sqrt{\ln 4 + 1} < x < \infty$. ∎

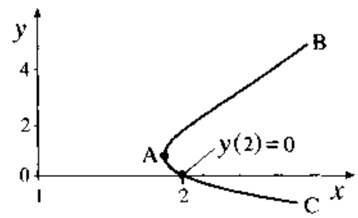


Figure 1. Graph of the relation (38).

EXAMPLE 3. Application to General Linear First-Order Equation. We've already solved the general *linear* first-order equation

$$\frac{dy}{dx} + p(x)y = q(x) \tag{39}$$

in Section 1.2, but let us see if we can solve it again, using the ideas in this section. First, put (39) into the form $M dx + N dy = 0$ by writing it as

$$[p(x)y - q(x)]\, dx + dy = 0. \tag{40}$$

Thus, $M = p(x)y - q(x)$ and $N = 1$, so $M_y = p(x)$ and $N_x = 0$. Hence $M_y \neq N_x$, so (40) is not exact [unless $p(x) = 0$]. Since

$$\frac{M_y - N_x}{N} = \frac{p(x) - 0}{1} = p(x) = \text{ function of } x \text{ alone,}$$

$$\frac{M_y - N_x}{M} = \frac{p(x) - 0}{p(x)y - q(x)} \neq \text{ function of } y \text{ alone,}$$

we can find an integrating factor that is a function of x alone, but not one that is a function of y alone. We leave it for the exercises to show that the integrating factor is

$$\sigma(x) = e^{\int p(x)\,dx},$$

and that the final solution (this time obtainable in explicit form) is

$$y(x) = e^{-\int p\,dx}\left(\int e^{\int p\,dx} q\,dx + C\right), \tag{41}$$

as we found in Section 1.2. ∎

Closure. Summary of the method of exact differentials:

To check your solution, a differential of $F(x, y) = C$ should give back the original equation $Mdx + Ndy = 0$.

1. Express the equation in the differential form $M(x,y)dx + N(x,y)dy = 0$. If M, N, M_y, and N_x are continuous in the x, y region of interest, check the exactness condition (10). If it is satisfied, the equation is exact, and its solution is $F(x, y) = C$, with F found from (11a) and (11b).

2. If the equation is not exact, see if $(M_y - N_x)/N$ is a function of x alone. If it is, an integrating factor $\sigma(x)$ can be found from (26). Multiply the given equation $Mdx + Ndy = 0$ through by that $\sigma(x)$ so the new equation is exact, then proceed as outlined in step 1.

3. If $(M_y - N_x)/N$ is not a function of x alone, see if $(M_y - N_x)/M$ is a function of y alone. If it is, an integrating factor $\sigma(y)$ can be found from (28). Multiply $Mdx + Ndy = 0$ through by that $\sigma(y)$ so the new equation is exact, then proceed as outlined in step 1.

4. If $M_y \neq N_x$, $(M_y - N_x)/N$ is not a function of x alone, and $(M_y - N_x)/M$ is not a function of y alone, then perhaps an integrating factor σ can be found that is a function of both x and y. Some examples of this type are included in the exercises.

Thus far, we've studied three types of first-order equation: the linear equation $y' + p(x)y = q(x)$ (Section 1.2), separable equations $y' = X(x)Y(y)$ (Section 1.4), and equations that are exact or can be made exact by the methods of this secction. Are these cases mutually exclusive? No. For instance, a subset of linear equations is also separable, namely, if $q(x)$ is zero or if $p(x)$ is a constant times $q(x)$. Further, *every* separable equation is exact if we write it in the form

$$X(x)dx - \frac{1}{Y(y)}dy = 0,$$

and every linear equation can be made exact (as we did in Example 3). These results are indicated schematically in Fig. 2.

We see from Fig. 2 that in principle it would suffice to study only equations that are exact [or can be made exact by $\sigma(x)$ or $\sigma(y)$] since that set *includes* linear and separable equations. However, it is important and traditional to study these cases separately — the linear equation because its theory is so complete and because it is so prominent in applications, and separable equations because the separation-of-variables solution method is so simple and, like the linear equation, it is so important in applications.

In fact, given a first-order differential equation, we suggest first checking to see if it is separable. If it is, solve by separation of variables. If not, see if it is linear or exact, whichever of these methods you prefer. If it is none of these, see if you can make it exact.

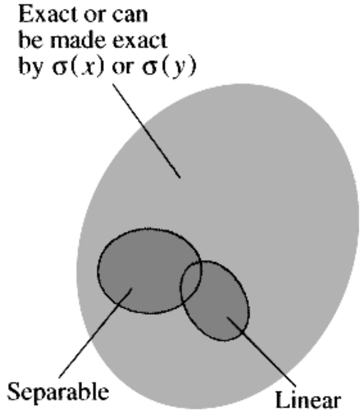


Figure 2. Schematic of the sets of first-order equations that are exact (or can be made exact), separable, or linear.

EXERCISES 1.7

1. Exact Equations. Show that the equation is exact, and obtain its solution. You may leave the answer in implicit form. If an initial condition is specified, also obtain a particular solution satisfying that condition.

(a) $3dx - dy = 0$
(b) $xdx - 4ydy = 0$; $y(0) = -1$
(c) $4\cos 2x\,dx - e^{-5y}dy = 0$
(d) $(y^2e^x + 1)dx + 2ye^x dy = 0$; $y(0) = -3$
(e) $(e^x + y)dx + (x - \sin y)dy = 0$
(f) $(x - 2y)dx + (y - 2x)dy = 0$
(g) $(\sin y + y\cos x)dx + (\sin x + x\cos y)dy = 0$; $y(\pi/2) = 2\pi$
(h) $e^y dx + (xe^y - 1)dy = 0$
(i) $2xydx + [(y+1)e^y + x^2]dy = 0$
(j) $(ye^{xy} + 1)dx + xe^{xy}dy = 0$; $y(2) = 0$
(k) $2xy\ln y\,dx + [x^2(\ln y + 1) + 2y]dy = 0$ $(y > 0)$

2. Describe a way to make up exact equations, such as those in Exercise 1, and give an example to illustrate your procedure.

3. Determine whatever conditions, if any, must be satisfied by the constants $a, b, \ldots, f, A, B, \ldots, F$ for the equation to be exact.

(a) $(ax + by + c)dx + (Ax + By + C)dy = 0$
(b) $(ax^2 + by^2 + cxy + dx + ey + f)dx + (Ax^2 + By^2 + Cxy + Dx + Ey + F)dy = 0$

4. An Integrating Factor Needed. Find an integrating factor $\sigma(x)$ or $\sigma(y)$, if possible, and use it to solve the given differential equation. If neither is possible, state that.

(a) $3ydx + dy = 0$
(b) $ydx + x\ln x\,dy = 0$ $(x > 0)$
(c) $y\ln y\,dx + (x + y)dy = 0$ $(y > 0)$
(d) $dx + (x - e^{-y})dy = 0$
(e) $dx + xdy = 0$
(f) $(ye^{-x} + 1)dx + (xe^{-x})dy = 0$
(g) $xy\,dx + \sin x\cos y\,dy = 0$
(h) $\sin x\,dx + y\cos x\,dy = 0$
(i) $(3x - 2y)dx - xdy = 0$
(j) $2xy\,dx + (y^2 - x^2)dy = 0$
(k) $(y\ln y + 2xy^2)dx + (x + x^2y)dy = 0$

5. First-Order Linear Equation. Use the integrating factor

$\sigma(x) = e^{\int p(x)\,dx}$ of (40) to derive the general solution (41). That is, fill in the missing steps between (40) and (41).

6. Cases Requiring $\sigma(x,y)$. Show that the following equations are not exact, nor do they admit an integrating factor that is a function of x alone or of y alone. If possible, find an integrating factor in the form $\sigma(x,y) = x^a y^b$, where a and b are suitably chosen constants. You need not solve the equation, just find σ; if such a σ cannot be found, state that.

