CHAPTER ONE

Matrix Two-Person Games

1.1 The Basics

Problems

1.1 There are 100 bankers lined up in each of 100 rows. Pick the richest banker in each row. Javier is the poorest of those. Pick the poorest banker in each column. Raoul is the richest of those. Who is richer: Javier or Raoul?

1.1 Answer: Think of this as a 100 × 100 game matrix and we are looking for the upper and lower values except that we are really doing it for the transpose of the matrix.

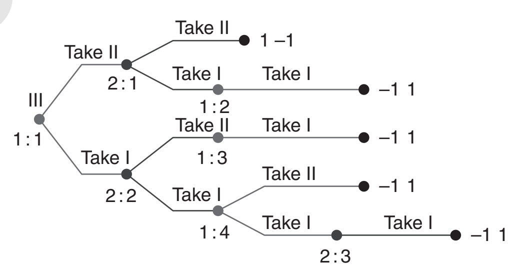
If we take the maximum in each row and then Javier is the minimum maximum, Javier is v^+. If we take the minimum in each column, and Raoul is the maximum of those, then Raoul is the maximum minimum, or v^-. Thus, Javier is richer.

Another way to think of this is that the poorest rich guy is wealthier than the richest poor guy. Common sense.

1.2 In a Nim game start with 4 pennies. Each player may take 1 or 2 pennies from the pile. Suppose player I moves first. The game ends when there are no pennies left and the player who took the last penny pays 1 to the other player.

(a) Draw the game as we did in 2 × 2 Nim.

1.2.a Answer: The game tree is



(b) Write down all the strategies for each player and then the game matrix.

Solutions Manual to Accompany Game Theory: An Introduction, Second Edition. E.N. Barron.

1.2.b Answer:
Using the notation from the figure, we may list the strategies for player I as follows:

1. Go to 2 : 1; if at 1 : 2 take 1. [Same as Take 2. (there are no more choices for I after that)]
2. Go to 2 : 2; if at 1 : 3 take 1; if at 1 : 4 take 2. [Same as Take 1, then if 2 are left, take 1.]
3. Go to 2 : 2; if at 1 : 3 take 1; if at 1 : 4 take 1. [Same as Take 1, then if 2 are left, take 2.]

For player II, the strategies are as follows:

1. If at 2 : 1 take 2; if at 2 : 2 take 2. [Same as If there are 3 left, take 2.]
2. If at 2 : 1 take 2; if at 2 : 2 take 1.
3. If at 2 : 1 take 1; if at 2 : 2 take 2.
4. If at 2 : 1 take 1; if at 2 : 2 take 1; if at 2 : 3 take 1.

The game matrix is

I/II	1	2	3	4
1	1	1	−1	−1
2	−1	−1	−1	−1
3	−1	1	−1	1

(c) Find v^+, v^-. Would you rather be player I or player II?

1.2.c Answer: Since $v^+ = -1$, $v^- = -1$, this game has a value of -1. Player II can always win by playing as follows: If player I takes 2 pennies, then II should take 1. If player I takes 1 penny, then II should take 2 pennies. No matter what player I does, player II wins.

1.3 In the game rock-paper-scissors both players select one of these objects simultaneously. The rules are as follows: paper beats rock, rock beats scissors, and scissors beats paper. The losing player pays the winner $1 after each choice of object. If both choose the same object the payoff is 0.

(a) What is the game matrix?

1.3.a Answer: The rock-paper-scissors game matrix with the rules of the problem is

I/II	Rock	Paper	Scissors
Rock	0	−1	1
Paper	1	0	−1
Scissors	−1	1	0

(b) Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.3.b Answer: $v^+ = 1$, $v^- = -1$. No saddle point in pure strategies since $v^+ > v^-$.

1.4 Each of two players must choose a number between 1 and 5. If a player's choice = opposing player's choice +1, she loses $2; if a player's choice ≥ opposing player's choice +2, she wins $1. If both players choose the same number the game is a draw.

(a) What is the game matrix?

1.4.a Answer: The game matrix is

I/II	1	2	3	4	5
1	0	2	−1	−1	−1
2	−2	0	2	−1	−1
3	1	−2	0	2	−1
4	1	1	−2	0	2
5	1	1	1	−2	0

(b) Find v^+ and v^- and determine whether a saddle point exists in pure strategies, and if so, find it.

1.4.b Answer: $v^+ = 1,\ v^- = -1$, no pure saddle point.

1.5 Each player displays either one or two fingers and simultaneously guesses how many fingers the opposing player will show. If both players guess either correctly or incorrectly, the game is a draw. If only one guesses correctly, he wins an amount equal to the total number of fingers shown by both players. Each pure strategy has two components: the number of fingers to show, the number of fingers to guess. Find the game matrix, v^+, v^-, and optimal pure strategies if they exist.

1.5 Answer: For each player we let (i, j) be the pure strategy in which the player shows i fingers, and guesses the other player will show j fingers. The matrix is

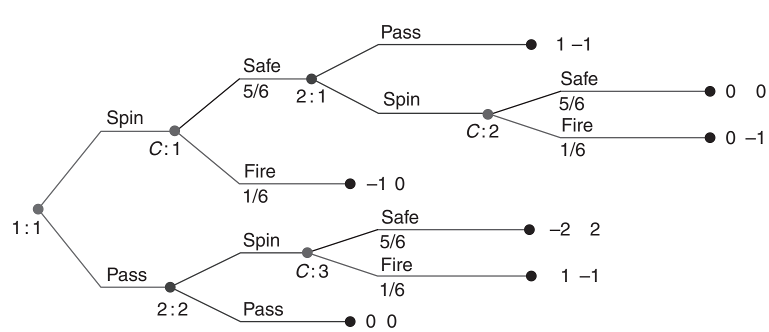
I/II	(1, 1)	(1, 2)	(2, 1)	(2, 2)
(1, 1)	0	2	−3	0
(1, 2)	−2	0	0	3
(2, 1)	3	0	0	−4
(2, 2)	0	−3	4	0

Since $v^+ = 2$, $v^- = -2$, there are no pure optimal strategies.

1.6 In the Russian roulette Example 1.5 suppose that if player I spins and survives and player II decides to pass, then the net gain to I is \$1000 and so I gets all of the additional money that II had to put into the pot in order to pass. Draw the game tree and find the game matrix. What are the upper and lower values? Find the saddle point in pure strategies.

1.6 Answer: The game tree stays the same but the payoff at the end of the Spin-Safe-Pass branch becomes 1, instead of $\frac{1}{2}$.

Here is the game tree:



This is a Gambit generated tree. Any node labeled C is a Chance node. A node labeled with a number indicates that player is making a move. The numbers below the branches in chance moves are the probability that branch is taken. The payoffs at the end of the tree are the payoffs to each player. In a zero sum game, the payoff to player II is the negative of the payoff to player I.

Going through the same calculations as before, we get the game matrix

$$A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3}, & -\frac{1}{36} & -\frac{1}{36} \\ -\frac{3}{2} & 0 & -\frac{3}{2} & 0 \end{bmatrix}.$$

The upper and lower values are $v^- = -\frac{1}{36} = v^+$. There is a pure saddle at row 1, column 3. Even if player I takes the entire pot there will be a saddle at row 1, column 3, both players should spin.

1.7 Let x be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & x \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ x & 0 \end{bmatrix}.$$

Show that no matter what x is, each matrix has a pure saddle point.

1.7 Answer: Consider first the game with A. To calculate v^-, we take the minimum of x and 0, written $\min\{x, 0\}$, and the minimum of 1, 2 which is 1. Then

$$v^- = \max\{\min\{x, 0\}, 1\} = 1$$

since $\min\{x, 0\} \leq 0$ no matter what x is. Similarly,

$$v^+ = \min\{1, \max\{x, 2\}\} = 1 \text{ since } \max\{x, 2\} \geq 2.$$

Thus, $v^- = v^+ = 1$ and there is a pure saddle at row 2, column 1.

For matrix B, we have $v^- = \max\{1, \min\{x, 0\}\} = 1$, $v^+ = \min\{\max\{2, x\}, 1\} = 1$, and there is a pure saddle at row 1, column 2, no matter what x is.

