1 Formal Language Theory

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1 Introduction

This chapter provides a gentle introduction to formal language theory, aimed at readers with little background in formal systems. The motivation is natural language processing (NLP), and the presentation is geared towards NLP applications, with linguistically motivated examples, but without compromising mathematical rigor.

The text covers elementary formal language theory, including: regular languages and regular expressions; languages vs. computational machinery; finite state automata; regular relations and finite state transducers; context-free grammars and languages; the Chomsky hierarchy; weak and strong generative capacity; and mildly context-sensitive languages.

2 Basic Notions

Formal languages are defined with respect to a given *alphabet*, which is a finite set of symbols, each of which is called a *letter*. This notation does not mean, however, that elements of the alphabet must be "ordinary" letters; they can be any symbol, such as numbers, or digits, or words. It is customary to use 'Σ' to denote the alphabet. A finite sequence of letters is called a *string*, or a *word*. For simplicity, we usually forsake the traditional sequence notation in favor of a more straightforward representation of strings.

Example 1 (Strings). Let $\Sigma = \{0, 1\}$ be an alphabet. Then all binary numbers are strings over Σ. Instead of $\langle 0, 1, 1, 0, 1 \rangle$ we usually write 01101. If $\Sigma = \{a, b, c, d, \ldots, y, z\}$ is an alphabet, then *cat*, *incredulous*, and *supercalifragilisticexpialidocious* are strings, as are *tac*, *qqq*, and *kjshdflkwjehr*.

The *length* of a string w is the number of letters in the sequence, and is denoted $|w|$. The unique string of length 0 is called the *empty string* and is usually denoted ϵ (but sometimes λ).

The Handbook of Computational Linguistics and Natural Language Processing, First Edition.
Edited by Alexander Clark, Chris Fox and Shalom Lappin.

Let $w_1 = \langle x_1, \ldots, x_n \rangle$ and $w_2 = \langle y_1, \ldots, y_m \rangle$ be two strings over the same alphabet Σ. The *concatenation* of w_1 and w_2, denoted $w_1 \cdot w_2$, is the string $\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$. Note that the length of $w_1 \cdot w_2$ is the sum of the lengths of w_1 and w_2: $|w_1 \cdot w_2| = |w_1| + |w_2|$. When it is clear from the context, we sometimes omit the '$\cdot$' symbol when depicting concatenation.

Example 2 (Concatenation). Let $\Sigma = \{a, b, c, d, \ldots, y, z\}$ be an alphabet. Then *master* $\cdot$ *mind* = *mastermind*, *mind* $\cdot$ *master* = *mindmaster*, and *master* $\cdot$ *master* = *mastermaster*. Similarly, *learn* $\cdot$ *s* = *learns*, *learn* $\cdot$ *ed* = *learned*, and *learn* $\cdot$ *ing* = *learning*.

Notice that when the empty string ϵ is concatenated with any string w, the resulting string is w. Formally, for every string w, $w \cdot \epsilon = \epsilon \cdot w = w$.

We define an *exponent* operator over strings in the following way: for every string w, w^0 (read: w raised to the power of zero) is defined as ϵ. Then, for $n > 0$, w^n is defined as $w^{n-1} \cdot w$. Informally, w^n is obtained by concatenating w with itself n times. In particular, $w^1 = w$.

Example 3 (Exponent). If $w =$ *go*, then $w^0 = \epsilon$, $w^1 = w =$ *go*, $w^2 = w^1 \cdot w = w \cdot w =$ *gogo*, $w^3 =$ *gogogo*, and so on.

A few other notions that will be useful in the sequel: the *reversal* of a string w is denoted w^R and is obtained by writing w in the reverse order. Thus, if $w = \langle x_1, x_2, \ldots, x_n \rangle$, $w^R = \langle x_n, x_{n-1}, \ldots, x_1 \rangle$.

Example 4 (Reversal). Let $\Sigma = \{a, b, c, d, \ldots, y, z\}$ be an alphabet. If w is the string *saw*, then w^R is the string *was*. If $w =$ *madam*, then $w^R =$ *madam* $= w$. In this case we say that w is a *palindrome*.

Given a string w, a *substring* of w is a sequence formed by taking contiguous symbols of w in the order in which they occur in w: w_c is a substring of w if and only if there exist (possibly empty) strings w_l and w_r such that $w = w_l \cdot w_c \cdot w_r$. Two special cases of substrings are *prefix* and *suffix*: if $w = w_l \cdot w_c \cdot w_r$ then w_l is a prefix of w and w_r is a suffix of w. Note that every prefix and every suffix is a substring, but not every substring is a prefix or a suffix.

Example 5 (Substrings). Let $\Sigma = \{a, b, c, d, \ldots, y, z\}$ be an alphabet and $w =$ *indistinguishable* a string over Σ. Then ϵ, *in*, *indis*, *indistinguish*, and *indistinguishable* are prefixes of w, while ϵ, *e*, *able*, *distinguishable* and *indistinguishable* are suffixes of w. Substrings that are neither prefixes nor suffixes include *distinguish*, *gui*, and *is*.

Given an alphabet Σ, the set of all strings over Σ is denoted by Σ^* (the reason for this notation will become clear presently). Notice that no matter what the Σ is, as long as it includes at least one symbol, Σ^* is always infinite. A *formal language* over an alphabet Σ is any subset of Σ^*. Since Σ^* is always infinite, the number of formal languages over Σ is also infinite.

As the following example demonstrates, formal languages are quite unlike what one usually means when one uses the term "language" informally. They

are essentially sets of strings of characters. Still, all natural languages are, at least superficially, such string sets. Higher-level notions, relating the strings to objects and actions in the world, are completely ignored by this view. While this is a rather radical idealization, it is a useful one.

Example 6 (Languages). Let $\Sigma = \{a, b, c, \ldots, y, z\}$. Then Σ^* is the set of all strings over the Latin alphabet. Any subset of this set is a language. In particular, the following are formal languages:

- Σ^*;
- the set of strings consisting of consonants only;
- the set of strings consisting of vowels only;
- the set of strings each of which contains at least one vowel and at least one consonant;
- the set of palindromes: strings that read the same from right to left and from left to right;
- the set of strings whose length is less than 17 letters;
- the set of single-letter strings;
- the set {*i, you, he, she, it, we, they*};
- the set of words occurring in Joyce's Ulysses (ignoring punctuation etc.);
- the empty set.

Note that the first five languages are infinite while the last five are finite.

We can now lift some of the string operations defined above to languages. If L is a language then the *reversal* of L, denoted L^R, is the language $\{w \mid w^R \in L\}$, that is, the set of reversed L-strings. *Concatenation* can also be lifted to languages: if L_1 and L_2 are languages, then $L_1 \cdot L_2$ is the language defined as $\{w_1 \cdot w_2 \mid w_1 \in L_1 \textit{ and } w_2 \in L_2\}$: the concatenation of two languages is the set of strings obtained by concatenating some word of the first language with some word of the second.

Example 7 (Language operations). Let $L_1 = \{$*i, you, he, she, it, we, they*$\}$ and $L_2 = \{$*smile, sleep*$\}$. Then $L_1^R = \{$*i, uoy, eh, ehs, ti, ew, yeht*$\}$ and $L_1 \cdot L_2 = \{$*ismile, yousmile, hesmile, shesmile, itsmile, wesmile, theysmile, isleep, yousleep, hesleep, shesleep, itsleep, wesleep, theysleep*$\}$.

In the same way we can define the *exponent* of a language: if L is a language then L^0 is the language containing the empty string only, $\{\epsilon\}$. Then, for $i > 0$, $L^i = L \cdot L^{i-1}$, that is, L^i is obtained by concatenating L with itself i times.

Example 8 (Language exponentiation). Let L be the set of words {*bau, haus, hof, frau*}. Then $L^0 = \{\epsilon\}$, $L^1 = L$ and $L^2 =$ {*baubau, bauhaus, bauhof, baufrau, hausbau, haushaus, haushof, hausfrau, hofbau, hofhaus, hofhof, hoffrau, fraubau, frauhaus, frauhof, fraufrau*}.

The language obtained by considering any number of concatenations of words from L is called the *Kleene closure* of L and is denoted L^*. Formally, $L^* = \bigcup_{i=0}^{\infty} L^i$,

which is a terse notation for the union of L^0 with L^1, then with L^2, L^3 and so on ad infinitum. When one wants to leave L^0 out, one writes $L^+ = \bigcup_{i=1}^{\infty} L^i$.

Example 9 (Kleene closure). Let $L = \{dog, cat\}$. Observe that $L^0 = \{\epsilon\}$, $L^1 = \{dog, cat\}$, $L^2 = \{catcat, catdog, dogcat, dogdog\}$, etc. Thus L^* contains, among its infinite set of strings, the strings ϵ, *cat, dog, catcat, catdog, dogcat, dogdog, catcatcat, catdogcat, dogcatcat, dogdogcat*, etc.

As another example, consider the alphabet $\Sigma = \{a, b\}$ and the language $L = \{a, b\}$ defined over Σ. L^* is the set of all strings over a and b, which is exactly the definition of Σ^*. The notation for Σ^* should now become clear: it is simply a special case of L^*, where $L = \Sigma$.

3 Language Classes and Linguistic Formalisms

Formal languages are sets of strings, subsets of Σ^*, and they can be specified using any of the specification methods for sets (of course, since languages may be infinite, stipulation of their members is in the general case infeasible). When languages are fairly simple (not arbitrarily complex), they can be characterized by means of *rules*. In the following sections we define several mechanisms for defining languages, and focus on the *classes* of languages that can be defined with these mechanisms. A formal mechanism with which formal languages can be defined is a *linguistic formalism*. We use L (with or without subscripts) to denote languages, and $\mathcal{L}$ to denote classes of languages.

Example 10 (Language class). Let $\Sigma = \{a, b, c, \ldots, y, z\}$. Let $\mathcal{L}$ be the set of all the *finite* subsets of Σ^*. Then $\mathcal{L}$ is a language class.

When classes of languages are discussed, some of the interesting properties to be investigated are *closures* with respect to certain operators. The previous section defined several operators, such as concatenation, union, Kleene closure, etc., on languages. Given a particular (binary) operation, say union, it is interesting to know whether a class of languages is *closed under* this operation. A class of languages $\mathcal{L}$ is said to be closed under some operation '$\bullet$' if and only if, whenever two languages L_1 and L_2 are in the class ($L_1, L_2 \in \mathcal{L}$), the result of performing the operation on the two languages is also in this class: $L_1 \bullet L_2 \in \mathcal{L}$.

Closure properties have a theoretical interest in and by themselves, but they are especially important when one is interested in processing languages. Given an efficient computational implementation for a class of languages (for example, an algorithm that determines *membership*: whether a given string indeed belongs to a given language), one can use the operators that the class is closed under, and still preserve computational efficiency in processing. We will see such examples in the following sections.

