CHAPTER 1

HOW LONG WILL MY NUMBER LAST?

Equation #1: Leonardo Fibonacci (1170–1250)

$$t = \frac{1}{r} \ln\left[\frac{c}{c - Wr}\right]$$

Leonardo had a problem. A close friend had invested some money a few years earlier in a local Italian bank, in Pisa, that promised him steady interest of 4% per month. (Yes, I wish I got 4% interest per month. I don't even get that *per year* nowadays. Sounds shady to me.) Anyway, rather than sitting by and letting the money rapidly grow and compound over time, Lenny's friend started withdrawing large and irregular sums of money from the account every few months. These sums were soon exceeding the interest he was earning and the whole process was eating heavily into his capital. To make a long story short, Leonardo—known to be quite good with numbers—was approached by this friend and asked how long the money would last if he kept up these withdrawals. Reasonable question, no?

Now, if Leonardo had been me, he'd have pulled out his handy Hewlett-Packard (HP) business calculator, entered the cash flows, pushed the relevant buttons and quickly had the answer. In fact, with any calculator these sorts of questions can be answered quite easily using the technique known as *present value analysis*—something all finance professors teach their students on the first day of class. Later, I'll explain this important process in some detail.

Unfortunately, Leonardo didn't have access to an HP business calculator that performed the necessary compound interest calculations. (He didn't have a calculator at all because they hadn't been invented yet.) You see, Leonardo was asked this question *more than 800 years ago*, in the early part of the 13th century. But to answer the question—which he certainly did—he actually invented a technique we know today as present value analysis. Yes, the one I mentioned we teach our students.

You might have heard of Leonardo by his more formal name: Leonardo Pisano *filius* ("family" in Latin) Bonacci, a.k.a. Fibonacci to the rest of the world, and probably the most famous mathematician of the Middle Ages.

In fact, Fibonacci helped solve his friend's problem—writing the first commercial mathematics textbook in recorded history in the process—and introduced a revolutionary methodology for solving complicated questions involving interest rates. Let me repeat: his technique, with only slight refinements, is still used and taught to college and university students 800 years later. Now *that* is academic immortality! (He published and his name hasn't perished yet.)

Everyone owes a debt of gratitude to Fibonacci. Had it not been for him, we would probably still be using Roman numerals in our day-to-day calculations. He helped introduce and popularize the usage of the Hindu–Arabic number system—the 10 digits from zero to nine—in the Western world by illustrating how much easier they were for doing commercial mathematics. Imagine calculating square roots or performing long division with Roman numerals. (Okay: What is XMLXVI times XVI?) Well, you can thank Leonardo.

Leonardo Fibonacci was the first financial engineer, or "quant" (translation: highly compensated, scary-smart guys and gals who use advanced mathematics to analyze financial markets) and he didn't work on Wall Street or Bay Street. He worked in the city of Pisa. More on his well-known work, and lesser-known life, later.

The Spending Rate: A Burning Question

Let's translate Fibonacci's mostly hypothetical 800-year-old puzzles into a problem with more recent implications. Imagine you're thinking about retiring and have managed to save $300,000 in your retirement account. For now, I'll stay away from discussing taxes and the exact administrative classification of the account. (I'll revisit this case in Chapter Seven, where I'll add more realistic details.) Allow me to further assume you're entitled to a retirement pension income of $25,000 per year. This is the sum total of your (government) Social Security plus other (corporate) pension plans—but the $25,000 is not enough. You need at least $55,000 per year to maintain your current standard of living. This leaves a gap of $30,000 per year, which you hope to fill with your $300,000 nest egg. The pertinent question, then, becomes: Is the $300,000 enough to fill the budget deficit of $30,000 per year? If not, how long will the money last?

As you probably suspected, your $300,000 nest egg is likely not enough. Think about it this way: the ratio of $30,000 per year (the income you want to generate) divided by the original $300,000 (your nest egg) is 10%. There is no financial instrument I'm aware of—and I've spent the last 20 years of my life searching for one—that can generate a consistent, guaranteed and reliable 10% per year. If you don't want to risk any of your hard-earned nest egg in today's volatile economic environment, the best you can hope for is about 3% after inflation is accounted for, and even that is pushing it. Sure, you might think you're earning 5% guaranteed by a bank, or 5% in dividends or 5% in bond coupons, but an inflation rate of 2% will erode the true return to a mere 3%. Needless to say, 3% will only generate $9,000 per year in interest from your $300,000 nest egg. That is a far cry from (actually $21,000 short of) the extra $30,000 you wanted to extract from the nest egg.