(a) $ydx + (x - x^2y)dy = 0$
(b) $(x + y^2)dx + (x - y)dy = 0$
(c) $(3xy + 2y^2)dx + (3x^2 + 4xy)dy = 0$

7. Nonuniqueness of σ. Of course, if σ is an integrating factor of a given equation then so is any nonzero constant times σ. But, integrating factors may be nonunique beyond an arbitrary scale factor. To illustrate, show that the equation

$$2ydx + 3xdy = 0 \tag{7.1}$$

has integrating factors $\sigma(x) = x^{-1/3}$ and also $\sigma(x,y) = 1/xy$. You need not derive these; just verify them.

8. Integrating Factors for Separable Equations. Show that

$$P(x)Q(y)dx + R(x)S(y)dy = 0 \tag{8.1}$$

has an integrating factor $\sigma(x,y) = 1/[R(x)Q(y)]$ and, after multiplying (8.1) by that integrating factor, that the solution can be found from

$$\int \frac{P(x)}{R(x)}\,dx + \int \frac{S(y)}{Q(y)}\,dy = 0. \tag{8.2}$$

NOTE: Actually, (8.1) is a *separable* equation, and using the integrating factor $\sigma(x,y) = 1/[R(x)Q(y)]$ simply amounts to separating the variables. Having thus shown the connection between separable equations and the method of integrating factors, we suggest that if an equation is separable it is simplest to just separate the variables and integrate, rather than to invoke the integrating factor method. For instance, in Exercise 4 the equations in parts (a), (b), (e), (g), and (h) could have been solved more readily by separation of variables than by the integrating mactor method. Note also that a special case of (8.1) is $P(x)dx + S(y)dy = 0$, which is exact.

9. Solve, using the methods of this section. HINT: First re-express the equation in differential form.

(a) $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$ (b) $\dfrac{dr}{d\theta} = -\dfrac{r^2\cos\theta}{2r\sin\theta + 1}$

(c) $t\dfrac{dv}{dt} = 2te^v + 1$ (d) $(x\cos y + x^2)\dfrac{dy}{dx} = \sin y$

10. If $Mdx + Ndy = 0$ and $Pdx + Qdy = 0$ are exact, does it follow that $(M+P)dx + (N+Q)dy = 0$ is exact? Explain.

11. We solved (1) by using the fact that (2) is exact. Alternatively, observe that although (1) is neither separable nor first-order linear, it *is* first-order linear if we change our viewpoint and regard x as a function of y. Use that idea to solve for $x(y)$ and verify that your solution agrees with (4).

12. Grade This. Asked to solve

$$(3x - 2y)dx - xdy = 0, \tag{12.1}$$

a student writes this: "If we can find an $F(x,y)$ such that (12.1) is $dF = F_x dx + F_y dy = 0$, then the general solution of (12.1) is $F(x,y) = C$. Integrating $F_x = 3x - 2y$ gives $F(x,y) = 3x^2/2 - 2xy + A(y)$, and then $F_y = 0 - 2x + A'(y) = -x$ gives $A'(y) = x$ and $A(y) = xy + B$. Then, $F(x,y) = 3x^2/2 - 2xy + xy + B =$ constant gives the general solution as $3x^2/2 - xy = C$." Grade that response, based on 10 points, and explain your grade.

13. Thermodynamics; the Entropy of an Ideal Gas. Consider an *ideal gas*, namely, a gas for which

$$pv = RT, \tag{13.1}$$

in which p is the pressure, v is the specific volume (i.e., the volume per mole), T is the absolute temperature, and R is the universal gas constant. The first law of thermodynamics for one mole of an ideal gas can be expressed in differential form as

$$\begin{aligned} dq &= pdv + c_v\,dT \\ &= RT\frac{dv}{v} + c_v(T)\,dT \end{aligned} \tag{13.2}$$

in which dq is the heat input and the known function $c_v(T)$ is the specific heat at constant volume. (Here, the independent variables are v and T rather than the generic x and y.) Show that the right-hand side of (13.2) is not an exact differential. Show that an integrating factor that is a function of v does not exist, but that an integrating factor that is a function of T does exist, $\sigma(T) = 1/T$, so that

$$\frac{dq}{T} = R\frac{dv}{v} + \frac{c_v(T)}{T}dT \tag{13.3}$$

is an exact differential, which we will call $ds(v,T)$; s is the *entropy* and we have just shown that it can be defined by the integral

$$s(v,T) = \int \frac{dq}{T}. \qquad (13.4)$$

The latter formula is fundamental in the study of thermodynamics.

1.8 SOLUTION BY SUBSTITUTION

The integral $I = \int \frac{x^2\,dx}{(x^3+5)^2}$ may look difficult, but with the substitution $u = x^3 + 5$ we obtain $I = \frac{1}{3}\int u^{-2}du$, which readily gives $I = -1/3u + C$. Finally, replace u by x^3+5 and obtain $I = -1/[3(x^3+5)] + C$. The same idea of *substitution* can be used to solve differential equations.

1.8.1 Bernoulli's equation.

The differential equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \qquad (1)$$

in which n is a constant (not necessarily an integer), is called **Bernoulli's equation** after the Swiss mathematician **James Bernoulli**.[1] If n is 0 or 1, then (1) is linear and readily solved, so our interest is in the case where n is neither 0 nor 1.

Following Leibniz, change the dependent variable from y to v by the substitution

$$v(x) = y(x)^{1-n}, \qquad (2)$$

keeping x as the independent variable. To substitute (2) into (1), we will need y and dy/dx in terms of v and dv/dx:

It is always important to be clear as to which variables are the independent and dependent variables.

$$y(x) = v(x)^{1/(1-n)} \quad \text{[from (2)]} \qquad (3a)$$

and

$$\frac{dy}{dx} = \frac{1}{1-n} v^{\left(\frac{1}{1-n}-1\right)} \frac{dv}{dx} \quad \text{[by chain differentiation of (3a)]} \qquad (3b)$$

[1]James (1654–1705), his brother John (1667–1748), and John's son Daniel (1700–1782) are the best known of the eight members of the Bernoulli family who were mathematicians and scientists. James proposed equation (1) as a challenge to the mathematicians of his day in 1695 and solved it himself in 1696. Other solutions were put forward by his brother John and by *Gottfried Leibniz* (1646–1716), and it is Leibniz's substitution method that we will discuss. The Bernoulli equation (1) is not related to the Bernoulli (energy) equation that one studies in a course in fluid mechanics.

so (1) becomes

$$\frac{1}{1-n}v^{n/(1-n)}\frac{dv}{dx}+p(x)\,v^{1/(1-n)}=q(x)\,v^{n/(1-n)}. \tag{4}$$

Finally, multiplying (4) by $(1-n)v^{-n/(1-n)}$ gives

$$\boxed{\frac{dv}{dx}+(1-n)p(x)\,v=(1-n)q(x).} \tag{5}$$

The upshot is that the substitution (2) works — in the sense that it reduces the nonlinear equation (1) to a simpler one, the *linear* equation (5). We can solve (5) for $v(x)$, then return from $v(x)$ to $y(x)$ by (3a).

First, put (6) in the form (1) by multiplying through by $1/x$.

EXAMPLE 1. Solve

$$x\frac{dy}{dx}-y=-2xy^2. \tag{6}$$

The latter is a Bernoulli equation with $p(x)=-1/x$, $q(x)=-2$, and $n=2$. Then, (2) gives $v=1/y$. Assuming that $y\neq 0$, for $v=1/y$ to be meaningful, (5) is the linear equation

$$\frac{dv}{dx}+\frac{1}{x}v=2, \tag{7}$$

with solution

$$v(x)=x+\frac{C}{x}. \tag{8}$$

But $v=1/y$, so (8) gives $1/y=x+C/x$, and hence

$$y(x)=\frac{x}{x^2+C} \tag{9}$$

is the solution of (6).

