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Introduction

The parallel developments of the finite element methods (FEM) in the 1950's [1, 2] and the engineering applications of the stochastic processes in the 1940's [3, 4] provided a combined numerical analysis tool for the studies of dynamics of structures and structural systems under random loadings. There are books on statistical dynamics of structures [5, 6] and books on structural dynamics with chapter(s) dealing with random response analysis [7, 8]. In addition, there are various monographs and lecture notes on the subject. However, a systematic treatment of the stochastic structural dynamics applying the FEM seems to be lacking. The present book is believed to be the first relatively in-depth and systematic treatment of the subject that applies the FEM to the field of stochastic structural dynamics.

Before the introduction to the concept and theory of stochastic quantities and their applications with the FEM in subsequent chapters, the two FEM employed in the investigations presented in the present book are outlined in this chapter. Specifically, Section 1.1 is concerned with the derivation of the temporally stochastic element equation of motion applying the displacement formulation. The consistent element stiffness and mass matrices of two beam elements, each having two nodes are derived. One beam element is uniform and the other is tapered. The corresponding temporally and spatially stochastic element equation of motion is derived in Section 1.2. The element equations of motion based on the mixed formulation are introduced in Section 1.3. Consistent element matrices for a beam of uniform cross-sectional area are obtained. This beam element has two nodes, each of which has two degrees-of-freedom (dof). This beam element is applied to show that stiffness matrices derived from the displacement and mixed formulations are identical. The incremental variational principle and element matrices based on the mixed formulation for nonlinear

structures are presented in Section 1.4. Section 1.5 deals with constitutive relations and updating of configurations and stresses. Closing remarks for this chapter are provided in Section 1.6.

1.1 Displacement Formulation-Based Finite Element Method

Without loss of generality and as an illustration, the displacement formulation based element equations of motion for temporally stochastic linear systems are presented in this section. These equations are similar in form to those under deterministic excitations. It is included in Sub-section 1.1.1 while application of the technique for the derivation of element matrices of a two-node beam element of uniform cross-section is given in Sub-section 1.1.2. The tapered beam element is presented in Sub-section 1.1.3.

1.1.1 Derivation of element equations of motion

The Rayleigh-Ritz (RR) method approximates the displacement by a linear set of admissible functions that satisfy the geometric boundary conditions and are p times differentiable over the domain, where p is the number of boundary conditions that the displacement must satisfy at every point of the boundary of the domain. The admissible functions required by the RR method are constructed employing the finite element displacement method with the following steps:

- (a) idealization of the structure by choosing a set of imaginary reference or node points such that on joining these node points by means of imaginary lines a series of finite elements is formed;
- (b) assigning a given number of dof, such as displacement, slope, curvature, and so on, to every node point; and
- (c) constructing a set of functions such that every one corresponds to a unit value of one dof, with the others being set to zero.

Having constructed the admissible functions, the element matrices are then determined. For simplicity, the damping matrix of the element will be disregarded. Thus, in the following the definition of consistent element mass and stiffness matrices in terms of deformation patterns usually referred to as shape functions is given.

Assuming the displacement $u(x,t)$ or simply u at the point x (for example, in the three-dimensional case it represents the local co-ordinates r, s and t at the point) within the e 'th element is expressed in matrix form as

$$u(x,t) = N(x)q(t) , \quad (1.1)$$

where $N(\mathbf{x})$ or simply N is a matrix of element shape functions, and $\mathbf{q}(t)$ or $\mathbf{q}$ a matrix of nodal dof with reference to the local axes, also known as the vector of nodal displacements or generalized displacements.

The matrix of strain components $\boldsymbol{\varepsilon}$ thus takes the form

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q} , \quad (1.2)$$

where $\mathbf{B}$ is a differential of the shape function matrix N .

The matrix of stress components $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} , \quad (1.3)$$

where $\mathbf{D}$ is the elastic matrix.

Substituting Eq. (1.2) into (1.3) gives

$$\boldsymbol{\sigma} = \mathbf{D}\mathbf{B}\mathbf{q} . \quad (1.4)$$

In order to derive the element equations of motion for a conservative system, the Hamilton's principle can be applied

$$L = T - (U + W) , \quad (1.5)$$

where T and $(U + W)$ are the kinetic and potential energies, respectively.

It may be appropriate to note that for a non-conservative system or system with non-holonomic boundary conditions, the modified Hamilton's principle [9] or the virtual power principle [10, 11] may be applied. Non-holonomic systems are those with constraint equations containing velocities which cannot be integrated into relations in co-ordinates or displacements only. An example of a non-holonomic system is the bicycle moving down an inclined plane in which enforcing no slipping at the contact point gives rise to non-holonomic constraint equations. Another example is a disk rolling on a horizontal plane. In this case enforcing no slipping at the contact point also give rise to non-holonomic constraint equations.

The kinetic energy density of the element is defined as

$$dT = \frac{1}{2} \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} dV \quad (1.6)$$

where ρ is the density of the material, dV is the incremental volume, and the over-dot denotes the differentiation with respect to time t .

By making use of Eq. (1.6), the kinetic energy of the element becomes

$$T = \frac{1}{2} \iiint_V \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} dV . \quad (1.7)$$

The strain energy density for a linear elastic body is defined as

$$dU = \frac{1}{2} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \frac{1}{2} \boldsymbol{\varepsilon}^T D \boldsymbol{\varepsilon} dV. \quad (1.8)$$

The potential energy for a linearly elastic body can be expressed as the sum of internal work, the strain energy due to internal stress, and work done by the body forces and surface tractions. Thus,

$$U + W = \iiint_V dU - \iiint_V \mathbf{u}^T \bar{Q} dV - \iint_S \mathbf{u}^T \bar{Y} dS, \quad (1.9)$$

where S now is the surface of the body on which surface tractions $\bar{Y}$ are prescribed. The last two integrals on the right-hand side (rhs) of Eq. (1.9) represent the work done by the external random forces, the body forces $\bar{Q}$ and surface tractions $\bar{Y}$. In the last equation the over-bar of a letter designates the quantity is specified.

Applying Eq. (1.8), the total potential of the element from Eq. (1.9) becomes

$$U + W = \frac{1}{2} \iiint_V \boldsymbol{\varepsilon}^T D \boldsymbol{\varepsilon} dV - \iiint_V \mathbf{u}^T \bar{Q} dV - \iint_S \mathbf{u}^T \bar{Y} dS. \quad (1.10)$$

Substituting Eqs. (1.7) and (1.10) into (1.5), the functional of a linearly elastic element,

$$L = \frac{1}{2} \iiint_V (\rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} - \boldsymbol{\varepsilon}^T D \boldsymbol{\varepsilon} + 2\mathbf{u}^T \bar{Q}) dV + \iint_S \mathbf{u}^T \bar{Y} dS. \quad (1.11)$$

On substituting Eqs. (1.1) through (1.3) into the last equation and using the matrix relation $(XY)^T = Y^T X^T$, the Lagrangian becomes

$$L = \frac{1}{2} \iiint_V (\rho \dot{\mathbf{q}}^T N^T N \dot{\mathbf{q}} - \mathbf{q}^T B^T D B \mathbf{q} + 2\mathbf{q}^T N^T \bar{Q}) dV + \iint_S \mathbf{q}^T N^T \bar{Y} dS. \quad (1.12)$$

Applying Hamilton's principle, it leads to

$$\int_{t_1}^{t_2} \left(\delta \dot{q}^T \iiint_V \rho N^T N dV \dot{q} - \delta q^T \iiint_V B^T D B dV q + \delta q^T \iiint_V N^T \bar{Q} dV + \delta q^T \int_S N^T \bar{Y} dS \right) dt = 0. \quad (1.13)$$

Integrating the first term inside the brackets on the left-hand side (lhs) of Eq. (1.13) by parts with respect to time t results

$$\begin{aligned} & \int_{t_1}^{t_2} \delta \dot{q}^T \iiint_V \rho N^T N dV \dot{q} dt \\ &= \left[\delta q^T \iiint_V \rho N^T N dV \dot{q} \right]_{t_1}^{t_2} \\ & \quad - \int_{t_1}^{t_2} \delta q^T \iiint_V \rho N^T N dV \ddot{q} dt. \end{aligned} \quad (1.14)$$

According to Hamilton's principle, the tentative displacement configuration must satisfy given conditions at times t_1 and t_2 , that is,

$$\delta q(t_1) = 0, \quad \delta q(t_2) = 0.$$

Hence, the first term on the rhs of Eq. (1.14) vanishes.

Substituting Eq. (1.14) into (1.13) and rearranging, it becomes

$$\begin{aligned} & \int_{t_1}^{t_2} \delta q^T \left(\iiint_V \rho N^T N dV \ddot{q} + \iiint_V B^T D B dV q \right. \\ & \quad \left. - \iiint_V N^T \bar{Q} dV - \int_S N^T \bar{Y} dS \right) dt = 0. \end{aligned} \quad (1.15)$$

As the variations of the nodal displacements δq are arbitrary, the expressions inside the parentheses must be equal to zero in order that Eq. (1.15) is satisfied. Therefore, the equation of motion for the e 'th element in matrix form is

$$m\ddot{q} + kq = f, \quad (1.16)$$

where the element mass and stiffness matrices are defined, respectively as

$$m = \iiint_V \rho N^T N dV, \quad k = \iiint_V B^T DB dV,$$

and the element random load matrix

$$f = \iiint_V N^T \bar{Q} dV + \iint_S N^T \bar{Y} dS.$$

Applying the generalized co-ordinate form of displacement model the displacement can be expressed as

$$u = \Phi \zeta, \quad (1.17)$$

where Φ is a matrix of function of variables x and ζ is the vector of generalized co-ordinates, also known as generalized displacement amplitudes. The coefficient matrix may be determined by introducing the nodal co-ordinates successively into Eq. (1.17) such that the vector u and matrix Φ become the nodal displacement vector q and coefficient matrix C , respectively. That is,

$$q = C\zeta. \quad (1.18)$$

Hence, the generalized displacement amplitude vector

$$\zeta = C^{-1}q, \quad (1.19)$$

where C^{-1} is the inverse of the coefficient matrix also known as the transformation matrix and is independent of the variables x .

Substituting Eq. (1.19) into (1.17) one has

$$u = \Phi C^{-1}q. \quad (1.20)$$

Comparing Eqs. (1.1) and (1.20), one has the shape function matrix

$$N = \Phi C^{-1}. \quad (1.21)$$

On application of Eqs. (1.16) and (1.21), the element mass, stiffness and load matrices can be evaluated.

To provide a more concrete illustration of the shape function matrix and a better understanding of the steps in the derivation of element mass and stiffness matrices, a uniform beam element is considered in the next sub-section.

1.1.2 Mass and stiffness matrices of uniform beam element

The uniform beam element considered in this sub-section has two nodes, each of which has two dof. The latter include nodal transverse displacement, and rotation or angular displacement about an axis perpendicular to the plane containing the beam and the transverse displacement. For simplicity, the theory of the Euler beam is assumed. The cross-sectional area A and second moment of area I are constant. Let ρ and E be the density and modulus of elasticity of the beam. The bending beam element is shown in Figure 1.1 where the edge displacements and angular displacements are included. The convention adopted in the figure is sagging being positive.

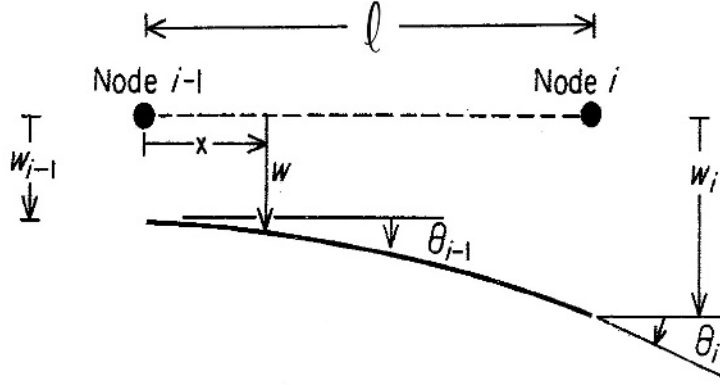


Figure 1.1 Uniform beam element with edge displacements.