You have no choice. In retirement you will have to eat into your principal.

Here's a side note. In my personal experience talking to retirees and soon-to-be retirees, I find this realization is one of the most difficult concepts they must accept. Some people simply refuse to spend principal and instead submit to a reduced standard of living. Principal is sacred and they agree to live on and adapt to interest income. But in today's low-interest-rate environment, once you account for income taxes, living on interest only will eventually lead to a greatly reduced standard of living over time.

Once you accept that actually depleting your nest egg is necessary, the next—and much more relevant—question becomes: If I start depleting capital, how long before there is nothing left? After all, if you eat into the $300,000 there's a chance it might be gone, especially if you live a long time. This is exactly where Leonardo Fibonacci's insight and technique come in handy, and why I've bequeathed to him Equation #1.

Time to roll up the sleeves and get to work. Let's plug some numbers into Equation #1 and see what Fibonacci has to say.

(You might want to quickly flip back to the equation at the start of the chapter.)

Notice the right-hand side lists three variables (or inputs) that can affect the outcome. The first is the letter *W*, which represents the size of your nest egg, $300,000. The second variable is *c*, which captures the amount you would like to spend or consume, *above and beyond* any retirement pension you might be receiving. (This was $30,000 in the earlier example.) Although your spending takes place continuously (daily, weekly) it adds up to the value of *c*, per year. Think of it as a rate. The final variable, *r*, the trickiest to estimate, is the interest rate your nest egg is earning while it's being depleted, expressed in inflation-adjusted terms. That was the 3% number I mentioned earlier. Now all that's left is to compute the *natural* logarithm of the ratio, denoted by *ln[]* in the first equation.

Natural logarithms are close cousins of common logarithms. Both buttons appear on any good business calculator, but the latter uses a base of 10 and the former a base of 2.7183. If you're unfamiliar with *natural* logarithms—or it has been a while since high-school mathematics—you can find a crash course on natural logs and how they differ from common logs in the appendix to this book.

For now, you can think (very crudely) of the natural logarithm as a process that *shrinks* numbers down to a compact size that is much easier to work with. Later you can worry about how exactly this shrinking works.

Back to the first equation. The mathematics proceeds as follows. The ratio inside the square brackets is 1.42857 written to five digits. In words, it's the desired annual consumption rate of $30,000 (above and beyond the pension income you are receiving) divided by the same consumption rate, *minus* the nest egg value ($300,000), times the investment rate (3%). Sounds wordy? I agree. That's why I—and most financial quants—prefer equations to words. But we're not done yet. Looking back at the right-hand side of the equation, now take the natural logarithm of 1.42857, which leads to 0.35667 written to five digits. Finally, and for the last step, divide this number by 0.03, which is the interest rate, and *voilà*, $t = 11.9$ years.

In words, here is the harsh truth. Keep up this lifestyle, and you'll be broke by the beginning of the 12th year of retirement spending. Not a good outcome, although you will still get your $25,000 pension for the rest of your life, which may (or may not) be enough. But the nest egg is blown. Don't feign surprise. You knew that a yearly $30,000 withdrawal (i.e., spending from the nest egg) would be too much if all you're earning is 3%. But what if you lower the withdrawal rate? Again, Fibonacci's equation, Equation #1, divulges exactly how many years of income you'll gain if you cut down on your planned spending.

Let's do this with revised values. Assume you're consuming (above the pension income you are receiving) $25,000 per year (instead of $30,000 per year), and earning the same 3% per year interest rate. In this case the item in the square brackets is 1.5625; its natural logarithm is 0.44629. When you divide by 0.03 you arrive at $t = 14.9$ years, a gain of almost three years. Here it is in words. Cut down on your planned spending by $5,000 per year and the money will last three years longer.

This standard of living is still not sustainable because there's a very good chance you'll be living in retirement more than 15 years. Okay, what if you reduce your annual (additional) withdrawals from the nest egg to \$20,000? In this case the numbers are 1.81818 in the square brackets, then 0.59784 after you take natural logs and finally $t = 19.9$ years, eight years better than the original.

