Part I: Theory and Analysis

Chapter 1
Introduction—Oscillators and Synchronization

Oscillation is among the simplest of dynamic behaviors to describe mathematically and has thus been conveniently used in modeling a wide variety of physical phenomena ranging from mechanical vibration to quantum mechanical behavior and even neurological systems. Certainly not the least of these is the area of electronic circuits. Many years ago, van der Pol created his classical model of an oscillator including the nonlinear saturation effects that determine the amplitude of the steady-state oscillation [9]. Soon afterward, Adler provided a simple theory of what is now known as injection locking, and coupled oscillators became a valuable design resource for the electronics engineer and the antenna designer [10]. Moreover, circuit theorists were able to apply these principles to long chains and closed rings of coupled oscillators to model biological behaviors such as intestinal and colorectal myoelectrical activity in humans [11, 12].

1.1 Early Work in Mathematical Biology and Electronic Circuits

Biologists, in trying to understand how neurons coordinate the movements of animals, have defined what is known as a "central pattern generator," or "CPG" for short. A CPG in this context is a group of neurons that produce rhythmic or periodic signals without sensory input. Biologists have found that CPGs are conveniently modeled mathematically if treated as a set of oscillators that are coupled to each other, most often using nearest-neighbor coupling but sometimes using more elaborate coupling schemes. Taking this viewpoint and performing the subsequent mathematical analysis has enabled biologists to fruitfully study the manner in which vertebrates (such as the lamprey) coordinate their muscles in locomotion (swimming) and how bipeds (such as you and I) do so in walking or running. The muscles are controlled by signals from a CPG [13, 14]. Electronics engineers have also found oscillators to be useful but more as a component of a man-made system than as a model of a naturally occurring one as in biology. Legend has it that the first electronic oscillator was made by accident in trying to construct an amplifier and encountering unwanted feedback that produced oscillatory behavior. In any case, to deliberately make an oscillator, one starts with an amplifier and provides a feedback path that puts some of the amplifier output into its input whence it is amplified and again returned to the input, and so on. The feedback signal is arranged to arrive at the input in-phase with the preexisting signal at that point so the feedback is regenerative. Thus, the amplitude of the circulating signal would continue to increase indefinitely. However, the amplification or gain of practical amplifiers decreases as the signal amplitude increases. Thus, an equilibrium is quickly reached where the amplitude is just right so the amplifier gain balances the losses in the loop. Then the oscillation amplitude stops increasing and becomes constant. This equilibrium occurs at a particular frequency of oscillation depending on the frequency response of the amplifier and the phase characteristics of the feedback path. Thus, the amplitude and frequency become stable and constant. These can be controlled by changing the circuit component values.

Before long it was realized that an oscillator could also be controlled by injecting a signal from outside the circuit into the feedback loop. This, in a sense, adds energy to the circuit at the injection frequency, making it easier for the circuit to sustain oscillation at that frequency. Therefore, if the injected signal is strong enough, the oscillator will oscillate, not at its natural or free running frequency but, rather, at the injection signal frequency, and the oscillator is said to be "injection locked." If the injection signal comes from another oscillator similar to the one being injected and the coupling is bidirectional, the pair is said to be "mutually injection locked."

If many oscillators are mutually injection locked by providing signal paths between them, mutual coupling paths, they can be made to oscillate as a synchronized ensemble. The ensemble properties of such a system are both interesting and useful, and it is this aspect that so intrigued the mathematical biologists. However, some years ago, it was noted by antenna design engineers that these ensemble properties may be exploited in providing driving signals for phased-array antennas. This is because the phases of the oscillators in a coupled group are coordinated and form useful distributions across the oscillator array.

These phase distributions will be discussed in great detail in the remainder of this book, but, for now, we only note that in, for example, using a linear array of mutually injection locked oscillators coupled to nearest neighbors, one may create linear phase progressions across the array by merely changing the free-running frequencies of the end oscillators of the array anti-symmetrically; that is, one up in frequency and the other down by the same amount. Such a linear distribution of signal phases, when used to excite the elements of a linear array of radiating antenna elements, produces a radiated beam whose direction depends on the phase slope. This slope is determined by the amount by which the free-running frequencies of the end oscillators are changed. Electronic oscillators can be designed so that their free-running frequencies are determined by the bias applied to a varactor in the circuit. These are called voltage-controlled oscillators or "VCOs." So we have now described an antenna wherein the beam direction is controlled by a DC bias voltage, a very convenient and useful arrangement that is, in large part, the subject of this book.

1.2 van der Pol's Model

Although having published some related earlier results, in the fall of 1934, Balthasar van der Pol, of the Natuurkuedig Laboratorium der N. V. Philips' Gloeilampenfabricken in Eindhoven, published in the *Proceedings of the Institute of Radio Engineers*, what has become a classic paper on his analyses of the nonlinear behavior of triode vacuum-tube based electronic oscillators [9]. The beauty of his work lies in the fact that he included in his model only the degree of complexity necessary to produce the important phenomena observed. Thus, his mathematical description remained reasonably tractable, permitting detailed analytical, and more recently computational, study of all the salient behaviors of such circuits.

An important aspect that was missing from the earlier, linear treatments was that of gain saturation. Recall that it is this saturation of the gain that produces a stable steady-state amplitude of oscillation. van der Pol included this as a negative damping of his oscillator which depends quadratically on the oscillation amplitude and becomes positive for sufficiently large amplitude. He also allowed for a driving signal with a frequency different from the resonant frequency of the oscillator. The inclusion of these two features in his model will enable us to use it to describe in this book both the steady-state and the transient behavior of coupled oscillator arrays.