Applying Eq. (1.17) so that the transverse displacement at a point inside the beam element can be written as

$$u = w = \Phi \zeta, \quad \Phi = [1 \quad x \quad x^2 \quad x^3] \quad (1.22a, b)$$

$$\zeta^T = [\zeta_1 \quad \zeta_2 \quad \zeta_3 \quad \zeta_4]. \quad (1.22c)$$

Consider the nodal values. At $x = 0$, $w = w_{i-1}$ and $\theta = \partial w / \partial x = \theta_{i-1}$ so that upon application of Eq. (1.22a) one has

$$\mathbf{w}_{i-1} = [1 \ 0 \ 0 \ 0] \boldsymbol{\zeta}, \quad \boldsymbol{\theta}_{i-1} = [0 \ 1 \ 0 \ 0] \boldsymbol{\zeta}. \quad (1.23a, b)$$

Similarly, at $x = \ell$, $w = w_i$ and $\theta = \theta_i$ so that upon application of Eq. (1.22a) it leads to

$$\mathbf{w}_i = [1 \ \ell \ \ell^2 \ \ell^3] \boldsymbol{\zeta}, \quad \boldsymbol{\theta}_i = [0 \ 1 \ 2\ell \ 3\ell^2] \boldsymbol{\zeta}. \quad (1.23c, d)$$

Re-writing Eq. (1.23) in matrix form as in Eq. (1.18), one has

$$\mathbf{q} = \begin{pmatrix} \mathbf{w}_{i-1} \\ \boldsymbol{\theta}_{i-1} \\ \mathbf{w}_i \\ \boldsymbol{\theta}_i \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix}. \quad (1.24)$$

Thus, the inverse of matrix C becomes

$$C^{-1} = \frac{1}{\ell^3} \begin{bmatrix} \ell^3 & 0 & 0 & 0 \\ 0 & \ell^3 & 0 & 0 \\ -3\ell & -2\ell^2 & 3\ell & -\ell^2 \\ 2 & \ell & -2 & \ell \end{bmatrix}. \quad (1.25)$$

Making use of Eqs. (1.22b) and (1.25), the shape function matrix by Eq. (1.21) is obtained as

$$\mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \end{bmatrix}, \quad (1.26)$$

in which

$$\begin{aligned} N_{11} &= 1 - 3\xi^2 + 2\xi^3, & N_{12} &= x(1 - 2\xi + \xi^2), & \xi &= \frac{x}{\ell}, \\ N_{13} &= 3\xi^2 - 2\xi^3, & N_{14} &= -x(\xi - \xi^2). \end{aligned}$$

Substituting Eq. (1.26) into the equation for element mass matrix defined in Eq. (1.16), one can show that

$$\mathbf{m} = \rho A \int_0^\ell \mathbf{N}^T \mathbf{N} dx = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ \cdot & 4\ell^2 & 13\ell & -3\ell^2 \\ \text{symmetric} & \cdot & 156 & -22\ell \\ \cdot & \cdot & \cdot & 4\ell^2 \end{bmatrix}. \quad (1.27)$$

Similarly, the element stiffness matrix is obtained as

$$\mathbf{k} = \frac{EI}{\ell^6} \int_0^\ell \mathbf{B}^T \mathbf{B} dx = \frac{2EI}{\ell^3} \begin{bmatrix} 6 & 3\ell & -6 & 3\ell \\ \cdot & 2\ell^2 & -3\ell & \ell^2 \\ \text{symmetric} & \cdot & 6 & -3\ell \\ \cdot & \cdot & \cdot & 2\ell^2 \end{bmatrix}, \quad (1.28)$$

in which

$$\mathbf{B} = \frac{\partial^2 \mathbf{N}}{\partial x^2} = [B_{11} \ B_{12} \ B_{13} \ B_{14}], \quad B_{11} = 12x - 6\ell,$$

$$B_{12} = 6x\ell - 4\ell^2, \quad B_{13} = -12x + 6\ell, \quad B_{14} = 6x\ell - 2\ell^2.$$

1.1.3 Mass and stiffness matrices of higher order taper beam element

The tapered beam element considered in this sub-section has two nodes, each of which has four dof. The latter include nodal displacement, rotation or angular displacement, curvature, and shear dof. This is the higher order tapered beam element first developed and presented by the author [12].

The tapered beam element of length ℓ , shown in Figure 1.2, is assumed to be of homogeneous and isotropic material. Its cross-sectional area and second moment of area are, respectively given by

$$A(x) = c_1 b(s) d(x), \quad I(x) = c_2 b(s) d^3(x), \quad (1.29)$$

where c_1 and c_2 depend on the shape of the beam cross-section. For an elliptic-type closed curve cross-section, they are given by [13]

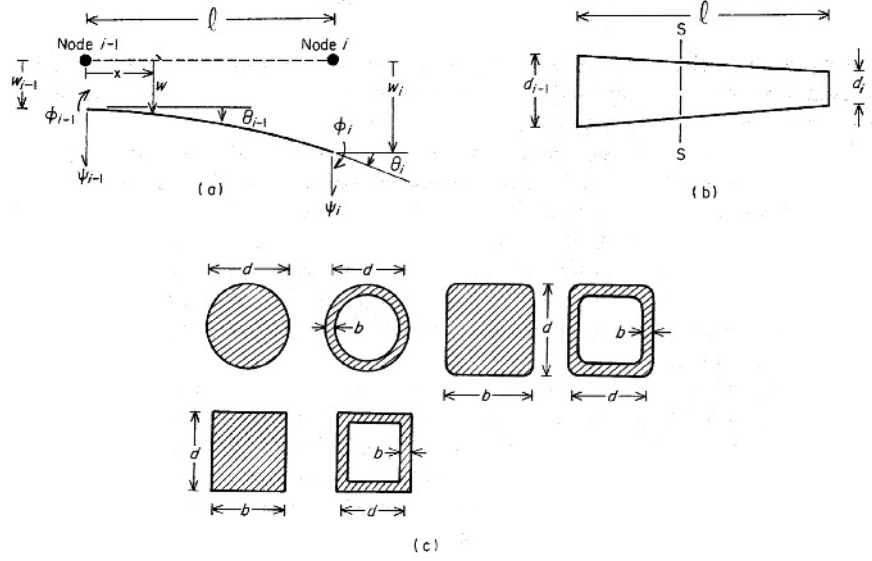


Figure 1.2 Linearly tapered beam element: (a) beam element with edge forces; (b) tapered beam element; (c) cross-section at section S-S in (b).

$$c_1 = \frac{\Gamma\left(\frac{1}{\mu_1} + 1\right) \Gamma\left(\frac{1}{\mu_2} + 1\right)}{\Gamma\left(\frac{1}{\mu_1} + \frac{1}{\mu_2} + 1\right)}, \quad c_2 = \frac{\Gamma\left(\frac{1}{\mu_1} + 1\right) \Gamma\left(\frac{3}{\mu_2} + 1\right)}{12 \Gamma\left(\frac{1}{\mu_1} + \frac{3}{\mu_2} + 1\right)}, \quad (1.30a, b)$$

in which $\Gamma(\cdot)$ is the gamma function, and μ_1 and μ_2 are real positive numbers which need not be integers. When $\mu_1 = \mu_2 = 1$, the cross-section is a triangle and in this case the factor $1/12$ in c_2 should be replaced by $1/9$. When $\mu_1 = \mu_2 = 2$, the cross-section is an ellipse. As μ_1 and μ_2 each approaches infinity, it is a rectangle.

The cross-sectional dimensions, $b(x)$ and $d(x)$, vary linearly along the length of the element so that

$$b(x) = b_{i-1} \left[1 + (\alpha - 1) \frac{x}{\ell} \right], \quad d(x) = d_{i-1} \left[1 + (\beta - 1) \frac{x}{\ell} \right], \quad (1.31a, b)$$

where $\alpha = b_i/b_{i-1}$ and $\beta = d_i/d_{i-1}$ are the taper ratios for the beam element.

Substituting Eq. (1.31) into (1.29) leads to

$$A(x) = A_{i-1} \left(1 + \gamma_1 \xi + \gamma_2 \xi^2 \right), \quad (1.32a)$$

$$I(x) = I_{i-1} \left(1 + \delta_1 \xi + \delta_2 \xi^2 + \delta_3 \xi^3 + \delta_4 \xi^4 \right), \quad (1.32b)$$

$$\xi = \frac{x}{\ell}, \quad \gamma_1 = (\alpha - 1) + (\beta - 1), \quad \gamma_2 = (\alpha - 1)(\beta - 1),$$

$$\delta_1 = (\alpha - 1) + 3(\beta - 1), \quad \delta_2 = 3(\alpha - 1)(\beta - 1) + 3(\beta - 1)^2,$$

$$\delta_3 = 3(\alpha - 1)(\beta - 1)^2 + (\beta - 1)^3, \quad \delta_4 = (\alpha - 1)(\beta - 1)^3,$$

A_{i-1} and I_{i-1} are respectively the cross-sectional area and second moment of area associated with Node $i - 1$.

It should be noted that in applying Eq. (1.32) to hollow beams, of square or circular cross-section, for instance, either the ratio b/d must be small or the ratio b/d must be constant because in Eq. (1.29) for a square hollow cross-section $c_1 = 4$ and $c_2 = (2/3)[1 + (b/d)^2]$, and for a circular hollow cross-section $c_1 = \pi$ and $c_2 = (\pi/8)[1 + (b/d)^2]$.

With the cross-sectional area and second moment of area defined, the element mass and stiffness matrices can be derived accordingly. To this end let the transverse displacement of the beam element be

$$w = \sum_{j=1}^8 \zeta_j x^{j-1}, \quad \text{or} \quad w = \Phi \zeta, \quad (1.33)$$

where the row and column vectors are respectively

$$\Phi = [1 \ x \ x^2 \ x^3 \ x^4 \ x^5 \ x^6 \ x^7], \quad \zeta = [\zeta_1 \ \zeta_2 \ \zeta_3 \ \zeta_4 \ \zeta_5 \ \zeta_6 \ \zeta_7 \ \zeta_8]^T.$$

Equation (1.33) can be identified as Eq. (1.17) in which the displacement function u is replaced by w . Thus, the nodal displacement vector in Eq. (1.18) for the present tapered beam element becomes

$$q = [w_{i-1} \ \theta_{i-1} \ \phi_{i-1} \ \psi_{i-1} \ w_i \ \theta_i \ \phi_i \ \psi_i]^T, \quad \phi_i = \frac{\partial^2 w_i}{\partial x^2}, \quad \psi_i = \frac{\partial^3 w_i}{\partial x^3}.$$

The corresponding coefficient matrix in Eq. (1.18) is obtained as [12]

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 & \ell^4 & \ell^5 & \ell^6 & \ell^7 \\ 0 & 1 & 2\ell & 3\ell^2 & 4\ell^3 & 5\ell^4 & 6\ell^5 & 7\ell^6 \\ 0 & 0 & 2 & 6\ell & 12\ell^2 & 20\ell^3 & 30\ell^4 & 42\ell^5 \\ 0 & 0 & 0 & 6 & 24\ell & 60\ell^2 & 120\ell^3 & 210\ell^4 \end{bmatrix}. \quad (1.34)$$

The inverse of matrix C can be found to be [12]

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ -35/\ell^4 & -20/\ell^3 & -5/\ell^2 & -2/3\ell & 35/\ell^4 & -15/\ell^3 & 5/2\ell^2 & -1/6\ell \\ 84/\ell^5 & 45/\ell^4 & 10/\ell^3 & 1/\ell^2 & -84/\ell^5 & 39/\ell^4 & -7/\ell^3 & 1/2\ell^2 \\ -70/\ell^6 & -36/\ell^5 & -15/2\ell^4 & -2/3\ell^3 & 70/\ell^6 & -34/\ell^5 & 13/2\ell^4 & -1/2\ell^3 \\ 20/\ell^7 & 10/\ell^6 & 2/\ell^5 & 1/6\ell^4 & -20/\ell^7 & 10/\ell^6 & -2/\ell^5 & 1/6\ell^4 \end{bmatrix}.$$

With this inverse matrix and operating on Eq. (1.21) one can obtain the shape function matrix for the present higher order tapered beam element as

$$N = [N_{11} \ N_{12} \ N_{13} \ N_{14} \ N_{15} \ N_{16} \ N_{17} \ N_{18}], \quad (1.35)$$

where the shape functions are defined by

$$N_{11} = 1 - 35\xi^4 + 84\xi^5 - 70\xi^6 + 20\xi^7,$$

$$N_{12} = \xi \ell (1 - 20\xi^3 + 45\xi^4 - 36\xi^5 + 10\xi^6),$$

$$N_{13} = \frac{1}{2}(\xi \ell)^2 (1 - 10\xi^2 + 20\xi^3 - 15\xi^4 + 4\xi^5),$$

$$N_{14} = \frac{1}{6}(\xi \ell)^3 (1 - 4\xi + 6\xi^2 - 40\xi^3 + \xi^4),$$

$$N_{15} = 35\xi^4 - 84\xi^5 + 70\xi^6 - 20\xi^7,$$

$$N_{16} = \xi \ell (39\xi^4 - 15\xi^3 - 34\xi^5 + 10\xi^6),$$

$$N_{17} = \frac{1}{2}(\xi \ell)^2 (5\xi^2 - 14\xi^3 + 13\xi^4 - 4\xi^5),$$

$$N_{18} = \frac{1}{6}(\xi \ell)^3 (3\xi^2 - \xi - 3\xi^3 + \xi^4).$$

By making use of Eqs. (1.28) and (1.16), one can find the mass and stiffness matrices of the tapered beam element. These element matrices are given in Appendix 1A.

1.2 Element Equations of Motion for Temporally and Spatially Stochastic Systems

The displacement based FEM presented in Section 1.1 can straightforwardly be extended to temporally and spatially stochastic systems. Without loss of generality and for easy understanding in the following presentation the notation applied in the last section is adopted in this section.

Consider now the elastic matrix in Eq. (1.2) is replaced by the following spatially stochastic elastic matrix,

$$D_h = D + \tilde{D}, \quad (1.36)$$

in which D is the deterministic elastic matrix while the second term on the rhs is the spatially stochastic component of the elastic matrix whose ensemble average is zero such that the element stiffness matrix with spatially stochastic elastic component becomes

$$k_h = k + \tilde{k}, \quad (1.37)$$

where the element stiffness matrices associated with the deterministic and spatially stochastic components are, respectively

$$\mathbf{k} = \iiint_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV, \quad \tilde{\mathbf{k}} = \iiint_V \mathbf{B}^T \tilde{\mathbf{D}} \mathbf{B} dV. \quad (1.38a, b)$$

To provide a simple example, suppose the modulus of elasticity of the material is spatially stochastic such that it can be written as

$$\mathbf{E}_h = \mathbf{E} + \tilde{\mathbf{E}}, \quad (1.39)$$

where $\mathbf{E}$ is the deterministic component of the modulus of elasticity whereas the second term on the rhs of Eq. (1.39) is the spatially stochastic component of the modulus of elasticity with zero ensemble in the spatial domain.