COMMENT. Since we assumed that $y\neq 0$, we must check $y=0$ separately. In fact, $y(x)=0$ does satisfy (6), and it cannot be obtained from (9) by any (finite) choice of C. Thus, besides the one-parameter family of solutions (9) we have the additional solution $y(x)=0$. ∎

The sequence of steps in solving a differential equation by the method of substitution is as follows: *Find a substitution, if possible, that converts the given differential equation to one we can solve. Make the substitution, obtain the new differential equation, solve it, and return to the original variables.*

1.8.2 Homogeneous equations. An equation $y' = f(x, y)$ is **homogeneous** if $f(x, y)$ can be expressed as a function of the ratio y/x alone, in which case we can express the equation as

$$\boxed{\frac{dy}{dx} = F\left(\frac{y}{x}\right).} \tag{10}$$

For instance,

$$\frac{dy}{dx} = \frac{2x + 2y}{3x + y} \tag{11}$$

is homogeneous because if we divide the numerator and denominator on the right-hand side by x we can express (11) as

$$\frac{dy}{dx} = \frac{2 + 2\frac{y}{x}}{3 + \frac{y}{x}} = F\left(\frac{y}{x}\right). \tag{12}$$

However, the equation

$$\frac{dy}{dx} = \frac{x + y + 2x^2}{x + 4y} = \frac{1 + \frac{y}{x} + 2x}{1 + 4\frac{y}{x}} \tag{13}$$

is *not* homogeneous because the right-hand side is a function of y/x and x, not of y/x alone.

CAUTION: Earlier, we defined a linear equation $y' + p(x)y = q(x)$ to be homogeneous if the forcing function $q(x)$ is zero. That was a different use of the word homogeneous and is not relevant in the present discussion.

If the equation is homogeneous, of the form (10), it seems natural to let y/x be a single variable, say v, so $v = y/x$. That is, $v(x) = y(x)/x$, or

$$\boxed{y(x) = xv(x).} \tag{14}$$

To put (14) into (10) we need dy/dx, so differentiate (14):

$$\frac{dy}{dx} = v(x) + x\frac{dv}{dx}. \tag{15}$$

Using (14) and (15), equation (10) becomes

$$v + x\frac{dv}{dx} = F(v)$$

or

$$\boxed{x\frac{dv}{dx} = F(v) - v,} \tag{16}$$

which is simple because it is separable. Thus,

$$\boxed{\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}.} \tag{17}$$

As discussed in Section 1.4, if v_0 is any root of $F(v) - v = 0$ then $v(x) = v_0$ is an additional solution of (16), in addition to the solutions found from (17). This point will come up, below, in COMMENT 2.

Evaluate the integrals in (17) and, in the result, replace v by y/x.

EXAMPLE 2. Solve the homogeneous equation (11),

$$\frac{dy}{dx} = \frac{2x+2y}{3x+y} = \frac{2+2(y/x)}{3+(y/x)} = F\left(\frac{y}{x}\right). \tag{18}$$

In this case $F(v) = \dfrac{2+2v}{3+v}$, so (16) is

$$\begin{aligned} x\frac{dv}{dx} &= \frac{2+2v}{3+v} - v \\ &= -\frac{v^2+v-2}{3+v}. \end{aligned} \tag{19}$$

If $v^2+v-2 \neq 0$, separation of variables (and partial fractions) gives

$$-\int \frac{(3+v)\,dv}{v^2+v-2} = \int \frac{dx}{x}, \tag{20a}$$

$$\frac{1}{3}\int \frac{dv}{v+2} - \frac{4}{3}\int \frac{dv}{v-1} = \int \frac{dx}{x}, \tag{20b}$$

$$\frac{1}{3}\ln|v+2| - \frac{4}{3}\ln|v-1| = \ln|x| + A, \tag{20c}$$

$$\ln\left|\frac{(v+2)^{1/3}}{x(v-1)^{4/3}}\right| = A, \tag{20d}$$

$$\left|\frac{(v+2)^{1/3}}{x(v-1)^{4/3}}\right| = e^A, \tag{20e}$$

$$\frac{(v+2)^{1/3}}{x(v-1)^{4/3}} = \pm\, e^A \equiv B, \tag{20f}$$

$$(v+2) = B^3x^3(v-1)^4. \tag{20g}$$

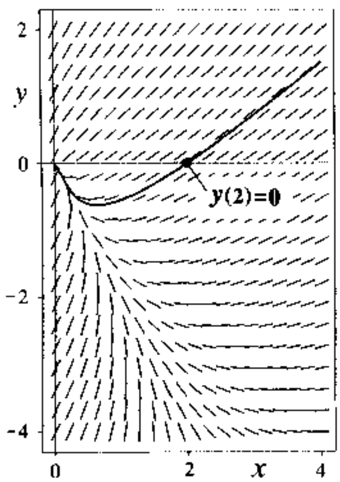


Figure 1. Solution of (18) with the initial condition $y(2) = 0$, together with the direction field.

Putting $v = y/x$, renaming B^3 as C and simplifying gives the solution in implicit form as

$$y + 2x = C(y-x)^4. \tag{21}$$

If we have an initial condition $y(2) = 0$, for instance, (21) gives $C = 1/4$. The corresponding solution, and the direction field, are plotted in Fig. 1.

COMMENT 1. Is C arbitrary in (21)? We need to track A, B, C in (20). In (20c), A is arbitrary ($-\infty < A < \infty$). Consequently, $B = \pm\, e^A$ is arbitrary but nonzero, because $e^A \neq 0$. Finally, since B is arbitrary but nonzero and $C = B^3$, then C is arbitrary but nonzero in (21). Conclusion: C in (21) is arbitrary but nonzero.

COMMENT 2. Besides the family of solutions given by (21), we must see if there are any additional solutions from the roots of $v^2 + v - 2 = 0$, which we assumed was nonzero when we proceeded from (19) to (20a). The roots are $v = 1$ and $v = -2$, which, recalling that $v = y/x$, correspond to straight-line solutions $y = x$ and $y = -2x$ of (18). Of these, $y = -2x$ is contained within (21) if we allow $C = 0$, but $y = x$ cannot be obtained from (21) by any finite choice of C. We conclude that the solutions of (18) are those defined implicitly by (21) (with $-\infty < C < \infty$), plus the line $y = x$. It is tempting to not fuss with such details as whether C in (21) is arbitrary, or arbitrary but nonzero, and whether there are any additional solutions from the roots of $v^2 + v - 2 = 0$, but if we did not fuss with those details we would have missed the solution $y = x$. That loss would be fatal if an initial condition were prescribed on that line.

A number of solutions of (18) are plotted in Fig. 2. ∎

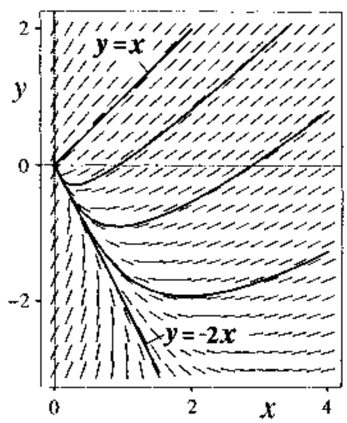


Figure 2. Representative solutions of (18), together with the direction field.

Closure. In earlier sections we attacked the differential equation $y' = f(x, y)$ by considering only special cases that are tractable: equations that are linear, separable, or exact (or can be made exact by an integrating factor that is a function of x or y). Those cases by no means cover all possible equations $y' = f(x, y)$, but they do cover a great many equations that arise in applications. Similarly, for the method of substitution one tries to develop substitutions that work for various types of equations. In this section we considered only two: Bernoulli equations and homogeneous equations. Additional types are included in the exercises.

EXERCISES 1.8

1. Bernoulli Equations. Show that the equation is of Bernoulli type; that is, identify $p(x)$, $q(x)$, and n. Then, solve it accordingly. If the equation happens to also be separable you can solve it by separation of variables as well, if you wish, to check your results.

(a) $xy' - 2y = x^3y^2$ (b) $y' + 2y = -6e^{3x}y^{3/2}$
(c) $(1+x)y' + 2y = 2\sqrt{y}$ (d) $xy' = y - xy^2$
(e) $\sqrt{y}\,(3y' + y) = x$ (f) $y' = y^2$
(g) $y' = x\sqrt{y}$ (h) $xy' - y = -12x^3y^2$

2. Inventing Leibniz's Substitution. In case you regard a substitution such as Leibniz's substitution (2) as a "miracle," let us illustrate how that this (and other substitutions) might be developed in a reasonable and systematic way. To begin, we observe that the difficulty with (1) is the y^n term. Thus, try letting y^n be a new variable, for surely the substitution $v = y^n$ will simplify the right-hand side of (1). But it is possible that while simplifying the right-hand side it might complicate the left-hand side.