Consider the oscillator of Fig. 1-1 and let Y_L be a resonant parallel combination of an inductor, a capacitor, and a resistor. Application of Kirchhoff's current law to the node at the top of Y_L, using phasors with $e^{j\omega t}$ time dependence, yields

$$j\omega I_D + \left(\frac{1}{L} + \frac{j\omega}{R} - \omega^2 C\right)V = 0 \quad (1.2\text{-}1)$$

Now, van der Pol recognized that the active device current, i_d, would be a nonlinear function of the node voltage and modeled that nonlinear function in the time domain as

$$i_D(t) = -\varepsilon\left(g_1 v(t) - g_3 v^3(t)\right) \quad (1.2\text{-}2)$$

using the constants ε, g_1, and g_3 for consistency with Section 7.5 where the van der Pol model is revisited in the context of circuit parameter extraction. Thus we have that

$$\frac{d}{dt} i_D(t) = -\varepsilon g_1 \frac{d}{dt} v(t) + 3\varepsilon g_3 v^2(t) \frac{d}{dt} v(t) \quad (1.2\text{-}3)$$

or in phasor notation,

$$j\omega I_D = -j\omega\varepsilon\left(g_1 - 3g_3 V^2\right)V \quad (1.2\text{-}4)$$

where capital letters denote phasors. Substituting this into Eq. (1.2-1) yields

$$-j\omega\varepsilon\left(g_1 - 3g_3 V^2\right)V + \left(\frac{1}{L} + \frac{j\omega}{R} - \omega^2 C\right)V = 0 \quad (1.2\text{-}5)$$

which may be rewritten in the form

$$\left[-j\omega\varepsilon\left(g_1 - 3g_3 V^2\right) + \frac{1}{L} + \frac{j\omega}{R} - \omega^2 C\right]V = j\omega Y V = 0 \quad (1.2\text{-}6)$$

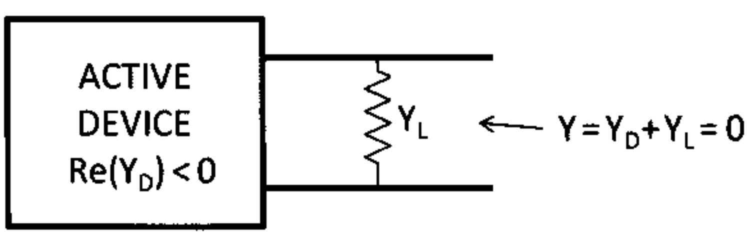

Fig. 1-1. An oscillator as a negative admittance.

where

$$Y = -\varepsilon\left(g_1 - 3g_3V^2\right) + \frac{1}{j\omega L} + \frac{1}{R} + j\omega C \tag{1.2-7}$$

Now, expanding this admittance in a Taylor series about the resonant frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{1.2-8}$$

results in

$$\begin{aligned} Y &= -\varepsilon\left(g_1 - 3g_3V^2\right) + \frac{1}{j\omega L} + \frac{1}{R} + j\omega C \\ &\approx -\varepsilon\left(g_1 - 3g_3V^2\right) + \frac{2jQ}{\omega_0 R}\left(\omega - \omega_0\right) \end{aligned} \tag{1.2-9}$$

where

$$Q = \omega_0 RC \tag{1.2-10}$$

is the traditional quality factor of the oscillator. Use of this expression for the admittance is how we will introduce the van der Pol model into our analysis of an injection locked oscillator below.

1.3 Injection Locking (Adler's Formalism) and Its Spectra (Locked and Unlocked)

To analytically describe the injection locking phenomenon, an oscillator can be viewed as an admittance with a negative real part connected to a resonant load admittance with a positive real part as shown in Fig. 1-1. Using this representation, we proceed now to develop a differential equation for the dynamic behavior of the phase of the oscillation.

The voltage across the load admittance can be written in time-varying phasor form as

$$V = A(t)e^{j\theta(t)} \tag{1.3-1}$$

where,

$$\theta(t) = \omega_0 t + \varphi(t) \tag{1.3-2}$$

Note that V may also be written

$$V = e^{j[\theta(t) - j\ln A(t)]} \tag{1.3-3}$$

Kurokawa [15] suggested that the time derivative of this phasor be written in the form

$$\frac{dV}{dt} = j\left[\omega_0 + \frac{d\varphi}{dt} - j\frac{d}{dt}\ln A\right]V \tag{1.3-4}$$

and that the quantity in brackets be identified as the "instantaneous frequency," ω_{inst}. That is,

$$\frac{dV}{dt} = j\omega_{inst}V \tag{1.3-5}$$

where

$$\omega_{inst} = \left[\omega_0 + \frac{d\varphi}{dt} - j\frac{d}{dt}\ln A\right] \tag{1.3-6}$$

The negative admittance of the device, Y_D, is a function of both the frequency and the amplitude of the oscillating voltage across it. The oscillator operates at the frequency and amplitude that makes this negative admittance equal to the negative of the load admittance, Y_L, so that the total admittance is zero. Following Chang, Shapiro, and York [16], we may expand the admittance in a Taylor series about this operating point in the form

$$Y(\omega_{inst}, A) = Y_L + Y_D(\omega_0, A_0) + (\omega_{inst} - \omega_0)\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0} + \cdots \tag{1.3-7}$$

where we have neglected the amplitude dependence of Y_D. Multiplying by V, we obtain Kirchhoff's current law at the top node of Fig. 1-1.

$$Y(\omega_{inst}, A)V =$$

$$Y_L V + Y_D(\omega_0, A_0)V + (\omega_{inst} - \omega_0)\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0} V + \cdots = 0 \tag{1.3-8}$$

In steady state, the oscillator will oscillate with frequency ω_0 and amplitude A_0 making the derivative term zero. Then the load current cancels the oscillator current for a total of zero current exiting the node. However, if a signal is injected at the node from an external source, this equilibrium is changed to

$$I_{inj} + Y(\omega_{inst}, A)V = I_{inj} + Y_L V + Y_D(\omega_0, A_0)V + (\omega_{inst} - \omega_0)\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0} V + \cdots = 0 \quad (1.3\text{-}9)$$

Inserting Eq. (1.3-6) for the instantaneous frequency results in

$$I_{inj} + Y_D(\omega_0, A_0)V + Y_L V + \left[\frac{d\varphi}{dt} - j\frac{d}{dt}\ln A\right]\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0} V = 0 \quad (1.3\text{-}10)$$

or

$$\left[\frac{d\varphi}{dt} - j\frac{d}{dt}\ln A\right] + \frac{Y(\omega_0, A_0)}{\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0}} + \frac{I_{inj}}{\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0} V} = 0 \quad (1.3\text{-}11)$$

We will now substitute the negative admittance appropriate to the van der Pol oscillator model and analyze the oscillator assuming that a current, I_{inj}, is injected.