With reference to Eq. (1.38b), the spatially stochastic component of the stiffness matrix can be written as

$$\mathbf{k}_h = \mathbf{k} + \tilde{\mathbf{k}} = \mathbf{k} + \mathbf{r}\boldsymbol{\varphi}, \quad (1.40)$$

where the spatially stochastic component of the stiffness matrix is $\mathbf{r}\boldsymbol{\varphi}$.

Substituting Eq. (1.40) into (1.16), the element equations of motion for the temporally and spatially stochastic system becomes

$$\mathbf{m}\ddot{\mathbf{q}} + (\mathbf{k} + \mathbf{r}\boldsymbol{\varphi})\mathbf{q} = \mathbf{f}. \quad (1.41)$$

For systems with other spatially stochastic material properties, similar element equations of motion can be obtained accordingly. Note that the spatially stochastic matrix $\mathbf{r}$ can have large stochastic variation. This is different from that applying the SFEM or PFEM in which the spatially stochastic variation is limited to a small quantity.

Applying Eq. (1.41) for the entire system, the assembled equation of motion can be constructed in the usual manner.

1.3 Hybrid Stress-Based Element Equations of Motion

The main objective of this section is to provide the element equations of motion by applying the hybrid stress FEM pioneered by Pian [14]. In addition, it is shown by way of derivation of the element mass and stiffness matrices that the hybrid stress-based FEM can give results identical to those obtained by the displacement formulation-based FEM. The hybrid stress-based formulation is presented in Sub-section 1.3.1 whereas the derivation of the element matrices is included in Sub-section 1.3.2.

1.3.1 Derivation of element equations of motion

The Hellinger-Reissner's variational principle is adopted in this sub-section

$$\begin{aligned} \pi_{HR} = & \int \int \int_V \left(\sigma^T \mathfrak{L} u - \frac{1}{2} \sigma^T C \sigma \right) dV - \int \int \int_V b^T u dV \\ & - \int \int_{S_u} (\Gamma \sigma)^T (u - \bar{u}) dS - \int \int_{S_t} \tau^T u dS, \end{aligned} \quad (1.42)$$

where σ is the stress vector, u is the displacement vector, C is the compliance matrix, b is the body force vector, τ is the prescribed traction vector on boundary S_t , $\bar{u}$ is the prescribed displacement on boundary S_u , $\mathfrak{L}$ is the linear differential operator to derive strain from displacement, and Γ is the linear differential operator to evaluate surface traction from stress.

In dynamic problems, one can introduce the kinetic energy term to Eq. (1.42) such that a new functional is formed

$$\begin{aligned} T - \pi_{HR} = & \frac{1}{2} \int \int \int_V \rho \dot{u}^T \dot{u} dV - \int \int \int_V \left(\sigma^T \mathfrak{L} u - \frac{1}{2} \sigma^T C \sigma \right) dV \\ & + \int \int \int_V b^T u dV + \int \int_{S_u} (\Gamma \sigma)^T (u - \bar{u}) dS + \int \int_{S_t} \tau^T u dS. \end{aligned} \quad (1.43)$$

It is observed that the lhs of Eq. (1.43) can be identified as the Lagrangian in Eq. (1.11). Therefore, Hamilton's principle can be applied. A formal presentation is included in Section 1.4 for nonlinear dynamic problems.

Returning to the linear element equation of motion, the assumed displacement field and assumed stress field are, respectively, given by

$$u = Nq, \quad \sigma = P\beta, \quad (1.44a, b)$$

where β , different from that in Eq. (1.31b), is the vector of stress parameters, q and N are defined in Eq. (1.1), and P is the stress shape function matrix.

Substituting Eq. (1.44) into (1.43) and some manipulation, one has

$$T - \pi_{HR} = \pi_{DHR} = \frac{1}{2} \dot{q}^T m \dot{q} - \frac{1}{2} \beta^T H \beta - \beta^T G q - f^T q, \quad (1.45)$$

where π_{DHR} is similar to L in Eq. (1.11), f the nodal force vector, and

$$H = \iiint_V P^T C P dV, \quad G = \iiint_V P^T \mathcal{Q} N dV, \quad m = \iiint_V \rho N^T N dV,$$

in which H and G are known as, respectively, the generalized stiffness matrix and leverage matrix.

For π_{DHR} in Eq. (1.45) to have an extremum value, one has

$$\delta \pi_{DHR} = \frac{\partial \pi_{DHR}}{\partial t} \delta t + \frac{\partial \pi_{DHR}}{\partial \beta} \delta \beta + \frac{\partial \pi_{DHR}}{\partial q} \delta q = 0, \quad (1.46)$$

in which it is understood that the division of a quantity by a vector is not permitted in matrix operations. However, it is used as an abbreviation for the partial differentiation of all the elements or entries in the vector concerned. Since δt , $\delta \beta$, and δq are all arbitrary, Eq. (1.46) holds only if the following equations are satisfied

$$\frac{\partial \pi_{DHR}}{\partial t} = 0, \quad \frac{\partial \pi_{DHR}}{\partial \beta} = 0, \quad \frac{\partial \pi_{DHR}}{\partial q} = \frac{\partial \pi_{DHR}}{\partial t} \left(\frac{\partial t}{\partial q} \right) = 0. \quad (1.47a, b, c)$$

When Eq. (1.47a) is satisfied, it simultaneously satisfies Eq. (1.47c). Thus, Eqs. (1.47b) and (1.47a) give, respectively

$$-H\beta - Gq = 0, \quad \ddot{q}^T m - \beta^T G - f^T = 0. \quad (1.48a, b)$$

From Eq. (1.48a), one has

$$\beta = -H^{-1}Gq. \quad (1.49)$$

Substituting Eq. (1.49) into the transpose of Eq. (1.48b) yields

$$m\ddot{q} + G^T H^{-1} G q - f = 0, \quad m^T = m.$$

This equation becomes Eq. (1.16) with the definition,

$$k = G^T H^{-1} G. \quad (1.50)$$

Once the nodal displacement vector is determined, it is substituted into Eq. (1.49) which is substituted, in turn, into Eq. (1.44b) to recover the stress vector.

1.3.2 Mass and stiffness matrices of uniform beam element

To provide an understanding of the steps involved in the derivation of element mass and stiffness matrices by applying the hybrid stress or mixed formulation, and for illustration as well as for simplicity, the beam element of uniform cross-

sectional area A and length ℓ as shown in Figure 1.1 is considered in this sub-section. Its material is assumed to be isotropic and homogeneous. It has two nodes, each of which has two dof as in Sub-section 1.1.2. Thus, the shape function matrix in Eq. (1.44a) is identical to that in Eq. (1.26). The assumed stress shape functions and stress parameters are related to the stress by the following equation

$$\sigma = P\beta = [1 \quad x \quad x^2]\beta, \quad \beta = [\beta_1 \quad \beta_2 \quad \beta_3]^T.$$

Applying the definitions in Eq. (1.45), the element mass matrix is identical to that given by Eq. (1.27) since it is the same beam element with identical shape functions. Similarly, the generalized stiffness matrix becomes

$$H = \frac{1}{EI} \int_0^\ell \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} [1 \quad x \quad x^2] dx = \frac{\ell}{60EI} \begin{bmatrix} 60 & 30\ell & 20\ell^2 \\ . & 20\ell^2 & 15\ell^3 \\ \text{symmetric} & . & 12\ell^4 \end{bmatrix}. \quad (1.51)$$

The inverse of the generalized stiffness matrix is found as

$$H^{-1} = \frac{3EI}{\ell} \begin{bmatrix} 3 & -12/\ell & 10/\ell^2 \\ . & 64/\ell^2 & -60/\ell^3 \\ \text{symmetric} & . & 60/\ell^4 \end{bmatrix}. \quad (1.52)$$

The leverage matrix is

$$G = \int_0^\ell \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \frac{\partial^2 N}{\partial x^2} dx = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & \ell \\ \ell & \ell^2/6 & -\ell & 5\ell^2/6 \end{bmatrix}. \quad (1.53)$$

By making use of Eqs. (1.52), (1.53), (1.50), and after some manipulation, the element stiffness matrix is found to be identical to that given by Eq. (1.28).

Now, consider a lower order stress shape function matrix so that

$$\sigma = P\beta = [1 \quad x]\beta, \quad \beta = [\beta_1 \quad \beta_2]^T.$$

In this case, the generalized stiffness matrix becomes

$$H = \frac{1}{EI} \int_0^\ell \begin{pmatrix} 1 \\ x \end{pmatrix} [1 \quad x] dx = \frac{\ell}{6EI} \begin{bmatrix} 6 & 3\ell \\ 3\ell & 2\ell^2 \end{bmatrix}. \quad (1.54)$$

The inverse of this matrix is

$$H^{-1} = \frac{2EI}{\ell^3} \begin{bmatrix} 2\ell^2 & -3\ell \\ -3\ell & 6 \end{bmatrix}. \quad (1.55)$$

The corresponding leverage matrix is

$$G = \int_0^\ell \begin{pmatrix} 1 \\ x \end{pmatrix} \frac{\partial^2 N}{\partial x^2} dx = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & \ell \end{bmatrix}. \quad (1.56)$$

By making use of Eqs. (1.55), (1.56), and (1.50), one arrives at the identical element stiffness matrix defined by Eq. (1.28) which agrees with that presented in [15, 16]. This confirms the fact that the displacement and hybrid stress formulations are equivalent [17].

The two and three stress parameters of the assumed stress fields give an identical element stiffness matrix because they satisfy the Tong, Pian and Chen condition [18, 19],

$$n_s \geq n_d - n_r, \quad (1.57)$$

where n_s is the number of assumed stress modes, n_d is the number of generalized displacements, and n_r is the number of zero-eigenvalues or rigid-body modes. In the three stress parameter case, the number of assumed stress modes $n_s = 3$, the number of generalized displacements $n_d = 4$, and the number of rigid-body modes $n_r = 2$. Therefore, Eq. (1.57) is satisfied. Similarly, for the two stress parameter case, $n_s = 2$, $n_d = 4$, and $n_r = 2$. Thus, Eq. (1.57) is also satisfied and therefore identical element stiffness matrix is obtained.

1.4 Incremental Variational Principle and Mixed Formulation-Based Nonlinear Element Matrices

In this section the incremental mixed formulation for nonlinear element equations of motion and derivation of the nonlinear mass and stiffness matrices

for lower order flat triangular shell elements are presented. The formulation and derivation of element matrices closely follow those presented in [20, 21]. As the detailed presentation has been provided in the latter reference, the following is an outline of [21] with some changes of symbols in the present section.

In addition to other advantages over the displacement formulation, the two main ones are: (a) the elements derived are free of the locking phenomenon, and (b) the strains and stresses are continuous through the application of Eq. (1.72).

In the following, Sub-section 1.4.1 deals with the formulation and linearization of the nonlinear incremental variational principle while Sub-section 1.4.2 includes an outline of the derivation of shell element matrices.

1.4.1 Incremental variational principle and linearization

Consider the dynamic counterpart of the Hellinger-Reissner variational principle for every element of the nonlinear structural systems [22] where the kinetic energy of the system is defined by

$$\delta \int_{t_1}^{t_2} (T - \pi_{HR}) d\tau = 0, \quad (1.58)$$

π_{HR} is the Hellinger-Reissner's functional, and the remaining symbols have

$$T = \frac{1}{2} \iiint_V \rho \dot{\mathbf{u}}^T \dot{\mathbf{u}} dV, \quad (1.59)$$

their usual meaning. Note that the integrand in Eq. (1.58) is π_{DHR} in Eq. (1.45).

For nonlinear dynamic systems the incremental approach is adopted in the response analysis. Therefore, in the following the incremental variational principle and linearization are first presented.

As pointed out in [20, 21], the fundamental difficulty in any nonlinear analysis is the unknown configuration of a body at time $(t+\Delta t)$. In obtaining an approximate solution, the static and kinematic variables in the current configuration C^t of the incremental formulation are assumed to be known. Their values in an unknown neighbouring configuration $C^{t+\Delta t}$ at a later time $(t+\Delta t)$ are determined from the known solutions. In the present study the starting point of such an incremental analysis is the incremental or modified Hellinger-Reissner variational principle. It has two independently assumed fields, the incremental generalized displacement and the incremental strain fields. The incremental Hellinger-Reissner variational principle can be written as [21]

$$\begin{aligned} \Delta\pi_{HR}(\Delta u, \Delta e) = & \frac{1}{2} \int_{V^t} \left[-(\Delta e)^T D(\Delta e) + 2\sigma^T(\Delta e^u) \right. \\ & \left. + 2(\Delta e)^T D(\Delta e^u) \right] dV - W^{t+\Delta t}, \end{aligned} \quad (1.60)$$

where the integration is performed over the reference volume V^t ,

Δu is the vector of assumed incremental displacement,

Δe is the vector of assumed incremental Green strain,

Δe^u is the vector of incremental Washizu strain calculated from the vector Δu , see Eq. (1.61) in the following,

D is the time-dependent material elastic matrix so that $\Delta S = D\Delta e$ with ΔS being the incremental second Piola-Kirchhoff (PK2) stress vector, see later in the following,

σ is the Cauchy or true stress vector at time t , and

$W^{t+\Delta t}$ is the work-equivalent term corresponding to prescribed body-force and surface traction in configuration $C^{t+\Delta t}$.