Is *this* good enough? Is it *long* enough? Well, that's for you to decide. Hopefully you get the point of how to "use" the equation and can generate your own values for how long the money will last. In fact, you might want to try changing the interest rate, ***r***, which I took to be 3%. For example, if you believe (I don't) you can earn a guaranteed, safe 4% real interest rate per year, Equation #1 will result in the value of $t = 22.9$ years, if you withdraw $c = \$20{,}000$ per year and start with a $W = \$300{,}000$ nest egg. In contrast, if all you can earn guaranteed is $r = 1.5\%$ per year, your money will last $t = 17$ years. Personally, I'd lean toward using even lower values in this equation.

The input choices are infinite (no pun intended), so to help you get a better sense of the resulting values I've attached two tables with a range of output numbers. Table 1.1 assumes an investment return of 1.5%, adjusted for inflation, while Table 1.2 assumes a higher (3%) investment return, also adjusted for inflation. Again, you might think these are rather small numbers but remember these numbers are net of inflation, or what I call "real rates." If your bank is paying you 3% on your savings account, but inflation erodes 2% per year, then all you're really earning is (approximately) 1%.

Some might argue this "equation"—nominal interest earned, minus inflation rate, equals real interest—is more important than all seven equations mentioned in this entire book! If you're wondering, the person responsible for this insight is Irving Fisher, the early

Table 1.1 In How Many Years Will the Money Run Out If You Are Earning 1.5% Interest?

Real Spending Rate ($) / Nest Egg ($) →	100,000	200,000	300,000	400,000	500,000
10,000	10.8	23.8	39.9	61.1	92.4
15,000	7.0	14.9	23.8	34.1	46.2
20,000	5.2	10.8	17.0	23.8	31.3
25,000	4.1	8.5	13.2	18.3	23.8
30,000	3.4	7.0	10.8	14.9	19.2
35,000	2.9	6.0	9.2	12.5	16.1
40,000	2.5	5.2	8.0	10.8	13.8
45,000	2.3	4.6	7.0	9.5	12.2
50,000	2.0	4.1	6.3	8.5	10.8

20th-century American economist and champion of Equation #4. No rush. We'll get to his story.

Back to the tables. The columns represent the size of your nest egg (*W*), and the rows represent the annual spending rate (consumption above any pension income)—also adjusted for inflation. Think of them as today's dollars.

Looking at Table 1.1, if you start retirement with $300,000 in a bank account earning 1.5% interest *every* year and you plan to withdraw $35,000 *every* year, then according to Equation #1 the money will run out in exactly 9.2 years. This is 110 months of income. That's it!

In contrast, if you reduce your spending withdrawals to $20,000 and start with the same $300,000 nest egg, your money will last 17 years. Sounds like a lot of time, but note if you retire at 65 this strategy will last (only) until you're 82.

As you'll see later in Chapter Two, when we explore patterns of longevity and mortality in retirement, there are better-than-even odds you'll still be alive at age 82. So even $20,000 from a nest egg of $300,000 (a 6.66% initial spending rate) is too high, unless you're willing to (only) live on the pension income of $25,000 once the money runs out of the nest egg. You might be willing to take that chance and trade off more money earlier in retirement in exchange for a reduced standard of living later in retirement—when you're less likely to be alive—but again, that's your choice to make after you know the numbers and odds. We will return to this economic tradeoff in Irving Fisher's Chapter Four.

Now, let's say you have $1 million in your bank account (earning 1.5%) and you plan to withdraw $50,000 per year. How long will the money last? Those values aren't directly in the table. What do you do? Well, in this case you should be able to use the equation directly (which is actually the point of this book).

Alternatively, you'll notice this equation *scales* in W and c. In other words, you can divide both W and c numbers by any number and the results don't change. So whether you have $1 million and are withdrawing $50,000, or you have $500,000 and are withdrawing $25,000, or you have $2 million and are withdrawing $100,000, they're the equivalent mathematical problem. (Although, personally, I'd obviously like to have the $2 million.) In all cases, the ratio of withdrawal-to-wealth is 1/20. Look carefully at Equation #1: only the ratio matters. On a side note, mathematicians love equations that scale. It helps do something called "reduce the dimensionality" of a problem, and eliminates the need for unnecessary information, so they tend to get excited about these things. Yes, geeky, I know.

Either way, Table 1.1 tells us that in this particular case, the money will last 23.8 years, assuming an interest rate of 1.5%.