1.8 If we have a game with matrix A and we modify the game by adding a constant C to every element of A, call the new matrix $A + C$, is it true that $v^+(A + C) = v^+(A) + C$?

1.8 Answer: This is true since

$$v^+(A + C) = \min_{1 \leq j \leq m} \max_{1 \leq i \leq n} (a_{ij} + C) = \min_{1 \leq j \leq m} \max_{1 \leq i \leq n} a_{ij} + C = v^+(A) + C.$$

It is also true that $v^-(A + C) = v^-(A) + C$.

(a) If it happens that $v^-(A + C) = v^+(A + C)$, will it be true that $v^-(A) = v^+(A)$, and conversely?

1.8.a Answer: It is true that $v^-(A + C) = v^+(A + C) \Leftrightarrow v^-(A) = v^+(A)$. This follows from the first part.

(b) What can you say about the optimal pure strategies for $A + C$ compared to the game for just A?

1.8.b Answer: The previous parts of this problem imply that $A + C$ has a pure saddle point if and only if A has a saddle point. Since

$$a_{ij^*} + C \leq a_{i^*j^*} + C \leq a_{i^*j} + C \Leftrightarrow a_{ij^*} \leq a_{i^*j^*} \leq a_{i^*j},$$

we conclude that $A + C$ has a saddle point at (i^*, j^*) if and only if A has a saddle point at (i^*, j^*).

1.9 Consider the square game matrix $A = (a_{ij})$ where $a_{ij} = i - j$ with $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, n$. Show that A has a saddle point in pure strategies. Find them and find $v(A)$.

1.9 Answer: We have

$$v^+(A) = \min_{1\leq j\leq n} \max_{1\leq i\leq n} (i - j) = \min_{1\leq j\leq n} (n - j) = n - n = 0$$

and

$$v^-(A) = \max_{1\leq i\leq n} \min_{1\leq j\leq n} (i - j) = \max_{1\leq j\leq n} (i - n) = n - n = 0.$$

Thus, $v = 0$ with a pure saddle point at (n, n).

1.10 Player I chooses 1, 2, or 3 and player II guesses which number I has chosen. The payoff to I is |I's number − II's guess|. Find the game matrix. Find v^- and v^+.

1.10 Answer: The game matrix is

I/II	1	2	3
1	0	1	2
2	1	0	1
3	2	1	0

It is easy to see that $v^- = 0$, $v^+ = 1$ and we have no pure saddle point.

1.11 In the Cat versus Rat game, determine v^+ and v^- without actually writing out the matrix. It is a 16×16 matrix.

1.11 Answer: $v^+ = 1$, $v^- = 0$ because there is always at least one 1 and one 0 in each row and column.

1.12 In a football game, the offense has two strategies: run or pass. The defense also has two strategies: defend against the run, or defend against the pass. A possible game matrix is

$$A = \begin{bmatrix} 3 & 6 \\ x & 0 \end{bmatrix}.$$

This is the game matrix with the offense as the row player I. The numbers represent the number of yards gained on each play. The first row is run, the second is pass. The first column is defend the run and the second column is defend the pass. Assuming that $x > 0$, find the value of x so that this game has a saddle point in pure strategies.

1.12 Answer: Since $v^- = \max\{3, \min\{x, 0\}\} = 3$ and $v^+ = \min\{\max\{x, 3\}, 6\}$. In order to have a pure saddle, we need $v^+ = 3$ that requires $\max\{x, 3\} = 3$ and so $0 < x \leq 3$. Thus, any $0 < x \leq 3$ will produce a pure saddle at row 1, column 1.

1.13 Suppose A is a 2×3 matrix and A has a saddle point in pure strategies. Show that it must be true that either one column dominates another, or one row dominates the other, or both. Then find a matrix A that is 3×3 and has a saddle point in pure strategies, but no row dominates another and no column dominates another.

1.13 Answer: Denote the game matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}.$$

Without loss of generality, we may as well assume that the saddle is at row 1, column 1, $v^- = v^+ = a_{11}$. Since (1, 1) is a saddle, we have

$$a_{i1} \le a_{11} \le a_{1j},\ i = 2,\ j = 2, 3.$$

In particular, $a_{21} \le a_{11}$. If it is also true that $a_{22} \le a_{12}$ and $a_{23} \le a_{13}$ then row 1 dominates row 2 and we are done. Thus, we need only to suppose that $a_{22} > a_{12}$. Then, from the saddle point inequalities,

$$a_{22} > a_{12} \ge a_{11} \ge a_{21}.$$

But then $a_{12} \ge a_{11}$ and $a_{22} > a_{21}$ says that column 2 is dominated by column 1.

Now consider the 3×3 matrix

$$A = \begin{bmatrix} 4 & -2 & 0 \\ 3 & 1 & 1 \\ 0 & 2 & \frac{1}{2} \end{bmatrix}.$$

Then $v^- = v^+ = 1$ and there is a saddle at row 2, column 3, but no row or column dominates another.

1.2 The von Neumann Minimax Theorem

Problems

1.14 Let $f(x, y) = x^2 + y^2$, $C = D = [-1, 1]$. Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

1.14 Answer: We have

$$v^+ = \min_{-1 \le y \le 1} \max_{-1 \le x \le 1} (x^2 + y^2) = \min_{-1 \le y \le 1} (1 + y^2) = 1,$$

and

$$v^- = \max_{-1 \le x \le 1} \min_{-1 \le y \le 1} (x^2 + y^2) = \max_{-1 \le x \le 1} (x^2 + 0) = 1.$$

1.15 Let $f(x, y) = y^2 - x^2$, $C = D = [-1, 1]$.

(a) Find $v^+ = \min_{y \in D} \max_{x \in C} f(x, y)$ and $v^- = \max_{x \in C} \min_{y \in D} f(x, y)$.

1.15.a Answer: The upper value is

$$v^+ = \min_{-1\le y\le 1}\max_{-1\le x\le 1}(y^2 - x^2) = \min_{-1\le y\le 1} y^2 = 0,$$

and the lower value is

$$v^- = \max_{-1\le x\le 1}\min_{-1\le y\le 1}(y^2 - x^2) = \max_{-1\le x\le 1}(0) = 0,$$

since if y chooses first, y can choose $y = x$. Thus, $v^+ = 0$, $v^- = 0$.

(b) Show that $(0, 0)$ is a pure saddle point for $f(x, y)$.

1.15.b Answer: We have $f(0, 0) = 0$, and

$$f(x, 0) = -x^2 \le f(0, 0) = 0 \le f(0, y) = y^2, \quad \forall -1 \le x, y \le 1.$$

1.16 Let $f(x, y) = (x - y)^2$, $C = D = [-1, 1]$. Find $v^+ = \min_{y\in D}\max_{x\in C} f(x, y)$ and $v^- = \max_{x\in C}\min_{y\in D} f(x, y)$.

1.16 Answer: $v^+ = 1$, $v^- = 0$. Here is why. For $v^- = \max_x \min_y (x - y)^2$, y can be chosen to be $y = x$ to get a minimum of zero. For $v^+ = \min_y \max_x (x - y)^2$, x wants to be as far away from y as possible. So, if $y < 0$, then $x = 1$, and if $y > 0$, then $x = -1$, so

$$\max_{-1\le x\le 1}(x - y)^2 = \begin{cases} (1 + y)^2 & \text{if } y > 0; \\ (1 - y)^2 & \text{if } y \le 0. \end{cases}$$

The minimum of this over $y \in [-1, 1]$ is 1, so $v^+ = 1$. You can see this with the Maple commands

```
> f:=y->piecewise(y<0,(1-y)^2,y>=0,(1+y)^2);
> plot(f(y),y=-1..1,view=[-1..1,0..3]);
```

Observe that the function $f(x, y)$ is not concave–convex.

1.17 Show that for any matrix $A_{n\times m}$, the function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by $f(x, y) = \sum_{i=1}^n \sum_{j=1}^m a_{ij} x_i y_j = x A y^T$ is convex in $y = (y_1, \ldots, y_m)$ and concave in $x = (x_1, \ldots, x_n)$. In fact, it is **bilinear**.