In Eq. (1.60), the component form of the incremental Washizu strain vector Δe^u is given by

$$\begin{aligned} \Delta e^u_{ij} &= \Delta \epsilon_{ij} + \Delta \eta_{ij}, \\ \Delta \epsilon_{ij} &= \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}), \quad \Delta \eta_{ij} = \frac{1}{2}\Delta u_{k,i}\Delta u_{k,j}, \end{aligned} \quad (1.61)$$

where the Einstein summation convention for indices has been adopted for the integer k and the differentiation is with respect to the reference co-ordinates, x_i^t with $i = 1, 2, 3$, at time t .

In the hybrid strain or mixed formulation FEM it is assumed that within every element one can write

$$\Delta u = N\Delta q, \quad \Delta e = P\Delta \beta, \quad (1.62a, b)$$

where Δq is the vector of incremental generalized nodal displacements and $\Delta \beta$ the vector of incremental strain parameters, while N and P are the matrices of displacement interpolation functions and strain interpolation functions. Substituting Eq. (1.62) into Eq. (1.60) gives

$$\Delta\pi_{HR}(\Delta q, \Delta \beta) = U - W^{t+\Delta t}(\Delta q), \quad (1.63)$$

where

$$\begin{aligned}
 U = \frac{1}{2} \int_{V_e} & \left[-(\Delta\beta)^T P^T D P (\Delta\beta) + 2 \sigma^T B_L \Delta q \right. \\
 & + (\Delta q)^T (B_{NL})^T \sigma_1^T B_{NL} (\Delta q) + 2 (\Delta\beta)^T P^T D B_L \Delta q \\
 & \left. + 2 (\Delta e)^T D (\Delta \eta) \right] dV ,
 \end{aligned}$$

V_e is the volume of an element at the reference configuration,

σ_1 is the matrix that contains the Cauchy stress components at t ,

B_L is the linear strain-displacement matrix, and

B_{NL} is the nonlinear strain-displacement matrix.

The last term within the square brackets in the last equation contains the third order product of Δq and $\Delta\beta$ and therefore will be disregarded when linearizing Eq. (1.60) or Eq. (1.63).

Note that in arriving at Eq. (1.63) the following two relations have been applied

$$\Delta \epsilon = (B_L)(\Delta q), \quad \sigma^T \Delta \eta = \frac{1}{2} (\Delta q)^T (B_{NL})^T (\sigma_1)^T (B_{NL})(\Delta q). \quad (1.64a, b)$$

To proceed further one can define

$$H = \int_{V_e} P^T D P dV, \quad G = \int_{V_e} P^T D B_L dV, \quad (1.65a, b)$$

$$k_{NL} = \int_{V_e} (B_{NL})^T \sigma_1 B_{NL} dV, \quad F_1 = \int_{V_e} (B_L)^T \sigma dV, \quad (1.65c, d)$$

and substitutes Eq. (1.65) into Eq. (1.63) to give

$$\Delta \pi_{HR}(\Delta q, \Delta \beta) = \sum_{i=1}^5 \pi_e^{(i)}, \quad (1.66)$$

in which

$$\pi_e^{(1)} = -\frac{1}{2} (\Delta\beta)^T H (\Delta\beta), \quad \pi_e^{(2)} = F_1(\Delta q), \quad \pi_e^{(3)} = \frac{1}{2} (\Delta q)^T k_{NL} (\Delta q),$$

$$\pi_e^{(4)} = (\Delta\beta)^T G (\Delta q), \quad \pi_e^{(5)} = -[F(t+\Delta t)]^T (\Delta q),$$

with $F(t+\Delta t)$, the externally generalized nodal force vector in $C^{t+\Delta t}$ associated

with the $W^{t+\Delta t}(\Delta q)$ term on the rhs of Eq. (1.63).

Applying stationarity to Eq. (1.66) with respect to $\Delta \beta$ yields

$$\Delta \beta = H^{-1} G(\Delta q) . \quad (1.67)$$

Substituting Eq. (1.67) into Eq. (1.66) and applying stationarity with respect to Δq , it results in the following equilibrium equation

$$(k_L + k_{NL})(\Delta q) = F(t+\Delta t) - F_1 , \quad (1.68)$$

where the linear or small displacement element stiffness matrix

$$k_L = G^T H^{-1} G , \quad (1.69)$$

and the matrix k_{NL} on the lhs of Eq. (1.68) and defined by Eq. (1.65c) is the element initial stress stiffness matrix.

On the rhs of Eq. (1.68), F_1 is the pseudo-force vector. One may rewrite $F(t+\Delta t) - F_1$ as $\Delta F + F(t) - F_1$. Thus, $F(t+\Delta t) - F_1$ consists of ΔF , the incremental external force from the current time t to the next time step $(t+\Delta t)$, and $F(t) - F_1$, the equilibrium imbalance at time t . If the equilibrium at time t is satisfied in an average sense, $F(t) - F_1$ vanishes and $F(t+\Delta t) - F_1$ reduces to the increment of external force ΔF from t to $(t+\Delta t)$. Note that Eq. (1.60) differs from those presented by Boland and Pian [23], and Saleeb *et al.* [24]. The variational principle applied in Refs. [23, 24] was based on the hybrid stress formulation and it contained one additional term, which is, in the present notation,

$$\int_{V^t} [(\Delta e)^T D(e - e^u)] dV \quad (1.70)$$

where Δe and D are the same quantities in Eq. (1.60), but e is the vector of Almansi strain accumulated from Δe . The vector e^u is the Almansi strain vector calculated from the total displacements u . In other words,

$$e^u = \frac{1}{2} (u_{i,j} + u_{j,i} - u_{k,i} u_{k,j}) .$$

The aforementioned additional term, Eq. (1.70), accounts for the compatibility mismatch due to inaccurate total displacements and strains. Boland and Pian [23] argued that the term should not be expected to vanish. On the other hand, the numerical experiments of Saleeb *et al.* [24] showed that, though totally discarding the term resulted in convergence difficulties, including the term in only the first iteration of every load step yielded essentially the same results as

those having the term under all circumstances. In the present formulation, the compatibility mismatch term vanishes as a consequence of the presently employed hybrid strain formulation. Specifically, $(\Delta e)^T D(e - e'')$ results in third and fourth order products of strain terms after applying the linearization to the incremental variational principle. Thus, these third and fourth product terms are eliminated after linearization.

The above element level formulas are applied to every element. Once all the element matrices have been determined, they are assembled to form the global equilibrium equation. This equation is then solved for the displacement increments.

Finally, at the element level the incremental Washizu strain can be obtained by using Eqs. (1.62b) and (1.67) as

$$\Delta e = P(\Delta \beta) = PH^{-1}G(\Delta q) , \quad (1.71)$$

whereas the incremental PK2 stress can be found by the following equation

$$\Delta S = D\Delta e . \quad (1.72)$$

1.4.2 Linear and nonlinear element stiffness matrices

This section is concerned with the presentation of an outline of the derivation of element stiffness matrices. The element geometry and co-ordinate systems are introduced first. The assumed incremental displacement field and the assumed incremental strain field within an element are considered subsequently. These are followed by the derivation of the linear and initial stress stiffness matrices. A simplified version of the stiffness matrices from those derived with the nonlinear formulation is included.

1.4.2.1 Element geometry and co-ordinate systems

The finite elements under consideration are the three-node flat triangular shell finite elements. An example is shown in Figure 1.3. The three nodes are located at the three corners of the mid-surface of the element. A local rectangular co-ordinate system is attached to Node 1, with its r -axis coinciding with the side 1-2, its t -axis being parallel to the normal of the element and its s -axis perpendicular to the r - t plane. With such a co-ordinate system, the r and s co-ordinates of Nodes 1, 2, and 3 are: $(0,0)$, $(r_2, 0)$ and (r_3, s_3) , respectively. A director orthogonal frame with components d_r , d_s and d at any points on the mid-surface is also considered. In the undeformed configuration, the director d

coincides with the normal to the mid-surface of the shell element. However, as the shell deforms, the director is, in general, not normal to the mid-surface. Thus, the director orthogonal frame differs from point to point and from the rectangular co-ordinate system. The director orthogonal frame serves as a basis of measuring and recording the change of orientation of points situated on the mid-surface. As indicated in Figure 1.3 the nodal dof are:

- u being the displacement in the r -direction and should not be confused with the assumed displacement vector defined in Eq. (1.62a),
- v being the displacement in the s -direction,
- w being the displacement in the t -direction,
- θ_r being the rotation component about the r -axis,
- θ_s being the rotation component about the s -axis, and
- θ_t being the rotation about the t -axis, or the drilling dof (ddof).

The right-handed screw rule is adopted in the present formulation.

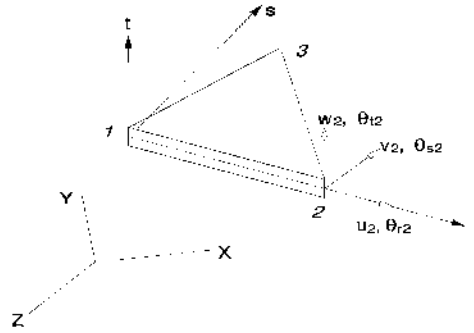


Figure 1.3 Shell element in local and global co-ordinate systems.

1.4.2.2 Assumed incremental displacement field within an element

The isoparametric approach is employed. The local co-ordinates (r, s, t) of an arbitrary point within the element can be written as

$$\begin{pmatrix} r^t \\ s^t \\ t^t \end{pmatrix} = \sum_{i=1}^3 \xi_i \begin{pmatrix} \bar{r}_i^t \\ \bar{s}_i^t \\ 0 \end{pmatrix} + \eta^t \sum_{i=1}^3 \xi_i d_i^t, \quad (1.73)$$

where the superscript t denotes time t and the over bars indicate variables that are defined on the mid-surface. Recall that rotational or angular displacements as well as directors and their increments are defined on the mid-surface. Therefore, over-bars will not be placed on them for conciseness. The symbol d_i^t , with $i = 1, 2, 3$, denotes the director of Node i at time t , while ξ_i are the natural or area co-ordinates of the triangle satisfying

$$0 \leq \xi_i \leq 1, \quad \sum_{i=1}^3 \xi_i = 1, \quad (1.74a,b)$$

and η^t is the co-ordinate along the director direction satisfying

$$-\frac{h^t}{2} \leq \eta^t \leq \frac{h^t}{2} \quad (1.74c)$$

with h^t being the thickness of the shell at time t . Unless stated otherwise, h^t is considered constant over the entire element. The first summation of Eq. (1.73) represents the position of the mid-surface while the second summation indicates that the director orthogonal frame is interpolated in exactly the same way as the mid-surface r and s co-ordinates. This scheme of interpolation is commonly known as the continuum consistent interpolation [25].

The incremental displacements of any point within the element from time t to $t+\Delta t$ can be expressed as

$$\begin{pmatrix} \Delta u^t \\ \Delta v^t \\ \Delta w^t \end{pmatrix} = \sum_{i=1}^3 \xi_i \begin{pmatrix} \Delta \bar{u}_i^t \\ \Delta \bar{v}_i^t \\ \Delta \bar{w}_i^t \end{pmatrix} + \eta^t \sum_{i=1}^3 \xi_i (\Delta d_i^t). \quad (1.75)$$

In this equation the first summation term contains incremental displacements of any point located on the mid-surface. The second summation term includes the change in orientation of director of any point on the mid-surface which is interpolated from Δd_i^t , the increment of director of Node i from time t to $t+\Delta t$. By following the steps in Refs. [20, 21], one can obtain the shape function matrix through the following equation

$$\begin{bmatrix} \Delta u^t & \Delta v^t & \Delta w^t \end{bmatrix}^T = N \begin{bmatrix} \Delta \bar{u}_1^{(0)} & \Delta \theta_1^{(0)} & \dots & \Delta \bar{u}_3^{(0)} & \Delta \theta_3^{(0)} \end{bmatrix}^T, \quad (1.76)$$

where the displacement shape function matrix for the shell element is

$$N = \begin{bmatrix} N_{11} & N_{12} & N_{13} \end{bmatrix}_{3 \times 18}, \quad (1.77)$$

in which the incremental displacement row vectors

$$\Delta \bar{u}_i^{(0)} = [\Delta \bar{u}_i^t \quad \Delta \bar{v}_i^t \quad \Delta \bar{w}_i^t], \quad \Delta \theta_i^{(0)} = [\Delta \theta_{ri}^t \quad \Delta \theta_{si}^t \quad \Delta \theta_{ti}^t], \quad i = 1, 2, 3,$$

and the 3×6 sub-matrix N_{1i} is defined by

$$N_{1i} = \begin{bmatrix} \xi_i & 0 & 0 & \eta^t \xi_i \Lambda_{i(11)}^t & \eta^t \xi_i \Lambda_{i(12)}^t & \bar{p}_i \\ 0 & \xi_i & 0 & \eta^t \xi_i \Lambda_{i(21)}^t & \eta^t \xi_i \Lambda_{i(22)}^t & \bar{q}_i \\ 0 & 0 & \xi_i & \eta^t \xi_i \Lambda_{i(31)}^t - \bar{p}_i & \eta^t \xi_i \Lambda_{i(32)}^t - \bar{q}_i & 0 \end{bmatrix}, \quad (1.78)$$

where the 3×2 matrix Λ_i^t is

$$\Lambda_i^t = \begin{bmatrix} \Lambda_{i(11)}^t & \Lambda_{i(12)}^t \\ \Lambda_{i(21)}^t & \Lambda_{i(22)}^t \\ \Lambda_{i(31)}^t & \Lambda_{i(32)}^t \end{bmatrix} = -\Omega_i^t (\Gamma_i^s)^t, \quad (1.79)$$

where the skew-symmetric matrix

$$\Omega_i^t = \begin{bmatrix} 0 & -d_{ti}^t & d_{si}^t \\ d_{ti}^t & 0 & -d_{ri}^t \\ -d_{si}^t & d_{ri}^t & 0 \end{bmatrix}, \quad (1.80)$$

and $(\Gamma_i^s)^t$ consists of the first two columns of the exponential mapping Γ_i^t which is an orthogonal matrix associated with Node i at time t . The exponential mapping satisfies the following relation

$$d_i^t = \Gamma_i^t e_3, \quad (1.81)$$

with e_3 being the unit vector along the t -axis. That is, $e_3 = [0, 0, 1]^T$. The lhs of Eq. (1.81) is the current position of the director at Node i and is known from configuration updating [20, 21].