(a) Try it. Let $v = y^n$ and show that it does not work.

(b) Not discouraged, try $v = y^r$ instead, where this time the exponent r is not prescribed in advance. Make that substitution in (1) and choose r, if possible, so that the equation for $v(x)$ is simple, such as linear or separable. Show that these steps lead to the choice $r = 1 - n$ and hence to Leibniz's substitution (2).

3. Homogeneous. (a) Solve $(2x - y)y' = x - 2y$ and find a particular solution for each of the initial conditions $y(2) = 0$, $y(2) = 4 - 2\sqrt{3}$, $y(2) = 1$, $y(2) = 7$, $y(2) = 4 + 2\sqrt{3}$, $y(2) = 8$ and determine its interval of existence. Obtain computer-generated graphs of those solutions.
(b) Solve $y' = (4x^2 + 3y^2)/2xy$ with the initial condition $y(1) = 2$, and determine the interval of existence.
(c) Solve $y' = (xy + 2y^2)/x^2$ $(x > 0)$, find a particular solution satisfying $y(1) = 2$, and determine its interval of existence. NOTE: The differential equation is both homogeneous and a Bernoulli equation. Solve it both ways and show that your results are the same.
(d) Solve $y' = e^{y/x} + y/x$ $(x > 0)$ with the initial condition $y(1) = 0$, and determine the solution's interval of existence.
(e) Solve $y' = \tan(y/x) + y/x$ $(x > 0)$ with the initial condition $y(1) = \pi/6$, and determine the solution's interval of existence.

4. Almost Homogeneous. The equation

$$y' = \frac{ax + by + c}{dx + ey + f}, \tag{4.1}$$

in which $a, b, \dots, f$ are constants, is "almost homogeneous" in the sense that it would be homogeneous if c and f were not present.

(a) Change variables from $x, y(x)$ to $X, Y(X)$ according to the "translation"

$$x = X + h, \quad y = Y + k, \tag{4.2}$$

and choose the constants h and k so as to knock out the c and the f. Show that the result is

$$\frac{dY}{dX} = \frac{aX + bY}{dX + eY}, \tag{4.3}$$

with $h = (bf - ce)/(ae - bd)$ and $k = (cd - af)/(ae - bd)$, provided that $ae - bd \neq 0$.
(b) Use the idea in part (a) to solve $y' = (y+1)/(2x - y - 3)$. (This problem is continued in Exercise 10.)
(c) Similarly, solve $y' = (x - y - 4)/(x + y - 4)$.
(d) Similarly, solve $y' = (y + x + 2)/(y - x)$.
(e) As stated in part (a), the change of variables (4.2) fails if $ae - bd = 0$. Devise a substitution that *will* work in that case, and use it to solve $y' = (2x + 4y + 1)/(4x + 8y - 2)$.

5. The equation $y' = x^3 e^{y/x} + y/x$ is not homogeneous because of the x^3. Show that the change of variables $v = y/x$ from $y(x)$ to $v(x)$ works nevertheless, and use it to solve for $y(x)$.

6. To solve equations of the form

$$\boxed{y' = f(ax + by)} \tag{6.1}$$

it seems reasonable to try the substitution $v = ax + by$; that is,

$$v(x) = ax + by(x). \tag{6.2}$$

(a) Show that (6.2) simplifies (6.1) to the separable equation

$$v' = bf(v) + a. \tag{6.3}$$

Use this idea to solve the following equations.

(b) $y' = (2x + y)^2 - 2$
(c) $(x - y)y' = 4$
(d) $y' = (x + y + 2)/(x + y)$

7. Riccati's Equation. The equation

$$\boxed{y' = f(x, y); \quad y(a) = b} \tag{7.1}$$

is called **Riccati's equation** after the Italian mathematician *Jacopo Francesco Riccati* (1676–1754). The latter is made difficult by the y^2 term, which makes the equation nonlinear [unless $p(x)$ is zero]. The key to being able to solve a Riccati equation is to find any one solution, hopefully by inspection. For suppose $y = \eta(x)$ is any particular solution of (7.1). Then, show that by changing the dependent variable from y to v according to

$$y = \eta(x) + \frac{1}{v} \tag{7.2}$$

the Riccati equation (7.1) is converted to

$$v' + \left[2p(x)\eta(x) + q(x)\right] v = -p(x), \tag{7.3}$$

which is *linear*. This solution method was discovered by *Leonhard Euler* (1707–1783) in 1760. Note that if $r(x) = 0$ in (7.1), then an obvious choice for a particular solution is simply $y = \eta(x) = 0$, although in that case (7.1) is also a Bernoulli equation.

8. Read Exercise 7 and use the method described there to solve the following Riccati equations. HINT: In part (b) try $\eta(x)$ in the form ax and determine an a that works.

(a) $y' = 4y + y^2$ (b) $y' = y^2 - 2xy + 1 + x^2$
(c) $y' = e^{-x}y^2 - y$ (d) $y' = x^2y^2 - y$

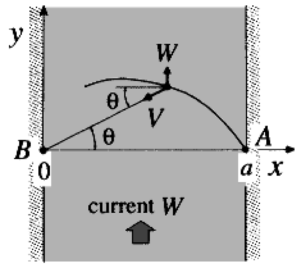
9. Smorgasbord. You don't need to solve these. Just identify any solution method that will work and give any substitutions that will be needed. You may find ideas in the preceding exercises.

(a) $xy' = \sqrt{xy} + y$ (b) $y' = (y + x)^3$
(c) $(2x - y)y' = y$ (d) $y' = x(y - x)^2 + 1$
(e) $(xy)' = y^{3.2}$ (f) $y' = x^3y^2 - y$
(g) $y^2y' = (x + y)^2$ (h) $yy' = x - y$
(i) $y' - y^2 = 3ye^x$ (j) $y' = 4 + (y - 5x)^4$

10. Continuation of Exercise 4(b). For the differential equation in Exercise 4(b), use computer graphics to plot representative solution curves, being sure to include the two straight line solutions.

ADDITIONAL EXERCISES

11. Swimming in a Current. A river is a miles wide and has a current W mi/hr. Two swimmers, Maifeng and Yuan, are to race, from point A on one bank to point B on the opposite bank (see the figure). They are equally fast, able to swim



with a constant speed V relative to the water, so the outcome will be determined by strategy. Yuan elects to swim so as to always aim at the destination B, as indicated in the figure, and Maifeng elects to aim upstream so as to swim straight across from A to B. Assume that they swim faster than the current, so $V > W$.

(a) Compute and compare the crossing time T for each of them. The calculation for Yuan is the harder so we'll get you started. Yuan's x and y velocity components are

$$x' = -V\cos\theta = -V\frac{x}{\sqrt{x^2+y^2}}, \tag{11.1a}$$

$$y' = W - V\sin\theta = W - V\frac{y}{\sqrt{x^2+y^2}}, \tag{11.1b}$$

and dividing (11.1b) by (11.1a) gives

$$\frac{dy}{dx} = \frac{y - r\sqrt{x^2+y^2}}{x} \tag{11.2}$$

with $r = W/V$. Solve (11.2) and show that the path traversed by Yuan is

$$y(x) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-r} - \left(\frac{x}{a}\right)^{1+r}\right]. \tag{11.3}$$

Put (11.3) into (11.1a) and get a differential equation for $x(t)$ alone. Solve that equation for $x(t)$, and compute Yuan's crossing time. Then compute Maifeng's time — which should be a much simpler calculation. You should obtain the two times

$$T = \frac{a}{V}\frac{1}{\sqrt{1-r^2}} \quad \text{and} \quad \frac{a}{V}\frac{1}{1-r^2}, \tag{11.4}$$

but we leave it for you to determine which one corresponds to Yuan and which one to Maifeng. As a partial check on (11.4) show that both results are correct in the limit as $r \to 0$.

(b) Show that Maifeng wins if $W < V$ $(r < 1)$, as assumed, but that if $W \geq V$ $(r \geq 1)$ then neither swimmer can reach point B.

NOTE: This problem is based on one given in Ralph Palmer Agnew's dated-but-still-excellent text *Differential Equations* (NY: McGraw Hill, 1942).