Here's another set of values. Table 1.2 displays the left-hand side of Equation #1, but under the assumption (or input) that the interest rate on your money is $r = 3\%$ every single year, as opposed to the $r = 1.5\%$ used in Table #1. Notice that the numbers in Table 1.2 are uniformly (always) larger than the numbers in Table 1.1, and the money runs

Table 1.2 In How Many Years will the Money Run Out If You Are Earning 3% Interest?

Real Spending Rate ($) / Nest Egg ($) →	100,000	200,000	300,000	400,000	500,000
10,000	11.9	30.5	76.8	∞	∞
15,000	7.4	17.0	30.5	53.6	∞
20,000	5.4	11.9	19.9	30.5	46.2
25,000	4.3	9.1	14.9	21.8	30.5
30,000	3.5	7.4	11.9	17.0	23.1
35,000	3.0	6.3	9.9	14.0	18.7
40,000	2.6	5.4	8.5	11.9	15.7
45,000	2.3	4.8	7.4	10.3	13.5
50,000	2.1	4.3	6.6	9.1	11.9

out later because the interest rate is 3% versus 1.5%. Hopefully the impact of increasing the projected interest rate makes intuitive sense.

As I encouraged earlier—and like all the other equations displayed in this book—you are now free to plug in your own withdrawal assumptions and interest rates.

The one thing you might wonder about is the odd-looking symbols in the upper right-hand corner of Table 1.2. They're not stray symbols or typos, but actually represent the mathematical symbol for infinity. Don't be scared. That is good news. Under these conditions the nest egg money will never run out.

Here's an explanation for why the answer is defined to be infinity, in some cases. Skip ahead if you want. Look carefully at the three cells in which the infinity symbols appear and their corresponding row and column coordinates. In particular, when your nest egg is $500,000 the 3% interest rate will generate $15,000 in annual interest. This exceeds the $10,000 you would like to extract every year. So instead of the nest egg shrinking over time, it will continue to grow! Ergo, the money will never run out. In fact, if you withdraw or consume $15,000 from the account per year, exactly the interest you are earning, the account will continue at the same $500,000 value forever. The same concept applies to the $400,000 case, where the 3% interest will generate $12,000—more than the $10,000. In all these cases the denominator within the logarithm will either be zero or negative. The logarithm of infinity (if you divide something by zero) is infinity, and the logarithm of a negative number is simply undefined. So before you use the formula on your calculator, make sure you are spending (withdrawing) more money than the interest you are earning. Otherwise, Fibonacci's equation might lead to gibberish.

Here's the bottom line with infinites. May we all be lucky enough to have large enough nest eggs relative to our withdrawal rate that Fibonacci's equation results in infinity. Most of us, unfortunately, will retire to a reality reflected in the lower left-hand corner of these tables.

Fibonacci's Fabulous Flash of Finance

The name Fibonacci is widely recognized among the bookish masses for something known as the Fibonacci series (or Fibonacci numbers), which has nothing to do with retirement finance or stock trading and more to do with sexually active rabbits. More on this later, but first let me describe Fibonacci's contributions to commercial mathematics.

To begin with, Leonardo Pisano—a.k.a. Fibonacci—wrote a very famous book called *Liber Abaci* (Latin for "Book of Calculations") whose first edition appeared around the year 1202. He revised the book a number of times (that is, he rewrote it) over the next 30 years, and only a few of these revisions are available today. The book itself was written in Latin (which I don't speak or understand). But in the year 2003—exactly eight centuries after the first edition was written—*Liber Abaci* was translated by Professor Laurence Sigler into English, a language I do speak.

To be clear, *Liber Abaci* is a textbook and probably the first textbook of its kind. It has many chapters, diagrams, theorems, proofs and many, many problems—you could think of them as homework assignments—which Fibonacci solved in painstaking detail. It is these problems and their solution methodology that are the main gems of the book. I posed one of those problems at the beginning of this chapter.

To understand the context of Fibonacci's contribution to commercial mathematics, consider the era in which he lived. In the 13th century, the city of Pisa—and most of what is today northern Italy—was at the commercial center of the world. Think of Hong Kong, London and New York squeezed into a few hundred miles. The region had 28 different vibrant cities, each issuing their own currency for trading purposes. This lively economic environment was ideal for the sorts of commercial problems Fibonacci posed and solved in *Liber Abaci*. He didn't have to make up the stories. He lived them.