Recall that near ω_0 van der Pol's model gives

$$Y = -\varepsilon\left(g_1 - 3g_3V^2\right) + \frac{2jQ}{\omega_0 R_{osc}}(\omega - \omega_0) \quad (1.3\text{-}12)$$

so that

$$\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0} = \frac{2jQ}{\omega_0 R_{osc}} \quad (1.3\text{-}13)$$

Taking the real part of (1.3-11) using (1.3-13) yields

$$\frac{d\varphi}{dt} + \mathrm{Re}\left(\frac{I_{inj}}{\frac{2jQ}{\omega_0 R_{osc}} V}\right) = 0 \tag{1.3-14}$$

Letting $V_{inj} = R_{osc} I_{inj}$, we obtain

$$\frac{d\varphi}{dt} + \frac{\omega_0}{2Q} \mathrm{Im}\left(\frac{V_{inj}}{V}\right) = 0 \tag{1.3-15}$$

Using phasor notation for the injection signal $V_{inj} = A_{inj} e^{j\theta_{inj}}$, and using Eq. (1.3-2), we obtain

$$\frac{d\theta}{dt} = \omega_0 + \frac{\omega_0}{2Q} \frac{A_{inj}}{A} \mathrm{Im}\, e^{j(\theta_{inj} - \theta)} = \omega_0 + \frac{\omega_0}{2Q} \frac{A_{inj}}{A} \sin\left(\theta_{inj} - \theta\right) \tag{1.3-16}$$

Defining $\frac{\omega_0}{2Q} \frac{A_{inj}}{A} = \Delta\omega_{lock}$, the so-called "locking range," we have

$$\frac{d\theta}{dt} = \omega_0 + \Delta\omega_{lock} \sin\left(\theta_{inj} - \theta\right) \tag{1.3-17}$$

known as Adler's equation [10]. Taking the imaginary part of Eq. (1.3-11) leads in the same manner to a differential equation for the amplitude dynamics, but treatment of that aspect will be postponed until Chapter 7, which deals with nonlinear analysis of oscillator arrays. For clarity and simplicity in the initial description of the array properties, the amplitude variation will be assumed negligible. If you are particularly interested, however, you may wish to consult Nogi et al. [17], Meadows et al. [18] , and Seetharam et al. [19], which discuss some aspects of amplitude behavior.

Although the differential equation given by Eq. (1.3-17) is first order, it is nonlinear. Remarkably, however, it can nevertheless be solved analytically. Once the solution is obtained, it can be used to describe the dynamic behavior of the locking process and, very interestingly, the spectrum of the oscillations under both locked and unlocked conditions. We begin by solving Eq. (1.3-17) and then proceed to exhibit the spectral properties of the solution.

First, we define

$$\psi = \theta - \theta_{inj} = \left(\varphi - \varphi_{inj}\right) + \left(\omega_0 - \omega_{inj}\right) t \tag{1.3-18}$$

so that Eq. (1.3-17) may be written

$$\frac{d\psi}{dt} = -\Delta\omega_{lock}\left(\sin\psi + \frac{\Delta\omega_{inj}}{\Delta\omega_{lock}}\right) \tag{1.3-19}$$

where $\Delta\omega_{inj} = \omega_{inj} - \omega_0$. Now defining $K = \dfrac{\Delta\omega_{inj}}{\Delta\omega_{lock}}$ and $\tau = \Delta\omega_{lock} t$, we have the deceptively simple looking differential equation

$$\frac{d\psi}{d\tau} = -\left(\sin\psi + K\right) \tag{1.3-20}$$

Integrating from an initial time, τ_0, to an arbitrary subsequent time, τ,

$$\int_{\psi(\tau_0)}^{\psi(\tau)} \frac{d\psi}{\left(\sin\psi + K\right)} = -\int_{\tau_0}^{\tau} d\tau \tag{1.3-21}$$

we arrive at

$$\tau = \tau_0 - \int_{\psi(\tau_0)}^{\psi(\tau)} \frac{d\psi}{\left(\sin\psi + K\right)} \tag{1.3-22}$$

and it remains to carry out the integration. Using the substitution

$$u = \tan\left(\frac{\psi}{2}\right) \tag{1.3-23}$$

the integral may be cast in the form

$$\frac{1}{K}\int_{u_0}^{u} \frac{2du}{u^2 + \frac{2u}{K} + 1} \tag{1.3-24}$$

where

$$u_0 = \tan\left(\frac{\psi(\tau_0)}{2}\right) \tag{1.3-25}$$

By factoring the denominator of the integrand and expanding in partial fractions, the integral, Eq. (1.3-24), can be expressed in terms of the natural logarithm function in the form

$$\frac{1}{K}\int_{u_0}^{u}\frac{2du}{u^2+\frac{2u}{K}+1}=\frac{1}{\sqrt{1-K^2}}\ln\left(\frac{u-u_2}{u-u_1}\right)\Bigg|_{u_0}^{u} \tag{1.3-26}$$

where u_1 and u_2 are the roots of the quadratic in the denominator of the integrand. That is, Eq. (1.3-22) becomes

$$\int_{\psi(\tau_0)}^{\psi(\tau)}\frac{d\psi}{(\sin\psi+K)}=$$

$$\frac{1}{\sqrt{1-K^2}}\ln\left(\frac{K\tan\left(\frac{\psi}{2}\right)+\left(1-\sqrt{1-K^2}\right)}{K\tan\left(\frac{\psi}{2}\right)+\left(1+\sqrt{1-K^2}\right)}\right)\Bigg|_{\psi(\tau_0)}^{\psi(\tau)}=\tau_0-\tau \tag{1.3-27}$$