1.9 NUMERICAL SOLUTION BY EULER'S METHOD

The preceding sections have been devoted mostly to analytical solution methods: for instance, solving the linear equation $y' + p(x)y = q(x)$ by using an integrating factor, the use of separation of variables, exactness, and so on. Our use of direction fields and phase line analysis in those sections has been more qualitative. Here, we complement those analytical and qualitative approaches with a brief introduction to *quantitative* methods. By a quantitative method we mean one that uses a numerical algorithm to generate the solution numerically, approximately and only at discrete points, the calculations normally being carried out on a computer rather than by hand.

1.9.1 Euler's method. We consider the IVP

$$\boxed{y' = f(x,y); \quad y(a) = b} \tag{1}$$

Our chief interest is the case in which $y' = f(x,y)$ is nonlinear, because we have an exact solution for the linear equation $y' + p(x)y = q(x)$.

for $y(x)$.

To motivate the simplest numerical solution method, Euler's method, consider a specific example,

$$y' = y + 2x - x^2; \quad y(0) = 1 \quad (0 \leq x < \infty) \tag{2}$$

which has the exact solution (Exercise 1)

$$y(x) = x^2 + e^x. \tag{3}$$

Of course, we don't *need* to solve (2) numerically because we know its solution, given by (3), but our aim here is only to illustrate the method. In fact, it is good to begin with a problem for which we do know the exact solution because then we can compare our numerical results with that solution.

In Fig. 1 we display the direction field defined by $f(x, y) = y + 2x - x^2$, and the exact solution $y(x)$ given by (3). In graphical terms, Euler's method amounts to using the direction field as a "road map" to develop an approximate solution to (2) in a step-by-step manner. Beginning at the initial point P, namely $(0, 1)$, we strike out in the direction dictated by the lineal element at that point. As seen from the figure, the farther we move along that line, from the starting point P to a stopping point Q, the more we can expect our path to deviate from the exact solution. Thus, the idea is not to move very far. Stopping at $x = 0.5$, for the sake of illustration, we then revise our direction according to the slope of the lineal element there, at Q. Moving in that new direction until $x = 1$, we revise our direction again at R, and so on, in x increments of 0.5.

Thus, our strategy is this: First, **discretize** the problem (1) by seeking the solution $y(x)$ not everywhere on the x interval, but only at discrete points $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_1 + h$, and so on, where h is our chosen **step size** for the calculation; in Fig. 1, $h = 0.5$. That is, rather than seek the function $y(x)$ we seek only the discrete approximate values $y_1, y_2, \ldots$ at $x_1, x_2, \ldots$, respectively. According to

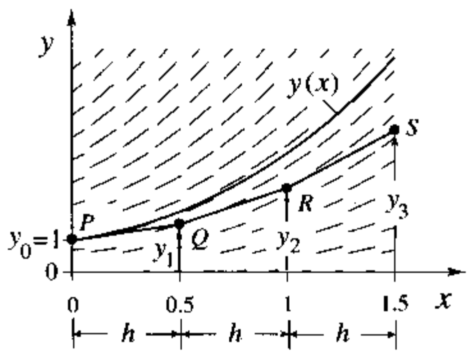


Figure 1. Direction field motivation of Euler's method, for the initial value problem (2); "going with the flow."

the algorithm known as **Euler's method**, we compute y_{n+1} as the preceding value y_n plus the slope $f(x_n, y_n)$ at (x_n, y_n) times the step size h:

$$y_{n+1} = y_n + f(x_n, y_n)h, \qquad (n = 0, 1, 2, \dots) \tag{4}$$

From Fig. 1, we see that the values y_1, y_2, ... generated by the Euler algorithm (4) do not, in general, fall on the solution curve and are only approximate.

in which f is the function on the right-hand side of the given differential equation (1), $x_0 = a$, $y_0 = b$, h is the chosen step size, and $x_n = x_0 + nh$.

Euler's method is also known as the **tangent-line method** because each straight-line segment of the approximate solution, emanating from (x_n, y_n), is tangent to the exact solution curve through that point.

We will illustrate the calculation, using the IVP (2).

EXAMPLE 1. Application of Euler's Method to (2). In (2), $f(x, y) = y + 2x - x^2$, $x_0 = 0$, and $y_0 = 1$. With $h = 0.5$, Euler's method (4) gives

$$y_1 = y_0 + \left(y_0 + 2x_0 - x_0^2\right) h = 1 + (1 + 0 - 0)(0.5) = 1.5,$$
$$y_2 = y_1 + \left(y_1 + 2x_1 - x_1^2\right) h = 1.5 + (1.5 + 1 - 0.25)(0.5) = 2.625,$$
$$y_3 = y_2 + \left(y_2 + 2x_2 - x_2^2\right) h = 2.625 + (2.625 + 2 - 1)(0.5) = 4.4375,$$

and so on. ∎

We can see from Example 1 that Euler's method is simple and readily implemented, even by hand calculation.

Evidently, the greater the step size the less accurate the results, in general. For instance, we can see that the first point Q in Fig. 1 deviates more and more from the exact solution as the step size is increased — that is, as the segment PQ is extended. Conversely, we expect the approximate solution to approach the exact solution curve as $h \to 0$. We are encouraged in this expectation by the results shown in Table 1 for the IVP (2), obtained by Euler's method with step sizes $h = 0.5, 0.1$, and 0.02; we've included the exact solution given by (3) in the final column, for comparison. To keep the tabulation short, many intermediate y_n and $y(x)$ values were omitted for the cases $h = 0.1$ and $h = 0.02$. For instance, for $h = 0.1$ we omitted y_1 through y_4, y_6 through y_9, and y_{11} through y_{14}.

Table 1. Comparison of numerical solution of (2) using Euler's method, with the exact solution (3).

x	$h = 0.5$	$h = 0.1$	$h = 0.02$	Exact $y(x)$
0	$y_0 = 1$	$y_0 = 1$	$y_0 = 1$	$y(0) = 1$
0.5	$y_1 = 1.5$	$y_5 = 1.7995$	$y_{25} = 1.8778$	$y(0.5) = 1.8987$
1.0	$y_2 = 2.625$	$y_{10} = 3.4344$	$y_{50} = 3.6578$	$y(1.0) = 3.7183$
1.5	$y_3 = 4.4375$	$y_{15} = 6.1095$	$y_{75} = 6.5975$	$y(1.5) = 6.7317$

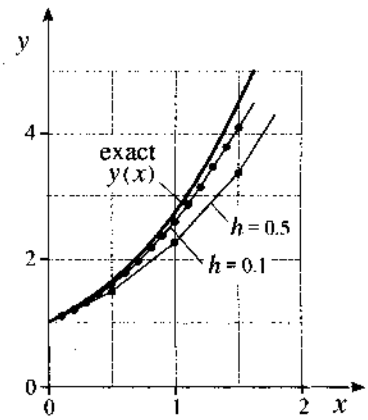


Figure 2. Plot of the results reported in Table 1. Note that the approximate solution approaches the exact solution as h is decreased.

If we scan across each of the bottom three rows of the tabulation, we see that the approximate solution values do appear to be converging to the exact solution as $h \to 0$ (though we cannot be certain of that, no matter how much we reduce h). Besides tabulating the results, above, we have also plotted them in Fig. 2 for $h = 0.5$ and 0.1.

1.9.2 Convergence of Euler's method. Two questions follow: Does the method really give convergence to the exact solution as $h \to 0$ and, if so, how fast? By the method being **convergent** we mean that *for any fixed x value in the x interval of interest the sequence of y values, obtained from (4) using smaller and smaller step size h, tends to the exact solution $y(x)$ of (1), at that point x, as $h \to 0$.*

In Table 1, this amounts to moving from left to right across a given row. Understand this definition.

There are two sources of error in the numerical solution. One is the tangent-line approximation upon which the method is based, and the other is an accumulation of numerical **roundoff errors** that result from the machine carrying only a finite number of significant figures, after which it rounds (or chops) off. Typically, roundoff errors are negligible, and they will be ignored in this discussion.

The single-step error. Although we are interested in the accumulation of error after many steps have been carried out, to reach a given x, it is logical to begin by investigating the error incurred *in a single step*, from x_n to x_{n+1}.