1.17 Answer: Let $x, \xi \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} f(\alpha x + \beta\xi, y) &= (\alpha x + \beta\xi) A y^T \\ &= \alpha(x A y^T) + \beta(\xi A y^T) \\ &= \alpha f(x, y) + \beta f(\xi, y). \end{aligned}$$

This proves $x \mapsto f(x, y)$ is linear. Similarly, $y \mapsto f(x, y)$ is also linear.

1.18 Show that for any real-valued function $f = f(x, y)$, $x \in C$, $y \in D$, where C and D are any old sets, it is always true that

$$\max_{x\in C}\min_{y\in D} f(x, y) \le \min_{y\in D}\max_{x\in C} f(x, y).$$

1.18 Answer: We have for any $x \in C$,

$$\min_{y \in D} f(x, y) \leq f(x, y) \Rightarrow \max_{x \in C} \min_{y \in D} f(x, y) \leq \max_{x \in C} f(x, y).$$

The right-hand side is a function of y. The left-hand side is a fixed number, v^-, always below the right-hand side for any y. Thus, the minimum of the right-hand side is

$$\max_{x \in C} \min_{y \in D} f(x, y) = v^- \leq \min_{y \in D} \max_{x \in C} f(x, y) = v^+.$$

1.19 Verify that if there is $x^* \in C$ and $y^* \in D$ and a real number v so that

$$f(x^*, y) \geq v, \ \forall y \in D, \ \text{ and } \ f(x, y^*) \leq v, \ \forall x \in C,$$

then

$$v = f(x^*, y^*) = \max_{x \in C} \min_{y \in D} f(x, y) = \min_{y \in D} \max_{x \in C} f(x, y).$$

1.19 Answer: Under the assumptions,

$$f(x^*, y) \geq v, \ \forall y \in D, \Rightarrow \min_{y \in D} f(x^*, y) \geq v.$$

Then

$$\max_{x \in C} \min_{y \in D} f(x, y) \geq \min_{y \in D} f(x^*, y) \geq v.$$

Similarly,

$$f(x, y^*) \leq v, \ \forall\, x \in C \Rightarrow \max_{x \in C} f(x, y^*) \leq v,$$

and then

$$\min_{y \in D} \max_{x \in C} f(x, y) \leq \max_{x \in C} f(x, y^*) \leq v.$$

Since min max ≥ max min, we have from

$$\min_{y \in D} \max_{x \in C} f(x, y) \leq v \leq \max_{x \in C} \min_{y \in D} f(x, y)$$

that we must have equality throughout.

1.20 Suppose that $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is strictly concave in $x \in [0, 1]$ and strictly convex in $y \in [0, 1]$ and continuous. Then there is a point (x^*, y^*) so that

$$\min_{y \in [0,1]} \max_{x \in [0,1]} f(x, y) = f(x^*, y^*) = \max_{x \in [0,1]} \min_{y \in [0,1]} f(x, y).$$

In fact, define $y = \varphi(x)$ as the function so that $f(x, \varphi(x)) = \min_y f(x, y)$. This function is well defined and continuous by the assumptions. Also define the function

$x = \psi(y)$ by $f(\psi(y), y) = \max_x f(x, y)$. The new function $g(x) = \psi(\varphi(x))$ is then a continuous function taking points in [0, 1] and resulting in points in [0, 1]. There is a theorem, called the **Brouwer fixed-point theorem**, which now guarantees that there is a point $x^* \in [0, 1]$ so that $g(x^*) = x^*$. Set $y^* = \varphi(x^*)$. Verify that (x^*, y^*) satisfies the requirements of a saddle point for f.

1.20 **Answer:** Use the definitions of $y^* = \varphi(x^*)$ and $x^* = \psi(y^*)$. We have

$$f(x^*, y^*) = f(\psi(\varphi(x^*)), \varphi(x^*)) = \max_x f(x, \varphi(x^*)) \geq f(x, \varphi(x^*)) = f(x, y^*),$$

for all $x \in [0, 1]$, and

$$\begin{aligned} f(x^*, y^*) &= f(\psi(\varphi(x^*)), \varphi(x^*)) = \min_y f(\psi(\varphi(x^*)), y) \leq f(\psi(\varphi(x^*)), y) \\ &= f(x^*, y), \end{aligned}$$

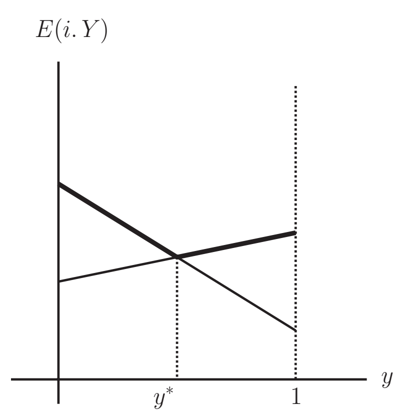
for all $y \in [0, 1]$. Putting these together we have $f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y)$ for all $x, y \in [0, 1]$.

1.4 Solving 2 × 2 Games Graphically

Problems

1.21 Following the same procedure as that for player I, look at $E(i, Y), i = 1, 2$ with $Y = (y, 1 - y)$. Graph the lines $E(1, Y) = y + 4(1 - y)$ and $E(2, Y) = 3y + 2(1 - y), 0 \leq y \leq 1$. Now, how does player II analyze the graph to find Y^*?

1.21 **Answer:** Since player II wants to guarantee that player I gets the smallest maximum, look at the line segments that are the highest and then choose the y^* that gives the smallest maximum payoff. The point of intersection of the two lines is where the y^* will be located and the corresponding horizontal coordinate will be the value of the game.



The result for this problem is $3y + 2(1 - y) = y + 4(1 - y)$ which implies $y^* = \frac{1}{2}$ and $v(A) = \frac{5}{2}$.

1.22 Find the value and optimal X^* for the games with matrices

$$\textbf{(a)} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \qquad \textbf{(b)} \begin{bmatrix} 3 & 1 \\ 5 & 7 \end{bmatrix}$$

What, if anything, goes wrong with (b) if you use the graphical method?

1.22 Answer: For (a), the lines for player I cross where $x - (1 - x) = 2(1 - x)$, which gives $x^* = \frac{3}{4}$. For player II, the lines cross where $y = -y + 2(1 - y)$ which gives $y^* = \frac{1}{2}$. Therefore, the solution of the game in mixed strategies is

$$X^* = \left(\frac{3}{4}, \frac{1}{4}\right), \quad Y^* = \left(\frac{1}{2}, \frac{1}{2}\right), \quad value(A) = \frac{1}{2}.$$

For part (b), the matrix has a saddle point at row 2, column 1, so the optimal strategies won't be mixed strategies. If we didn't spot the pure saddle point and applied the graphical method anyway, we would get for player II, the two lines cross where $3y + 1 - y = 5y + 7(1 - y)$, which gives $y^* = \frac{3}{2} > 1$. The second line lies above the first line for the range $0 \le y \le 1$.

1.23 Curly has two safes, one at home and one at the office. The safe at home is a piece of cake to crack and any thief can get into it. The safe at the office is hard to crack and a thief has only a 15% chance of doing it. Curly has to decide where to place his gold bar (worth 1). On the other hand, if the thief hits the wrong place he gets caught (worth -1 to the thief and $+1$ to Curly). Formulate this as a two-person zero sum matrix game and solve it using the graphical method.

1.23 Answer: Let Curly be the row player and the thief be the column player. The matrix is

I/II	Home	Office
Home	−1	1
Office	1	0.7

The payoff of 0.7 to Curly if he puts the gold bar at the office and the thief hits the office is obtained by calculating $(+1) \times 0.85 + (-1) \times 0.15 = 0.7$. The two lines for the expected payoffs of player I cross where $E((x, 1 - x), 1) = E((x, 1 - x), 2)$, which become $-x + (1 - x) = x + 0.7(1 - x)$. The solution is $x^* = \frac{3}{23}$.

Similarly for player II, the lines cross where $E(1, (y, 1 - y)) = E(2, (y, 1 - y))$, or $-y + (1 - y) = y + 0.7(1 - y)$.

The mixed strategy solution is $X^* = Y^* = \left(\frac{3}{23}, \frac{20}{23}\right)$, $v = \frac{17}{23}$.