Recall that the natural logarithm function is related to the inverse hyperbolic tangent function by

$$\ln\left(\frac{1+x}{1-x}\right)=2\tanh^{-1}(x) \tag{1.3-28}$$

if $0\le x^2<1$. Upon using Eq. (1.3-28) in Eq. (1.3-27), we obtain

$$\tau=\tau_0+\frac{2}{\sqrt{1-K^2}}\tanh^{-1}\left(\frac{\sqrt{1-K^2}}{K\tan\left(\frac{\psi}{2}\right)+1}\right)\Bigg|_{\psi(\tau_0)}^{\psi(\tau)} \tag{1.3-29}$$

Provided that $K^2<1$. This condition is equivalent to

$$|\Delta\omega_{inj}|<|\Delta\omega_{lock}| \tag{1.3-30}$$

which means that the injection signal frequency is within one locking range of the free-running frequency of the oscillator corresponding to the so-called "locked" condition. If $K^2 \geq 1$, the oscillator is said to be "unlocked" and the solution given by Eq. (1.3-27) becomes

$$\tau = \tau_0 - \frac{2}{\sqrt{K^2 - 1}} \tan^{-1}\left(\frac{\sqrt{K^2 - 1}}{K \tan\left(\frac{\psi}{2}\right) + 1} \right) \Bigg|_{\psi(\tau_0)}^{\psi(\tau)} \tag{1.3-31}$$

Now, rewriting Eqs. (1.3-29) and (1.3-31) explicitly evaluated at the limits and rearranging a bit results in

$$\frac{1}{2}\sqrt{1-K^2}\left(\tau - \tau_0\right) = \tanh^{-1}\left(\frac{\sqrt{1-K^2}}{K \tan\left(\frac{\psi(\tau)}{2}\right) + 1} \right) - \tanh^{-1}\left(\frac{\sqrt{1-K^2}}{K \tan\left(\frac{\psi(\tau_0)}{2}\right) + 1} \right) \tag{1.3-32}$$

and

$$-\frac{1}{2}\sqrt{K^2 - 1}\left(\tau - \tau_0\right) = \tan^{-1}\left(\frac{\sqrt{K^2 - 1}}{K \tan\left(\frac{\psi(\tau)}{2}\right) + 1} \right) - \tan^{-1}\left(\frac{\sqrt{K^2 - 1}}{K \tan\left(\frac{\psi(\tau_0)}{2}\right) + 1} \right) \tag{1.3-33}$$

We now make use of the following pair of identities.

$$\tanh^{-1}(x) - \tanh^{-1}(x_0) = \tanh^{-1}\left(\frac{x - x_0}{1 - x x_0} \right) \tag{1.3-34}$$

$$\tan^{-1}(x) - \tan^{-1}(x_0) = \tan^{-1}\left(\frac{x - x_0}{1 + x x_0} \right) \tag{1.3-35}$$

Applying these to Eqs. (1.3-32) and (1.3-33), respectively, we obtain

$$\tanh\left[\frac{1}{2}\sqrt{1-K^2}\,(\tau-\tau_0)\right]= \frac{-\left[\tan\left(\frac{\psi(\tau)}{2}\right)-\tan\left(\frac{\psi(\tau_0)}{2}\right)\right]}{K+K\tan\left(\frac{\psi(\tau)}{2}\right)\tan\left(\frac{\psi(\tau_0)}{2}\right)+\tan\left(\frac{\psi(\tau)}{2}\right)+\tan\left(\frac{\psi(\tau_0)}{2}\right)} \tag{1.3-36}$$

$$\tan\left[\frac{1}{2}\sqrt{1-K^2}\,(\tau-\tau_0)\right]= \frac{\left[\tan\left(\frac{\psi(\tau)}{2}\right)-\tan\left(\frac{\psi(\tau_0)}{2}\right)\right]}{K+K\tan\left(\frac{\psi(\tau)}{2}\right)\tan\left(\frac{\psi(\tau_0)}{2}\right)+\tan\left(\frac{\psi(\tau)}{2}\right)+\tan\left(\frac{\psi(\tau_0)}{2}\right)} \tag{1.3-37}$$

These equations may now be solved for $\psi(\tau)$. The results are

$$\psi(\tau)= 2\tan^{-1}\left\{\frac{\left[\tan\left(\frac{\psi(\tau_0)}{2}\right)-\tanh\left[\frac{1}{2}\sqrt{1-K^2}\,(\tau-\tau_0)\right]\right]\left[K+\tan\left(\frac{\psi(\tau_0)}{2}\right)\right]}{1+\tanh\left[\frac{1}{2}\sqrt{1-K^2}\,(\tau-\tau_0)\right]\left[1+K\tan\left(\frac{\psi(\tau_0)}{2}\right)\right]}\right\} \tag{1.3-38}$$

$$\psi(\tau)= 2\tan^{-1}\left\{\frac{\left[\tan\left(\frac{\psi(\tau_0)}{2}\right)+\tan\left[\frac{1}{2}\sqrt{K^2-1}\,(\tau-\tau_0)\right]\right]\left[K+\tan\left(\frac{\psi(\tau_0)}{2}\right)\right]}{1-\tan\left[\frac{1}{2}\sqrt{K^2-1}\,(\tau-\tau_0)\right]\left[1+K\tan\left(\frac{\psi(\tau_0)}{2}\right)\right]}\right\} \tag{1.3-39}$$

These represent the exact analytic solution of Eq. (1.3-20) giving the dynamic behavior of the phase of an externally injection locked oscillator for all time subsequent to τ_0. While they are actually the same solution, Eq. (1.3-38) is conveniently applied when $K^2 < 1$, and Eq. (1.3-39) is conveniently applied when $K^2 > 1$. When $K^2 = 1$, Eqs. (1.3-38) and (1.3-39) are identical.