Here is one of the many problems that Fibonacci posed, which may seem like just another problem but is actually the intellectual inspiration for Equation #1 in this book.

> *On Problems of Travellers and Also Similar Problems:* A Certain man proceeding to Lucca on business to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money and he spent 12 denari. Then, he returned to Pisa; doubled his money and spent 12 denari, and it is proposed that he had nothing left. It is sought how much he had in the beginning. . . . (Sigler translation, chapter 12, page 372)

With a little bit of imagination you can translate Fibonacci's 800-year-old traveler into a modern-day retiree who starts retirement with an unknown sum of money. That's the variable to be solved. The money is invested in a bank account that doubles its value every dozen years, an interest rate of approximately 6% per year. Now, at the end of each year, the retiree withdraws one denari (or dollar, euro, peso) from the bank account, and spends it. This growth and spending process continues for three dozen years (i.e., 36 withdrawals), at which point the money runs out. Fibonacci's question is: How much money did the retiree—who probably wants to spend his time traveling—begin with? Remember, this problem was posed 800 years ago, in the year 1200. Retirement challenges might not be as contemporary as you think.

Fibonacci formulated this problem algebraically, but his genius was in going one step further than the man who invented algebra itself, the Persian scholar Muhammed Al-Kwarizimi in 820 AD.

In Fibonacci's words, here's how he did it:

> Because it is proposed that he always doubled his money, it is clear that 2 will be made from one. Whence it is seen what fraction 1 is of 2, namely ½, which thus is written three times because of the three trips that he made: ½ ½ ½, and the 2 is multiplied by the 2 and the other twos that are under the fraction; there will be 8 of which you take ½ namely 4, of which you take ½ namely 2, and of the two you take ½, namely 1. After this you add the 4 to the 2 and the 1, there will be 7 that you multiply by the 12 denari, which he spent; there will be 84 that you divide by the 8. The quotient will be **ten and one half denari**, and the man had this money. . . . (Sigler translation, chapter 12, page 373)

Did you catch it? Did you see the flash of genius? On the third line, Fibonacci invented interest discounting. Granted, he didn't have the greatest talent for explaining his inventions. Truthfully, the prose sounds more biblical rebuke than textbook pedagogy, but most scholars agree that multiplying the third cash flow by the triple fraction ½ ½ ½ and the second cash flow by ½ ½ and the first cash flow by ½, he gets the credit for introducing the world to the present value factor.

He allowed money to travel from the future into the present and back again. In fact, to ensure everyone understood his clever technique, the next 15 pages of *Liber Abaci* introduce ever more complicated traveler problems with cash flows and interest rates of different sizes using something he called the "method of trips." It seems he recognized the centrality of this technique to commercial transactions, and—as many students would demand from their modern-day instructor—solved many similar problems so it was clear to all his readers.

In sum, Fibonacci's genius was that he broke down complicated compound interest calculations—taking place across different periods of time—by bringing cash flows back to the present and manipulating those values while eliminating the messy time dimension.

So, for example, when you are told that a dollar today is worth more than a dollar next year, that is a present value statement. Or, when a bond that pays $100,000 at the end of the year is currently trading for only $96,000, the reason is present value.

In fact, in the latter part of *Liber Abaci*, Fibonacci used this technique to quantify the impact of compounding periods—annual versus quarterly, for example—involving various debt instruments and even pensions. Indeed, 800 years later, problems such as *On a Soldier Receiving Three Hundred Bezants for His Fief* (page 392), *On a Ton of Pisan Cheese* (page 137) and *On Two Men Who Had a Company* *in Constantinople* (page 393) could all serve as excellent homework assignments and exam questions on financial mathematics. As Professor William Goetzmann from Yale—who has done much to alert scholars to Fibonacci's contribution to finance—wrote in a survey article, "He was not only a brilliant analyst of the business problems of his day, but also a very early financial engineer whose work played a major role in Europe's distinctive capital market development in the late Middle Ages and the Renaissance."

Back to the motivating question of this chapter: How long will the money last? Fibonacci's answer is as follows.

The present value of your retirement withdrawals, from the date of retirement until the date the bank account is exhausted, must exactly equal the sum of money you started out with. This is the equation to be solved. Locating the time at which the money runs out boils down to locating a present value—as a function of time—that is equal to the initial nest egg.