1.24 Let z be an unknown number and consider the matrices

$$A = \begin{bmatrix} 0 & z \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ z & 0 \end{bmatrix}.$$

(a) Find $v(A)$ and $v(B)$ for any z.

1.24.a Answer: No matter what z is the lower value is $v^-(A) = \max\{1, \min\{z, 0\}\} = 1$ and the upper value is $v^+(A) = \min\{1, \max\{2, z\}\} = 1$, so there a saddle at row 2, column 1, and $v(A) = 1$.

Similarly, $v^-(B) = \max\{1, \min\{z, 0\}\} = 1$, and $v^+(B) = \min\{\max\{2, z\}, 1\} = 1$, so $v(B) = 1$ and there is a pure saddle at row 1, column 2.

(b) Now consider the game with matrix $A + B$. Find a value of z so that $v(A + B) < v(A) + v(B)$ and a value of z so that $v(A + B) > v(A) + v(B)$. Find the values of $A + B$ using the graphical method. This problem shows that the value is not a linear function of the matrix.

1.24.b Answer: We know $v(A) = v(B) = 1$ and so $v(A) + v(B) = 2$.

Now pick $z = 3$. Then $A + B = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ and the graphical method gives $X^* = Y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $v(A + B) = 3 > v(A) + v(B)$.

Next pick $z = -1$. Then $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and the graphical method gives $X^* = Y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $v(A + B) = 1 < v(A) + v(B)$.

1.25 Suppose that we have the game matrix

$$A = \begin{bmatrix} 13 & 29 & 8 \\ 18 & 22 & 31 \\ 23 & 22 & 19 \end{bmatrix}.$$

Why can this be reduced to $B = \begin{bmatrix} 18 & 31 \\ 23 & 19 \end{bmatrix}$? Now solve the game graphically.

1.25 Answer: Column 2 may be eliminated by dominance: Any $\frac{9}{13} \le \lambda \le \frac{3}{4}$ will make

$$\begin{aligned} 13\lambda + 8(1 - \lambda) &\le 29, \\ 18\lambda + 31(1 - \lambda) &\le 22, \\ 23\lambda + 19(1 - \lambda) &\le 22. \end{aligned}$$

Once column 2 is gone, row 1 may be dropped. Then we apply the graphical method to get $X^* = \left(0, \frac{4}{17}, \frac{13}{17}\right)$ and $Y^* = \left(\frac{12}{17}, 0, \frac{5}{17}\right)$. The value of the game is $v = \frac{37}{17}$.

1.26 Two brothers, Curly and Shemp, inherit a car worth 8000 dollars. Since only one of them can actually have the car, they agree they will present sealed bids to buy the car from the other brother. The brother that puts in the highest sealed bid gets the car. They must bid in 1000 dollar units. If the bids happen to be the same, then they flip a coin to determine ownership and no money changes hands. Curly can bid only up to 5000, while Shemp can bid up to 8000.

Find the payoff matrix with Curly as the row player and the payoffs the expected net gain (since the car is worth 8000). Find v^-, v^+ and use dominance to solve the game.

1.26 Answer: The matrix is 6×9 with the bids $0, 1, \ldots, 5$ for Curly and $0, 1, \ldots, 8$ for Shemp.

Curly/Shemp	0	1	2	3	4	5	6	7	8
0	4	1	2	3	4	5	6	7	8
1	7	4	2	3	4	5	6	7	8
2	6	6	4	3	4	5	6	7	8
3	5	5	5	4	4	5	6	7	8
4	4	4	4	4	4	5	6	7	8
5	3	3	3	3	3	4	6	7	8

For example, if they bid exactly the same amount, the expected payoff to Curly is $\frac{1}{2}8000 = 4000$. If Curly bids 3 and Shemp bids 6 then Shemp gets the car and pays Curly 6000. Shemp's net gain is 2000. This can be thought of as a constant sum game.

It is immediate that $v^- = v^+ = 4$ and the saddle point is Curly should bid 3000 and Shemp should bid 3000, leading to an expected payoff to Curly of 4000. Of course, Shemp also has an expected payoff of 4000.

1.5 Graphical Solution of $2 \times m$ and $n \times 2$ Games

Problems

1.27 In the 2×2 Nim game, we saw that $v^+ = v^- = -1$. Reduce the game matrix using dominance.

1.27 Answer: The game matrix is

Player I/player II	1	2	3	4	5	6
1	1	1	−1	1	1	−1
2	−1	1	−1	−1	1	−1
3	−1	−1	−1	1	1	1

Column 3 immediately dominates every other column. Then it doesn't matter what row player I chooses because the payoff is always -1.

1.28 Consider the matrix game

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

(a) Find $v(A)$ and the optimal strategies.

1.28.a Answer: The graphical method gives $2x = 2(1-x) \Rightarrow x^* = \frac{1}{2}$. Thus, $v(A) = 1$, $X^* = Y^* = \left(\frac{1}{2}, \frac{1}{2}\right)$.

(b) Show that $X^* = (\frac{1}{2}, \frac{1}{2})$, $Y^* = (1, 0)$ is not a saddle point for the game even though it does happen that $E(X^*, Y^*) = v(A)$.

1.28.b Answer: A direct calculation gives $v(A) = 1 = E(X^*, Y^*)$. However, $E(X, Y^*) = 2x$, where $X = (x, 1-x)$, $0 \le x \le 1$, and it is not true that $2x < v(A) = 1$ for all x in that range. This means $Y^* = (1, 0)$ is not optimal.

1.29 Use the methods of this section to solve the games:

$$\textbf{(a)}\begin{bmatrix} 4 & -3 \\ -9 & 6 \end{bmatrix}, \qquad \textbf{(b)}\begin{bmatrix} 4 & 9 \\ 6 & 2 \end{bmatrix}, \qquad \textbf{(c)}\begin{bmatrix} -3 & -4 \\ -7 & 2 \end{bmatrix}.$$

1.29 Answer: (**a**) $X^* = \left(\frac{15}{22}, \frac{7}{22}\right)$; (**b**) $Y^* = \left(\frac{7}{9}, \frac{2}{9}\right)$; (**c**) $Y^* = \left(\frac{6}{10}, \frac{4}{10}\right)$.

To see where these come from since they are all 2 × 2 games without pure saddle points, simply find where the two payoff lines cross for each player.

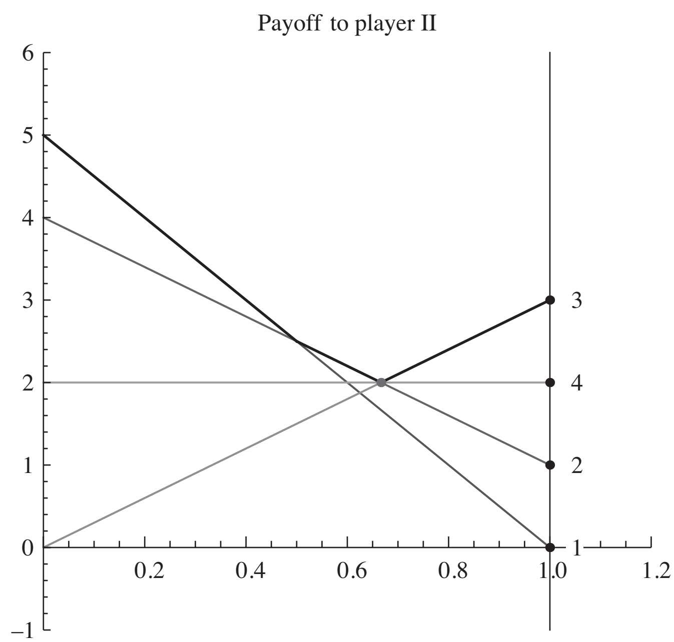
For (a) we must solve $4x - 9(1-x) = -3x + 6(1-x) \Rightarrow x = \frac{15}{22}$, and $4y - 3(1-y) = -9y + 6(1-y)$ that gives $y = \frac{9}{22}$. Then plugging in to either payoff line, we get $v = -\frac{3}{22}$. The other parts are similar.

1.30 Use (convex) dominance and the graphical method to solve the game with matrix

$$A = \begin{bmatrix} 0 & 5 \\ 1 & 4 \\ 3 & 0 \\ 2 & 2 \end{bmatrix}.$$

1.30 Answer: We may drop row 4 since it is (weakly) dominated by a convex combination of rows 2 and 3. In fact, $2 \le \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 3$, and $2 \le \frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 0$.