The membership problem is one of the fundamental questions of interest concerned with language classes. As we shall see, the more expressive the class, the harder it is to determine membership in languages of this class. Algorithms that determine membership are called *recognition* algorithms; when a recognition

algorithm additionally provides the structure that the formalism induces on the string in question, it is called a *parsing* algorithm.

4 Regular Languages

4.1 Regular expressions

The first linguistic formalism we discuss is *regular expressions*. These are expressions over some alphabet Σ, augmented by some special characters. We define a mapping, called *denotation*, from regular expressions to sets of strings over Σ, such that every well-formed regular expression denotes a set of strings, or a language.

DEFINITION 1. *Given an alphabet Σ, the set of **regular expressions** over Σ is defined as follows:*

- *$\emptyset$ is a regular expression;*
- *ϵ is a regular expression;*
- *if $a \in \Sigma$ is a letter, then a is a regular expression;*
- *if r_1 and r_2 are regular expressions, then so are $(r_1 + r_2)$ and $(r_1 \cdot r_2)$;*
- *if r is a regular expression, then so is $(r)^*$;*
- *nothing else is a regular expression over Σ.*

Example 11 (Regular expressions). Let Σ be the alphabet $\{a, b, c, \ldots, y, z\}$. Some regular expressions over this alphabet are $\emptyset$, a, $((c \cdot a) \cdot t)$, $(((m \cdot e) \cdot (o)^*) \cdot w)$, $(a + (e + (i + (o + u))))$, $((a + (e + (i + (o + u)))))^*$, etc.

DEFINITION 2. *Given a regular expression r, its **denotation, $[\![r]\!]$**, is a set of strings defined as follows:*

- *$[\![\emptyset]\!] = \{\}$, the empty set;*
- *$[\![\epsilon]\!] = \{\epsilon\}$, the singleton set containing the empty string;*
- *if $a \in \Sigma$ is a letter, then $[\![a]\!] = \{a\}$, the singleton set containing a only;*
- *if r_1 and r_2 are two regular expressions whose denotations are $[\![r_1]\!]$ and $[\![r_2]\!]$, respectively, then $[\![(r_1 + r_2)]\!] = [\![r_1]\!] \cup [\![r_2]\!]$ and $[\![(r_1 \cdot r_2)]\!] = [\![r_1]\!] \cdot [\![r_2]\!]$;*
- *if r is a regular expression whose denotation is $[\![r]\!]$ then $[\![(r)^*]\!] = [\![r]\!]^*$.*

Example 12 (Regular expressions). Following are the denotations of the regular expressions of the previous example:

$\emptyset$	$\emptyset$
ϵ	$\{\epsilon\}$
a	$\{a\}$
$((c \cdot a) \cdot t)$	$\{c \cdot a \cdot t\}$
$(((m \cdot e) \cdot (o)^*) \cdot w)$	$\{mew, meow, meoow, meooow, meoooow, \ldots\}$
$(a + (e + (i + (o + u))))$	$\{a, e, i, o, u\}$
$((a + (e + (i + (o + u)))))^*$	*the set containing all strings of 0 or more vowels*

Regular expressions are useful because they facilitate specification of complex languages in a formal, concise way. Of course, finite languages can still be specified by enumerating their members; but infinite languages are much easier to specify with a regular expression, as the last instance of the above example shows.

For simplicity, we omit the parentheses around regular expressions when no confusion can be caused. Thus, the expression $((a + (e + (i + (o + u)))))^*$ is written as $(a + e + i + o + u)^*$. Also, if $\Sigma = \{a_1, a_2, \ldots, a_n\}$, we use Σ as a shorthand notation for $a_1 + a_2 + \cdots + a_n$. As in the case of string concatenation and language concatenation, we sometimes omit the '$\cdot$' operator in regular expressions, so that the expression $c \cdot a \cdot t$ can be written *cat*.

Example 13 (Regular expressions). Given the alphabet of all English letters, $\Sigma = \{a, b, c, \ldots, y, z\}$, the language Σ^* is denoted by the regular expression Σ^*. The set of all strings which contain a vowel is denoted by $\Sigma^* \cdot (a + e + i + o + u) \cdot \Sigma^*$. The set of all strings that begin in "*un*" is denoted by $(un)\Sigma^*$. The set of strings that end in either "*tion*" or "*sion*" is denoted by $\Sigma^* \cdot (s + t) \cdot (ion)$. Note that all these languages are infinite.

The class of languages which can be expressed as the denotation of regular expressions is called the class of *regular languages*.

DEFINITION 3. *A language L is* ***regular*** *iff there exists a regular expression r such that* $L = [\![r]\!]$.

It is a mathematical fact that some languages, subsets of Σ^*, are not regular. We will encounter such languages in the sequel.

4.2 Properties of regular languages

The class of regular languages is interesting because of its "nice" properties, which we review here. It should be fairly easy to see that regular languages are closed under union, concatenation, and Kleene closure. Given two regular languages, L_1 and L_2, there must exist two regular expressions, r_1 and r_2, such that $[\![r_1]\!] = L_1$ and $[\![r_2]\!] = L_2$. It is therefore possible to form new regular expressions based on r_1 and r_2, such as $r_1 \cdot r_2$, $r_1 + r_2$ and r_1^*. Now, by the definition of regular expressions and their denotations, it follows that the denotation of $r_1 \cdot r_2$ is $L_1 \cdot L_2$: $[\![r_1 \cdot r_2]\!] = L_1 \cdot L_2$. Since $r_1 \cdot r_2$ is a regular expression, its denotation is a regular language, and hence $L_1 \cdot L_2$ is a regular language. Hence the regular languages are closed under concatenation. In exactly the same way we can prove that the class of regular languages is closed under union and Kleene closure.

One of the reasons for the attractiveness of regular languages is that they are known to be closed under a wealth of useful operations: intersection, complementation, exponentiation, substitution, homomorphism, etc. These properties come in handy both in practical applications that use regular languages and in mathematical proofs that concern them. For example, several formalisms extend regular expressions by allowing one to express regular languages using not only the three

basic operations, but also a wealth of other operations (that the class of regular languages is closed under). It is worth noting that such "good behavior" is not exhibited by more complex classes of languages.

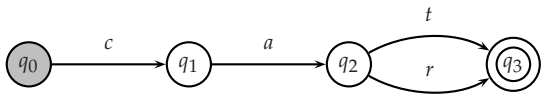
4.3 Finite state automata

Regular expressions are a declarative formalism for specifying (regular) languages. We now present languages as entities generated by a *computation*. This is a very common situation in formal language theory: many language classes are associated with computing machinery that generates them. The dual view of languages (as the denotation of some specifying formalism and as the output of a computational process) is central in formal language theory.

The computational device we define in this section is *finite state automata* (FSA). Informally, they consist of a finite set of *states* (sometimes called *nodes* or *vertices*), connected by a finite number of *transitions* (also called *edges* or *links*). Each of the transitions is labeled by a letter, taken from some finite alphabet Σ. A computation starts at a designated state, the *start* state or *initial* state, and it moves from one state to another along the labeled transitions. As it moves, it prints the letter which labels the transition. Thus, during a computation, a string of letters is printed out. Some of the states of the machine are designated *final* states, or *accepting* states. Whenever the computation reaches a final state, the string that was printed so far is said to be *accepted* by the machine. Since each computation defines a string, the set of all possible computations defines a set of strings or, in other words, a language. We say that this language is *accepted* or *generated* by the machine.

DEFINITION 4. *A **finite state automaton** is a five-tuple $\langle Q, q_0, \Sigma, \delta, F\rangle$, where Σ is a finite set of **alphabet** symbols, Q is a finite set of **states**, $q_0 \in Q$ is the **initial state**, $F \subseteq Q$ is a set of **final** states, and $\delta : Q \times \Sigma \times Q$ is a relation from states and alphabet symbols to states.*

Example 14 (Finite state automata). Finite state automata are depicted graphically, with circles for states and arrows for the transitions. The initial state is shaded and the final states are depicted by two concentric circles. The finite state automaton $A = \langle Q, \Sigma, q_0, \delta, F\rangle$, where $Q = \{q_0, q_1, q_2, q_3\}$, $\Sigma = \{c, a, t, r\}$, $F = \{q_3\}$, and $\delta = \{\langle q_0, c, q_1\rangle, \langle q_1, a, q_2\rangle, \langle q_2, t, q_3\rangle, \langle q_2, r, q_3\rangle\}$, is depicted graphically as follows:



To define the *language* generated by an FSA, we first extend the transition relation from single edges to *paths* by extending the transition relation δ to its reflexive transitive closure, $\hat{\delta}$. This relation assigns a string to each path (it also assumes that an empty path, decorated by ϵ, leads from each state to itself). We focus on paths that lead from the initial state to some final state. The strings that decorate these paths are said to be *accepted* by the FSA, and the *language* of the FSA is the set of all these strings. In other words, in order for a string to be in the

language of the FSA, there must be a path in the FSA which leads from the initial state to some final state decorated by the string. Paths that lead to non-final states do not define accepted strings.

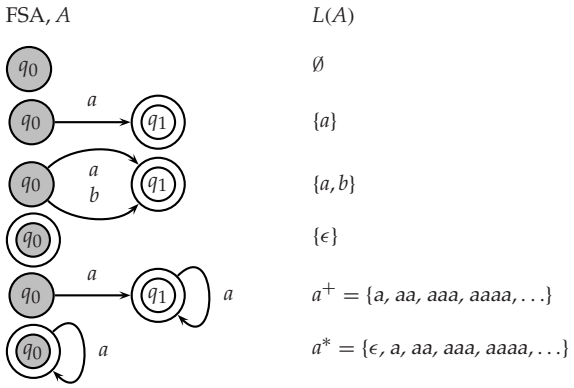
DEFINITION 5. *Given an FSA $A = \langle Q, q_0, \Sigma, \delta, F\rangle$, the reflexive transitive closure of the transition relation δ is $\hat{\delta}$, defined as follows:*

- *for every state $q \in Q$, $(q, \epsilon, q) \in \hat{\delta}$;*
- *for every string $w \in \Sigma^*$ and letter $a \in \Sigma$, if $(q, w, q') \in \hat{\delta}$ and $(q', a, q'') \in \delta$, then $(q, w \cdot a, q'') \in \hat{\delta}$.*

*A string w is **accepted** by A if and only if there exists a state $q_f \in F$ such that $\hat{\delta}(q_0, w) = q_f$. The **language** of A is the set of all the strings accepted by it: $L(A) = \{w \mid$ there exists $q_f \in F$ such that $\hat{\delta}(q_0, w) = q_f\}$.*

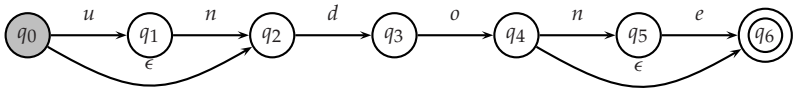
Example 15 (Language accepted by an FSA). For the finite state automaton of Example 14, $\hat{\delta}$ is the following set of triples: $\langle q_0, \epsilon, q_0\rangle$, $\langle q_1, \epsilon, q_1\rangle$, $\langle q_2, \epsilon, q_2\rangle$, $\langle q_3, \epsilon, q_3\rangle$, $\langle q_0, c, q_1\rangle$, $\langle q_1, a, q_2\rangle$, $\langle q_2, t, q_3\rangle$, $\langle q_2, r, q_3\rangle$, $\langle q_0, ca, q_2\rangle$, $\langle q_1, at, q_3\rangle$, $\langle q_1, ar, q_3\rangle$, $\langle q_0, cat, q_3\rangle$, $\langle q_0, car, q_3\rangle$. The language of the FSA is thus {*cat, car*}.