That is, the known director is given by

$$d_i^t = R(\Delta\theta) d_i^{t-\Delta t}, \quad (1.82)$$

where the transformation matrix

$$R(\Delta\theta) = \cos(|\Delta\theta|) I_3 + \frac{\sin(|\Delta\theta|)}{|\Delta\theta|} (\Delta\theta^s), \quad (1.83)$$

in which $\Delta\theta = (\Delta\theta_i)^{t-\Delta t}$, and $|\cdot|$ denotes the magnitude of the enclosed vector, I_3 is the unity matrix of order 3 while $\Delta\theta^s$ is a skew-symmetric matrix constructed from $\Delta\theta$,

$$\Delta\theta^s = \begin{bmatrix} 0 & -\Delta\theta_t & \Delta\theta_s \\ \Delta\theta_t & 0 & -\Delta\theta_r \\ -\Delta\theta_s & \Delta\theta_r & 0 \end{bmatrix}. \quad (1.84)$$

Applying Eq. (1.81) the matrix Γ_i^t can be determined and therefore Λ_i^t in Eq. (1.79) can be found.

Substituting Eq. (1.79) into (1.78) the displacement shape functions can be obtained. Note that in Eq. (1.78)

$$\begin{aligned} \bar{p}_1 &= (a_{31}\xi_3 - a_{12}\xi_2)\xi_1, & \bar{p}_2 &= (a_{12}\xi_1 - a_{23}\xi_3)\xi_2, \\ \bar{p}_3 &= (a_{23}\xi_2 - a_{31}\xi_1)\xi_3, & \bar{q}_1 &= (b_{31}\xi_3 - b_{12}\xi_2)\xi_1, \\ \bar{q}_2 &= (b_{12}\xi_1 - b_{23}\xi_3)\xi_2, & \bar{q}_3 &= (b_{23}\xi_2 - b_{31}\xi_1)\xi_3, \end{aligned}$$

with the coefficients on the rhs being defined as

$$\begin{aligned} 2a_{12} &= \ell_{12} \cos\gamma_{12}, & 2b_{12} &= \ell_{12} \sin\gamma_{12}, \\ 2a_{23} &= \ell_{23} \cos\gamma_{23}, & 2b_{23} &= \ell_{23} \sin\gamma_{23}, \\ 2a_{31} &= \ell_{31} \cos\gamma_{31}, & 2b_{31} &= \ell_{31} \sin\gamma_{31}, \end{aligned}$$

in which, as shown in Figure 1.4, ℓ_{ij} is the length of the side of the triangular element joining Nodes i and j .

In closing, it should be noted that because of the incorporation of the ddof in the foregoing formulation, the displacement shape function matrix defined by Eq. (1.78) makes the present finite element sub-parametric since different schemes are employed for the geometry and displacements.

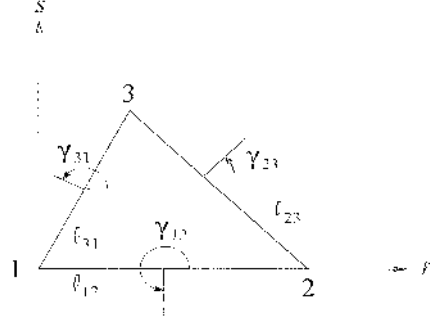


Figure 1.4 Triangular element on the r - s plane of the local co-ordinate system.

1.4.2.3 Assumed incremental strain field within an element

The assumed strain field for any point within the element is defined as

$$\begin{bmatrix} \Delta \epsilon_r & \Delta \epsilon_s & \Delta \epsilon_{rs} & \Delta \epsilon_{st} & \Delta \epsilon_{tr} & \Delta \epsilon_{rs}^s \end{bmatrix}^T = P \begin{bmatrix} \Delta \beta_1 & \Delta \beta_2 & \dots & \Delta \beta_9 & \Delta \beta_{10} \end{bmatrix}^T, \quad (1.85)$$

where the strain shape function matrix is defined by

$$P = \begin{bmatrix} \begin{bmatrix} P_5^{(1)} & P_5^{(2)} \end{bmatrix}_{5 \times 9} & (0)_{5 \times 1} \\ [0]_{1 \times 9} & 1 \end{bmatrix}_{6 \times 10}, \quad (1.86a)$$

in which the sub-matrix $[P_5^{(1)} \ P_5^{(2)}]_{5 \times 9}$ is the strain shape function matrix without the ddof. The two component sub-matrices are

$$P_5^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \eta^t & 0 & 0 \\ 0 & 1 & 0 & 0 & \eta^t & 0 \\ 0 & 0 & 1 & 0 & 0 & \eta^t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 6}, \quad (1.86b)$$

$$P_5^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -s_3(1-2\xi_2) & s_3(2\xi_2+2\xi_3-1) & 0 \\ -r_3(1-2\xi_2) & r_{32}(2\xi_2+2\xi_3-1) & r_2(1-2\xi_3) \end{bmatrix}_{5 \times 3}, \quad (1.86c)$$

with $r_{32} = r_3 - r_2$, and the superscript s on the lhs of Eq. (1.85) denoting the skew-symmetric part of the strain tensor. Note that the incremental assumed strain components $\Delta \varepsilon_{rs}$, $\Delta \varepsilon_{st}$, and $\Delta \varepsilon_{tr}$ employ the engineering definition. Of the strain parameters, $\Delta \beta_1$ through $\Delta \beta_3$ are associated with membrane strains, $\Delta \beta_4$ through $\Delta \beta_6$ bending strains, $\Delta \beta_7$ through $\Delta \beta_9$ transversal strains, and $\Delta \beta_{10}$ the skew-symmetric strain component.

1.4.2.4 Linear element stiffness matrix k_L

With reference to Eq. (1.68), the element stiffness matrix is the sum of linear and initial stress stiffness matrices, k_L and k_{NL} . The outline of the derivation of the linear stiffness matrix k_L is presented in the following.

Recall that the element linear stiffness matrix is defined by Eq. (1.69) in which the inverse of the generalized stiffness matrix H and the leverage matrix G are required. These, in turn, require the strain shape function matrix P , and the strain and displacement matrix B_L for a given material elastic matrix D . Since P is obtained by using Eq. (1.86a) and therefore B_L is required.

Symbolically, one can write

$$B_L = [B_{L_1} \quad B_{L_2} \quad B_{L_3}], \quad (1.87)$$

where the sub-matrices are defined by

$$B_{L_i} = B_{L_{mi}} + B_{L_{bi}} + B_{L_{si}} + B_{L_{ti}}, \quad i = 1, 2, 3, \quad (1.88a)$$

in which the first, second, third, and fourth terms on the rhs are respectively associated with the membrane, bending, shear, and torsional components of the element linear stiffness matrix. The membrane and bending parts are

$$B_{L_{mi}} = \begin{bmatrix} B_{L_i}^m \\ [0]_{3 \times 6} \end{bmatrix}, \quad B_{L_{bi}} = \begin{bmatrix} B_{L_i}^b \\ [0]_{3 \times 6} \end{bmatrix}, \quad (1.88b, c)$$

in which $[0]_{3 \times 6}$ is a 3×6 null matrix and

$$B_{Li}^m = \begin{bmatrix} \xi_{i,r} & 0 & 0 & 0 & 0 & \bar{p}_{i,r} \\ 0 & \xi_{i,s} & 0 & 0 & 0 & \bar{q}_{i,s} \\ \xi_{i,s} & \xi_{i,r} & 0 & 0 & 0 & \bar{p}_{i,s} + \bar{q}_{i,r} \end{bmatrix},$$

$$B_{Li}^b = \begin{bmatrix} 0 & 0 & 0 & \eta^t \xi_{i,r} \Lambda_{i(11)}^t & \eta^t \xi_{i,r} \Lambda_{i(12)}^t & 0 \\ 0 & 0 & 0 & \eta^t \xi_{i,s} \Lambda_{i(21)}^t & \eta^t \xi_{i,s} \Lambda_{i(22)}^t & 0 \\ 0 & 0 & 0 & b_{34}^b & b_{35}^b & 0 \end{bmatrix},$$

where the elements or entries have been defined in Eq. (1.78) except

$$b_{34}^b = \eta^t \left(\xi_{i,r} \Lambda_{i(21)}^t + \xi_{i,s} \Lambda_{i(11)}^t \right), \quad b_{35}^b = \eta^t \left(\xi_{i,r} \Lambda_{i(22)}^t + \xi_{i,s} \Lambda_{i(12)}^t \right),$$

$$\xi_{1,r} = -1/r_2^t, \quad \xi_{1,s} = (r_3^t - r_2^t)(r_2^t s_3^t)^{-1}, \quad \xi_{2,r} = 1/r_2^t,$$

$$\xi_{2,s} = -r_3^t (r_2^t s_3^t)^{-1}, \quad \xi_{3,r} = 0, \quad \xi_{3,s} = 1/s_3^t,$$

$$\bar{p}_{i,s} = \frac{\partial \bar{p}_i}{\partial s}, \quad \bar{p}_{i,r} = \frac{\partial \bar{p}_i}{\partial r}, \quad \bar{q}_{i,s} = \frac{\partial \bar{q}_i}{\partial s}, \quad \bar{q}_{i,r} = \frac{\partial \bar{q}_i}{\partial r}.$$

The shear component on the rhs of Eq. (1.88a) is defined as

$$B_{Lsi} = \begin{bmatrix} [0]_{3 \times 6} \\ B_{Li}^s \end{bmatrix}, \quad B_{Li}^s = \begin{bmatrix} 0 & 0 & \xi_{i,s} & b_{44}^s & b_{45}^s & 0 \\ 0 & 0 & \xi_{i,r} & b_{54}^s & b_{55}^s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1.88d, e)$$

where

$$b_{44}^s = \eta^t \xi_{i,s} \Lambda_{i(31)}^t - \bar{p}_{i,s} + \xi_{i,r} \Lambda_{i(21)}^t,$$

$$b_{45}^s = \eta^t \xi_{i,s} \Lambda_{i(32)}^t - \bar{q}_{i,s} + \xi_i \Lambda_{i(22)}^t, \quad b_{54}^s = \eta^t \xi_{i,r} \Lambda_{i(31)}^t - \bar{p}_{i,r} + \xi_i \Lambda_{i(11)}^t,$$

$$b_{55}^s = \eta^t \xi_{i,r} \Lambda_{i(32)}^t - \bar{q}_{i,r} + \xi_i \Lambda_{i(12)}^t.$$

The torsional component by the displacement formulation is defined by

$$B_{L_t} = \frac{1}{2} \left[\begin{array}{c} \frac{r_3 - r_2}{r_2 s_3}, \frac{1}{r_2}, 0, 0, 0, b_6^d, -\frac{r_3}{r_2 s_3}, -\frac{1}{r_2}, \\ 0, 0, 0, b_{12}^d, \frac{1}{s_3}, 0, 0, 0, 0, b_{18}^d \end{array} \right]_{1 \times 18}, \quad (1.88f)$$

in which

$$b_6^d = 2\xi_1 + (\bar{p}_{1,s} - \bar{q}_{1,r}), \quad b_{12}^d = 2\xi_2 + (\bar{p}_{2,s} - \bar{q}_{2,r}),$$

$$b_{18}^d = 2\xi_3 + (\bar{p}_{3,s} - \bar{q}_{3,r}).$$

With Eqs. (1.86a) and (1.87) defined, the element generalized stiffness matrix and the leverage matrix can be evaluated. Thus, the linear element stiffness matrix k_L can be determined. It is convenient to express it as

$$k_L = k_{L_m} + k_{L_b} + k_{L_s} + k_t, \quad (1.89)$$

where the component matrices are understood to be of the same order and in proper locations inside the matrices. They are defined by

$$k_{L_m} = (G_{L_m})^T H^{-1} G_{L_m}, \quad k_{L_b} = (G_{L_b})^T H^{-1} G_{L_b}, \quad (1.90a, b)$$

$$k_{L_s} = (G_{L_s})^T H^{-1} G_{L_s}, \quad k_t = \frac{1}{2} G_e h^t \int_A (B_{L_t})^T B_{L_t} dA, \quad (1.90c, d)$$

in which G_e is the shear modulus of elasticity of the material, H is evaluated by Eq. (1.65a), and G by Eq. (1.65b) such that

$$G_{L_m} = \int_V P^T D B_{L_m} dV, \quad G_{L_b} = \int_V P^T D B_{L_b} dV, \quad (1.91a, b)$$

$$G_{L_s} = \int_V P^T D B_{L_s} dV, \quad B_{L_m} = [B_{L_{m1}} \ B_{L_{m2}} \ B_{L_{m3}}], \quad (1.91c, d)$$

$$B_{L_b} = [B_{L_{b1}} \ B_{L_{b2}} \ B_{L_{b3}}], \ B_{L_s} = [B_{L_{s1}} \ B_{L_{s2}} \ B_{L_{s3}}], \ B_{L_t} = [B_{L_{t1}} \ B_{L_{t2}} \ B_{L_{t3}}], \quad (1.91e, f, g)$$

where the sub-matrices have been defined in Eqs. (1.88a) through (1.88d). The order of every matrix in Eqs. (1.91c) through (1.91g) is appropriately adjusted to 6×18 . For completeness, the explicit expressions for the element stiffness matrix defined by Eq. (1.89) are presented in Appendix 1B.