The lines corresponding to rows 2 and 3 intersect at $y^* = \frac{2}{3}$. That is, $E(2, Y) = y + 4(1-y) = E(3, Y) = 3y$ and so $y^* = \frac{2}{3}$. The value of the game is $v = 2$. Since we use only rows 2 and 3, it is easy to calculate from the graph that the saddle point for player I is $X^* = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right)$.



1.31 The third column of the matrix

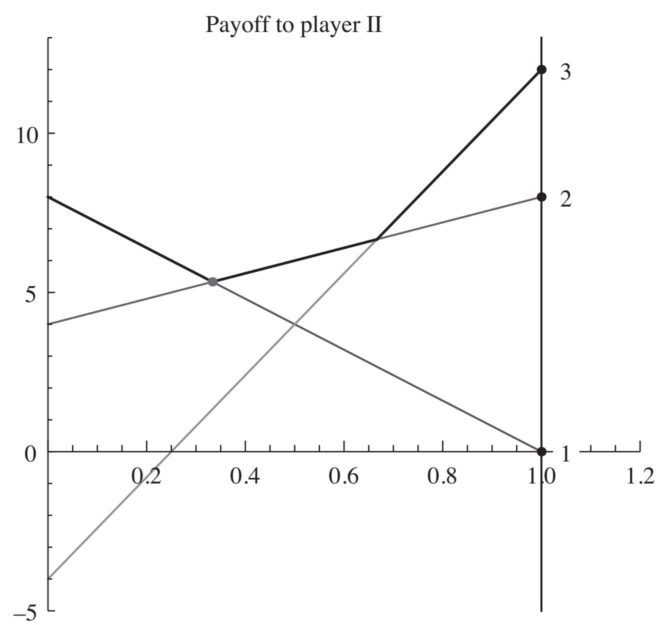
$$A = \begin{bmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{bmatrix}$$

is dominated by a convex combination. Reduce the matrix and solve the game.

1.31 Answer: Any $\frac{3}{8} \leq \lambda \leq \frac{7}{16}$ will work for a convex combination of columns 2 and 1. To see why

$$0 \cdot \lambda + 8 \cdot (1 - \lambda) \leq 5 \Rightarrow \frac{3}{8} \leq \lambda,$$
$$8 \cdot \lambda + 4 \cdot (1 - \lambda) \leq 6 \Rightarrow \lambda \leq \frac{1}{2},$$
$$12 \cdot \lambda + (-4) \cdot (1 - \lambda) \leq 3 \Rightarrow \lambda \leq \frac{7}{16}.$$

The reduced matrix is $\begin{bmatrix} 0 & 8 \\ 8 & 4 \\ 12 & -4 \end{bmatrix}$. The graph for this matrix for player II is



The solution of the original game is, therefore, $X^* = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, $Y^* = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$, and $v(A) = \frac{16}{3}$.

1.32 Four army divisions attack a town along two possible roads. The town has three divisions defending it. A defending division is dug in and hence equivalent to two attacking divisions. Even one division attacking an undefended road captures the town. Each commander must decide how many divisions to attack or defend each road. If the attacking commander captures a road to the town, the town falls. Score 1 to the attacker if the town falls and −1 if it doesn't.

(a) Find the payoff matrix with payoff the attacker's probability of winning the town.

1.32.a Answer: Call the two roads to the town a, b. Each force has strategy comprised of two numbers (i, j), where i is equal to the number of divisions assigned to a and j is equal to the number of divisions assigned to b. The game matrix is then

Attack/Defend	(3, 0)	(2, 1)	(1, 2)	(0, 3)
(4, 0)	−1	−1	1	1
(3, 1)	1	−1	1	1
(2, 2)	1	−1	−1	1
(1, 3)	1	1	−1	1
(0, 4)	1	1	−1	−1

(b) Find the value of the game and the optimal saddle point.

1.32.b Answer: We may reduce the matrix by dominance to the 2×2 game

Attack/Defend	(2, 1)	(1, 2)
(3, 1)	−1	1
(1, 3)	1	−1

Clearly, each row and column should be played with probability $\frac{1}{2}$. The solution of the original game is then

$$X^* = \left(0, \frac{1}{2}, 0, \frac{1}{2}, 0\right), \quad Y^* = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right), \quad v = 0.$$

1.33 Consider the matrix game $A = \begin{bmatrix} a_4 & a_3 & a_3 \\ a_1 & a_6 & a_5 \\ a_2 & a_4 & a_3 \end{bmatrix}$, where $a_1 < a_2 < \cdots < a_5 < a_6$. Use dominance to solve the game.

1.33 Answer: Column 2 is dominated by column 3, then row 3 is dominated by row 1. The reduced matrix is $\begin{bmatrix} a_4 & a_3 \\ a_1 & a_5 \end{bmatrix}$, which does not have a pure saddle. The resulting solution is obtained where the payoff lines cross; for example, $a_4 x + a_1(1-x) = a_3 x + a_5(1-x) \Rightarrow x^* = \frac{a_5 - a_1}{a_4 + a_5 - a_1 - a_3}$. We have,

$$v(A) = \frac{a_4 a_5 - a_1 a_3}{g}, \quad X^* = \left(\frac{a_5 - a_1}{g}, \frac{a_4 - a_3}{g}, 0\right),$$

$$Y^* = \left(\frac{a_5 - a_3}{g}, 0, \frac{a_4 - a_1}{g}\right),$$

where $g = a_4 + a_5 - a_1 - a_3$.

1.34 Aggie and Baggie are fighting a duel each with one lemon meringue pie starting at 20 paces. They can each choose to throw the pie at either 20 paces, 10 paces, or

0 paces. The probability either player hits the other at 20 paces is $\frac{1}{3}$; at 10 paces it is $\frac{3}{4}$ and at 0 paces it is 1. If they both hit or both miss at the same number of paces, the game is a draw. If a player gets a pie in the face, the score is -1, the player throwing the pie gets $+1$.

Set this up as a matrix game and solve it.

1.34 Answer: The strategies for each player are the number of paces at which to fire the pie. Here is the game matrix.

Aggie/Baggie	20	10	0
20	0	$-\frac{1}{6}$	$-\frac{1}{3}$
10	$\frac{1}{6}$	0	$\frac{1}{2}$
0	$\frac{1}{3}$	$-\frac{1}{2}$	0

For example, suppose Aggie decides to hurl at 10, while Baggie decides she will wait until 0. The expected payoff to Aggie is

$$\text{Prob(Aggie hits at 10)}(+1) + \mathit{Prob}\text{(Aggie misses at 10 and Baggie hits at 0)}(-1)$$
$$= \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

Now it is easy to calculate that $v^- = v^+ = 0$, and this zero is achieved at row 2, column 2. Thus, both players should hurl the pie at 10 paces, and the game will be a draw. It is an important part of this problem that both players will choose simultaneously the paces at which they will fire. It would be a different game if they chose by turns.

1.35 Consider the game with matrix

$$A = \begin{bmatrix} -2 & 3 & 5 & -2 \\ 3 & -4 & 1 & -6 \\ -5 & 3 & 2 & -1 \\ -1 & -3 & 2 & 2 \end{bmatrix}.$$

Someone claims that the strategies $X^* = \left(\frac{1}{9}, 0, \frac{8}{9}, 0\right)$ and $Y^* = \left(0, \frac{7}{9}, \frac{2}{9}, 0\right)$ are optimal.

(a) Is that correct? Why or why not?

1.35.a Answer: The given strategies are not optimal because $\max_i E(i, Y) = \frac{31}{9}$ and $\min_j E(X, j) = -\frac{42}{9}$. Another way to see it is to note that since rows 1 and 3 are used with positive probability, Theorem A.8 tells us that if X^* is optimal we must have $E(1, Y^*) = E(3, Y^*) = v$, which you can check easily is not true. Similarly, since columns 2 and 3 are used with positive probability, it must be true that $E(X^*, 2) = E(X^*, 3)$ for X^* to be optimal. But that fails also. Neither X^* nor Y^* are optimal.