Example 16 (Finite state automata). Following are some simple FSA and the languages they generate.



We now slightly amend the definition of finite state automata to include what is called ϵ-*moves*. By our original definition, the transition relation δ is a relation from states and alphabet symbols to states. We extend δ such that its second coordinate is now $\Sigma \cup \{\epsilon\}$, that is, any edge in an automaton can be labeled either by some alphabet symbol or by the special symbol ϵ, which as usual denotes the empty word. The implication is that a computation can move from one state to another over an ϵ-transition without printing out any symbol.

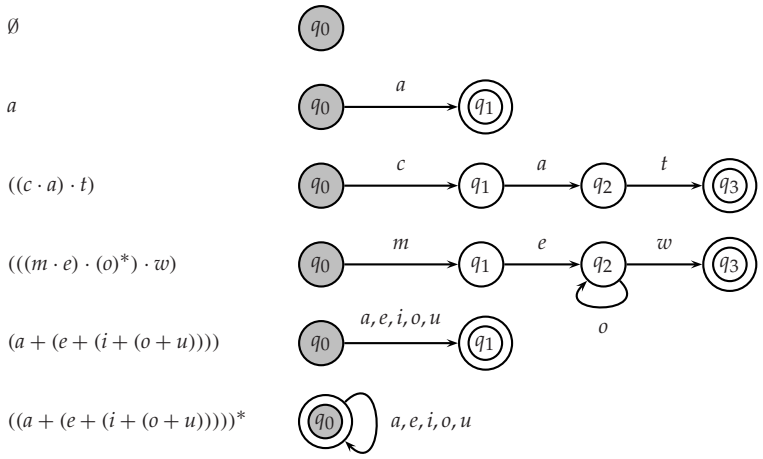
Example 17 (Automata with ϵ-moves). The language accepted by the following automaton is {*do, undo, done, undone*}:



Finite state automata, just like regular expressions, are devices for defining formal languages. The major theorem of regular languages states that the class of languages which can be generated by FSA is exactly the class of regular languages. Furthermore, there are simple and efficient algorithms for "translating" a regular expression to an equivalent automaton and vice versa.

THEOREM 1. *A language L is regular iff there exists an FSA A such that $L = L(A)$.*

Example 18 (Equivalence of finite state automata and regular expressions). For each of the regular expressions of Example 12 we depict an equivalent automaton below:



4.4 Minimization and determinization

The finite state automata presented above are non-deterministic. By this we mean that when the computation reaches a certain state, the next state is not uniquely determined by the next alphabet symbol to be printed. There might very well be more than one state that can be reached by a transition that is labeled by some symbol. This is because we defined automata using a transition relation, δ, which is not required to be functional. For some state q and alphabet symbol a, δ might include the two pairs $\langle q, a, q_1 \rangle$ and $\langle q, a, q_2 \rangle$ with $q_1 \neq q_2$. Furthermore, when we extended δ to allow ϵ-transitions, we added yet another dimension of non-determinism: when the machine is in a certain state q and an ϵ-arc leaves q, the computation must "guess" whether to traverse this arc.

DEFINITION 6. *An FSA $A = \langle Q, q_0, \Sigma, \delta, F \rangle$ is **deterministic** iff it has no ϵ-transitions and δ is a function from $Q \times \Sigma$ to Q.*

Much of the appeal of finite state automata lies in their efficiency; and their efficiency is in great part due to the fact that, given some deterministic FSA A and a string w, it is possible to determine whether or not $w \in L(A)$ by "walking" the path labeled w, starting with the initial state of A, and checking whether the walk leads to a final state. Such a walk takes time that is proportional to the length of w, and is completely independent of the number of states in A. We therefore say that

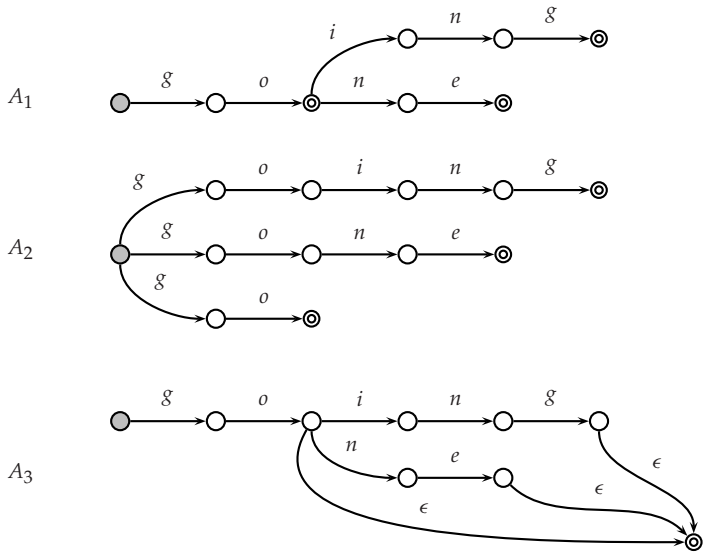
the *membership* problem for FSA can be solved in *linear* time. But when automata are non-deterministic, an element of guessing is introduced, which may impair the efficiency: no longer is there a single walk along a single path labeled w, and some control mechanism must be introduced to check that all possible paths are taken.

Non-determinism is important because it is sometimes much easier to construct a non-deterministic automaton for some language. Fortunately, we can rely on two very important results: every non-deterministic finite state automaton is equivalent to some deterministic one; and every finite state automaton is equivalent to one that has a minimum number of nodes, and the minimal automaton is unique. We now explain these results.

First, it is important to clarify what is meant by *equivalent*. We say that two finite state automata are equivalent if and only if they accept the same language.

DEFINITION 7. *Two FSA A_1 and A_2 are* ***equivalent*** *iff* $L(A_1) = L(A_2)$.

Example 19 (Equivalent automata). The following three finite state automata are equivalent: they all accept the set {*go, gone, going*}.



Note that A_1 is deterministic: for any state and alphabet symbol there is at most one possible transition. A_2 is not deterministic: the initial state has three outgoing arcs all labeled by g. The third automaton, A_3, has ϵ-arcs and hence is non-deterministic. While A_2 might be the most readable, A_1 is the most compact as it has the fewest nodes.

Given a non-deterministic FSA A, it is always possible to construct an equivalent deterministic automaton, one whose next state is fully determined by the current state and the alphabet symbol, and which contains no ϵ-moves. Sometimes this construction yields an automaton with more states than the original,

non-deterministic one (in the worst case, the number of states in the deterministic automaton can be exponential in the size of the non-deterministic one). However, the deterministic automaton can then be minimized such that it is guaranteed that no deterministic finite state automaton generating the same language is smaller. Thus, it is always possible to determinize and then minimize a given automaton without affecting the language it generates.

THEOREM 2. *For every FSA A (with n states) there exists a deterministic FSA A' (with at most 2^n states) such that $L(A) = L(A')$.*

THEOREM 3. *For every regular language L there exists a minimal FSA A such that no other FSA A' such that $L(A) = L(A')$ has fewer states than A. A is unique (up to isomorphism).*

4.5 Operations on finite state automata

We know from Section 4.3 that finite state automata are equivalent to regular expressions; we also know from Section 4.2 that the regular languages are closed under several operations, including union, concatenation, and Kleene closure. So, for example, if L_1 and L_2 are two regular languages, there exist automata A_1 and A_2 which accept them, respectively. Since we know that $L_1 \cup L_2$ is also a regular language, there must be an automaton which accepts it as well. The question is, can this automaton be constructed using the automata A_1 and A_2? In this section we show how simple operations on finite state automata correspond to some operators on languages.

We start with concatenation. Suppose that A_1 is a finite state automaton such that $L(A_1) = L_1$, and similarly that A_2 is an automaton such that $L(A_2) = L_2$. We describe an automaton A such that $L(A) = L_1 \cdot L_2$. A word w is in $L_1 \cdot L_2$ if and only if it can be broken into two parts, w_1 and w_2, such that $w = w_1 \cdot w_2$, and $w_1 \in L_1$, $w_2 \in L_2$. In terms of automata, this means that there is an accepting path for w_1 in A_1 and an accepting path for w_2 in A_2; so if we allow an ϵ-transition from all the final states of A_1 to the initial state of A_2, we will have accepting paths for words of $L_1 \cdot L_2$. The finite state automaton A is constructed by combining A_1 and A_2 in the following way: its set of states, Q, is the union of Q_1 and Q_2; its alphabet is the union of the two alphabets; its initial state is the initial state of A_1; its final states are the final states of A_2; and its transition relation is obtained by adding to $\delta_1 \cup \delta_2$ the set of ϵ-moves described above: $\{\langle q_f, \epsilon, q_{0_2}\rangle \mid q_f \in F_1\}$ where q_{0_2} is the initial state of A_2.

In a very similar way, an automaton A can be constructed whose languages is $L_1 \cup L_2$ by combining A_1 and A_2. Here, one should notice that for a word to be accepted by A it must be accepted either by A_1 or by A_2 (or by both). The combined automaton will have an accepting path for every accepting path in A_1 and in A_2. The idea is to add a new initial state to A, from which two ϵ-arcs lead to the initial states of A_1 and A_2. The states of A are the union of the states of A_1 and A_2, plus

the new initial state. The transition relation is the union of δ_1 with δ_2, plus the new ϵ-arcs. The final states are the union of F_1 and F_2.

An extension of the same technique to construct the Kleene closure of an automaton is rather straightforward. However, all these results are not surprising, as we have already seen in Section 4.2 that the regular languages are closed under these operations. Thinking of languages in terms of the automata that accept them comes in handy when one wants to show that the regular languages are closed under other operations, where the regular expression notation is not very suggestive of how to approach the problem. Consider the operation of *complementation*: if L is a regular language over an alphabet Σ, we say that the complement of L is the set of all the words (in Σ^*) that are not in L, and write $\overline{L}$ for this set. Formally, $\overline{L} = \Sigma^* \setminus L$. Given a regular expression r, it is not clear what regular expression r' is such that $[\![r']\!] = \overline{[\![r]\!]}$. However, with automata this becomes much easier.