We will now proceed to study the spectral properties of this solution. It will be expedient to return to the logarithmic representation in Eq. (1.3-27). For the locked condition we have

$$\ln\left\{\left(\frac{K\tan\left(\frac{\psi(\tau)}{2}\right)+\left(1-\sqrt{1-K^2}\right)}{K\tan\left(\frac{\psi(\tau)}{2}\right)+\left(1+\sqrt{1-K^2}\right)}\right)\left(\frac{K\tan\left(\frac{\psi(\tau_0)}{2}\right)+\left(1+\sqrt{1-K^2}\right)}{K\tan\left(\frac{\psi(\tau_0)}{2}\right)+\left(1-\sqrt{1-K^2}\right)}\right)\right\}= \quad (1.3\text{-}40)$$
$$(\tau_0-\tau)\sqrt{1-K^2}$$

Exponentiating both sides yields

$$\left\{\left(\frac{K\tan\left(\frac{\psi(\tau)}{2}\right)+\left(1-\sqrt{1-K^2}\right)}{K\tan\left(\frac{\psi(\tau)}{2}\right)+\left(1+\sqrt{1-K^2}\right)}\right)\left(\frac{K\tan\left(\frac{\psi(\tau_0)}{2}\right)+\left(1+\sqrt{1-K^2}\right)}{K\tan\left(\frac{\psi(\tau_0)}{2}\right)+\left(1-\sqrt{1-K^2}\right)}\right)\right\}= \quad (1.3\text{-}41)$$
$$e^{-(\tau-\tau_0)\sqrt{1-K^2}}$$

For simplicity of notation, the second factor in the curly brackets, being a constant that depends on the initial conditions, will be defined to be $1/C_0$. Thus,

$$\left(\frac{K\tan\left(\frac{\psi(\tau)}{2}\right)+\left(1-\sqrt{1-K^2}\right)}{K\tan\left(\frac{\psi(\tau)}{2}\right)+\left(1+\sqrt{1-K^2}\right)}\right)=C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}} \quad (1.3\text{-}42)$$

Now solving for $\psi(\tau)$, we obtain

$$\psi(\tau)=2\tan^{-1}\left[\frac{\sqrt{1-K^2}}{K}\left(\frac{1+C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}{1-C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}\right)-\frac{1}{K}\right] \quad (1.3\text{-}43)$$

Recall that

$$\tan^{-1}(x) = \frac{j}{2}\ln\left(\frac{j+x}{j-x}\right) \tag{1.3-44}$$

So that Eq. (1.3-43) may be written in the form

$$\psi(\tau) = j\ln\left\{\frac{j + \frac{\sqrt{1-K^2}}{K}\left(\frac{1+C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}{1-C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}\right) - \frac{1}{K}}{j - \frac{\sqrt{1-K^2}}{K}\left(\frac{1+C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}{1-C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}\right) + \frac{1}{K}}\right\} \tag{1.3-45}$$

Again exponentiating both sides, we obtain

$$e^{j\psi(\tau)} = \left\{\frac{j - \frac{\sqrt{1-K^2}}{K}\left(\frac{1+C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}{1-C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}\right) + \frac{1}{K}}{j + \frac{\sqrt{1-K^2}}{K}\left(\frac{1+C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}{1-C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}\right) - \frac{1}{K}}\right\} \tag{1.3-46}$$

This can be rearranged as

$$e^{j\psi(\tau)} = \left\{\frac{\left(jK+1-\sqrt{1-K^2}\right)-\left(jK+1+\sqrt{1-K^2}\right)C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}{\left(jK-1+\sqrt{1-K^2}\right)-\left(jK-1-\sqrt{1-K^2}\right)C_0e^{-(\tau-\tau_0)\sqrt{1-K^2}}}\right\} \tag{1.3-47}$$

Equation (1.3-47) gives the dynamic behavior of the oscillator voltage as the phase evolves from $\psi(\tau_0)$ to $\psi(\tau)$. This behavior is exponential, not oscillatory, and the steady-state value of the phase at infinite time is $-\sin^{-1}(K)$. Returning to Eq. (1.3-1) and using Eq. (1.3-18), we find that the oscillator voltage in steady state is

$$V_{ss} = A(t)e^{j\theta(t)} = Ae^{j(\psi+\theta_{inj})} = Ae^{j(-\sin^{-1}(K)+\varphi_{inj}+\omega_{inj}t)} \tag{1.3-48}$$

Thus, the spectrum is a single line at frequency ω_{inj} and there is a steady-state phase difference between the oscillator signal and the injection signal of $\sin^{-1}(K)$.

Suppose we allow K to become larger than unity in magnitude. In such a case, the injection signal frequency lies outside the locking range around the free running frequency and the oscillator will be in the "unlocked" condition described by Eq. (1.3-39). Now, however, the spectral properties of the solution become more interesting. We follow an approach suggested by Armand [20]. In this situation, Eq. (1.3-47) becomes

$$e^{j\psi(\tau)} = \left\{ \frac{\left(jK+1-j\sqrt{K^2-1}\right)-\left(jK+1+j\sqrt{K^2-1}\right)C_0 e^{-j\sqrt{K^2-1}(\tau-\tau_0)}}{\left(jK-1+j\sqrt{K^2-1}\right)-\left(jK-1-j\sqrt{K^2-1}\right)C_0 e^{-j\sqrt{K^2-1}(\tau-\tau_0)}} \right\} \tag{1.3-49}$$

or

$$e^{j\psi(\tau)} = \left\{ \frac{A_1 - A_2 C_0 e^{-jT}}{B_1 - B_2 C_0 e^{-jT}} \right\} \tag{1.3-50}$$

where

$$\begin{aligned} A_1 &= jK+1-j\sqrt{K^2-1} \\ A_2 &= jK+1+j\sqrt{K^2-1} \\ B_1 &= jK-1+j\sqrt{K^2-1} \\ B_2 &= jK-1-j\sqrt{K^2-1} \end{aligned} \tag{1.3-51}$$

$$T = \sqrt{K^2-1}\left(\tau-\tau_0\right) \tag{1.3-52}$$

and

$$C_0 = \left(\frac{K\tan\left(\frac{\psi(\tau_0)}{2}\right) + \left(1 - j\sqrt{K^2-1}\right)}{K\tan\left(\frac{\psi(\tau_0)}{2}\right) + \left(1 + j\sqrt{K^2-1}\right)} \right) \tag{1.3-53}$$