Notation: We denote the exact solution at x_n as $y(x_n)$ and the numerical solution there as y_n.

To distinguish between the exact and the approximate solutions, we will denote the exact solution at any computation point x_n as $y(x_n)$, and the approximate numerical solution there by a subscript notation, as y_n.

If we start out at x_n with the correct value, so $y(x_n) = y_n$, what is the error $y(x_{n+1}) - y_{n+1}$ after that one step? Write expressions for $y(x_{n+1})$ and y_{n+1}, and then subtract them. First, the Taylor series expansion of y about x_n gives

$$\begin{aligned} y(x_{n+1}) &= y(x_n) + y'(x_n)(x_{n+1} - x_n) + \frac{y''(x_n)}{2!}(x_{n+1} - x_n)^2 + \cdots \\ &= y(x_n) + f(x_n, y(x_n))h + \frac{y''(x_n)}{2!}h^2 + \cdots \\ &= y_n + f(x_n, y_n)h + \frac{y''(x_n)}{2!}h^2 + \cdots . \end{aligned} \tag{5}$$

In the final equality in (5) we use the fact that we begin our step, at x_n, with y_n equal to the exact solution there, $y(x_n)$.

Next, the Euler algorithm gives

$$y_{n+1} = y_n + f(x_n, y_n)h, \tag{6}$$

and if we subtract (6) from (5), we obtain the **single-step error** (SSE), as

$$\text{SSE} = y(x_{n+1}) - y_{n+1} = \frac{y''(x_n)}{2!}h^2 + \frac{y'''(x_n)}{3!}h^3 + \cdots \sim \frac{y''(x_n)}{2}h^2 \tag{7}$$

The single-step error, denoted here as SSE, is the error at x_{n+1} if $y(x_n) = y_n$.

as $h \to 0$. We are interested in how fast SSE $\to 0$ as $h \to 0$ and (7) tells us that it tends to zero proportional to h^2 so we write, more simply than (7),

$$\boxed{\text{SSE} = O(h^2).} \tag{8}$$

According to (8), the SSE for Euler's method is of order h^2.

In words, (8) means that the single-step error SSE tends to zero proportional to h^2. That is, SSE $\sim Ch^2$ for some nonzero constant C, as $h \to 0$. The notation used in (8) is the standard **big oh notation** used to express *order of magnitude*. Read (8) as follows: SSE is of order h^2 as $h \to 0$.

The error and convergence. Of ultimate interest, however, is not the error incurred in a single step, but the error that accumulates over *all* the steps (from x_0 to any given point x_N). That error is the difference between the exact solution and the computed solution at x_N, and we will simply call it the error E:

$$E \equiv y(x_N) - y_N. \tag{9}$$

Let us illustrate, using Table 1. For $h = 0.02$, for instance, the error at $x = 1$ is $y(1) - y_{50} = 3.7183 - 3.6578 = 0.0605$, and at $x = 1.5$ it is $y(1.5) - y_{50} = 6.7317 - 6.5975 = 0.1342$.

We can estimate E, at least insofar as its order of magnitude, as the single-step error SSE times the number of steps N. Since SSE $= O(h^2)$, that idea gives

$$\begin{aligned} E = O(h^2) \cdot N = O(h^2)\frac{Nh}{h} &= O(h^2)\frac{x_N - x_0}{h} \\ &= O(h)(x_N - x_0) = O(h). \end{aligned} \tag{10}$$

Regarding the last step in (10), the big oh notation is insensitive to nonzero scale factors. For instance, both $\sin x = x - \frac{x^3}{3!} + \cdots = O(x)$ and $\sin 8x = 8x - \frac{8^3 x^3}{3!} + \cdots = O(x)$, as $x \to 0$. We do not distinguish between $O(8x)$ and $O(x)$.

In the last equality in (10) we absorbed the $x_N - x_0$ factor into the $O(h)$ because $x_N - x_0$ is simply a constant, and the big oh notation is insensitive to (nonzero) constant scale factors. Thus,

$$\boxed{E = O(h),} \tag{11}$$

which tells us how fast the numerical solution converges to the exact solution, at any fixed x location, as $h \to 0$: $E \sim Ch$, as $h \to 0$, for some constant C. It tends to zero proportional to h.

Our steps in (10) were formal, not rigorous, but the result (11) is correct and indicates that the Euler method (4) is convergent because $E = O(h)$ does tend to zero as $h \to 0$. More generally, if, for a given method, $E = O(h^p)$ as $h \to 0$, then the method is convergent if $p > 0$ (because $h^p \to 0$ as $h \to 0$ if $p > 0$), and the method is said to be **of order p**. With $p = 1$ in (11), we see that **Euler's method is a first-order method**.

The order of the method and the order of the differential equation are distinct and unrelated.

How do we know how small to choose h in a given application? As a rule of thumb, repeat the calculation, reducing h until the results settle down to the desired accuracy. The foregoing is the same idea normally used in summing an infinite series, adding more terms until successive partial sums settle down to the desired number of significant figures.

1.9.3 Higher-order methods. Though convergent and readily implemented, Euler's method may be too inaccurate because it is only a first-order method. That is, since the error at any given x point is proportional to h to the first power, we must make h extremely small if the error is to be extremely small, and if h is extremely small then the number of computational steps is extremely large. Consequently, it is common to favor higher-order methods.

If a method is of order p, then $E = O(h^p)$ as $h \to 0$ or, equivalently,

$$E(h) \sim Ch^p \tag{12}$$

as $h \to 0$.

There are higher-order methods available, such as the **Runge–Kutta methods** of orders 2, 3, and 4, which are abbreviated as rk2, rk3, and rk4, respectively. As their names indicate, they are of orders $p = 2, 3$, and 4, respectively. We will not give those methods here, but merely highlight the dramatic increase in accuracy afforded by such higher-order methods, in Table 2. There, we tabulate the values of the numerical solution of the IVP (2), obtained using the first-order Euler method and the second- and fourth-order Runge–Kutta methods, at $x = 1$, for step sizes of $h = 0.1$ and 0.01. We've tabulated the first eleven digits and have indicated by bold fonts as many digits as agree with those in the exact solution. We see from

The accuracy increases if for a given method we decrease h, or if for a given h we turn to higher-order methods.

Table 2. Increasing the order of the method.

	Euler	rk2	rk4
$h = 0.1$	**3.**4343682140	**3.7**059185568	**3.71827**63403
$h = 0.01$	**3.**6877656911	**3.718**1513781	**3.718281827**8
Exact $y(1)$	**3.7182818285**	**3.7182818285**	**3.7182818285**

the table that the accuracy of fourth-order methods is quite impressive. In fact, rk4 is one of the most widely used methods.

Closure. The Euler method is given by (4). It is readily implemented, either using a hand-held calculator or running it on a computer. The method is convergent but is only of first order, so it is generally not very accurate unless the step size h is made extremely small. Thus, for "serious computation," higher-order methods such as rk4 are normally used and are available within CAS software. Of course, in a given application we may not even have a very accurate differential equation model in the first place and/or may not know the physical parameters very accurately, in which case there may be little point in demanding extreme accuracy from the numerical algorithm. By the way, when studying the calculus, one is not in a position to appreciate the many important applications of Taylor series. They are invaluable in developing numerical solution methods for differential equations, as we begin to see here in Section 1.9.2.

EXERCISES 1.9

1. Filling in a Gap. Derive the particular solution (3) of the IVP (2).

2. By Hand. Use the Euler method to compute, by hand, y_1, y_2, and y_3 for the specified IVP using $h = 0.2$.

(a) $y' = -y;\quad y(0) = 1$
(b) $y' = 2xy;\quad y(0) = 4$
(c) $y' = 1 + 2xy^2;\quad y(-1) = 2$
(d) $y' = 2xe^y;\quad y(1) = -1$
(e) $y' = x^2 - y^2;\quad y(3) = 0$
(f) $y' = x\sin y;\quad y(0) = 1$
(g) $y' = 5y - 2\sqrt{y};\quad y(1) = 3$
(h) $y' = \sqrt{x+y};\quad y(0) = 3$

3. By Computer. Use computer software to solve the given IVP for $y(x)$ by Euler's method, with a step size of 0.001. Print your Euler-computed values y_{100}, y_{200}, and y_{300} (i.e., at $x = 1, 2, 3$) as well as the exact solution (which is given) at those same points, for comparison.