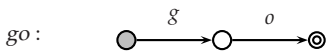
Assume that a finite state automaton A is such that $L(A) = L$. Assume also that A is deterministic. To construct an automaton for the complemented language, all one has to do is change all final states to non-final, and all non-final states to final. In other words, if $A = \langle Q, \Sigma, q_0, \delta, F\rangle$, then $\overline{A} = \langle Q, \Sigma, q_0, \delta, Q \setminus F\rangle$ is such that $L(\overline{A}) = \overline{L}$. This is because every accepting path in A is not accepting in $\overline{A}$, and vice versa.

Now that we know that the regular languages are closed under complementation, it is easy to show that they are closed under intersection: if L_1 and L_2 are regular languages, then $L_1 \cap L_2$ is also regular. This follows directly from fundamental theorems of set theory, since $L_1 \cap L_2$ can actually be written as $\overline{\overline{L_1} \cup \overline{L_2}}$, and we already know that the regular languages are closed under union and complementation. In fact, construction of an automaton for the intersection language is not very difficult, although it is less straightforward than the previous examples.

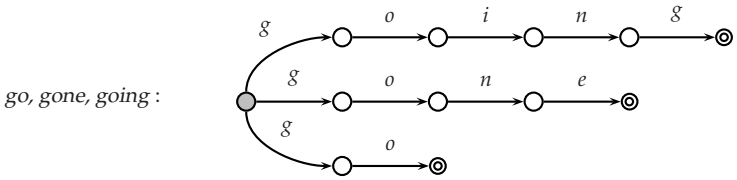
4.6 Applications of finite state automata in natural language processing

Finite state automata are computational devices that generate regular languages, but they can also be viewed as *recognizing* devices: given some automaton A and a word w, it is easy to determine whether $w \in L(A)$. Observe that such a task can be performed in time linear in the length of w, hence the efficiency of the representation is optimal. This reversed view of automata motivates their use for a simple yet necessary application of natural language processing: dictionary lookup.

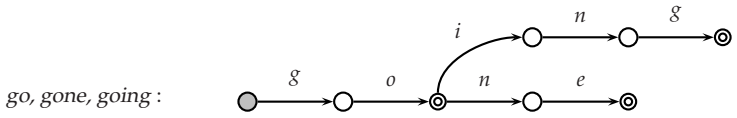
Example 20 (Dictionaries as finite state automata). Many NLP applications require the use of lexicons or dictionaries, sometimes storing hundreds of thousands of entries. Finite state automata provide an efficient means for storing dictionaries, accessing them, and modifying their contents. Assume that an alphabet is fixed (say, $\Sigma = \{a, b, \ldots, z\}$) and consider how a single word, say *go*, can be represented. As we have seen above, a naïve representation would be to construct an automaton with a single path whose arcs are labeled by the letters of the word *go*:



To represent more than one word, add paths to the FSA, one path for each additional word. For example, after adding the words *gone* and *going*, we obtain:



This automaton can then be determinized and minimized, yielding:



The organization of the lexicon as outlined above is extremely simplistic. A possible extension attaches to the final states of the FSA additional information pertaining to the words that decorate the paths to those states. Such information can include definitions, morphological information, translations, etc. FSA are thus suitable for representing various kinds of dictionaries, in addition to simple lexicons.

Regular languages are particularly appealing for natural language processing for two main reasons. First, it turns out that most phonological and morphological processes can be straightforwardly described using the operations that regular languages are closed under, in particular concatenation. With very few exceptions (such as the interdigitation word-formation processes of Semitic languages or the duplication phenomena of some Asian languages), the morphology of most natural languages is limited to simple concatenation of affixes, with some morphophonological alternations, usually on a morpheme boundary. Such phenomena are easy to model with regular languages, and hence are easy to implement with finite state automata. Second, many of the algorithms one would want to apply to finite state automata take time proportional to the length of the word being processed, independently of the size of the automaton. Finally, the various closure properties facilitate modular development of FSA for natural languages.

4.7 Regular relations

While finite state automata, which *define* (regular) languages, are sufficient for some natural language applications, it is often useful to have a mechanism for *relating* two (formal) languages. For example, a part-of-speech tagger can be viewed as an application that relates a set of natural language strings (the *source* language) to a set of part-of-speech tags (the *target* language). A morphological

analyzer can be viewed as a relation between natural language strings (the surface forms of words) and their internal structure (say, as sequences of morphemes). In this section we discuss a computational device, very similar to finite state automata, which defines a *relation* over two regular languages.

Example 21 (Relations over languages). Consider a simple part-of-speech tagger: an application which associates with every word in some natural language a tag, drawn from a finite set of tags. In terms of formal languages, such an application implements a relation over two languages. Assume that the natural language is defined over $\Sigma_1 = \{a, b, \ldots, z\}$ and that the set of tags is $\Sigma_2 = \{$*PRON, V, DET, ADJ, N, P*$\}$. Then the part-of-speech relation might contain the following pairs (here, a string over Σ_1 is mapped to a single element of Σ_2):

I	PRON	*the*	DET
know	V	*Cat*	N
some	DET	*in*	P
new	ADJ	*the*	DET
tricks	N	*Hat*	N
said	V		

As another example, assume that Σ_1 is as above, and Σ_2 is a set of part-of-speech and morphological tags, including {*-PRON, -V, -DET, -ADJ, -N, -P, -1, -2, -3, -sg, -pl, -pres, -past, -def, -indef*}. A morphological analyzer is a relation between a language over Σ_1 and a language over Σ_2. Some of the pairs in such a relation are:

I	I-PRON-1-sg	*the*	the-DET-def
know	know-V-pres	*Cat*	cat-N-sg
some	some-DET-indef	*in*	in-P
new	new-ADJ	*the*	the-DET-def
tricks	trick-N-pl	*Hat*	hat-N-sg
said	say-V-past		

Finally, consider the relation that maps every English noun in singular to its plural form. While the relation is highly regular (namely, adding "s" to the singular form), some nouns are irregular. Some instances of this relation are:

cat	*cats*	*hat*	*hats*
ox	*oxen*	*child*	*children*
mouse	*mice*	*sheep*	*sheep*
goose	*geese*		

Summing up, a regular relation is defined over *two* alphabets, Σ_1 and Σ_2. Of course, the two alphabets can be identical, but for many natural language applications they differ. If a relation in $\Sigma^* \times \Sigma^*$ is regular, its projections on both coordinates are regular languages (not all relations that satisfy this condition are regular; additional constraints must hold on the underlying mapping which we

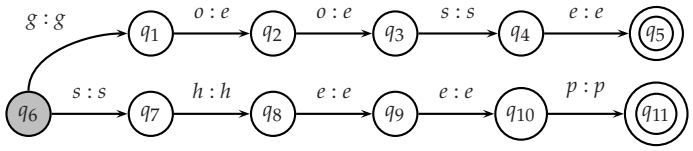
ignore here). Informally, a regular relation is a set of pairs, each of which consists of one string over Σ_1 and one string over Σ_2, such that both the set of strings over Σ_1 and that over Σ_2 constitute regular languages. We provide a precise characterization of regular relations via finite state transducers below.

4.8 Finite state transducers

Finite state automata are a computational device for defining regular languages; in a very similar way, *finite state transducers (FSTs)* are a computational device for defining regular relations. Transducers are similar to automata, the only difference being that the edges are not labeled by single letters, but rather by pairs of symbols: one symbol from Σ_1 and one symbol from Σ_2. The following is a preliminary definition that we will revise presently:

DEFINITION 8. *A **finite state transducer** is a six-tuple $\langle Q, q_0, \Sigma_1, \Sigma_2, \delta, F\rangle$, where Q is a finite set of states, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, Σ_1 and Σ_2 are alphabets, and δ is a subset of $Q \times \Sigma_1 \times \Sigma_2 \times Q$.*

Example 22 (Finite state transducers). Following is a finite state transducer relating the singular forms of two English words with their plural form. In this case, both alphabets are identical: $\Sigma_1 = \Sigma_2 = \{a, b, \ldots, z\}$. The set of nodes is $Q = \{q_1, q_2, \ldots, q_{11}\}$, the initial state is q_6 and the set of final states is $F = \{q_5, q_{11}\}$. The transitions from one state to another are depicted as labeled edges; each edge bears two symbols, one from Σ_1 and one from Σ_2, separated by a colon (:). So, for example, $\langle q_1, o, e, q_2\rangle$ is an element of δ.



Observe that each path in this device defines *two* strings: a concatenation of the left-hand-side labels of the arcs, and a concatenation of the right-hand-side labels. The upper path of the above transducer thus defines the pair *goose:geese*, whereas the lower path defines the pair *sheep:sheep*.

What constitutes a *computation* with a transducer? Similarly to the case of automata, a computation amounts to "walking" a path of the transducer, starting from the initial state and ending in some final state. Along the path, edges bear bi-symbol labels: one can view the left-hand-side symbol as an "input" symbol and the right-hand-side symbol as an "output" symbol. Thus, each path of the transducer defines a pair of strings, an input string (over Σ_1) and an output string (over Σ_2). This pair of strings is a member of the relation defined by the transducer.

DEFINITION 9. *Let $T = \langle Q, q_0, \Sigma_1, \Sigma_2, \delta, F\rangle$ be a finite state transducer. Define $\hat{\delta} \subseteq Q \times \Sigma_1^* \times \Sigma_2^* \times Q$ as follows:*

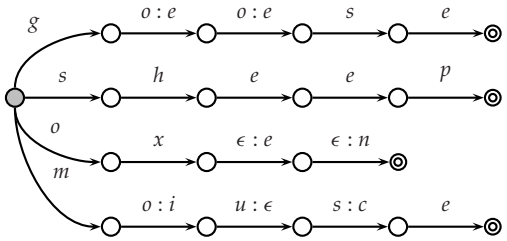
- *for each $q \in Q$, $\hat{\delta}(q, \epsilon, \epsilon, q)$;*
- *if $\hat{\delta}(q_1, w_1, w_2, q_2)$ and $\delta(q_2, a, b, q_3)$, then $\hat{\delta}(q_1, w_1 \cdot a, w_2 \cdot b, q_3)$.*

*Then a pair $\langle w_1, w_2\rangle$ is **accepted** (or **generated**) by T if and only if $\hat{\delta}(q_0, w_1, w_2, w_f)$ holds for some final state $q_f \in F$. The **relation defined by the transducer** is the set of all the pairs it accepts.*

As a shorthand notation, when an edge is labeled by two identical symbols, we depict only one of them and omit the colon.