Expanding Eq. (1.3-49) in a geometric series yields

$$e^{j\psi(\tau)} = \frac{A_1}{B_1} + \left\{ \frac{A_1}{B_1} - \frac{A_2}{B_2} \right\} \sum_{n=1}^{\infty} \left(\frac{B_2}{B_1} C_0 \right)^n e^{-jnT} \tag{1.3-54}$$

Now, the magnitude of the common ratio of the series is

$$\left| \frac{B_2}{B_1} C_0 \right|$$

$$= \left| \frac{jK - 1 - j\sqrt{K^2-1}}{jK - 1 + j\sqrt{K^2-1}} \right| \left| \frac{K\tan\left(\frac{\psi(\tau_0)}{2}\right) + \left(1 - j\sqrt{K^2-1}\right)}{K\tan\left(\frac{\psi(\tau_0)}{2}\right) + \left(1 + j\sqrt{K^2-1}\right)} \right| \tag{1.3-55}$$

$$= \sqrt{\frac{1 + \left(K - \sqrt{K^2-1}\right)^2}{1 + \left(K + \sqrt{K^2-1}\right)^2}}$$

This is less than unity for positive K, and the series converges for all T. If, on the other hand, K is negative, we instead expand the reciprocal of Eq. (1.3-49),

$$e^{-j\psi(\tau)} = \left\{ \frac{B_1 - B_2 C_0 e^{-jT}}{A_1 - A_2 C_0 e^{-jT}} \right\} = \frac{B_1}{A_1} + \left\{ \frac{B_1}{A_1} - \frac{B_2}{A_2} \right\} \sum_{n=1}^{\infty} \left(\frac{A_2}{A_1} C_0 \right)^n e^{-jnT} \tag{1.3-56}$$

and the magnitude of the common ratio is

$$\left|\frac{A_2}{A_1}C_0\right| = \left|\frac{jK+1+j\sqrt{K^2-1}}{jK+1-j\sqrt{K^2-1}}\right|\left|\frac{K\tan\left(\frac{\psi(\tau_0)}{2}\right)+\left(1-j\sqrt{K^2-1}\right)}{K\tan\left(\frac{\psi(\tau_0)}{2}\right)+\left(1+j\sqrt{K^2-1}\right)}\right| = \sqrt{\frac{1+\left(K+\sqrt{K^2-1}\right)^2}{1+\left(K-\sqrt{K^2-1}\right)^2}} \quad (1.3\text{-}57)$$

which is less than unity for K negative. Expressions (1.3-54) and (1.3-56) thus provide convergent series representations of the solution for the phase dynamics under unlocked conditions and we note that they are actually Fourier series. As such, the coefficients are the amplitudes of the harmonics of a line spectrum representing the oscillator signal. This spectrum has a well-known classic form that is easily observed experimentally using a spectrum analyzer and is depicted schematically in Fig. 1-2.

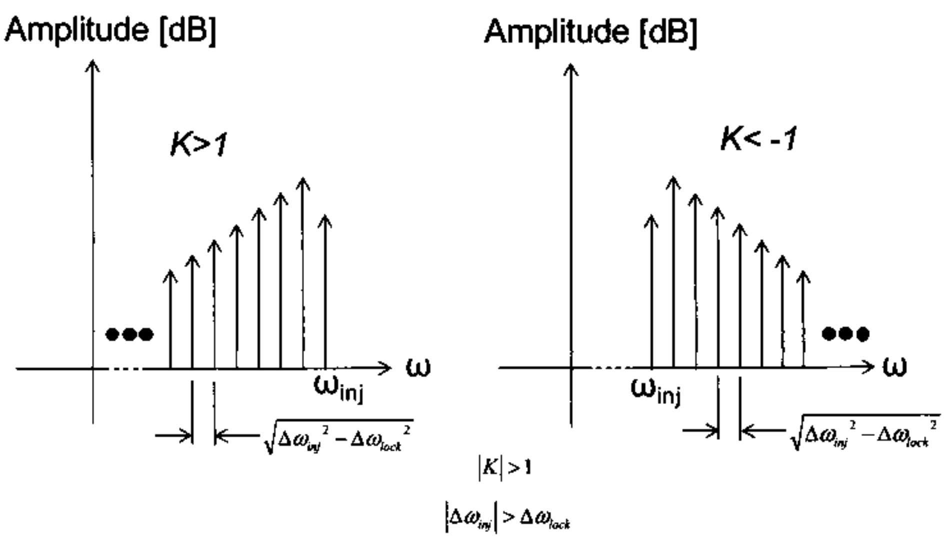

Fig. 1-2. Spectra of an unlocked injected oscillator.

(This K is Kurokawa's [15], which is the negative of Adler's [10] and Armand's [20].)

These mirror-image spectra have a number of interesting features. The most obvious feature is that they are one-sided, which may seem puzzling, but is a natural result of the analysis. Secondly, the amplitudes decrease linearly on a logarithmic scale as one progresses away from the injection frequency. This is a consequence of the geometric nature of the series representing the solution. Finally, the spacing between the spectral lines decreases with the proximity of the injection frequency to the oscillator free running frequency and, when the injection frequency differs from the free running frequency by exactly one locking range, the spacing goes to zero and the oscillator locks, reducing the spectrum to a single line at ω_{inj}.

Before we can legitimately call this analysis of injection locking complete, there remains one important issue to consider. The oscillator model shown in Fig. 1-1 exhibits a parallel resonance. It is, of course, possible to design an oscillator that exhibits a series resonance, and the question then becomes: How is this difference manifest in the formalism presented? This question has been studied in detail by Chang, Shapiro, and York [16]. They pointed out that the Taylor series for the admittance in the parallel resonant oscillator, Eq. (1.2-9), is identical in form to the Taylor expansion of the impedance in the series resonant case. We can see this by considering the series resonant oscillator shown in Fig. 1-3. In this case the resonant load, Z_L, on the active device is a series combination of an inductor, a capacitor, and a resistor.