Manipulating the First Equation

Equations have no feelings, so they can be manipulated and even abused without guilt or concern—as long as the rules of mathematics are obeyed. You might remember these operations from high-school algebra (or perhaps *you* felt abused by that experience). Either way, Equation #1 is ripe for manipulation. In particular you can *invert and solve* for the real interest rate ($\boldsymbol{r}$) you must earn during retirement so your initial retirement nest egg (W) lasts for a desired number of years (t), assuming you're planning to spend or withdraw (c) per year. Likewise, you can *invert and solve* for the nest egg (W) required so you can spend (c) for exactly (t) years. Remember, Equation #1 involves four variables so you can place any three of them on one side to solve for the fourth.

Table 1.3 How Much Money Do You Need for a 30-Year Retirement?

Real Interest Rate (%)	Real Spending per Year: $25,000	$50,000	$75,000	$100,000
0.5	$ 696,460	$ 1,392,920	$ 2,089,380	$ 2,785,840
1.0	$ 647,954	$ 1,295,909	$ 1,943,863	$ 2,591,818
1.5	$ 603,953	$ 1,207,906	$ 1,811,859	$ 2,415,812
2.0	$ 563,985	$ 1,127,971	$ 1,691,956	$ 2,255,942
2.5	$ 527,633	$ 1,055,267	$ 1,582,900	$ 2,110,534
3.0	$ 494,525	$ 989,051	$ 1,483,576	$ 1,978,101
3.5	$ 464,330	$ 928,660	$ 1,392,991	$ 1,857,321
4.0	$ 436,754	$ 873,507	$ 1,310,261	$ 1,747,014
4.5	$ 411,533	$ 823,066	$ 1,234,600	$ 1,646,133
5.0	$ 388,435	$ 776,870	$ 1,165,305	$ 1,553,740

If this sounds awfully abstract, here's a very specific and practical example of why manipulation pays off. How much money must you have saved up at retirement, if you want your money to last for exactly 30 years? Think of this as your retirement goal, or "the number," as Lee Eisenberg popularized it in his bestselling book (also called *The Number*). Fibonacci's equation gives us the answer. In this case, we are trying to solve for (*W*) in Equation #1, by isolating it from the other variables. (I'll spare you the algebraic details, but it's actually trivial. Ask your 15-year-old.) Table 1.3 provides a number of these results.

For example, in the first row and first column of Table 1.3, at the crossroads of 0.50% interest and $25,000 withdrawing/spending, you'll see the number $696,460. This means that if all you plan on spending each year—for the planned 30 years of retirement—is $25,000 and the money (while it is waiting to be spent) is earning 0.50% (yes, that is a small number, but have you checked your bank account lately?), then you need a nest egg of $696,460 at retirement. In contrast, if you go down one row and manage to earn 1% interest per year, you don't need as much to finance the $25,000 spending.

In that case, \$647,954 is enough. The extra half-percent interest rate will save you almost \$48,500.

All this information can be extracted from this chapter's champion, Equation #1. But hey, don't trust me. If you plug in (insert, substitute) the value of c = \$25,000 and r = 1% and W = \$647,954 into the right-hand side of the equation, out pops $t = 30$ years on the left-hand side. Confirmed!

Is It Really His?

Although I have given credit (and ownership) of this chapter's equation, Equation #1, to Fibonacci, I must be absolutely clear: he did *not* write down the equation as it's listed in the opening title to this chapter. In fact, were he alive today he would be hard-pressed to recognize the equation largely because logarithms—which are part of the equation—weren't actually invented for another 400 years (by John Napier) and natural logarithms didn't appear until a full century after that (with credit to Leonard Euler).

Fibonacci didn't write down any equations (as we use the term today) in *Liber Abaci*, and he certainly didn't manipulate equations or solve for variables using the notation taught in today's high schools or universities.

The reason I've given him credit for this equation—written using modern-day notation—is that this equation would be meaningless without the underlying concept of present value, precisely what Fibonacci deserves credit for inventing. So if he were alive today and asked me about Equation #1, it would probably take me a few minutes to explain my notation—and perhaps an hour more to explain natural logarithms. But I'm certain once I explained to him that the present value of your retirement income until the date your money runs out must equal your nest egg, he would say, "Hey, that was my idea about the trips and the denari." And he would be perfectly correct in claiming ownership; hence, he is the champion of this chapter.

Can We Really Know Interest Rates?