(b) If $X^* = \left(\frac{13}{33}, \frac{5}{33}, 0, \frac{15}{33}\right)$ is optimal and $v(A) = -\frac{26}{33}$, find Y^*.

1.35.b Answer: The optimal Y^* is $Y^* = \left(\frac{52}{99}, \frac{8}{33}, 0, \frac{23}{99}\right)$. This is obtained from solving the equations:

$$E(1, Y^*) = -2y_1 + 3y_2 + 5y_3 - 2y_4 = -\frac{26}{33},$$
$$E(2, Y^*) = 3y_1 - 4y_2 + y_3 - 6y_4 = -\frac{26}{33},$$
$$E(4, Y^*) = -y_1 - 3y_2 + 2y_3 + 2y_4 = -\frac{26}{33},$$
$$y_1 + y_2 + y_3 + y_4 = 1.$$

We use the fact that $E(i, Y^*) = v$ if $x_i^* > 0$.

1.36 In the baseball game Example 1.8, it turns out that an optimal strategy for player I, the batter, is given by $X^* = (x_1, x_2, x_3) = \left(\frac{2}{7}, 0, \frac{5}{7}\right)$ and the value of the game is $v = \frac{2}{7}$. It is amazing that the batter should never expect a curveball with these payoffs under this optimal strategy. What is the pitcher's optimal strategy Y^*?

1.36 Answer: The optimal strategy for the pitcher is $Y^* = (\frac{5}{7}, \frac{2}{7}, 0)$. To see why, since $x_1 > 0, x_3 > 0$, we must have $E(1, Y^*) = E(3, Y^*) = v = \frac{2}{7}$. This leads to the system of equations:

$$0.3y_1 + 0.25y_2 + 0.2y_3 = \frac{2}{7},$$
$$0.28y_1 + 0.3y_2 + 0.33y_3 = \frac{2}{7},$$
$$y_1 + y_2 + y_3 = 1.$$

This system has solution $y_1 = \frac{5}{7}, y_2 = \frac{2}{7}, y_3 = 0$. The pitcher should never throw a slider. The batter will get a hit with probability $\frac{2}{7}$.

1.37 In a football game, we use the matrix $A = \begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix}$. The offense is the row player. The first row is Run, the second is Pass. The first column is Defend against Run, the second is Defend against the Pass.

(a) Use the graphical method to solve this game.

1.37.a Answer: $X^* = \left(\frac{8}{11}, \frac{3}{11}\right), \quad Y^* = \left(\frac{6}{11}, \frac{5}{11}\right), \quad v(A) = \frac{48}{11}$.

(b) Now suppose the offense gets a better quarterback so the matrix becomes $A = \begin{bmatrix} 3 & 6 \\ 12 & 0 \end{bmatrix}$. What happens?

1.37.b Answer: Solving for the offense, we see that $3x + 12(1 - x) = 6x$, which gives $x^* = \frac{4}{5}$, so the optimal strategy is $X^* = \left(\frac{4}{5}, \frac{1}{5}\right)$. The value of the game is $v(A) = \frac{24}{5}$ and the optimal strategy for the defense is $Y^* = \left(\frac{2}{5}, \frac{3}{5}\right)$. If the offense gets a better quarterback, the team should Run more!

1.38 Two players Reinhard and Carla play a number game. Carla writes down a number 1, 2, or 3. Reinhard chooses a number (again 1, 2, or 3) and guesses that Carla has written down that number. If Reinhard guesses right he wins \$1 from Carla; if he guesses wrong, Carla tells him if his number is higher or lower and he gets to guess again. If he is right, no money changes hands but if he guesses wrong he pays Carla \$1.

(a) Find the game matrix with Reinhard as the row player and find the upper and lower values. A strategy for Reinhard is of the form

[first guess, guess if low, guess if high].

1.38.a Answer: Carla's strategies are: write down 1, 2, or 3. Reinhard's strategies will be written as $[a, b, c]$, where a is equal to the first number guessed, b is the number guessed if Carla says "lower," c is the number guessed if Carla says "higher." There are 27 strategies, some of which would be stupid, like repeating a number. Eliminating dominated strategies leaves Reinhard with the strategies

$$[1, -, 2], [1, -, 3], [2, 1, 3], [3, 1, -], [3, 2, -].$$

For instance, $[1, -, 2]$ means that Reinhard first guesses 1, Carla will not respond "lower," and then Reinhard responds with 2. The matrix, with Reinhard as the row player, is

I/II	1	2	3
[1, −, 2]	1	0	−1
[1, −, 3]	1	−1	0
[2, 1, 3]	0	1	0
[3, 1, −]	0	−1	1
[3, 2, −]	−1	0	1

It is easy to calculate the upper value is $v^+ = 1$ and the lower value is $v^- = 0$.

(b) Find the value of the game by first noticing that Carla's strategy 1 and 3 are symmetric as are $[1, -, 2]$, $[3, 2, -]$ and $[1, -, 3]$, $[3, 1, -]$ for Reinhard. Then modify the graphical method slightly to solve.

1.38.b Answer: To find the value of the game reason as follows. Both $[1, -, 2]$ $[3, 2, -]$ $[1, -, 3]$, $[3, 1, -]$ are essentially the same strategy and will, by symmetry, have the same probability of use. Furthermore, 1 and 3 should have the same probability of use by Carla. That means we are looking for optimal strategies of the form $X^* = (x_1, x_2, x_3, x_2, x_1)$ and $Y^* = (y_1, y_2, y_1)$. This requires that

$$2x_1 + 2x_2 + x_3 = 1 \quad \text{and} \quad 2y_1 + y_2 = 1.$$

Since Carla has only two unknowns, we could use the graphical method to solve. In fact, we graph the lines

$$A \cdot \begin{bmatrix} \dfrac{1-y_2}{2} \\ y_2 \\ \frac{1-y_2}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{1-3y_2}{2} \\ y_2 \\ \frac{1-3y_2}{2} \\ 0 \end{bmatrix}$$

with $0 \le y_2 \le 1$. We see that the intersection of the two lines $y_2 = \frac{1-3y_2}{2}$ gives the optimal strategy for Carla. Since $y_2 = \frac{1}{5}$, we get $Y^* = \left(\frac{2}{5}, \frac{1}{5}, \frac{2}{5}\right)$.

To find the optimal strategies for Reinhard, observe that the optimal strategy for Carla was obtained by using the second and third row of the matrix (or row 3 and row 4). That means we may drop the other rows to find X^*, noting that row 2 and row 4 would be played with the same probability. Automatically, $x_1 = 0$.

We look at the lines for $0 \le x_2 \le 1$,

$$\left(x_2 \quad \frac{1-x_2}{2}\right)\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left(x_2 \quad -x_2 + \frac{1-x_2}{2}\right)$$

and they cross at $x_2 = \frac{1}{5}$. This means $x_3 = \frac{3}{5}$ and $X^* = (0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$. The value of the game is $v = \frac{1}{5}$ to Reinhard.

1.39 We have an infinite sequence of numbers $0 < a_1 \le a_2 \le a_3 \le \cdots$. Each of two players chooses an integer independent of the other player. If they both happen to choose the same number k, then player I receives a_k dollars from player II. Otherwise, no money changes hand. Assume that $\sum_{k=1}^{\infty} \frac{1}{a_k} < \infty$.

(a) Find the game matrix (it will be infinite). Find v^+, v^-.

1.39.a Answer: The game matrix is

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & \cdots \\ 0 & 0 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Then since there is a 0 in every row and all the $a_k's > 0$, the maximum minimum must be $v^- = 0$. Since there is an $a_k > 0$ in every column, the maximum in every column is a_k. The minimum of those is a_1 and so $v^+ = a_1$.

(b) Find the value of the game if mixed strategies are allowed and find the saddle point in mixed strategies. Use Theorem A.8.