The output signal here is the current through this resonant series combination rather than the node voltage used in the parallel case. Application of Kirchhoff's voltage law around the oscillator loop yields

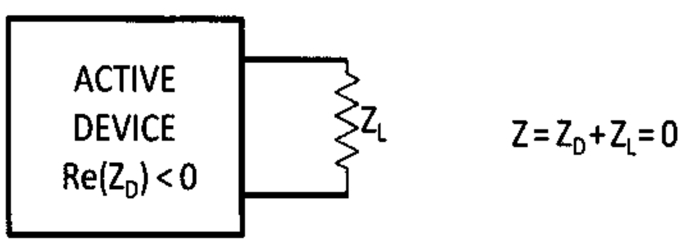

Fig. 1-3. An oscillator as a negative impedance.

$$V_D + \left(j\omega L + R + \frac{1}{j\omega C} \right) I = 0 \quad (1.3\text{-}58)$$

Using a van der Pol-type nonlinearity, the analog of Eq. (1.2-2) is

$$v_D(t) = -\varepsilon\left(r_1 i(t) - r_3 i^3(t) \right) \quad (1.3\text{-}59)$$

and the analog of Eq. (1.2-7) is

$$Z = -\varepsilon\left(r_1 - 3r_3 I^2 \right) + j\omega L + R + \frac{1}{j\omega C} = \frac{1}{Y} \quad (1.3\text{-}60)$$

Expanding Y in a Taylor series about the resonant frequency, we arrive at

$$Y \approx \frac{1}{R - \varepsilon\left(r_1 - 3r_3 I^2 \right)} - \frac{2jQ}{\omega_0 R}(\omega - \omega_0) \quad (1.3\text{-}61)$$

Comparing with Eq. (1.2-9) we see that the salient difference is the change in sign of the linear term in frequency. This in turn induces a change in the algebraic sign of the sine term in Eq. (1.3-17), resulting in

$$\frac{d\theta}{dt} = \omega_0 - \Delta\omega_{lock} \sin\left(\theta_{inj} - \theta \right) \quad (1.3\text{-}62)$$

and the remainder of the analysis proceeds as for the parallel resonant case above. We will further describe the implications of this when we consider more than one oscillator.

1.4 Mutual Injection Locking of Two Oscillators

Consider now two parallel resonant oscillators, identical except for free-running frequency, coupled together so that each injects a signal into the other. Such a system was considered by Stephan and Young [3] in which the coupling was due to free-space mutual coupling between radiating elements excited by the oscillators. We may describe this situation using Adler's Eq. (1.3-17) for each oscillator. That is,

$$\frac{d\theta_1}{dt} = \omega_{01} + \Delta\omega_{lock} \sin\left(\theta_2 - \theta_1 \right) \quad (1.4\text{-}1)$$

$$\frac{d\theta_2}{dt} = \omega_{02} + \Delta\omega_{lock} \sin(\theta_1 - \theta_2) \qquad (1.4\text{-}2)$$

where the subscripts identify the oscillators. Subtracting these equations yields

$$\frac{d(\theta_1 - \theta_2)}{dt} = (\omega_{01} - \omega_{02}) + 2\Delta\omega_{lock} \sin(\theta_2 - \theta_1) \qquad (1.4\text{-}3)$$

We now define

$$\tilde{\psi} = \theta_1 - \theta_2 \qquad (1.4\text{-}4)$$

$$\tilde{K} = \frac{\omega_{02} - \omega_{01}}{2\Delta\omega_{lock}} \qquad (1.4\text{-}5)$$

$$\tilde{\tau} = 2\Delta\omega_{lock} t \qquad (1.4\text{-}6)$$

so that Eq. (1.4-3) becomes

$$\frac{d\tilde{\psi}}{d\tilde{\tau}} = -\left(\sin\tilde{\psi} + \tilde{K}\right) \qquad (1.4\text{-}7)$$

which is identical with Eq. (1.3-20) except for the tildes, and all of the preceding results apply. Note that the locking range is replaced by twice the locking range in this equation. This happens because the injecting oscillator frequency is permitted to change under the influence of the oscillator being injected. The result is that the two oscillator frequencies can differ by nearly twice the locking range and still maintain lock. This is true because it will turn out that the steady-state oscillation frequency of the pair is the average of the two free-running frequencies, and we can show this as follows.

Recall that in steady state, if $\tilde{K} < 1$ so the oscillators are locked, $\tilde{\psi} = -\sin^{-1}\tilde{K}$, a constant, so its time derivative is zero. Further, from Eq. (1.4-4) we have

$$\theta_1 = \theta_2 + \tilde{\psi} \qquad (1.4\text{-}8)$$

so that, in steady state,

$$\frac{d\theta_1}{d\tilde{\tau}} = \frac{d\theta_2}{d\tilde{\tau}} + \frac{d\tilde{\psi}}{d\tilde{\tau}} = \frac{d\theta_2}{d\tilde{\tau}} \tag{1.4-9}$$

Therefore,

$$2\frac{d\theta_1}{d\tilde{\tau}} = \frac{d\theta_2}{d\tilde{\tau}} + \frac{d\theta_1}{d\tilde{\tau}} \tag{1.4-10}$$

or

$$\frac{d\theta_1}{d\tilde{\tau}} = \frac{d}{d\tilde{\tau}}\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{\omega_{01} + \omega_{02}}{2} \tag{1.4-11}$$

Similarly,

$$\frac{d\theta_2}{d\tilde{\tau}} = \frac{d}{d\tilde{\tau}}\left(\frac{\theta_1 + \theta_2}{2}\right) = \frac{\omega_{01} + \omega_{02}}{2} \tag{1.4-12}$$

Thus, we conclude that the steady-state frequency of the two oscillators, when mutually locked, that is, the "ensemble frequency," is the average of their free-running frequencies.

It now becomes clear how it is that the locking range for the two oscillators is twice that for one. One may visualize each oscillator differing from the ensemble frequency of the pair by one locking range so that the total difference between the free-running frequencies of the two oscillators is not one, but two, locking ranges. The term "ensemble frequency" has no relevance when one of the oscillators injection locks the other and is not influenced by the injected oscillator as discussed previously. In that case, as was demonstrated, the steady-state frequency is the injection frequency.