Now, the one thing that might bother users of this equation is the interest or investment rate. How in the world are you to figure out what your money will actually earn for the next few decades, be it in a bank account, mutual fund or anywhere else? Sure, today you might be getting 1.5%, but perhaps that will decline even more over time. What if the money is invested in the stock market, in which case $\boldsymbol{r}$ is truly random? How do you use the equation then?

Well, at the risk of getting ahead of ourselves, let me assure you that the seventh and final equation in this book—credited to the Russian mathematician Andrei Kolmogorov—is an extension of Equation #1. With that final equation we will enter a world in which you don't know exactly what your money will earn going forward, and you don't know how long you're going to live. It will bring together all the chapters and equations in this book—but I am getting way ahead of myself.

Back to Fibonacci's Life Story

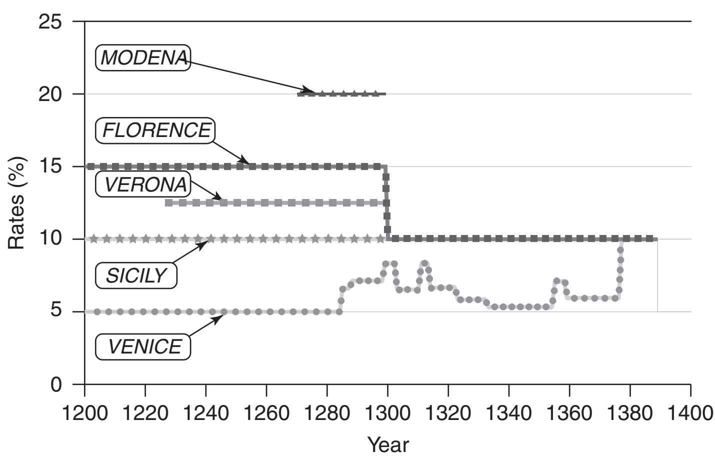
Very little is known about the life of Leonardo Fibonacci. The few facts that historians have gleaned about him come from a very short one-page biography he wrote about himself in the introduction to *Liber Abaci*, as well as some official documents from the city of Pisa.

His father was a customs official posted in Bugia, a sister city and trading post of Pisa on the Barbary coast of Africa, in today's Algeria. As a young child, Fibonacci was brought by his father from

Pisa (where he was likely born in 1170) to Bugia, where he had the opportunity, as an adult, to interact with merchants from Egypt, Syria and Greece. It is likely he traveled extensively in North Africa and had the opportunity to meet with other Middle Eastern scholars of the time, where he learned the Hindu–Arabic number system and perhaps Arabic itself. Fibonacci left Bugia in his early thirties, spending his later life in Pisa—a city which has now claimed him as its own, with the requisite Italian statue in a downtown piazza. Rightfully so, since Fibonacci is one of the two great scientific luminaries from Pisa (the other being Galileo Galilei, born 200 years later).

Recall that Fibonacci was located in the banking epicenter of the world. Figure 1.1 gives an indication of how rich and complex financial life was in 13th- and 14th-century Italy. Each city had its own currency and interest rate. No surprise, then, that Leonardo Fibonacci would be thinking of such matters.

Figure 1.1 Interest Rates: Italian Major Business Cities (1200–1400)



Data Source: Different data for interest rates for Italian cities gained from chapter 9, page 90–101, *A History of Interest Rates*, Fourth Edition, Sidney Homer, Richard Sylla.

Calculations by:
Minjie Zhang, Fall 2011

Historians speculate his *Liber Abaci* was copied and used as the basis for hundreds and possibly thousands of other derivative works during the Middle Ages. These textbooks formed the basic curriculum at so-called *abaci* schools, which taught commercial mathematics and finance to children around southern Europe. Think of them as the forerunners to today's business schools.

Without a doubt, Fibonacci's most lasting contribution to society was his promotion and advocacy of the Hindu–Arabic numeral system as an alternative to Roman numerals for commercial transactions. Oddly enough, this process wasn't as smooth as you might think. Getting merchants to adopt this way of keeping records and calculations wasn't easy. For the most part merchants conducted business with an abacus, some chalk and much hand waving, and they didn't see the need for modern-day numbers. In fact, the rival merchant city of Florence didn't take kindly to this innovation and actually banned (yes, outlawed) use of the so-called new numerals. Even 300 years later, in the middle of the 16th century, merchants in Frankfurt, Germany, had issues with Hindu–Arabic digits and introduced legislation to ban their use in commercial transactions in favor of the more conventional Roman numerals. (Can you believe it? Perhaps the abacus manufacturers lobbied for that one.)