1.39.b Answer: We use $E(i, Y^*) \le v \le E(X^*, j)$, $\forall\, i, j = 1, 2, \ldots$. We get for $X^* = (x_1, x_2, \ldots)$ and $E(X^*, j) = a_j x_j \ge v \Rightarrow x_j \ge \frac{v}{a_j}$. Similarly, for $Y^* = (y_1, y_2, \ldots)$, implies $y_j \le \frac{v}{a_j}$. Adding these inequalities results in

$$1 = \sum_i x_i \ge v \sum_i \frac{1}{a_i} \quad \text{and} \quad 1 = \sum_j y_j \le v \sum_j \frac{1}{a_j},$$

and we conclude that $v = \frac{1}{\sum_i \frac{1}{a_i}}$. We needed the facts that $\sum_i \frac{1}{a_i} < \infty$, and the fact $\sum_i \frac{1}{a_i} \ne 0$, since $a_k > 0$ for all $k = 1, 2, \ldots$.

Next $x_i \ge \frac{v}{a_i} > 0$, and

$$1 = \sum_{i=1}^{\infty} x_i \ge v \sum_{i=1}^{\infty} \frac{1}{a_i} = 1 \Rightarrow \sum_{i=1}^{\infty} \left[x_i - \frac{v}{a_i}\right] = 0$$

which means it must be true, since each term is nonnegative, $x_i = \frac{v}{a_i}, i = 1, 2, \ldots$. Similarly, $X^* = Y^*$, and the components of both optimal strategies are $\frac{v}{a_i}, i = 1, 2, \ldots$.

(c) Assume next that $\sum_{k=1}^{\infty} \frac{1}{a_k} = \infty$. Show that the value of the game is $v = 0$ and every mixed strategy for player I is optimal, but there is no optimal strategy for player II.

1.39.c Answer: Just as in the second part, we have for any integer $n > 1$, $a_j x_j \geq v$ implies

$$1 \geq \sum_{j=1}^{n} x_j \geq v \sum_{j=1}^{n} \frac{1}{a_j}.$$

Then

$$\frac{1}{\sum_{j=1}^{n} \frac{1}{a_j}} \geq v \geq 0.$$

Sending $n \to \infty$ on the left side and using the fact $\sum_{j=1}^{\infty} \frac{1}{a_j} = \infty$, we get $v = 0$. Note that we know ahead of time that $v \geq 0$ since

$$v \geq v^- = \max_{X \in S_\infty} \min_{j=1,2,\ldots} E(X, j) = \max_{X \in S_\infty} \min_{j=1,2,\ldots} x_j a_j \geq 0.$$

Let $X = (x_1, x_2, \ldots)$ be any mixed strategy for player I. Then, it is always true that $E(X, j) = x_j a_j \geq v = 0$ for any column j. By Theorem A.8 this says that X is optimal for player I. On the other hand, if Y^* is optimal for player II, then $E(i, Y^*) = a_i y_i \leq v = 0, i = 1, 2, \ldots$. Since $a_i > 0$, this implies $y_i = 0$ for every $i = 1, 2, \ldots$. But then $Y = (0, 0, \ldots)$ is not a strategy and we conclude player II does not have an optimal strategy. Since the space of strategies in an infinite sequence space is not closed and bounded, we are not guaranteed that an optimal mixed strategy exists by the minimax theorem.

1.40 Show that for any strategy $X = (x_1, \ldots, x_n) \in S_n$ and any numbers $b_1, \ldots, b_n$, it must be that

$$\max_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \max_{1 \leq i \leq n} b_i \quad \text{and} \quad \min_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \min_{1 \leq i \leq n} b_i.$$

1.40 Answer: Let $\max_i b_i = b_k$. Then $\sum_i x_i b_i - b_k = \sum_i x_i (b_i - b_k) = z$ since $\sum_i x_i = 1$. Now $b_i \leq b_k$ for each i, so $z \leq 0$. Its maximum value is achieved by taking $x_k = 1$ and $x_i = 0, i \neq k$. Hence, $\max_X \sum_i x_i b_i - b_k = 0$, which says $\max_X \sum_i x_i b_i = b_k = \max_i b_i$.

1.41 The properties of optimal strategies (Section A.2) show that $X^* \in S_n$ and $Y^* \in S_m$ are optimal if and only if $\min_j E(X^*, j) = \max_i E(i, Y^*)$. The common value will be the value of the game. Verify this.

1.41 Answer: Using Problem 1.40, we have

$$\begin{aligned} v &= \min_{Y\in S_m}\max_{X\in S_n} E(X, Y) = \min_{Y\in S_m}\max_{X\in S_n}\sum_{i=1}^{n} x_i E(i, Y) \\ &= \min_{Y\in S_m}\max_{1\le i\le n} E(i, Y) \end{aligned}$$

and

$$\begin{aligned} v &= \max_{X\in S_n}\min_{Y\in S_m} E(X, Y) = \max_{X\in S_n}\min_{Y\in S_m}\sum_{j=1}^{m} y_j E(X, j) \\ &= \max_{X\in S_n}\min_{1\le j\le m} E(X, j). \end{aligned}$$

Now, if (X^*, Y^*) is a saddle then $v = E(X^*, Y^*)$ and

$$\begin{aligned} \min_{Y\in S_m}\max_{1\le i\le n} E(i, Y) \le \max_{1\le i\le n} E(i, Y^*) \le v \\ \le \min_{1\le j\le m} E(X^*, j) \le \max_{X\in S_n}\min_{1\le j\le m} E(X^*, j). \end{aligned}$$

But we have seen that the two ends of this long inequality are the same. We conclude that

$$v = \max_{1\le i\le n} E(i, Y^*) = \min_{1\le j\le m} E(X^*, j).$$

Conversely, if $\max_{1\le i\le n} E(i, Y^*) = \min_{1\le j\le m} E(X^*, j) = a$, then we have the inequalities

$$E(i, Y^*) \le a \le E(X^*, j), \quad \forall\, i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m.$$

This immediately implies that (X^*, Y^*) is a saddle and $a = E(X^*, Y^*) =$ value of the game.

1.42 Show that if (X^*, Y^*) and (X^0, Y^0) are both saddle points for the game with matrix A, then so is (X^*, Y^0) and (X^0, Y^*). In fact, show that (X_λ, Y_β) where $X_\lambda = \lambda X^* + (1-\lambda)X^0$, $Y_\beta = \beta Y^* + (1-\beta)Y^0$ and λ, β any numbers in [0, 1], is also a saddle point. Thus if there are two saddle points, there are an infinite number.

1.42 Answer: By definition of saddle

$$E(X^0, Y^*) \le E(X^*, Y^*) \le E(X^*, Y^0)$$

and

$$E(X^*, Y^0) \le E(X^0, Y^0) \le E(X^0, Y^*).$$

Now put them together to get

$$E(X^*, Y^0) \le E(X^0, Y^0) \le E(X^0, Y^*) \le E(X^*, Y^*) \le E(X^*, Y^0)$$

and so all of them are equal. This implies, for example, that (X^*, Y^0) is also a saddle point since

$$E(X, Y^0) \le E(X^0, Y^0) = E(X^*, Y^0) = E(X^*, Y^*) \le E(X^*, Y), \quad \forall\, X, Y.$$

It is similar to see that (X^*, Y_β) and (X_λ, Y^*) are also saddle points.

Let (X, Y) be arbitrary strategies. Then using the bilinearity of $E(X, Y)$ and what we just showed,

$$\begin{aligned} E(X_\lambda, Y_\beta) &= \lambda E(X^*, Y_\beta) + (1-\lambda)E(X^0, Y_\beta) \\ &\le \lambda E(X^*, Y) + (1-\lambda)E(X^0, Y) = E(X_\lambda, Y), \quad \forall\, Y \in S_m \end{aligned}$$

and

$$\begin{aligned} E(X_\lambda, Y_\beta) &= \beta E(X_\lambda, Y^*) + (1-\beta)E(X_\lambda, Y^0) \\ &\ge \beta E(X, Y^*) + (1-\beta)E(X, Y^0) = E(X, Y_\beta), \quad \forall\, X \in S_n \end{aligned}$$

1.6 Best Response Strategies

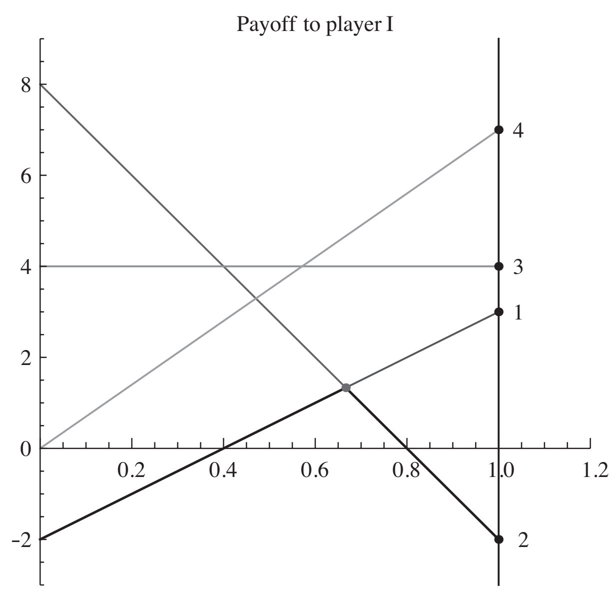
Problems

1.43 Consider the game with matrix

$$\begin{bmatrix} 3 & -2 & 4 & 7 \\ -2 & 8 & 4 & 0 \end{bmatrix}.$$

(a) Solve the game.