1.4.2.5 Element initial stress stiffness matrix k_{NL}

To obtain the element initial stress stiffness matrix k_{NL} , one first has the nonlinear strain-displacement matrix B_{NL} as

$$B_{NL} = \left[\frac{\partial N_{1i}}{\partial \mathbf{r}} \quad \frac{\partial N_{1i}}{\partial s} \quad \frac{\partial N_{1i}}{\partial \boldsymbol{\eta}} \right]_{9 \times 18}, \quad (1.92)$$

where N_{1i} with $i = 1, 2, 3$, are defined in Eq. (1.78). For instance,

$$\frac{\partial N_{1i}}{\partial \boldsymbol{\eta}} = \frac{\partial N_{1i}}{\partial \boldsymbol{\eta}^t} = \begin{bmatrix} 0 & 0 & 0 & \xi_i \Lambda_{i(11)}^t & \xi_i \Lambda_{i(12)}^t & 0 \\ 0 & 0 & 0 & \xi_i \Lambda_{i(21)}^t & \xi_i \Lambda_{i(22)}^t & 0 \\ 0 & 0 & 0 & \xi_i \Lambda_{i(31)}^t & \xi_i \Lambda_{i(32)}^t & 0 \end{bmatrix}.$$

It is noted that the partial differentiation in Eq. (1.92) is with respect to the local co-ordinates at time t or in configuration C^t . According to Eq. (1.65c), the evaluation of k_{NL} requires the Cauchy stress matrix σ_1 in addition to the nonlinear strain-displacement matrix B_{NL} . Hence, the Cauchy stress matrix

$$\sigma_1 = \begin{bmatrix} \sigma_{11} \mathbf{I}_3 & (\sigma_{12} + \sigma_{12}^s) \mathbf{I}_3 & \sigma_{31} \mathbf{I}_3 \\ (\sigma_{12} - \sigma_{12}^s) \mathbf{I}_3 & \sigma_{22} \mathbf{I}_3 & \sigma_{23} \mathbf{I}_3 \\ \sigma_{31} \mathbf{I}_3 & \sigma_{23} \mathbf{I}_3 & \mathbf{O}_3 \end{bmatrix} \quad (1.93)$$

with $\mathbf{O}_3$ being a 3×3 null matrix. Note that because of σ_{12}^s , matrix σ_1 is no longer symmetric. Consequently, k_{NL} is non-symmetric. However, in the present study σ_{12}^s is disregarded since its contribution has been assumed to be small. Further, in [26] Bufler has shown that for conservative systems tangent stiffness matrices are symmetric. Thus, Eq. (1.93) can be reduced to

$$\sigma_1 = \begin{bmatrix} \sigma_{11} \mathbf{I}_3 & \sigma_{12} \mathbf{I}_3 & \sigma_{31} \mathbf{I}_3 \\ \sigma_{12} \mathbf{I}_3 & \sigma_{22} \mathbf{I}_3 & \sigma_{23} \mathbf{I}_3 \\ \sigma_{31} \mathbf{I}_3 & \sigma_{23} \mathbf{I}_3 & \mathbf{O}_3 \end{bmatrix}. \quad (1.94)$$

Equations (1.71), (1.72) and (1.86a) indicate that, the membrane components of the incremental PK2 stress, ΔS are constant within an element. The bending components vary in the thickness direction only. However, the transverse components vary linearly within the element. Consequently, the PK2 and Cauchy stresses at time t , S and σ are functions of mid-surface position. Their updating requires storage of stresses at the three nodes.

Returning to the integration of Eq. (1.65), if one considers σ as mid-surface position dependent, it would produce very tedious expressions. Therefore, in the present study the transverse stress components of σ are considered constant in the element. That is, all the stress components of σ are computed and updated only at the centroid of every element. The justification for such an operation is that the membrane and bending stresses in shell structures are usually dominant. Such an approach reduces substantially the computation efforts involved and requires much less computer storage space. Note that, in general, every σ_{ij} of Eqs. (1.93) and (1.94) is a combination of membrane, bending and transverse shear components. At any point, including the centroid of the element, membrane and transverse shear components are obtained by setting η^t to zero for the purpose of evaluating the nonlinear stiffness matrix k_{NL} . The stress at the top surface of the shell element can be obtained by substituting $\eta^t = h^t/2$. The difference between the stresses at the top and the mid-surface of the element, after being divided by $h^t/2$, is the slope of bending stress component. Then, σ_{ij} is re-written as

$$\sigma_{ij} = \sigma_{ij(0)} + \eta^t d\sigma_{ij}, \quad d\sigma_{ij} = \frac{2}{h^t} (\sigma_{ij(+)} - \sigma_{ij(0)}), \quad (1.95a, b)$$

where $\sigma_{ij(0)}$ and $\sigma_{ij(+)}$ denote stresses at the middle and top surfaces of the shell element, respectively while $d\sigma_{ij}$ is the slope of the bending stress. Equation (1.95) is also valid for plastic deformation. In this case, $d\sigma_{ij}$ becomes zero because the non-layered approach has been employed in the present study. Of course, for the layered approach, $d\sigma_{ij}$ is not zero in general.

With B_{NL} and σ_1 now defined in Eqs. (1.92) and (1.94), they are substituted into Eq. (1.65c) so that k_{NL} can be obtained after evaluating the required integration. By applying the algebraic manipulation package MAPLE, explicit expressions for k_{NL} have been determined [20]. For brevity, these explicit expressions are not presented in the present book.

In closing, it should be mentioned that recovery of the linear element stiffness matrices from the nonlinear formulation developed in the foregoing has been made and presented [20, 21]. These recovered linear element stiffness matrices do not contain the directors and they have their usefulness to be employed for cases where the directors are not uniquely defined or are difficult to determine, typically at joints and discontinuities of shell structures. For reference the element stiffness matrices derived in the foregoing are called the director version of element stiffness matrices or simply the director version.

1.4.2.6 Consistent element mass matrices

In Refs. [20, 21] it was shown that the derivation of the consistent element mass matrix consists of three parts. The first part is concerned with the derivation of the translational component of the consistent element mass matrix m_u , the second part the rotational component without the ddof component of the consistent mass matrix m_r , and the third part the ddof component m_t . Symbolically, the consistent element mass matrix can be written as

$$\mathbf{m} = \mathbf{m}_u + \mathbf{m}_r + \mathbf{m}_t, \quad (1.96)$$

where the first two terms on the rhs are defined by

$$\mathbf{m}_u = \int_{A^t} \rho^t h^t \mathbf{N}_u^T \mathbf{N}_u dA, \quad \mathbf{m}_r = \int_{A^t} \mathbf{J}^t \mathbf{N}_r^T \mathbf{N}_r dA. \quad (1.97a, b)$$

This particular element is identical in both the director version and the non-director or simplified version provided in [27]. Note that the displacement shape function and angular displacement shape function matrices, and $\mathbf{J}^t$ in Eq. (1.97) are defined, respectively, by

$$N_u = [N_{u1} \quad N_{u2} \quad N_{u3}], \quad N_{ui} = \begin{bmatrix} \xi_i & 0 & 0 & 0 & 0 & \bar{p}_i \\ 0 & \xi_i & 0 & 0 & 0 & \bar{q}_i \\ 0 & 0 & \xi_i & -\bar{p}_i & -\bar{q}_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1.98a, b)$$

$$N_r = [N_{r1} \quad N_{r2} \quad N_{r3}], \quad N_{ri} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\xi_i \Lambda_{i(21)}^t & -\xi_i \Lambda_{i(22)}^t & 0 \\ 0 & 0 & 0 & \xi_i \Lambda_{i(11)}^t & \xi_i \Lambda_{i(12)}^t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (1.98c, d)$$

and $J^t = \rho^t (h^t)^3 / (12)$.

It is not difficult to recognize that the first three rows of Eq. (1.98b) are in fact constructed by applying Eq. (1.77) with the shell thickness co-ordinate η^t being set to zero because the displacement field is defined in terms of the mid-surface displacements.

It is observed that in the foregoing the derivation of consistent element mass matrices is with respect to the reference configuration C^t . Therefore, the mass matrices have to be updated at every time step since the density, thickness of the shell and element geometry change from one time step to another for large deformation problems. This approach is different from those in [28, 29], for example. In the latter two references the mass matrix was defined with respect to the undeformed configuration and was kept constant throughout the entire analysis. As pointed out in [20], for the updated Lagrangian formulation a consistent element mass matrix that was defined with respect to the reference state was obtained in [30]. However, numerical results were not included in the latter reference. In the present study, the option of updating the mass matrix has

been kept as in Refs. [20, 21]. Comparisons of numerical results using constant or updated mass matrices have been investigated and reported in [31].

For completeness, explicit expressions for the consistent mass matrix of the triangular shell element are included in Appendix 1C.

1.5 Constitutive Relations and Updating of Configurations and Stresses

The constitutive relations are first introduced in Sub-sections 1.5.1 and 1.5.2. These relations are for linear, elastic and isotropic materials with small or finite strains, and elasto-plastic materials with isotropic strain hardening. The latter case also includes deformations of small and finite strains. The updating of configurations and stresses are dealt with in Sub-section 1.5.3.

1.5.1 Elastic materials

The constitutive fourth order tensor for homogeneous, isotropic and linearly elastic materials undergoing deformation of small strain is

$$D_{ijkl} = \frac{E}{1 + \nu} \left[\frac{\nu}{1 - 2\nu} \delta_{ij} \delta_{kl} + \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \right] \quad (1.99)$$

where E is the Young's modulus of elasticity, ν Poisson's ratio and δ_{ij} the Kronecker delta. Equation (1.99) can be cast in matrix form as

$$D^c = \begin{bmatrix} c_{11} & c_{22} & c_{22} & 0 & 0 & 0 \\ c_{22} & c_{11} & c_{22} & 0 & 0 & 0 \\ c_{22} & c_{22} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{33} \end{bmatrix} \quad (1.100)$$

with the elements of this matrix being defined by

$$c_{11} = \frac{E}{1 + \nu} \left(\frac{1 - \nu}{1 - 2\nu} \right), \quad c_{22} = \frac{E}{1 + \nu} \left(\frac{\nu}{1 - 2\nu} \right), \quad c_{33} = \frac{1}{2} \left(\frac{E}{1 + \nu} \right).$$

The stress and strain vectors accompanying Eq. (1.100) are

$$\begin{aligned}\boldsymbol{\sigma} &= [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^T, \\ \boldsymbol{\varepsilon} &= [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31}]^T.\end{aligned}\quad (1.101a, b)$$

To include the skew-symmetric component of stress and strain tensors, D^c in Eq. (1.00) needs to be expanded to the following 7×7 matrix

$$D^s = \begin{bmatrix} (D^c)_{6 \times 6} & (0)_{6 \times 1} \\ [0]_{1 \times 6} & c_{33} \end{bmatrix}, \quad (1.102)$$

where $(0)_{6 \times 1}$ and $[0]_{1 \times 6}$ are null matrices of orders 6×1 and 1×6 , respectively. The accompanying stress and strain vectors are

$$\begin{aligned}\boldsymbol{\sigma} &= [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}, \sigma_{12}^s]^T, \\ \boldsymbol{\varepsilon} &= [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{31}, \varepsilon_{12}^s]^T,\end{aligned}\quad (1.103)$$

and the superscript s denotes the skew-symmetric components.

Finally, Eq. (1.102) should be reduced from general 3D applications to plate or shell analyses by imposing zero normal stress condition $\sigma_{33} = 0$. Hughes and Liu [32] proposed performing a transformation defined as

$$D = (T_c)^T D^s (T_c), \quad (1.104)$$

where the transformation matrix

$$T_c = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ t_1 & t_2 & t_4 & t_5 & t_6 & t_7 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad t_k = -\frac{D_{3k}^s}{D_{33}^s}, \quad (1.105a, b)$$

in which D_{3k}^s is the element located in the third row and k 'th column of D^s of Eq. (1.102).