But beyond making life easier by promoting digits and numbers, Fibonacci's contribution to the more specialized field of financial economics has only recently been recognized by historians. For a very long time his work was actually lost, and it was only in the 18th century that it was rediscovered. In other words, although Fibonacci is a household name to today's mathematicians and stock market speculators, there was a 500-year period when his name was almost forgotten.

Okay, Here Come the Rabbits

To most amateur mathematicians, Fibonacci is vaguely known for something called the Fibonacci series (a.k.a Fibonacci numbers). Basically, it's a collection of increasing numbers that go on forever, whose first 11 terms are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc. This sort of thing is called an *infinite series*. Do you see any pattern here? Can you guess the next number in this series? I'll show you momentarily.

As mentioned earlier, Fibonacci's classic book, *Liber Abaci*, includes many diverse questions meant to illustrate the power of the numerical techniques and algorithms he introduced. Among the assortment of commercial and interest-loan problems in one of the advanced chapters was this rather odd problem involving rabbits and sexual reproduction:

> *How Many Pairs of Rabbits Are Created by One Pair in One Year:* A certain man had one pair of rabbits together in a certain enclosed space, and one wishes to know how many are created from the pair in one year when it is the nature in a single month to bear another pair and in the second month those born to bear also . . . (Sigler translation, chapter 12, page 404)

Here is how to think about this whole sordid affair.

A pair of rabbits produces two pairs over the course of their life, one at the end of the first month of their life and then another at the end of the second month of their life, and then they die. Each pair of offspring does exactly the same.

You start with one pair at the beginning, which produces a pair by the end of the first month, leaving a total of two pairs. At the end of the second month the starting pair produces another pair—then dies—plus the pair created by the pair born in month one. So, at the end of the second month, there are three pairs. Then, at the end of the third month, the reproductive process leads to five pairs, resulting in eight pairs by the end of the fourth month, and 13 pairs by the end of month five. Remember that after two months of sexual activity the rabbits die, leaving only their offspring.

Anyway. This is exactly the pattern of numbers I presented above and has become known the world over as the Fibonacci series. Basically, each number in the series is the sum of the previous two numbers. Note that 1 + 2 = 3, that 2 + 3 = 5, that 3 + 5 = 8 and that 5 + 8 = 13, etc. Back to my earlier question: Can you now figure out the next number in the sequence? After the numbers 34 and 55 comes the number 89—the sum of the two previous numbers. Pretty basic. Then comes 144, then 233, then 377 by the end of the 12 months—quite the number of rabbits. The ratio of two adjacent Fibonacci numbers is 1.618, a.k.a. the Golden Ratio.

Why does this matter? Oddly enough, this question about rabbits, which seems completely *ad hoc* and out of context with the commercial and financial essence of this book, has become *the* most enduring legacy of Fibonacci. The Fibonacci sequence—which people have extended well beyond the first 12 numbers he listed—is more recognized and famous today than his work in popularizing Hindu–Arabic numerals, or even his work in financial mathematics. The Fibonacci series occurs naturally, from flower petals to pineapples, and these numbers have taken on a mystical and even religious role over the centuries. Stock market technicians believe that quoted prices follow Fibonacci sequences, and gamblers swear by Fibonacci when picking cards. Either way, I doubt Fibonacci had any inkling he would be remembered (mostly) for the sexually promiscuous rabbits he introduced on page 404 of *Liber Abaci*.

Fibonacci might be famous for his rabbit series, but his financial contribution to retirement income planning is immortal and of much greater significance.

He Retired Wealthy

In addition to his many scholarly talents, Fibonacci was financially shrewd, politically connected and quite influential in economic matters. His mathematical prowess, and the publication of *Liber Abaci* granted him a private audience with the Holy Roman Emperor, Frederick II, whom he greatly impressed by solving a variety of commercial mathematical problems. Toward the end of Fibonacci's life his fame traveled beyond Pisa, and in 1240, the proud city of his birth issued a proclamation granting him an annual pension of 20 Pisan pounds for life for service to the city. It is not clear whether he spent his retirement years trading stocks using technical analysis, but clearly he didn't run out of money before he ran out of life.