1.43.a Answer: We will use the graphical method. The graph is



The figure indicates that the two lines determining the optimal strategies will cross where $-2x + 3(1 - x) = 8x + (-2)(1 - x)$. This tells us that $x^* = \frac{2}{3}$ and then $v = -2\frac{2}{3} + 3\left(1 - \frac{2}{3}\right) = \frac{4}{3}$. We have $X^* = \left(\frac{2}{3}, \frac{1}{3}\right)$.

Next, since the two lines giving the optimal X^* come from columns 1 and 2. we may drop the remaining columns and solve for Y^* using the matrix $\begin{bmatrix} 3 & -2 \\ -2 & 8 \end{bmatrix}$. The two lines for player II cross where $3y - 2(1 - y) = -2y + 8(1 - y)$, or $y^* = \frac{2}{3}$. Thus, $Y^* = \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right)$.

(b) Find the best response for player I to the strategy $Y = (\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8})$.

1.43.b Answer: The best response for player I to $Y = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{8}\right)$ is $X = (0, 1)$. The reason is because if we look at the payoff for $X = (x, 1 - x)$, we get

$$E(X, Y^*) = X A Y^{*T} = 4 - \frac{23}{8}x.$$

The maximum of this over $0 \le x \le 1$ occurs when $x = 0$, which means the best response strategy for player I is $X^* = (0, 1)$, with expected payoff 4.

(c) What is II's best response to I's best response?

1.43.c Answer: Player I's best response is $X^* = (0, 1)$, which means always play row II. Player II's best response to that is always play column 1, $Y^* = (1, 0, 0, 0)$, resulting in a payoff to player I of -2.

1.44 An entrepreneur, named Victor, outside Laguna beach can sell 500 umbrellas when it rains and 100 when the sun is out along with 1000 pairs of sunglasses. Umbrellas cost him \$5 each and sell for \$10. Sunglasses wholesale for 2\$ and sell for \$5. The vendor has \$2500 to buy the goods. Whatever he doesn't sell is lost as worthless at the end of the day.

(a) Assume Victor's opponent is the weather set up a payoff matrix with the elements of the matrix representing his net profit.

1.44.a Answer: Weather has two strategies: Rain and Sun. Victor has two strategies: Assume Rain and Assume Sun.

If Victor buys for Rain, he buys 500 umbrellas at \$5 for an investment of 2500. If it does rain, he sells everything and earns $500 \cdot 10 = 5000$ for a net profit of 2500. The remaining numbers are calculated in a similar way.

The payoff matrix to Victor is then

V/W	Rain	Sun
Buy for Rain	2500	−1500
Buy for Sun	−1500	3500

Then using the graphical method we easily see that $X^* = \left(\frac{5}{9}, \frac{4}{9}\right)$, $v(A) = \frac{650}{9}$, and the optimal strategy for the weather is also $Y^* = \left(\frac{5}{9}, \frac{4}{9}\right)$.

(b) Suppose Victor hears the weather forecast and there is a 30% chance of rain. What should he do?

1.44.b Answer: Victor should play his best response strategy to $Y^0 = (0.3, 0.7)$, which is $X^* = (0, 1)$. He should assume that the sun will be out and buy for those conditions, giving him a net expected profit of $-0.3 \times 1500 + 0.7 \times 3000 = \$2000$.

1.45 You're in a bar and a stranger comes to you with a new pickup strategy.[1] The stranger proposes that you each call Heads or Tails. If you both call Heads, the stranger pays you \$3. If both call Tails, the stranger pays you \$1. If the calls aren't a match, then you pay the stranger \$2.

(a) Formulate this as a two-person game and solve it.

1.45.a Answer: The matrix is

You/Stranger	H	T
H	3	−2
T	−2	1

The saddle point is $X^* = \left(\frac{3}{8}, \frac{5}{8}\right) = Y^*$. This is obtained from solving $3x - 2(1-x) = -2x + (1-x)$ and $3y - 2(1-y) = -2y + (1-y)$. The value of this game is $v = -\frac{1}{8}$, so this is definitely a game you should not play.

(b) Suppose the stranger decides to play the strategy $\tilde{Y} = (\frac{1}{3}, \frac{2}{3})$. Find a best response and the expected payoff.

1.45.b Answer: If the stranger plays the strategy $\tilde{Y} = (\frac{1}{3}, \frac{2}{3})$, then your best response strategy is $\tilde{X} = (0, 1)$ that means you will call Tails all the time. The reason is because

$$E(X, \tilde{Y}) = (x, 1-x)A\,\tilde{Y}^T = -\frac{1}{3}x,$$

which is maximized at $x = 0$. This results in an expected payoff to you of zero. $\tilde{Y}$ is not a winning strategy for the stranger.

1.46 Suppose that the batter in the baseball game Example 1.8 hasn't done his homework to learn the percentages in the game matrix. So, he uses the strategy $X^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. What is the pitcher's best response strategy?

1.46 Answer: Since the expected payoff to player I using X^* is $X^*A = (0.28, 0.2933, 0.27)$, the smallest of these is 0.27. That means the best response for player II is to always play column 3. $Y = (0, 0, 1)$.

1.47 In general, if we have two payoff functions $f(x, y)$ for player I and $g(x, y)$ for player II, suppose that both players want to maximize their own payoff functions with the variables that they control. Then $y^* = y^*(x)$ is a best response of player II to x if

$$g(x, y^*(x)) = \max_y g(x, y), \quad y^* \in \arg\max_y g(x, y).$$

and $x^* = x^*(y)$ is a best response of player I to y if

$$f(x^*(y), y) = \max_x f(x, y), \quad x^* \in \arg\max_x f(x, y).$$

[1]This question appeared in a column by Marilyn Vos Savant in Parade Magazine on March 31, 2002.

(a) Find the best responses if $f(x, y) = (C - x - y)x$ and $g(x, y) = (D - x - y)y$, where C and D are constants.

1.47.a Answer: We can find the first derivatives and set to zero:

$$\frac{\partial f}{\partial x} = C - 2x - y = 0 \Rightarrow x^*(y) = \frac{C - y}{2},$$

and

$$\frac{\partial g}{\partial y} = D - x - 2y = 0 \Rightarrow y^*(x) = \frac{D - x}{2}.$$

These are the best responses since the second partials $f_{xx} = g_{yy} = -2 < 0$.

(b) Solve the best responses and show that the solution x^*, y^* satisfies $f(x^*, y^*) \geq f(x, y^*)$ for all x, and $g(x^*, y^*) \geq g(x^*, y)$ for all y.

1.47.b Answer: Best responses are $x = \frac{C-y}{2}$, $y = \frac{D-x}{2}$, which can be solved to give $x^* = \frac{(2C-D)}{3}$, $y^* = \frac{(2D-C)}{3}$.

Next,

$$f(x^*, y^*) = \frac{(D - 2C)^2}{9} \quad \text{and} \quad f(x, y^*) = \frac{(4C - 2D - 3x)x}{3}.$$

The maximum of $f(x, y^*)$ is $\frac{(D-2C)^2}{9}$, achieved at $x = \frac{2C-D}{3}$, which means it is true that $f(x^*, y^*) \geq f(x, y^*)$ for all x.