(a) $y' = -y,\ y(0) = 1;\ y(x) = e^{-x}$
(b) $y' = 0.1(x+1)e^y,\ y(0) = 0;\ y(x) = \ln\dfrac{20}{20 - 2x - x^2}$
(c) $y' = x^2 - y,\ y(0) = 2;\ y(x) = x^2 - 2x + 2$
(d) $y' = 0.01xy^2,\ y(0) = 5;\ y(x) = 200/(40 - x^2)$
(e) $y' = x^2/y^2,\ y(0) = 4;\ y(x) = (x^3 + 64)^{1/3}$
(f) $y' = \dfrac{y}{x-5} - 2y^2,\ y(0) = 1;\ y(x) = \dfrac{x-5}{x^2 - 10x - 5}$
(g) $y' = -y^2,\ y(0) = -0.2;\ y(x) = 1/(x-5)$

4. (a)–(g) **Convergence.** For the corresponding part of Exercise 3, use computer software to solve the given IVP for $y(x)$ at $x = 3$. Obtain computed values of $y(3)$ using step sizes of 0.1, 0.01, 0.001, and 0.0001, as well as the exact value of $y(3)$. Do your results appear to be consistent with Euler's method being a first-order method? Explain.

5. Negative Steps? Thus far we've taken the step h to be positive, and therefore we've developed solutions to the right of the initial point. Is Euler's method valid if we use a negative step, $h < 0$, and develop a solution to the left? Explain.

6. Variable Step Size? In this section we've taken the step size h to be a constant from one step to the next. Is there any reason why we could not vary h from one step to the next? Why might we *want* to use a variable step size? Explain.

7. (a) What is meant by discretizing an IVP?
(b) What is meant by a method being convergent?

8. Verifying Convergence for a Simple Example. For the simple IVP

$$y' = Ay;\quad y(0) = y_0, \tag{8.1}$$

in which A is a constant, we can actually show that its Euler solution converges to the exact solution $y(x) = y_0e^{Ax}$ as $h \to 0$. For (8.1), Euler's method gives

$$y_{n+1} = y_n + hAy_n = (1 + Ah)y_n;\quad y_0 \text{ given} \tag{8.2}$$

(a) Show that (8.2) gives the solution for y_n as

$$y_n = (1 + Ah)^n y_0. \tag{8.3}$$

(b) To see if the latter converges to the exact solution, consider a *fixed* point x. To arrive at x after n steps we must choose h to be x/n so (8.3) becomes

$$y_n = \left(1 + \frac{Ax}{n}\right)^n y_0. \tag{8.4}$$

With x fixed, h tending to zero corresponds to n tending to infinity. Show that the limit of (8.4) as $n \to \infty$ is indeed the exact solution y_0e^{Ax}. HINT: Recall from the calculus that

$$\lim_{h\to 0}\left(1 + \frac{1}{h}\right)^h = e.$$

9. Empirical Determination of the Order of Euler's Method. Suppose we do not know Euler's method is a first-order method, and want to determine its order empirically. Make up a simple "test equation" such as $y' = -y$, with initial condition $y(0) = 1$, so the known exact solution is $y(x) = e^{-x}$. Next, suppose we solve that IVP by Euler's method, for various h's, each time computing the solution at $x = 1$, at which point the known exact solution is $y(1) = 0.3678794412$. For $h = 0.1$, 0.01, 0.001, and 0.0001 the results at $x = 1$ are: $y|_{h=0.1} = 0.3486784401$, $y|_{h=0.01} = 0.3660323413$, $y|_{h=0.001} = 0.3676954248$, and $y|_{h=0.0001} = 0.3678610464$.

(a) From the computed values of y at $x = 1$ for $h = 0.1$ and $h = 0.01$, and the exact value of $y(1)$, use (12) to solve for p. That is, write $0.3678794412 - 0.3486784401 = C(0.1)^p$ and $0.3678794412 - 0.3660323413 = C(0.01)^p$, and solve those two equations for p. Show that $p = 1.01683$.
(b) Repeat the empirical evaluation of p, this time using the results for $h = 0.01$ and $h = 0.001$, and show that $p = 1.00163$.
(c) Finally, repeat the evaluation of p again, this time using

the results for $h = 0.001$ and $h = 0.0001$, and show that $p = 1.00016$.
NOTE: The results for p in parts (a), (b), (c) are not exact, but only approach the exact value $p = 1$ as $h \to 0$ because (12) is only an asymptotic result, valid as $h \to 0$.

10. Formula for Empirical Determination of the Order of Any Given Method. Exercise 9 pertained specifically to Euler's method. More generally, suppose we are using any numerical solution algorithm and want to determine its order empirically, for instance as a check against possible programming errors that would no doubt reduce its order. To do so, consider a simple test equation such as $y' = -y$ with initial condition $y(0) = 1$ and exact solution $y(x) = e^{-x}$. Suppose we use our algorithm to solve that problem, for two different h's, h_1 and h_2, obtaining the values $y|_{h_1}$ and $y|_{h_2}$, respectively, at $x = 1$. Then the errors are $E(h_1) = y(1) - y|_{h_1}$ and $E(h_2) = y(1) - y|_{h_2}$, respectively. From (12), show that

$$p \approx \frac{\ln\left[\dfrac{E(h_1)}{E(h_2)}\right]}{\ln\left[\dfrac{h_1}{h_2}\right]}. \tag{10.1}$$

11. Use of (10.1), Above, for Some Other Methods. First, read Exercise 10. A few well known higher-order methods are as follows: the **improved Euler method** (Heun's method), the fourth-order Runge–Kutta method **rk4**, and the Fehlberg fourth-fifth order Runge–Kutta method **rkf4-5**. Suppose we use each of these methods, which are available in CAS systems, to solve the test problem given in Exercise 10.

(a) If, at $x = 1$, the improved Euler method gives $y|_{h=0.001} = 0.367879502531$ and $y|_{h=0.0001} = 0.3678794417846$, use (10.1) to evaluate the order of the method. You will need to know the exact solution at $x = 1$, which is $y(1) = e^{-1} = 0.367879441171$.
(b) If the rk4 method gives $y|_{h=0.1} = 0.367879774412$ and $y|_{h=0.05} = 0.367879461148$, use (10.1) to evaluate the order of the method. (Of course, your answer should be close to 4.)
(c) If the rkf4-5 method gives $y(1)|_{h=0.1} = 0.367879437559$ and $y|_{h=0.05} = 0.367879441063$, use (10.1) to evaluate the order of the method.

ADDITIONAL EXERCISES

12. Convergence Theorem and Error Bound for Euler's Method. More informative than the formula $E = O(h)$ is the following:

THEOREM 1.9.1. Let y_n be the approximate solution of

$$y' = f(x, y); \quad y(a) = b \tag{12.1}$$

by Euler's method (4), and let $y(x)$ denote the exact solution. If $y'' = f_x + f_y y' = f_x + f_y f$ is continuous on the interval I of interest, $a \le x \le X$, and there are constants M and N such that

$$|f_y| \le M \text{ and } |f_x + f_y f| \le N \tag{12.2}$$

on I, then the error $e_n = y(x_n) - y_n$ at any *fixed* point $x_n = a + nh$ in I is bounded as follows:

$$|e_n| \le \frac{N}{2M}\left(e^{(x_n - a)M} - 1\right)h. \tag{12.3}$$

Before continuing, you may be wondering how the "x_n" in Theorem 1.9.1 can be a "fixed point," since it appears to increase with n. The idea is that once we choose a point x_n we keep it fixed by decreasing h and increasing n so that the nh in $x_n = a + nh$ remains constant. To continue, (12.3) is of the form $|e_n| \le Ch$ in which C is a constant, so $e_n \to 0$ at the fixed point x_n, as $h \to 0$, and it follows that Euler's method is indeed convergent.
Here is the problem: To illustrate (12.3), consider the Euler solution of the IVP (2), on $0 \le x \le 1.5$, for which results were given in Table 1. Use (12.3) to obtain a bound on the error at the fixed point $x_n = 1.5$, for each of the three h's used (namely, 0.5, 0.1, and 0.02), and verify that the actual errors (determined from Table 1) are indeed consistent with those bounds. You may use the fact that the solution is known to be $y(x) = x^2 + e^x$. HINT: Show that you can take $M = 1$ and $N = 2 + e^{1.5}$. Thus, show that (12.3) gives $|e_n| \le 11.284h$ at $x_n = 1.5$. NOTE: For a derivation of (12.3) see S. D. Conte, *Elementary Numerical Analysis*, 3rd ed (Auckland: McGraw-Hill International Book Company, 1980), Chapter 8. NOTE: This calculation does not prove (12.3), it only illustrates its use.