Now suppose that the coupling between the oscillators is accomplished via a transmission line so that there is a phase delay associated with the coupled signal. This coupling phase changes the phase relationship between the coupled signal and the oscillator that produced it and thus modifies the behavior of the oscillator pair. We can account for this in our formulation by inserting the coupling phase shift through the transmission line, Φ_{12}, into Eqs. (1.4-1) and (1.4-2), resulting in

$$\frac{d\theta_1}{dt} = \omega_{01} + \Delta\omega_{lock}\sin\left(\theta_2 - \theta_1 - \Phi_{12}\right) \tag{1.4-13}$$

$$\frac{d\theta_2}{dt} = \omega_{02} + \Delta\omega_{lock} \sin(\theta_1 - \theta_2 - \Phi_{12}) \tag{1.4-14}$$

where we have assumed that the transmission line is reciprocal so that the coupling phase is the same in both directions. Using trigonometric identities, Eqs. (1.4-13) and (1.4-14) may be rewritten in the form

$$\begin{aligned}\frac{d\theta_1}{dt} &= \left[\omega_{01} - \Delta\omega_{lock} \sin\Phi_{12} \cos(\theta_2 - \theta_1)\right] \\ &\quad + \left[\Delta\omega_{lock} \cos\Phi_{12}\right]\sin(\theta_2 - \theta_1)\end{aligned} \tag{1.4-15}$$

$$\begin{aligned}\frac{d\theta_2}{dt} &= \left[\omega_{02} - \Delta\omega_{lock} \sin\Phi_{12} \cos(\theta_1 - \theta_2)\right] \\ &\quad + \left[\Delta\omega_{lock} \cos\Phi_{12}\right]\sin(\theta_1 - \theta_2)\end{aligned} \tag{1.4-16}$$

Again by subtraction we obtain

$$\frac{d(\theta_1 - \theta_2)}{dt} = (\omega_{01} - \omega_{02}) - 2(\Delta\omega_{lock}\cos\Phi_{12})\sin(\theta_1 - \theta_2) \tag{1.4-17}$$

Comparing with Eq. (1.4-3) we see that the locking range has been modified by the cosine of the coupling phase. We define this effective locking range to be

$$\Delta\omega_{eff} = \Delta\omega_{lock}\cos\Phi_{12} \tag{1.4-18}$$

and using this in place of the unmodified locking range, the preceding theory may be applied to the case having nonzero coupling phase. One obvious consequence of this is that, if the coupling phase is 90 degrees (deg) or an odd multiple thereof, the effective locking range becomes zero and the two oscillators cannot be made to lock.

If, instead of subtracting Eqs. (1.3-15) and (1.3-16), we add them, we obtain

$$\frac{d(\theta_1 + \theta_2)}{dt} = (\omega_{01} + \omega_{02}) - 2(\Delta\omega_{lock} \sin\Phi_{12})\cos(\theta_1 - \theta_2) \tag{1.4-19}$$

and we note that the ensemble frequency Eq. (1.4-12) is replaced by

$$\omega_{ens} = \frac{(\omega_{01} + \omega_{02})}{2} - (\Delta\omega_{lock} \sin\Phi_{12})\cos(\theta_1 - \theta_2) \tag{1.4-20}$$

which varies sinusoidally with coupling phase. This variation of ensemble frequency with coupling phase has been studied in somewhat more detail by

Sancheti and Fusco in the context of an active radiator coupling with its image in a reflecting object [21, 22].

Before moving on to study arrays of oscillators we take a quick look at the stability of the behavior of two coupled oscillators. Much more detail on this subject may be found in Chapter 7. The stability of the solution can be assessed by assuming that the oscillators are evolving according to a solution of Eq. (1.4-17) and perturbing the phase difference away from that solution by a small amount, δ. This results in the following differential equation for the time dependence of the perturbation.

$$\frac{d\delta}{dt} = -\left[2\Delta\omega_{lock}\cos\Phi_{12}\cos(\theta_1-\theta_2)\right]\delta \qquad (1.4\text{-}21)$$

This equation has the solution

$$\delta(t) = e^{-\left[2\Delta\omega_{lock}\cos\Phi_{12}\cos(\theta_1-\theta_2)\right]t} \qquad (1.4\text{-}22)$$

The solution for the oscillator phase difference is stable against the perturbation, δ, if the exponent is negative. That is,

$$\cos\Phi_{12}\cos(\theta_1-\theta_2) > 0 \qquad (1.4\text{-}23)$$

This means that, if the magnitude of the coupling phase is less than 90 deg, the oscillators will lock such that their phases differ by less than 90 deg; while if the magnitude of the coupling phase is greater than 90 deg, the oscillators will lock such that their phases differ by more than 90 deg; that is, they will tend to oscillate out of phase. This behavior was predicted and observed by Stephan and Young [3] and formulated and studied in more detail by Humphrey and Fusco [23, 24] using an earlier theoretical construct they formulated for linear chains of coupled oscillators [25].

Conversely, for series resonant oscillators, the stability condition is

$$\cos\Phi_{12}\cos(\theta_1-\theta_2) < 0 \qquad (1.4\text{-}24)$$

and the behavior of the oscillators will be opposite that described above. These properties have been exploited by Lee and Dalman in switching pairs of coupled oscillators from symmetric to antisymmetric phase by changing the coupling phase [26]. All of these effects have been observed experimentally as reported by Chang, Shapiro, and York [16]. Thus, the optimum coupling phase for parallel resonant oscillators is an even multiple of 180 deg, while that for series resonant oscillators is an odd multiple of 180 deg.

Very recently, it was pointed out that a given oscillator can present either series or parallel resonance, depending upon where in the oscillator circuit the coupling is implemented [27].

1.5 Conclusion

In this chapter we have developed a theory of oscillator behavior that admits the possibility of coupling the oscillators together such that they can mutually injection lock and thus oscillate as a coherent ensemble. This behavior is central to the remainder of the book as it forms the basis of the applications to be discussed. In Chapter 2 this theoretical framework will be applied in describing the behavior of arrays containing many oscillators coupled together in linear and planar configurations. The coupling for the most part is with nearest neighbors only. More elaborate coupling schemes have been studied in mathematical biology but remain as a potentially fruitful but largely untapped resource in the arena of phased-array antennas.