The above definition of finite state transducers is not very useful: since each arc is labeled by exactly one symbol of Σ_1 and exactly one symbol of Σ_2, any relation that is implemented by such a transducer must relate only strings of exactly the same length. This should not be the case, and to overcome this limitation we extend the definition of δ to allow also ϵ-labels. In the extended definition, δ is a relation over Q, $\Sigma_1 \cup \{\epsilon\}$, $\Sigma_2 \cup \{\epsilon\}$ and Q. Thus a transition from one state to another can involve "reading" a symbol of Σ_1 without "writing" any symbol of Σ_2, or the other way round.

Example 23 (Finite state transducer with ϵ-labels). With the extended definition of transducers, we depict below an expanded transducer for singular–plural noun pairs in English.



Note that ϵ-labels can occur on the left or on the right of the ':' separator. The pairs accepted by this transducer are *goose:geese, sheep:sheep, ox:oxen*, and *mouse:mice*.

4.9 Properties of regular relations

The extension of automata to transducers carries with it some interesting results. First and foremost, finite state transducers define exactly the set of regular relations. Many of the closure properties of automata are valid for transducers, but some are not. As these properties bear not only theoretical but also practical significance, we discuss them in more detail in this section.

Given some transducer T, consider what happens when the labels on the arcs of T are modified such that only the left-hand symbol remains. In other words,

consider what is obtained when the transition relation δ is projected on three of its coordinates: Q, Σ_1, and Q only, ignoring the Σ_2 coordinate. It is easy to see that a finite state automaton is obtained. We call this automaton the *projection* of T to Σ_1. In the same way, we can define the projection of T to Σ_2 by ignoring Σ_1 in the transition relation. Since both projections yield finite state automata, they induce regular languages. Therefore the relation defined by T is a regular relation.

We can now consider certain operations on regular relations, inspired by similar operations on regular languages. For example, *union* is very easy to define. Recall that a regular relation is a subset of the Cartesian product of $\Sigma_1^* \times \Sigma_2^*$, that is, a set of pairs. If R_1 and R_2 are regular relations, then $R_1 \cup R_2$ is well defined, and it is straightforward to show that it is a regular relation. To define the union operation directly over transducers, extend the construction of FSA delineated in Section 4.5, namely add a new initial state with two edges labeled $\epsilon:\epsilon$ leading from it to the initial states of the given transducers. In a similar way, *concatenation* can be extended to regular relations: if R_1 and R_2 are regular relations then $R_1 \cdot R_2 = \{\langle w_1 \cdot w_2, w_3 \cdot w_4\rangle \mid \langle w_1, w_3\rangle \in R_1$ *and* $\langle w_2, w_4\rangle \in R_2\}$. Again, the construction for FSA can be straightforwardly extended to the case of transducers, and it is easy to show that $R_1 \cdot R_2$ is a regular relation.

Example 24 (Operations on finite state transducers). Let R_1 be the following relation, mapping some English words to their German counterparts: $R_1 = \{$*tomato:Tomate, cucumber:Gurke, grapefruit:Grapefruit, pineapple:Ananas, coconut:Koko*$\}$. Let R_2 be a similar relation: $R_2 = \{$*grapefruit:Pampelmuse, coconut:Kokusnuß*$\}$. Then: $R_1 \cup R_2 = \{$*tomato:Tomate, cucumber:Gurke, grapefruit:Grapefruit, grapefruit: Pampelmuse, pineapple:Ananas, coconut:Koko, coconut:Kokusnuß*$\}$.

A rather surprising fact is that regular relations are *not* closed under *intersection*. In other words, if R_1 and R_2 are two regular relations, then it very well might be the case that $R_1 \cap R_1$ is not a regular relation. It will take us beyond the scope of the material covered so far to explain this fact, but it is important to remember it when dealing with finite state transducers. For this reason exactly it follows that the class of regular relations is not closed under *complementation*: since intersection can be expressed in terms of union and complementation, if regular relations were closed under complementation they would have been closed also under intersection, which we know is not the case.

A very useful operation that is defined for transducers is *composition*. Intuitively, a transducer relates one word ("input") with another ("output"). When we have more than one transducer, we can view the output of the first transducer as the input to the second. The composition of T_1 and T_2 relates the input language of T_1 with the output language of T_2, bypassing the intermediate level (which is the output of T_1 and the input of T_2).

DEFINITION 10. *If R_1 is a relation from Σ_1^* to Σ_2^* and R_2 is a relation from Σ_2^* to Σ_3^* then the* ***composition*** *of R_1 and R_2, denoted $R_1 \circ R_2$, is a relation from Σ_1^* to Σ_3^* defined as $\{\langle w_1, w_3\rangle \mid$ there exists a string $w_2 \in \Sigma_2^*$ such that $w_1 R_1 w_2$ and $w_2 R_2 w_3\}$.*

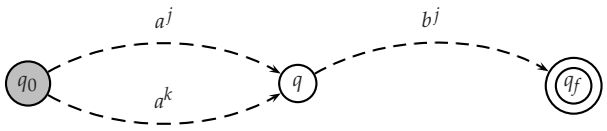
Example 25 (Composition of finite state transducers). Let R_1 be the following relation, mapping some English words to their German counterparts: $R_1 = \{$*tomato*: *Tomate, cucumber:Gurke, grapefruit:Grapefruit, grapefruit:Pampelmuse, pineapple:Ananas, coconut:Koko, coconut:Kokusnuß*$\}$. Let R_2 be a similar relation, mapping French words to their English translations: $R_2 = \{$*tomate:tomato, ananas*: *pineapple, pamplemousse:grapefruit, concombre:cucumber, cornichon: cucumber, noix-de-coco:coconut*$\}$. Then $R_2 \circ R_1$ is a relation mapping French words to their German translations (the English translations are used to compute the mapping, but are not part of the final relation): $R_2 \circ R_1 =$ $\{$*tomate:Tomate, ananas:Ananas, pamplemousse:Grapefruit, pamplemousse: Pampelmuse, concombre:Gurke, cornichon:Gurke, noix-de-coco:Koko, noix-de-coco:Kokusnuße*$\}$.

5 Context-Free Languages

5.1 Where regular languages fail

Regular languages and relations are useful for various applications of natural language processing, but there is a limit to what can be achieved with such means. We mentioned in passing that not *all* languages over some alphabet Σ are regular; we now look at what kind of languages lie beyond the regular ones.

To exemplify a non-regular language, consider a simple language over the alphabet $\Sigma = \{a, b\}$ whose members are strings that consist of some number, n, of 'a's, followed by *the same* number of 'b's. Formally, this is the language $L = \{a^n \cdot b^n \mid n > 0\}$. Assume towards a contradiction that this language is regular, and therefore a deterministic finite state automaton A exists whose language is L. Consider the language $L_i = \{a^i \mid i > 0\}$. Since every string in this language is a prefix of some string $(a^i \cdot b^i)$ of L, there must be a path in A starting from the initial state for every string in L_i. Of course, there is an infinite number of strings in L_i, but by its very nature, A has a finite number of states. Therefore there must be two different strings in L_i that lead the automaton to a single state. In other words, there exist two strings, a^j and a^k, such that $j \neq k$ but $\hat{\delta}(q_0, a^j) = \hat{\delta}(q_0, a^k)$. Let us call this state q. There must be a path labeled b^j leading from q to some final state q_f, since the string $a^j b^j$ is in L. This situation is schematically depicted below (the dashed arrows represent paths):



Therefore, there is also an accepting path $a^k b^j$ in A, and hence also $a^k b^j$ is in L, in contradiction to our assumption. Hence no deterministic finite state automaton exists whose language is L.

We have seen one language, namely $L = \{a^n \cdot b^n \mid n > 0\}$, which cannot be defined by a finite state automaton and therefore is not regular. In fact, there are several other such languages, and there is a well-known technique, the so-called *pumping lemma*, for proving that certain languages are not regular. If a language is not regular, then it cannot be denoted by a regular expression. We must look for alternative means of specification for non-regular languages.

5.2 Grammars

In order to specify a class of more complex languages, we introduce the notion of a *grammar*. Intuitively, a grammar is a set of rules that manipulate symbols. We distinguish between two kinds of symbols: *terminal* ones, which should be thought of as elements of the target language, and *non-terminal* ones, which are auxiliary symbols that facilitate the specification. It might be instructive to think of the non-terminal symbols as *syntactic categories*, such as *Sentence, Noun Phrase*, or *Verb Phrase*. However, formally speaking, non-terminals have no "special," external interpretation where formal languages are concerned. Similarly, terminal symbols might correspond to letters of some natural language, or to words, or to something else: they are simply elements of some finite set.

Rules can express the internal structure of "phrases," which should not necessarily be viewed as natural language phrases. A rule is a non-empty sequence of symbols, a mixture of terminals and non-terminals, with the only requirement that the first element in the sequence be a non-terminal one (alternatively, one can define a rule as an ordered pair whose first element is a non-terminal symbol and whose second element is a sequence of symbols). We write such rules with a special symbol, '$\rightarrow$,' separating the distinguished leftmost non-terminal from the rest of the sequence. The leftmost non-terminal is sometimes referred to as the *head* of the rule, while the rest of the symbols are called the *body* of the rule.

Example 26 (Rules). Assume that the set of terminals is {*the, cat, in, hat*} and the set of non-terminals is {*D, N, P, NP, PP*}. Then possible rules over these two sets include:

$D \rightarrow the$	$NP \rightarrow D\,N$
$N \rightarrow cat$	$PP \rightarrow P\,NP$
$N \rightarrow hat$	$NP \rightarrow NP\,PP$
$P \rightarrow in$	

Note that the terminal symbols correspond to words of English, and not to letters as was the case above.

Consider the rule $NP \rightarrow D\,N$. If we interpret NP as the syntactic category *noun phrase*, D as *determiner*, and N as *noun*, then what the rule informally means is that one possible way to construct a noun phrase is by concatenating a determiner with a noun. More generally, a rule specifies one possible way to construct a "phrase" of

the category indicated by its head: this way is by concatenating phrases of the categories indicated by the elements in the body of the rule. Of course, there might be more than one way to construct a phrase of some category. For example, there are two rules which define the structure of the category *NP* in Example 26: either by concatenating a phrase of category *D* with one of category *N*, or by concatenating an *NP* with a *PP*.

In Example 26, rules are of two kinds: the ones on the left have a single terminal symbol in their body, while the ones on the right have one or more non-terminal symbols, but no rule mixes both terminal and non-terminal symbols in its body. While this is a common practice where grammars for natural languages are concerned, nothing in the formalism requires such a format for rules. Indeed, rules can mix any combination of terminal and non-terminal symbols in their bodies.

Formal language theory defines rules and grammars in a much broader way than that which was discussed above, and the definition below is actually only a special case of rules and grammars. For various reasons that have to do with the format of the rules, this special case is known as *context-free* rules. This has nothing to do with the ability of grammars to refer to context; the term should not be taken mnemonically. In the next section we discuss other rule-based systems. In this section, however, we use the terms *rule* and *context-free rule* interchangeably, as we do for grammars, derivations, etc.