13. Extrapolation. Suppose we use Euler's method to find an approximation y_N to the exact solution $y(x_N)$ at some point x_N. The error $E = y(x_N) - y_N$ there satisfies

$$y(x_N) - y_N \sim Ch \tag{13.1}$$

as $h \to 0$. If we don't actually let $h \to 0$ but merely choose a small h and compute y_N, then the *asymptotic* formula (13.1) becomes the *approximate* formula

$$y(x_N) - y_N \approx Ch. \tag{13.2}$$

In (13.2) there are two unknowns, C and $y(x_N)$. We're not particularly interested in C, but are after the exact solution $y(x_N)$. For definiteness, consider the illustrative IVP (2):

(a) Suppose we wish to find the solution $y(x)$ at $x = 0.5$. If we run the Euler method with $h = 0.1$ through $N = 5$, and with $h = 0.02$ through $N = 25$, we obtain the results $y_5 = 1.7995$ and $y_{25} = 1.8778$, respectively (Table 1). Thus, (13.2) gives

$$y(0.5) - 1.7995 \approx C(0.1), \tag{13.3a}$$

$$y(0.5) - 1.8778 \approx C(0.02), \tag{13.3b}$$

which are two (approximate) equations in the two unknowns $y(0.5)$ and C. Solve (13.3a,b) for $y(0.5)$, and show that the result is much more accurate than the value 1.8778 obtained using $h = 0.02$.

NOTE: The latter is called an **extrapolation method** because if we know how the error dies out as $h \to 0$ [namely, according to (13.1) for Euler's method], and we run the method for two small but different h's, then we can "extrapolate" those two results to solve for $y(x_N)$. Though the method gives an improved result it does not yield the *exact* solution $y(x_N)$, except by coincidence, because we've used (13.2) and (13.4) which are approximate, not exact.

(b) Use this same procedure to obtain improved estimates of $y(1.0)$ and $y(1.5)$.

14. Possible Existence of a Critical Step Size h. First, read Exercise 8. Suppose $A = -50$, for instance, and $y_0 = 1$. Then the exact solution of (8.1) is $y(x) = e^{-50x}$, which is unity at $x = 0$ and tends rapidly to zero as x increases. Yet, we see from (8.3) that the Euler approximate solution y_n dies out as n increases *only* if $|1 - 50h| < 1$, that is, only if $-1 < 1 - 50h < 1$; otherwise it *grows* with n.

(a) Show from the two foregoing inequalities that we must have $0 < h < 0.04$ if we expect meaningful results.

(b) As a numerical experiment, use the computer to run the Euler solution of (8.1), with $A = -50$ and $y_0 = 1$, on $0 \le x \le 1$, using various values of h, and report your results. The upshot in this example is that even though Euler's method gives convergence to the exact solution at any fixed x as $h \to 0$, we must reduce h below some threshold before that convergence begins to be manifested, before it "kicks in."

(c) Show, from computer results, that if $h > 0.04$ then instead of y_n dying out as n increases, as it should, it exhibits a rapidly growing oscillation. This behavior in the output, a growing oscillation, with a sign change at each successive calculation point — even if the step size is changed — should always suggest to us that the results are not meaningful.

CHAPTER 1 REVIEW

This chapter was devoted to first-order differential equations. We began with the homogeneous linear equation

$$\frac{dy}{dx} + p(x)y = 0 \tag{1}$$

and obtained its general solution

$$y(x) = Ae^{-\int p(x)\,dx} \tag{2}$$

by separation of variables, then considered the general linear equation

$$\frac{dy}{dx} + p(x)y = q(x). \tag{3}$$

Using an integrating factor we obtained a general solution, in explicit form, as

$$y(x) = \frac{1}{\sigma(x)}\Big(\int \sigma(x)q(x)\,dx + C\Big), \tag{4}$$

in which

$$\sigma(x) = e^{\int p(x)\,dx} \tag{5}$$

is an integrating factor of (3) and C is an arbitrary constant.

If an initial condition $y(a) = b$ is appended to (3), we can evaluate the constant C in (4) and obtain

$$y(x) = \frac{1}{\sigma(x)}\Big(\int_a^x \sigma(s)q(s)\,ds + b\sigma(a)\Big). \tag{6}$$

The Existence and Uniqueness Theorem 1.2.1 assured us that the solution (6) exists and is unique at *least* on the broadest x interval, containing the initial point a, on which $p(x)$ and $q(x)$ are continuous.

The equation

$$\frac{dy}{dx} = f(x, y), \tag{7}$$

which in general is nonlinear, is much more difficult and we were able to solve it only in special cases. First, if $f(x, y)$ is of the separable form $X(x)Y(y)$, we can separate variables and obtain the solution from

$$\int \frac{dy}{Y(y)} = \int X(x)\,dx. \tag{8}$$

If we can do the integrations and solve the resulting equation for $y(x)$, then we have the solution in explicit form; otherwise (8) gives it in implicit form.

We can divide $y' = X(x)Y(y)$ by $Y(y)$ to obtain (8) only if $Y(y) \neq 0$, so the case where $Y(y) = 0$ must be treated separately. If $y = y_0$ is any root of $Y(y) = 0$, then $y(x) = y_0$ is a solution of $y' = X(x)Y(y)$, because it reduces the latter to the identity $0 = 0$, so we must include such constant solutions in addition to the one-parameter family of solutions given by (8).

After separable equations, we considered exact equations. Expressed in the differential form

$$M(x, y)dx + N(x, y)dy = 0, \tag{9}$$

(9) is exact if there exists a function $F(x, y)$ such that $Mdx + Ndy = dF$, in which case integration of $dF = 0$ gives the one-parameter family of solutions, in implicit form, as $F(x, y) = C$. If (9) is not exact, we may nevertheless be able to make it exact using an integrating factor that is a function only of x or only of y.

Finally, we studied the use of substitutions in converting a given differential equation to one that we can solve, such as one that is linear or separable. For instance, the nonlinear Bernoulli equation

$$\frac{dy}{dx} + p(x)y = q(x)y^n \tag{10}$$

can be converted to a linear equation for $v(x)$ by the substitution $v = y^{1-n}$, and a homogeneous equation

$$\frac{dy}{dx} = F\Big(\frac{y}{x}\Big) \tag{11}$$

can be converted to a separable equation for $v(x)$ by letting $v = y/x$.

The Existence and Uniqueness Theorem 1.5.1 for the IVP

$$\frac{dy}{dx} = f(x, y); \qquad y(a) = b \tag{12}$$

was less informative than the corresponding Theorem 1.2.1 for the linear case, for whereas Theorem 1.2.1 gave an explicit solution and predicted a minimum x interval on which that solution exists and is unique, Theorem 1.5.1 gave sufficient conditions ensuring existence but did not give the solution. Also, whereas Theorem 1.5.1 assured existence and uniqueness on "some" x interval, it did not indicate how broad that interval will be.

We closed our discussion of first-order equations with an introduction to numerical solution methods in Section 1.9, covering only the Euler algorithm for the approximate solution of the IVP $y' = f(x, y)$ with initial condition $y(a) = b$. The method generates the approximating sequence $y_1, y_2, \ldots$ according to the algorithm $y_{n+1} = y_n + f(x_n, y_n)h$ for $n = 0, 1, 2, \ldots$, where $y_0 = b$ and h is the chosen step size. The latter is a first-order method, which means that at a given computational x point, the error E is asymptotic to some constant times h to the first power as $h \to 0$. We mentioned higher-order methods such as the fourth-order Runge–Kutta method, but did not study them in this brief introduction.