For finite strain deformations, Refs. [22, 33, 34] suggested adding to Eq. (1.99) the following term,

$$D_{ijkl}^h = -\frac{1}{2}(\sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il} + \sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik}). \quad (1.106)$$

Note that this term is a consequence of transforming the Jaumann stress rate to the incremental PK2 stress. If cast into matrix form with Eq. (1.103) as the accompanying stress and strain vectors, Eq. (1.106) becomes

$$D^r = \begin{bmatrix} D_{11}^\beta & D_{12}^\beta \\ D_{21}^\beta & D_{22}^\beta \end{bmatrix}, \quad D_{11}^\beta = -2 \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}, \quad (1.107)$$

where

$$D_{12}^\beta = - \begin{bmatrix} \sigma_{12} & 0 & \sigma_{13} & 0 \\ \sigma_{12} & \sigma_{23} & 0 & 0 \\ 0 & \sigma_{23} & \sigma_{13} & 0 \end{bmatrix}, \quad D_{21}^\beta = (D_{12}^\beta)^T,$$

$$D_{22}^\beta = -\frac{1}{2} \begin{bmatrix} \sigma_{11} + \sigma_{22} & \sigma_{13} & \sigma_{23} & 0 \\ \sigma_{13} & \sigma_{22} + \sigma_{33} & \sigma_{12} & 0 \\ \sigma_{23} & \sigma_{12} & \sigma_{11} + \sigma_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The transformation rule of Eq. (1.104) applies to $(D^s + D^r)$, except that σ_{33} in Eq. (1.107) is zero.

Other approaches in dealing with finite strain problems include performing numerical integration on rate constitutive equations [32, 35], or on the Jaumann rate [28]. The approach in the present study is, however, simpler and more direct, compared with those in Refs. [28, 32, 35].

1.5.2 Elasto-plastic materials with isotropic strain hardening

Notable large strain elasto-plastic deformation theories include those by, for examples, Green and Naghdi [36], Nemat-Nasser [37] and Lee [38]. It seems that most of the controversies arise in cases where both the elastic and plastic parts of strain are large. However, if confined to applications of small elastic, but large plastic strain (thus, large total strain), it can be shown that all the different theories reduce to the so-called J_2 flow theory of plasticity [28, 22, 39, 33-35]. As the material of interest in the present study is metal with small elastic, but large plastic strain, in what follows, the J_2 flow theory is considered.

The small strain formulation of J_2 flow theory involves the stress deviator σ^D and the J_2 invariant of stresses. In Cartesian co-ordinates, they are defined as in [33],

$$\sigma_{ij}^D = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}, \quad J_2 = \frac{1}{2} \sigma_{ij}^D \sigma_{ij}^D. \quad (1.108)$$

Consequently, the constitutive fourth order tensor is now given by

$$D_{ijkl}^{ep} = D_{ijkl} - \left(\frac{E}{1+\nu} \right) \left(\frac{\alpha}{E^p} \right) \sigma_{ij}^D \sigma_{kl}^D, \quad (1.109)$$

where

$$E^p = \frac{2}{3} (\sigma^e)^2 \left(\frac{E - \frac{1-2\nu}{3} E_T}{E - E_T} \right), \quad (1.110)$$

in which E_T is the tangent modulus, ν Poisson's ratio, and $(\sigma^e)^2 = 3J_2$ the square of the effective stress while α , not to be confused with that in Eq. (1.31a), is a parameter having the value of either zero or unity. When $\alpha = 0$, it is associated with elastic loading or any unloading, and when $\alpha = 1$, it is associated with plastic loading. Whether the material is undergoing plastic loading or not, it can be determined through the following conditions

$$\alpha = \begin{cases} 1 & \dot{J}_2 = \dot{\sigma}_{ij}^D \dot{\sigma}_{ij} \geq 0 \quad \wedge \quad J_2 = (J_2)_{\max} \\ 0 & \dot{J}_2 < 0 \quad \vee \quad J_2 < (J_2)_{\max} \end{cases} \quad (1.111)$$

with $\wedge$ denoting the logical "and" and $\vee$ the logical "or".

The above small strain formulation can be extended to finite strain cases in a similar approach to that of Sub-section 1.5.1. This is achieved by adding Eq. (1.106) to (1.109). Then, the matrix form of the constitutive tensor becomes

$$D^{ep} = D^s + D^r + D^\alpha, \quad (1.112)$$

where D^s and D^r are defined in Eqs. (1.102) and (1.107), respectively.

The term D^α denotes the component matrix associated with α in Eq. (1.109). It is defined by

$$D^\alpha = -\lambda \begin{bmatrix} D_{11}^\alpha & D_{12}^\alpha \\ D_{21}^\alpha & D_{22}^\alpha \end{bmatrix}, \quad (1.113)$$

where

$$D_{11}^\alpha = \begin{bmatrix} \sigma_{11}^D \sigma_{11}^D & \sigma_{11}^D \sigma_{22}^D & \sigma_{11}^D \sigma_{33}^D \\ \sigma_{22}^D \sigma_{11}^D & \sigma_{22}^D \sigma_{22}^D & \sigma_{22}^D \sigma_{33}^D \\ \sigma_{33}^D \sigma_{11}^D & \sigma_{33}^D \sigma_{22}^D & \sigma_{33}^D \sigma_{33}^D \end{bmatrix}, \quad D_{21}^\alpha = \begin{bmatrix} \sigma_{12}^D \sigma_{11}^D & \sigma_{12}^D \sigma_{22}^D & \sigma_{12}^D \sigma_{33}^D \\ \sigma_{23}^D \sigma_{11}^D & \sigma_{23}^D \sigma_{22}^D & \sigma_{23}^D \sigma_{33}^D \\ \sigma_{31}^D \sigma_{11}^D & \sigma_{31}^D \sigma_{22}^D & \sigma_{31}^D \sigma_{33}^D \\ \sigma_{12}^s \sigma_{11}^D & \sigma_{12}^s \sigma_{22}^D & \sigma_{12}^s \sigma_{33}^D \end{bmatrix},$$

$$D_{12}^\alpha = \left(D_{21}^\alpha \right)^T, \quad D_{22}^\alpha = \begin{bmatrix} \sigma_{12}^s \sigma_{12}^s & \sigma_{12}^s \sigma_{23}^s & \sigma_{12}^s \sigma_{31}^s & \sigma_{12}^s \sigma_{12}^s \\ \sigma_{23}^s \sigma_{12}^s & \sigma_{23}^s \sigma_{23}^s & \sigma_{23}^s \sigma_{31}^s & \sigma_{23}^s \sigma_{12}^s \\ \sigma_{31}^s \sigma_{12}^s & \sigma_{31}^s \sigma_{23}^s & \sigma_{31}^s \sigma_{31}^s & \sigma_{31}^s \sigma_{12}^s \\ \sigma_{12}^s \sigma_{12}^s & \sigma_{12}^s \sigma_{23}^s & \sigma_{12}^s \sigma_{31}^s & \sigma_{12}^s \sigma_{12}^s \end{bmatrix},$$

$$\lambda = \left(\frac{E}{1 + \nu} \right) \frac{\alpha}{E^p}.$$

Equation (1.112) is the elasto-plastic material matrix for general 3D problems. When reducing the general 3D theory to thin or moderately thick plates or shells

two aspects are to be noted. First, it is usually assumed that the effects of transverse shear stresses on plastic behaviors can be disregarded [39]. Such an assumption leads to a simplified matrix D^α in which three of the four sub-matrices are

$$D_{21}^\alpha = \begin{bmatrix} \sigma_{12}\sigma_{11}^D & \sigma_{12}\sigma_{22}^D & \sigma_{12}\sigma_{33}^D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_{12}^s\sigma_{11}^D & \sigma_{12}^s\sigma_{22}^D & \sigma_{12}^s\sigma_{33}^D \end{bmatrix}, \quad (1.114a)$$

$$D_{12}^\alpha = (D_{21}^\alpha)^T, \quad D_{22}^\alpha = \begin{bmatrix} \sigma_{12}\sigma_{12} & 0 & 0 & \sigma_{12}\sigma_{12}^s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_{12}^s\sigma_{12} & 0 & 0 & \sigma_{12}^s\sigma_{12}^s \end{bmatrix}, \quad (1.114b, c)$$

and the sub-matrix D_{11}^α remains unchanged. Second, the transformation rule of Eq. (1.104) is applied to D^{ep} . In doing so, one can only set σ_{33} to zero while σ_{33}^D is not zero.

It may be appropriate to point out that in arriving at Eqs. (1.112) and (1.114), the von Mises yield criterion has been employed. The von Mises criterion is considered presently because the materials of interest in the present investigation are metals. It can be expressed in terms of stresses or stress resultants. The first approach allows for the spread of plasticity over the thickness of plates and shells, and is termed the *layered approach*. This approach has been adopted in [39]. The second approach, the *non-layered approach*, on the other hand, employs yield functions that are in terms of stress resultants. This latter approach assumes that at a point the entire cross-section becomes plastic simultaneously. Therefore, compared with the non-layered approach, the layered approach seems to be more realistic but requires a larger number of algebraic manipulations in forming the element stiffness matrix. However, Robinson [40] showed that the discrepancy between the two approaches was insignificant, while Ref. [25] seemed to suggest the application of a large number of elements with the non-layered approach.

In the present investigation, the non-layered approach is employed for two main considerations. First, it requires less computation to evaluate element stiffness matrices. Second, the D^{ep} matrix can be written in a simple and concise way, which enables one to obtain explicit expressions for the stiffness matrix that are of a manageable size. The second main consideration is satisfied because, in the non-layered approach, stress distribution along the thickness direction is simple to formulate. Prior to the occurrence of plasticity at a point, the membrane and bending stresses are, respectively, uniformly and linearly distributed along the thickness. If the yield criterion is satisfied, the membrane stress remains uniform over the thickness. But bending stresses become symmetric about the mid-surface. That is, the bending stresses takes equal magnitudes but opposite signs below and above the mid-surface. Consequently, the yield criterion should be expressed in terms of stress resultants. Such attempts include those proposed in Refs. [41, 40, 42, 43].

In 1948, Ilyushin applied the von Mises yield criterion to thin shells [42]. The idea was further developed by Shapiro [43]. Simo and Kennedy [41] then extended the Ilyushin-Shapiro two-surface yield condition to nonlinear shell analysis. The yield conditions of Simo and Kennedy [41] were written in terms of membrane forces and bending moments, and are capable of reflecting the coupling effect of membrane forces and bending moments on plastic behaviours. Because the yield conditions included two additional parameters that were deformation-path dependent, Simo and Kennedy [41] have constructed a complex return mapping algorithm, to bring stress points outside of the yield surface onto the surface. This technique, however, requires a very significant amount of computational effort. On the other hand, Robinson [40] had shown that the Ilyushin-Shapiro yield condition reduced to a non-parametric form without loss of accuracy and generality. Consequently, the corresponding return mapping algorithm is simpler to construct and is less expensive computationally. In what follows, the yield condition of Robinson [40] and its corresponding return mapping are introduced. The yield condition in Ref. [40] shall henceforth be called Ilyushin's or the Ilyushin-Robinson yield condition.

If σ_y is the yield stress of the material in simple tension and h the thickness of the shell at time t with the superscript t being disregarded in all the quantities considered here for conciseness, one defines the dimensionless membrane forces n_{11}^p , n_{22}^p and n_{12}^p , and the dimensionless bending moments m_{11}^p , m_{22}^p and m_{12}^p as

$$n_{ij}^p = \frac{N_{ij}^p}{\sigma_y h}, \quad m_{ij}^p = \frac{4M_{ij}^p}{\sigma_y h^2}, \quad (1.115a, b)$$

with the membrane forces N_{ij}^p and bending moments M_{ij}^p being related to stress components across the cross-section as

$$N_{ij}^p = \int_{-h/2}^{h/2} \sigma_{ij} d\eta, \quad M_{ij}^p = \int_{-h/2}^{h/2} \sigma_{ij} \eta d\eta. \quad (1.116a, b)$$

Therefore, the membrane forces and bending moments are defined over a cross-section with thickness h and unity width. The Ilyushin yield condition proposed by Robinson [40] is stated as

$$Q_t^p + Q_m^p + \frac{|Q_{tm}^p|}{\sqrt{3}} \leq 1, \quad (1.117)$$

where

$$Q_t^p = (n_{11}^p)^2 + (n_{22}^p)^2 - n_{11}^p n_{22}^p + 3(n_{12}^p)^2,$$

$$Q_m^p = (m_{11}^p)^2 + (m_{22}^p)^2 - m_{11}^p m_{22}^p + 3(m_{12}^p)^2,$$

$$Q_{tm}^p = n_{11}^p m_{11}^p + n_{22}^p m_{22}^p - \frac{1}{2} n_{11}^p m_{22}^p - \frac{1}{2} n_{22}^p m_{11}^p + 3 n_{12}^p m_{12}^p.$$

It should be noted that in the above yield condition transverse shear stresses have been disregarded. Robinson has shown that Eq. (1.117) is a very good approximation to the exact criterion and superior to the other linear approximations.