DEFINITION 11. *A **context-free grammar** is a four-tuple $G = \langle V, \Sigma, P, S \rangle$, where V is a finite set of **non-terminal symbols**, Σ is an alphabet of **terminal symbols**, $P \subseteq V \times (V \cup \Sigma)^*$ is a set of **rules** and $S \in V$ is the **start symbol**.*

Note that this definition permits rules with empty bodies. Such rules, which consist of a left-hand-side only, are called ϵ-rules, and are useful both for formal and for natural languages. Example 33 below makes use of an ϵ-rule.

Example 27 (Grammar). The set of rules depicted in Example 26 can constitute the basis for a grammar $G = \langle V, \Sigma, P, S \rangle$, where $V = \{D, N, P, NP, PP\}$, $\Sigma = \{$*the, cat, in, hat*$\}$, P is the set of rules, and the start symbol S is *NP*.

In the sequel we depict grammars by listing their rules only, as we did in Example 26. We keep a convention of using uppercase letters for the non-terminals and lowercase letters for the terminals, and we assume that the set of terminals is the smallest that includes all the terminals mentioned in the rules, and the same for the non-terminals. Finally, we assume that the start symbol is the head of the first rule, unless stated otherwise.

5.3 Derivation

In order to define the language denoted by a grammar we need to define the concept of *derivation*. Derivation is a relation that holds between two *forms*, each a sequence of grammar symbols (terminal and/or non-terminal).

DEFINITION 12. *Let $G = \langle V, \Sigma, P, S\rangle$ be a grammar. The set of* ***forms*** *induced by G is $(V \cup \Sigma)^*$. A form α* ***immediately derives*** *a form β, denoted by $\alpha \Rightarrow \beta$, if and only if there exist $\gamma_l, \gamma_r \in (V \cup \Sigma)^*$ such that $\alpha = \gamma_l A \gamma_r$ and $\beta = \gamma_l \gamma_c \gamma_r$, and $A \rightarrow \gamma_c$ is a rule in P. A is called the* ***selected symbol****.*

A form α immediately derives β if a single non-terminal symbol, A, occurs in α, such that whatever is to its left in α, the (possibly empty) sequence of terminal and non-terminal symbols γ_l, occurs at the leftmost edge of β; and whatever is to the right of A in α, namely the (possibly empty) sequence of symbols γ_r, occurs at the rightmost edge of β; and the remainder of β, namely γ_c, constitutes the body of some grammar rule of which A is the head.

Example 28 (Immediate derivation). Let G be the grammar of Example 27. The set of forms induced by G contains all the (infinitely many) sequences of elements from V and Σ, such as $\langle\rangle$, $\langle NP\rangle$, $\langle D\ cat\ P\ D\ hat\rangle$, $\langle D\ N\rangle$, $\langle the\ cat\ in\ the\ hat\rangle$, etc.

Let us start with a simple form, $\langle NP\rangle$. Observe that it can be written as $\gamma_l NP \gamma_r$, where both γ_l and γ_r are empty. Observe also that NP is the head of some grammar rule: the rule $NP \rightarrow D\ N$. Therefore, the form is a good candidate for derivation: if we replace the selected symbol NP with the body of the rule, while preserving its environment, we obtain $\gamma_l D\ N \gamma_r = D\ N$. Therefore, $\langle N\rangle \Rightarrow \langle D\ N\rangle$.

We now apply the same process to $\langle D\ N\rangle$. This time the selected symbol is D (we could have selected N, of course). The left context is again empty, while the right context is $\gamma_r = N$. As there exists a grammar rule whose head is D, namely $D \rightarrow the$, we can replace the rule's head by its body, preserving the context, and obtain the form $\langle the\ N\rangle$. Hence $\langle D\ N\rangle \Rightarrow \langle the\ N\rangle$.

Given the form $\langle the\ N\rangle$, there is exactly one non-terminal that we can select, namely N. However, there are two rules that are headed by N: $N \rightarrow cat$ and $N \rightarrow hat$. We can select either of these rules to show that both $\langle the\ N\rangle \Rightarrow \langle the\ cat\rangle$ and $\langle the\ N\rangle \Rightarrow \langle the\ hat\rangle$.

Since the form $\langle the\ cat\rangle$ consists of terminal symbols only, no non-terminal can be selected and hence it derives no form.

We now extend the immediate derivation relation from a single step to an arbitrary number of steps by considering the reflexive transitive closure of the relation.

DEFINITION 13. *The* ***derivation*** *relation, denoted '$\stackrel{*}{\Rightarrow}$,' is defined recursively as follows: $\alpha \stackrel{*}{\Rightarrow} \beta$ if $\alpha = \beta$, or if $\alpha \Rightarrow \gamma$ and $\gamma \stackrel{*}{\Rightarrow} \beta$.*

Example 29 (Extended derivation). In Example 28 we showed that the following immediate derivations hold: $\langle NP\rangle \Rightarrow \langle D\ N\rangle$; $\langle D\ N\rangle \Rightarrow \langle the\ N\rangle$; $\langle the\ N\rangle \Rightarrow \langle the\ cat\rangle$. Therefore, $\langle NP\rangle \stackrel{*}{\Rightarrow} \langle the\ cat\rangle$.

The derivation relation is the basis for defining the language denoted by a grammar. Consider the form obtained by taking a single grammar symbol, say $\langle A\rangle$; if this form derives a sequence of terminals, this string is a member of the language denoted by A. The language of a grammar G, $L(G)$, is the language denoted by its start symbol.

DEFINITION 14. *Let $G = \langle V, \Sigma, P, S\rangle$ be a grammar. The* ***language of a non-terminal*** *$A \in V$ is*

$$L_G(A) = \{a_1 \cdots a_n \mid a_i \in \Sigma \text{ for } 1 \leq i \leq n \text{ and } \langle A\rangle \stackrel{*}{\Rightarrow} \langle a_1, \ldots, a_n\rangle\}$$

The ***language of the grammar*** *G is $L(G) = L_G(S)$.*

Example 30 (Language of a grammar). Consider again the grammar G of Example 27. It is fairly easy to see that the language denoted by the non-terminal symbol D, $L_G(D)$, is the singleton set {*the*}. Similarly, $L_G(P)$ is {*in*} and $L_G(N) = \{cat, hat\}$. It is more difficult to define the languages denoted by the non-terminals *NP* and *PP*, although it should be straightforward that the latter is obtained by concatenating {*in*} with the former. We claim without providing a proof that $L_G(NP)$ is the denotation of the regular expression $(the \cdot (cat + hat) \cdot (in \cdot the \cdot (cat + hat))^*)$.

5.4 Derivation trees

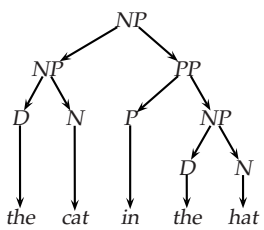
Sometimes two derivations of the same string differ only in the order in which they were applied. Consider again the grammar of Example 27. Starting with the form $\langle NP\rangle$ it is possible to derive the string *the cat* in two ways:

(1) $\langle NP\rangle \Rightarrow \langle D\ N\rangle \Rightarrow \langle D\ cat\rangle \Rightarrow \langle the\ cat\rangle$
(2) $\langle NP\rangle \Rightarrow \langle D\ N\rangle \Rightarrow \langle the\ N\rangle \Rightarrow \langle the\ cat\rangle$

Derivation (1) applies first the rule $N \to cat$ and then the rule $D \to the$ whereas derivation (2) applies the same rules in the reverse order. But since both use the same rules to derive the same string, it is sometimes useful to collapse such "equivalent" derivations into one. To this end the notion of *derivation trees* is introduced.

A derivation tree (sometimes called *parse* tree, or simply tree) is a visual aid in depicting derivations, and a means for imposing structure on a grammatical string. Trees consist of vertices and branches; a designated vertex, the *root* of the tree, is depicted on the top. Branches are connections between pairs of vertices. Intuitively, trees are depicted "upside down," since their root is at the top and their leaves are at the bottom. An example of a derivation tree for the string *the cat in the hat* with the grammar of Example 27 is given in Example 31.

Example 31 (Derivation tree).



Formally, a tree consists of a finite set of vertices and a finite set of branches (or arcs), each of which is an ordered pair of vertices. In addition, a tree has a designated vertex, the *root*, which has two properties: it is not the target of any arc, and every other vertex is accessible from it (by following one or more branches). When talking about trees we sometimes use family notation: if a vertex v has a branch leaving it which leads to some vertex u, then we say that v is the *mother* of u and u is the *daughter*, or *child*, of v. If u has two daughters, we refer to them as *sisters*. Derivation trees are defined with respect to some grammar G, and must obey the following conditions:

(1) every vertex has a *label*, which is either a terminal symbol, a non-terminal symbol, or ϵ;
(2) the label of the root is the start symbol;
(3) if a vertex v has an outgoing branch, its label must be a non-terminal symbol; furthermore, this symbol must be the head of some grammar rule; and the elements in the body of the same rule must be the labels of the children of v, in the same order;
(4) if a vertex is labeled ϵ, it is the only child of its mother.

A *leaf* is a vertex with no outgoing branches. A tree induces a natural "left-to-right" order on its leaves; when read from left to right, the sequence of leaves is called the *frontier*, or *yield*, of the tree.

Derivation trees correspond very closely to derivations. In fact, it is easy to show that a non-terminal symbol A derives a form α if and only if α is the yield of some parse tree whose root is A. In other words, whenever some string can be derived from a non-terminal, there exists a derivation tree for that string, with the same non-terminal as its root. However, sometimes there exist different derivations of the same string that correspond to a single tree. The tree representation collapses exactly those derivations that differ from each other only in the order in which rules are applied.

Sometimes, however, different derivations (of the same string!) correspond to different trees. This can happen only when the derivations differ in the rules which they apply. When more than one tree exists for some string, we say that the string is *ambiguous*. Ambiguity is a major problem when grammars are used for certain formal languages, in particular for programming languages. But for natural languages, ambiguity is unavoidable as it corresponds to properties of the natural language itself.

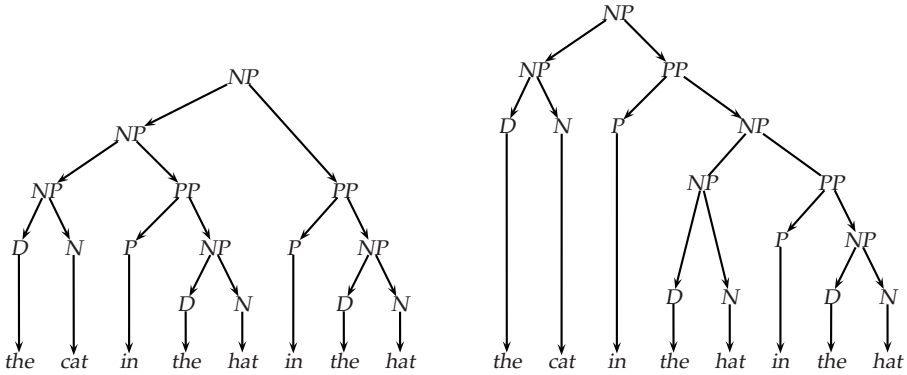
Example 32 (Ambiguity). Consider again the grammar of Example 27, and the string *the cat in the hat in the hat*. Intuitively, there can be (at least) two readings for this string: one in which a certain cat wears a hat-in-a-hat, and one in which a certain cat-in-a-hat is inside a hat. If we wanted to indicate the two readings with parentheses, we would distinguish between

((the cat in the hat) in the hat)

and

(the cat in (the hat in the hat))

This distinction in intuitive meaning is reflected in the grammar, and two different derivation trees, corresponding to the two readings, are available for this string:



Using linguistic terminology, in the left tree the second occurrence of the prepositional phrase *in the hat* modifies the noun phrase *the cat in the hat*, whereas in the right tree it only modifies the (first occurrence of) the noun phrase *the hat*. This situation is known as *syntactic* or *structural* ambiguity.

5.5 Expressiveness

Context-free grammars are more expressive than regular expressions. In Section 5.1 we claimed that the language $L = \{a^n b^n \mid n > 0\}$ is not regular; we now show a context-free grammar for this language. The grammar, $G = \langle V, \Sigma, P, S \rangle$, has two terminal symbols, $\Sigma = \{a, b\}$, and one non-terminal symbol, $V = \{S\}$. The idea is that whenever S is used recursively in a derivation (rule 1), the current form is extended by exactly one a on the left and one b on the right, hence the number of 'a's and 'b's must be equal.

Example 33 (A context-free grammar for $L = \{a^n b^n \mid n \geq 0\}$).

(1) $S \rightarrow a\, S\, b$
(2) $S \rightarrow \epsilon$

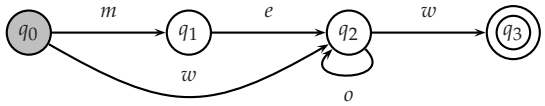
DEFINITION 15. *The class of languages that can be generated by context-free grammars is the class of* ***context-free languages****.*

The class of context-free languages properly contains the regular languages: given some finite state automaton which generates some language L, it is always possible to construct a context-free grammar whose language is L. We conclude this section with a discussion of converting automata to context-free grammars.

Let $A = \langle Q, q_0, \delta, F \rangle$ be a deterministic finite state automaton with no ϵ-moves over the alphabet Σ. The grammar we define to simulate A is $G = \langle V, \Sigma, P, S \rangle$,

where the alphabet Σ is that of the automaton, and where the set of non-terminals, V, is the set Q of the automaton states. The idea is that a single (immediate) derivation step with the grammar simulates a single arc traversal with the automaton. Since automata states are simulated by grammar non-terminals, it is reasonable to simulate the initial state by the start symbol, and hence the start symbol S is q_0. What is left, of course, are the grammar rules. These come in two varieties: first, for every automaton arc $\delta(q, a) = q'$ we stipulate a rule $q \to a\, q'$. Then, for every final state $q_f \in F$, we add the rule $q_f \to \epsilon$.

Example 34 (Simulating a finite state automaton by a grammar). Consider the automaton $\langle Q, q_0, \delta, F\rangle$ depicted below, where $Q = \{q_0, q_1, q_2, q_3\}$, $F = \{q_3\}$, and δ is $\{\langle q_0, m, q_1\rangle, \langle q_1, e, q_2\rangle, \langle q_2, o, q_2\rangle, \langle q_2, w, q_3\rangle, \langle q_0, w, q_2\rangle\}$:



The grammar $G = \langle V, \Sigma, P, S\rangle$ which simulates this automaton has $V = \{q_0, q_1, q_2, q_3\}$, $S = q_0$, and the set of rules:

(1) $q_0 \to m\, q_1$
(2) $q_1 \to e\, q_2$
(3) $q_2 \to o\, q_2$
(4) $q_2 \to w\, q_3$
(5) $q_0 \to w\, q_2$
(6) $q_3 \to \epsilon$

The string *meoow*, for example, is generated by the automaton by walking along the path $q_0 - q_1 - q_2 - q_2 - q_2 - q_3$. The same string is generated by the grammar with the derivation

$$\langle q_0\rangle \overset{1}{\Rightarrow} \langle m q_1\rangle \overset{2}{\Rightarrow} \langle me q_2\rangle \overset{3}{\Rightarrow} \langle meo q_2\rangle \overset{3}{\Rightarrow} \langle meoo q_2\rangle \overset{4}{\Rightarrow} \langle meoow q_3\rangle \overset{6}{\Rightarrow} \langle meoow\rangle$$

Since every regular language is also a context-free language, and since we have shown a context-free language that is not regular, we conclude that the class of regular languages is properly contained within the class of context-free languages.

Observing the grammar of Example 34, a certain property of the rules stands out: the body of each of the rules either consists of a terminal followed by a non-terminal or is empty. This is a special case of what are known as *right-linear* grammars. In a right-linear grammar, the body of each rule consists of a (possibly empty) sequence of terminal symbols, optionally followed by a single non-terminal symbol. Most importantly, no rule exists whose body contains more than one non-terminal; and if a non-terminal occurs in the body, it is in the final position. Right-linear grammars are a restricted variant of context-free grammars, and it can be shown that they generate all and only the regular languages.

5.6 Formal properties of context-free languages

Context-free languages are more expressive than regular languages; this additional expressive power comes with a price: given an arbitrary context-free grammar G and some string w, determining whether $w \in L(G)$ takes time proportional to the cube of the length of w, $O(|w|^3)$ (in the worst case). In addition, context-free languages are not closed under some of the operations that the regular languages are closed under.

It should be fairly easy to see that context-free languages are closed under union. Given two context-free grammars $G_1 = \langle V_1, \Sigma_1, P_1, S_1 \rangle$ and $G_2 = \langle V_2, \Sigma_2, P_2, S_2 \rangle$, a grammar $G = \langle V, \Sigma, P, S \rangle$ whose language is $L(G_1) \cup L(G_2)$ can be constructed as follows: the alphabet Σ is the union of Σ_1 and Σ_2, the non-terminal set V is a union of V_1 and V_2, plus a new symbol S, which is the start symbol of G. Then, the rules of G are just the union of the rules of G_1 and G_2, with two additional rules: $S \to S_1$ and $S \to S_2$, where S_1 and S_2 are the start symbols of G_1 and G_2 respectively. Clearly, every derivation in G_1 can be simulated by a derivation in G using the same rules exactly, starting with the rule $S \to S_1$, and similarly for derivations in G_2. Also, since S is a new symbol, no other derivations in G are possible. Therefore $L(G) = L(G_1) \cup L(G_2)$.

A similar idea can be used to show that the context-free languages are closed under concatenation: here we only need one additional rule, namely $S \to S_1\, S_2$, and the rest of the construction is identical. Any derivation in G will "first" derive a string of G_1 (through S_1) and then a string of G_2 (through S_2). To show closure under the Kleene-closure operation, use a similar construction with the added rules $S \to \epsilon$ and $S \to S\, S_1$.

However, it is possible to show that the class of context-free languages is not closed under intersection. That is, if L_1 and L_2 are context-free languages, then it is not guaranteed that $L_1 \cap L_2$ is context-free as well. From this fact it follows that context-free languages are not closed under complementation either. While context-free languages are not closed under intersection, they *are* closed under intersection with regular languages: if L is a context-free language and R is a regular language, then it is guaranteed that $L \cap R$ is context-free.

In the previous section we have shown a correspondence between two specification formalisms for regular languages: regular expressions and finite state automata. For context-free languages, we focused on a declarative formalism, namely context-free grammars, but they, too, can be specified using a computational model. This model is called *push-down automata*, and it consists of finite state automata augmented with unbounded memory in the form of a *stack*. Computations can use the stack to store and retrieve information: each transition can either push a symbol (taken from a special alphabet) onto the top of the stack, or pop one element off the top of the stack. A computation is successful if it ends in a final state with an empty stack. It can be shown that the class of languages defined by push-down automata is exactly the class of context-free languages.

5.7 Normal forms

The general definition of context-free grammars stipulates that the body of a rule may consist of any sequence of terminal and non-terminal symbols. However, it is possible to restrict the form of the rules without affecting the generative capacity of the formalism. Such restrictions are known as *normal forms* and are the topic of this section.

The best-known normal form is the Chomsky normal form (CNF): under this definition, rules are restricted to be of either of two forms. The body of any rule in a grammar may consist either of a single terminal symbol, or of exactly two non-terminal symbols (as a special case, empty bodies are also allowed). For example, the rules $D \rightarrow \mathit{the}$ and $NP \rightarrow D\,N$ can be included in a CNF grammar, but the rule $S \rightarrow a\,S\,b$ cannot.

Unlike the right-linear grammars defined in Section 5.5, which can only generate regular languages, CNF grammars are equivalent in their weak generative capacity to general context-free grammars: it can be proven that for every context-free language L there exists a CNF grammar G such that $L = L(G)$. In other words, CNF grammars can generate all the context-free languages.

The utility of normal forms is in their simplicity. When some property of context-free languages has to be proven, it is sometimes much simpler to prove it for the restricted version of the formalism (e.g., for CNF grammars only), because the result can then extend to the entire class of languages. Similarly, processing normal-form grammars may be simpler than processing the general class of grammars. Thus, the first parsing algorithms for context-free grammars were limited to grammars in CNF. In natural language grammars, a normal form can embody the distinction between "real" grammar rules and the lexicon; a commonly used normal form defines grammar rules to have either a single terminal symbol or any sequence of zero or more non-terminal symbols in their body (notice that this is a relaxation of CNF).

6 The Chomsky Hierarchy

6.1 A hierarchy of language classes

We focus in this section on grammars as formalisms which denote languages. We have seen two types of grammars: context-free grammars, which generate the class of context-free languages; and right-linear grammars, which generate the class of regular languages. Right-linear grammars are a special case of context-free grammars, where additional constraints are imposed on the form of the rules. More generally, constraining the form of the rules can constrain the expressive power of the formalism. Similarly, more freedom in the form of the rules can extend the expressiveness of the formalism.

One way to achieve this is to allow more than a single non-terminal symbol in the *head* of the rules or, in other words, restrict the application of rules to a

specified *context*. In context-free grammars, a rule can be applied during a derivation whenever its head, A, is an element in a form. In the extended formalism such a derivation is allowed only if the context of A in the form, that is, A's neighbors to the right and left, are as specified in the rule. Due to this reference to context, this formalism is known as *context-sensitive* grammars. A rule in a context-sensitive grammar has the form $\alpha_1 A \alpha_2 \rightarrow \alpha_1 \beta \alpha_2$, where α_1, α_2, and β are all (possibly empty) sequences of terminal and non-terminal symbols. The other components of context-sensitive grammars are as in context-free grammars.