The return mapping follows that described in [28] for general 3D problems. Noting that in the present study membrane and bending stresses are uncoupled, it is proposed to split the return mapping into two portions, one for the membrane stress and the other for bending stress. The portion for membrane stress is, in fact, a return mapping for plane stress problems. The portion for bending stress is identical to a return mapping for plate bending problems.

Assuming that $\Delta \epsilon^{ep}$ is the elasto-plastic strain increment which is divided into sub-increments $\Delta(\Delta \epsilon^{ep})$ according to [29]

$$\Delta(\Delta \epsilon^{ep}) = \gamma / \lambda_{ep} \quad (1.118)$$

with the elasto-plastic parameters

$$\gamma = \frac{\Delta \epsilon^{ep}}{\Delta \epsilon}, \quad \lambda_{ep} = 1 + \frac{\gamma}{30}. \quad (1.119a, b)$$

Therefore, the parameter γ indicates the elasto-plastic portion of total strain increment $\Delta \epsilon$. In Eqs. (1.118) and (1.119a, b), in fact, $\Delta \epsilon$ is divided into 30 equal sub-increments, λ_{ep} of which correspond to $\Delta \epsilon^{ep}$, the elasto-plastic portion of the strain increment. For each of the λ_{ep} sub-increments of the elasto-plastic strain, $\Delta(\Delta \epsilon^{ep})$, the updating formula is

$$\sigma_m \leftarrow \sigma_m + D_3^{ep} \Delta(\Delta \epsilon^{ep}), \quad (1.120)$$

where $\leftarrow$ denotes “assign to”. For the membrane stress portion, σ_m and $\Delta(\Delta \epsilon^{ep})$ are understood to be the membrane stresses and membrane strain sub-increment, respectively. For the bending moment portion, σ_m and $\Delta(\Delta \epsilon^{ep})$ are understood to be the bending moments and bending curvature sub-increment, respectively. The 3×3 matrix D_3^{ep} is obtained through the following relation

$$D_3^{ep} = (T_{c4})^T D_4^{ep} (T_{c4}), \quad (1.121)$$

where the 4×4 matrix D_4^{ep} is defined as

$$D_4^{ep} = D_4^c + D_4^r + D_4^\alpha, \quad (1.122)$$

in which square matrices D_4^r and D_4^α are of order 4 and can, in fact, be formed by two steps. First, they are constructed by selecting the first 4 rows and columns of D^r in Eq. (1.107), and D^α in Eq. (1.113), respectively. Second, the stresses σ_{ij} , and so on in Eqs. (1.107) and (1.113) are replaced with appropriate σ_m . The matrix D_4^c , is defined as

$$D_4^c = \lambda_\ell \begin{bmatrix} 1 - \nu & \nu & \nu & 0 \\ \nu & 1 - \nu & \nu & 0 \\ \nu & \nu & 1 - \nu & 0 \\ 0 & 0 & 0 & \frac{1 - 2\nu}{2} \end{bmatrix} \quad (1.123)$$

with the subscript $\ell = 1, 2$ such that λ_1 is associated with membrane stresses and λ_2 bending moments. These parameters are defined as

$$\lambda_1 = \frac{E}{(1 + \nu)(1 - 2\nu)} , \quad \lambda_2 = \frac{Eh^3}{12(1 + \nu)(1 - 2\nu)} .$$

It is observed that with $\ell = 1$, Eq. (1.123) is just the matrix formed by selecting the first four rows and first four columns of D^c in Eq. (1.100).

In closing this sub-section, it should be mentioned that the transformation matrix T_{c4} in Eq. (1.121) is given by

$$T_{c4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_1 & t_2 & t_4 \\ 0 & 0 & 1 \end{bmatrix} , \quad t_k = \frac{\left(D_4^{ep}\right)_{3k}}{\left(D_4^{ep}\right)_{33}} , \quad (1.124a, b)$$

where $(D_4^{ep})_{3k}$ is the element in the third row and k 'th column of D_4^{ep} .

1.5.3 Configuration and stress updatings

The updating of configuration and stresses at every time step is required in problems with large deformations. Therefore, in this sub-section steps of updating of configuration and stresses are provided.

1.5.3.1 Updating of configuration

This consists of the updating of mid-surface co-ordinates and directors. Mid-surface co-ordinates are updated by adding mid-surface displacements to mid-surface co-ordinates of the reference configuration. To this end, one makes use of Eqs. (1.73) and (1.75) with $\eta^t = 0$ such that

$$\begin{pmatrix} \bar{\mathbf{r}}^{t+\Delta t} \\ \bar{\mathbf{s}}^{t+\Delta t} \\ \bar{\mathbf{t}}^{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{r}}^t \\ \bar{\mathbf{s}}^t \\ \bar{\mathbf{t}}^t \end{pmatrix} + \begin{pmatrix} \Delta \bar{\mathbf{u}}^t \\ \Delta \bar{\mathbf{v}}^t \\ \Delta \bar{\mathbf{w}}^t \end{pmatrix} . \quad (1.125)$$

Similar updating for the global co-ordinates can be made. But for brevity, those in terms of global co-ordinates are not presented here.

In the present study, updating procedures for directors follow those proposed in Refs. [24, 44], for example. Assuming that incremental rotations $(\Delta\theta_i)^t$, $i = 1, 2, 3$, have been determined at the current time step t , the directors at the next time step or new position are obtained by making use of Eq. (1.82),

$$d_i^{t+\Delta t} = R(\Delta\theta) d_i^t, \quad (1.126)$$

where the transformation matrix $R(\Delta\theta)$ or simply written as R is defined by Eq. (1.83) except that now it is in configuration C^t so that $\Delta\theta = (\Delta\theta_i)^t$.

Having updated the directors in configuration $C^{t+\Delta t}$ and since the director angular velocity as well as the director angular acceleration in C^t are known, the director angular velocity and director angular acceleration fields are updated applying the following scheme [44]

$$\begin{aligned} \omega^{t+\Delta t} &= \frac{\gamma \Delta\theta}{\beta \Delta t} - R \left[\omega^t + \left(\frac{\gamma}{\beta} - 2 \right) \left(\omega^t + \frac{\Delta t \dot{\omega}^t}{2} \right) \right], \\ \dot{\omega}^{t+\Delta t} &= \frac{\omega^{t+\Delta t}}{\gamma \Delta t} - R \left[\frac{\omega^t}{\gamma \Delta t} + \left(\frac{1}{\gamma} - 1 \right) \dot{\omega}^t \right], \end{aligned} \quad (1.127a, b)$$

in which $\omega^t = d^t \times d(d^t)/dt$, β and γ are the parameters of the Newmark family of algorithms, not to be confused with that in Eqs. (1.31b) and (1.44b), and (1.119b). It should be noted that the numerical integration scheme in Eq. (1.127) is applied to the updating of the translational velocity and acceleration,

$$\begin{aligned} \dot{u}^{t+\Delta t} &= \frac{\gamma \Delta u}{\beta \Delta t} - \left[\dot{u}^t + \left(\frac{\gamma}{\beta} - 2 \right) \left(\dot{u}^t + \frac{\Delta t \ddot{u}^t}{2} \right) \right], \\ \ddot{u}^{t+\Delta t} &= \frac{\dot{u}^{t+\Delta t}}{\gamma \Delta t} - \left[\frac{\dot{u}^t}{\gamma \Delta t} + \left(\frac{1}{\gamma} - 1 \right) \ddot{u}^t \right], \end{aligned} \quad (1.128a, b)$$

where Δu is understood to be the incremental displacement vector at time t .

For the trapezoidal rule in which $\beta = 1/4$ and $\gamma = 1/2$, Eq. (1.127) reduces to the following

$$\begin{aligned} \omega^{t+\Delta t} &= \frac{2\Delta\theta}{\Delta t} - R \omega^t, \\ \dot{\omega}^{t+\Delta t} &= \frac{2\omega^{t+\Delta t}}{\Delta t} - R \left(\frac{2\omega^t}{\Delta t} + \dot{\omega}^t \right), \end{aligned} \quad (1.129a, b)$$

Other quantities such as the translational velocity and acceleration can be similarly obtained but are not included presently for brevity.

It is noted that in the limiting case of small rotations, $\Delta\theta \rightarrow 0$. Consequently, $\cos|\Delta\theta| \rightarrow 1$, $\sin(|\Delta\theta|)/|\Delta\theta| \rightarrow 1$ and $R \rightarrow I_3$. Naturally, in this case, the

directors needs not be updated and the updating scheme only has to use Eq. (1.128) for the translational velocity and acceleration vectors.

It should also be noted that the foregoing updating scheme, Eqs. (1.126) through (1.129) and so on, has embodied one implicit condition [24, 44]. That is, the incremental rotation vector $\Delta\theta$ is perpendicular to the director's reference position d_i^t . The physical interpretation is that the incremental rotational component along the director does not have any effects on the re-orientation of the director. Only those incremental rotational components lying on the plane perpendicular to the director can have an effect on bringing the director to a new position. Fox and Simo [45] proposed replacing the exponential mapping Γ_i^t of Eq. (1.81) with another mapping that in fact contained the product of two exponential mappings. One of these mappings was constructed from the ddof and the director. However, in [45] it was shown that such a new mapping was not uniquely defined. To preserve uniqueness *drill rotation constraint* has to be applied. This no doubt complicates the computation. It may be appropriate to note that no numerical results were included in [45]. As Ref. [46], the precursor of [45], pointed out, the aim was to identify the independent rotation field, which is the ddof in the present study, with the rotation of the continuum. This aim is realized in the foregoing formulation and derivation.

The present study adopts Eqs. (1.126) and (1.127) as the director updating scheme for two main considerations. First, it has a good physical basis. Second, it involves relatively less computational efforts.

Finally, as a part of configuration updating, the updating of density and thickness are included in the present investigation. Such an updating requires the calculation of the *relative* deformation gradient [22] which is defined as

$$\mathcal{F}_t^{t+\Delta t} = \begin{bmatrix} \frac{\partial(\Delta u^t)}{\partial \mathbf{r}^t} & \frac{\partial(\Delta u^t)}{\partial s^t} & \frac{\partial(\Delta u^t)}{\partial t^t} \\ \frac{\partial(\Delta v^t)}{\partial \mathbf{r}^t} & \frac{\partial(\Delta v^t)}{\partial s^t} & \frac{\partial(\Delta v^t)}{\partial t^t} \\ \frac{\partial(\Delta w^t)}{\partial \mathbf{r}^t} & \frac{\partial(\Delta w^t)}{\partial s^t} & \frac{\partial(\Delta w^t)}{\partial t^t} \end{bmatrix}, \quad (1.130)$$

where the displacement increments Δu^t , Δv^t , and Δw^t are defined by Eq. (1.75). Equation (1.130) expresses the deformations of the body in $C^{t+\Delta t}$ with respect to the reference configuration C^t . Thus, the density and thickness relations of

the shell element become, respectively,

$$\rho^{t+\Delta t} = \frac{\rho^t}{\det(\mathcal{F}_t^{t+\Delta t})}, \quad h^{t+\Delta t} = \frac{h^t \det(\mathcal{F}_t^{t+\Delta t}) A^t}{A^{t+\Delta t}}, \quad (1.131a, b)$$

where $\det(\cdot)$ denotes the *determinant of*, and the area of the shell element at t , A^t is given by

$$A^t = \frac{r_2^t s_3^t}{2}. \quad (1.131c)$$

1.5.3.2 Updating of Stresses

After solving the nodal displacement increments, the incremental strain and stress increments can be recovered by applying Eqs. (1.71) and (1.72). These stress increments are defined with respect to C^t , and therefore they can be added to the Cauchy stress σ^t since they are referred to the same reference state. The sum of σ^t and ΔS is equal to $S^{t+\Delta t}$. That is, $S^{t+\Delta t} = \sigma^t + \Delta S$, which is the PK2 stress vector in deformation state $C^{t+\Delta t}$ measured with respect to C^t . The transformation of $S^{t+\Delta t}$ to the Cauchy stress $\sigma^{t+\Delta t}$ is [22, 28]

$$\sigma^{t+\Delta t} = \frac{1}{\det(\mathcal{F}_t^{t+\Delta t})} \mathcal{F}_t^{t+\Delta t} S^{t+\Delta t} (\mathcal{F}_t^{t+\Delta t})^T, \quad (1.132)$$

where the deformation gradient $\mathcal{F}_t^{t+\Delta t}$ is defined by Eq. (1.130).

1.6 Concluding Remarks

While it is assumed that the readers have at least a minimum knowledge of a first course in the FEM, the basic theories and steps in the displacement formulation and the hybrid stress formulation have been included in this chapter.

To illustrate the steps in a more concrete way, the derivation of element mass and stiffness matrices for the uniform and tapered beam elements have been presented. An outline of the incremental hybrid strain or mixed formulation has also been included. Explicit expressions for the element mass and stiffness matrices have been derived and presented as appendices to this chapter.

Throughout this chapter the element equations of motion are emphasized.

No attempt has been made to include the assembled equations of motion for the structural systems. This is because assembled equations of motion are concerned with computer programming and it is a standard feature in any FEM package.

Finally, it is observed that the literature on FEM is vast and the topics presented in this chapter serve as a preparation for a better understanding of the concepts and detailed steps in subsequent chapters. However, Sections 1.4 and 1.5 contain more advanced topics and therefore they may be disregarded by the average or general readers.

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