As usual, the class of languages that can be generated by context-sensitive grammars is called the *context-sensitive languages*. Considering that every context-free grammar is a special case of context-sensitive grammars (with an empty context), it should be clear that every context-free language is also context-sensitive or, in other words, that the context-free languages are contained in the set of the context-sensitive ones. As it turns out, this containment is proper, and there are context-sensitive languages that are not context-free.

This establishes a *hierarchy* of classes of languages: the regular languages are properly contained in the context-free languages, which are properly contained in the context-sensitive languages. These, in turn, are known to be properly contained in the set of languages generated by the so-called *unrestricted* or *general phrase-structure* grammars (this set is called the *recursively enumerable languages*). Each of the language classes in this hierarchy is associated with a computational model: FSA and push-down automata for the regular and context-free languages respectively; linear bounded Turing machines for the context-sensitive languages; and Turing machines for the recursively enumerable languages.

This hierarchy of language classes is called the *Chomsky hierarchy of languages*, and is schematically depicted in Figure 1.1.

6.2 *The location of natural languages in the hierarchy*

The Chomsky hierarchy of languages reflects a certain order of complexity: in some sense, the lower the language class is in the hierarchy, the simpler are its possible constructions. Furthermore, lower language classes allow for more efficient processing (in particular, the recognition problem is tractable for regular and context-free languages, but not for higher classes). If formal grammars are used to express the structure of *natural* languages, then we must know the location of these languages in the hierarchy.

Chomsky presents a theorem that says "English is not a regular language" (1957: 21); as for context-free languages, he says "I do not know whether or not English is itself literally outside the range of such analyses" (1957: 34). For many years, however, it was well accepted that natural languages were beyond the expressive power of context-free grammars. This was only proven in the 1980s, when two natural languages (Dutch and a dialect of Swiss German) were shown to be trans-context-free (that is, beyond the expressive power of context-free grammars). Still, the constructions in natural languages that necessitate more than

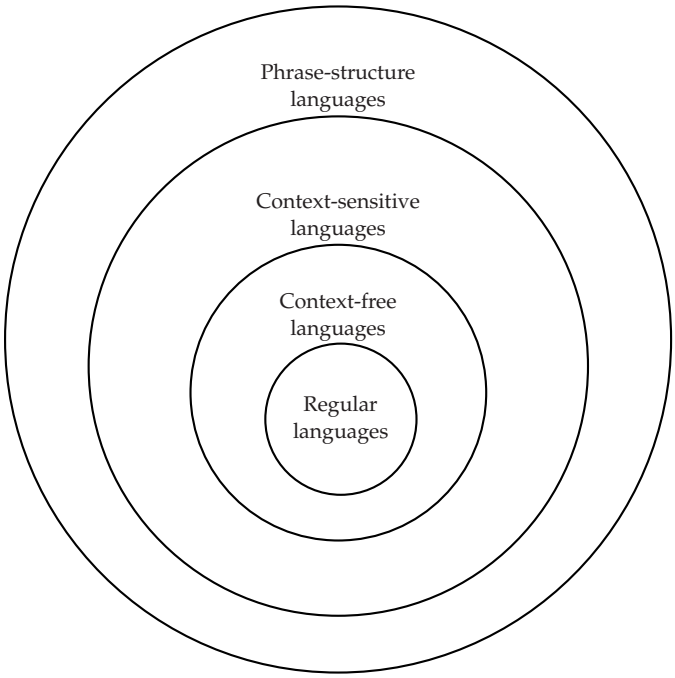


Figure 1.1 Chomsky's hierarchy of languages.

context-free power are few and very specific. (Most of these constructions boil down to patterns of the form $a^n b^m c^n d^m$, known as *cross-serial dependencies*; with some mathematical machinery, based mostly on closure properties of the context-free languages, it can be proven that languages that include such patterns cannot be context-free.) This motivated the definition of the class of *mildly context-sensitive languages*, which we discuss in Section 7.

6.3 Weak and strong generative capacity

So far we have only looked at grammars as generating sets of strings (i.e., languages), and ignored the structures that grammars impose on the strings in their languages. In other words, when we say that English is not a regular language we mean that no regular expression exists whose denotation is the set of all and only the sentences of English. Similarly, when a claim is made that some natural language, say Dutch, is not context-free, it should be read as saying that no context-free grammar exists whose language is Dutch. Such claims are propositions about the *weak generative capacity* of the formalisms involved: the weak generative capacity of regular expressions is insufficient for generating English; the weak generative capacity of context-free languages is insufficient for Dutch. Where natural languages are concerned, however, weak generative capacity might

not correctly characterize the relationship between a formalism (such as regular expressions or context-free grammars) and a language (such as English or Dutch). This is because one expects the formalism not only to be able to generate the strings in a language, but also to assign them "correct" structures.

In the case of context-free grammars, the structure assigned to strings is a derivation tree. Other linguistic formalisms may assign other kinds of objects to their sentences. We say that the *strong generative capacity* of some formalism is sufficient to generate some language if the formalism can (weakly) generate all the strings in the language, and also to assign them the "correct" structures. Unlike weak generative capacity, which is a properly defined mathematical notion, strong generative capacity is poorly defined, because no accepted definition of the "correct" structure for some string in some language exists.

7 Mildly Context-Sensitive Languages

When it was finally proven that context-free grammars are not even weakly adequate as models of natural languages, research focused on "mild" extensions of the class of context-free languages. In a seminal work, Joshi (1985) coined the term *mildly context-sensitive languages*, which is loosely defined as a class of languages that:

(1) properly contains all the context-free languages;
(2) can be parsed in polynomial time;
(3) can properly account for the constructions in natural languages that context-free languages fail to account for, such as cross-serial dependencies; and
(4) has the linear-growth property (this is a formal property that we ignore here).

One formalism that complies with these specifications (and which motivated their design) is *tree adjoining grammars* (TAGs). Motivated by linguistic considerations, TAGs extend the scope of locality in which linguistic constraints can be expressed. The elementary building blocks of the formalism are trees. Whereas context-free grammar rules enable one to express constraints among the mother in a local tree and its immediate daughters, the elementary trees of TAG facilitate the expression of constraints between arbitrarily distant nodes, as long as they are part of the same elementary tree. Two operations, *adjunction* and *substitution*, construct larger trees from smaller ones, so that the basic operations that take place during derivations are not limited to string concatenation. Crucially, these operations facilitate nesting of one tree within another, resulting in extended expressiveness.

The class of languages generated by tree adjoining grammars is naturally called the *tree adjoining languages*. It contains the context-free languages, and several trans-context-free ones, such as the language $\{a^n b^m c^n d^m \mid n, m \geq 0\}$. As usual, the added expressiveness comes with a price, and determining membership of a string w in a language generated by some TAG can only be done in time proportional to $|w|^6$.

Several linguistic formalisms were proposed as adequate for expressing the class of natural languages. Noteworthy among them are three formalisms: *head grammars*, *linear indexed grammars*, and *combinatory categorial grammars*. All three were developed independently with natural languages as their main motivation; and all three were proven to be (weakly) equivalent to TAG. The class of tree adjoining languages, therefore, may be just the correct formal class in which all natural languages reside.

8 Further Reading

Much of the material presented in this chapter can be found in introductory textbooks on formal language theory. Hopcroft and Ullman (1979, chapter 1) provide a formal presentation of formal language theory; just as rigorous, but with an eye to linguistic uses and applications, is the presentation of Partee et al. (1990, chapters 1–3). For the ultimate reference, consult the *Handbook of Formal Languages* (Rozenberg & Salomaa 1997).

A very good formal exposition of regular languages and the computing machinery associated with them is given by Hopcroft and Ullman (1979, chapters 2–3). Another useful source is Partee et al. (1990, chapter 17). Theorem 1 is due to Kleene (1956); Theorem 2 is due to Rabbin and Scott (1959); Theorem 3 is a corollary of the Myhil–Nerode theorem (Nerode 1958). The pumping lemma for regular languages is due to Bar-Hillel et al. (1961).

For natural language applications of finite state technology refer to Roche and Schabes (1997a), which is a collection of papers ranging from mathematical properties of finite state machinery to linguistic modeling using them. The introduction (Roche & Schabes 1997b) can be particularly useful, as will be Karttunen (1991). Kaplan and Kay (1994) is a classic work that sets the very basics of finite state phonology, referring to automata, transducers, and two-level rules. As an example of an extended regular expression language, with an abundance of applications to natural language processing, see Beesley and Karttunen (2003). Finally, Karttunen et al. (1996) is a fairly easy paper that relates regular expressions and relations to finite automata and transducers, and exemplifies their use in several language engineering applications.

Context-free grammars and languages are discussed by Hopcroft and Ullman (1979, chapters 4, 6) and Partee et al. (1990, chapter 18). The correspondence between regular languages and right-linear grammars is due to Chomsky and Miller (1958). A cubic-time parsing algorithm for context-free languages was first proposed by Kasami (1965); see also Younger (1967). Push-down automata were introduced by Oettinger (1961); see also Schützenberger (1963). Chomsky (1962) proved that they were equivalent to context-free grammars.

A linguistic formalism that is based on the ability of context-free grammars to provide adequate analyses for natural languages is generalized phrase-structure grammars, or GPSGs (Gazdar et al., 1985).

The Chomsky hierarchy of languages is due to Chomsky (1956, 1959). The location of the natural languages in this hierarchy is discussed in several papers, of which the most readable, enlightening, and amusing is Pullum and Gazdar (1982). Several other works discussing the non-context-freeness of natural languages are collected in Part III of Savitch et al. (1987). Rounds et al. (1987) inquire into the relations between formal language theory and linguistic theory, in particular referring to the distinction between weak and strong generative capacity. Works showing that natural languages cannot be described by context-free grammars include Bresnan et al. (1982) (Dutch), Shieber (1985) (Swiss German), and Manaster-Ramer (1987) (Dutch). Miller (1999) is dedicated to generative capacity of linguistic formalisms, where strong generative capacity is defined as the model theoretic semantics of a formalism.

Tree adjoining grammars were introduced by Joshi et al. (1975) and are discussed in several subsequent papers Joshi (1985; 1987; 2003). A polynomial-time parsing algorithm for TAG is given by Vijay-Shanker and Weir (1993) and Satta (1994). The three formalisms that are equivalent to TAG are head grammars (Pollard 1984), linear-indexed grammars (Gazdar 1988), and combinatory categorial grammars (Steedman 2000); they were proven equivalent by Vijay-Shanker and Weir (1994).