PART I

MATHEMATICS IN HISTORY

CHAPTER 1

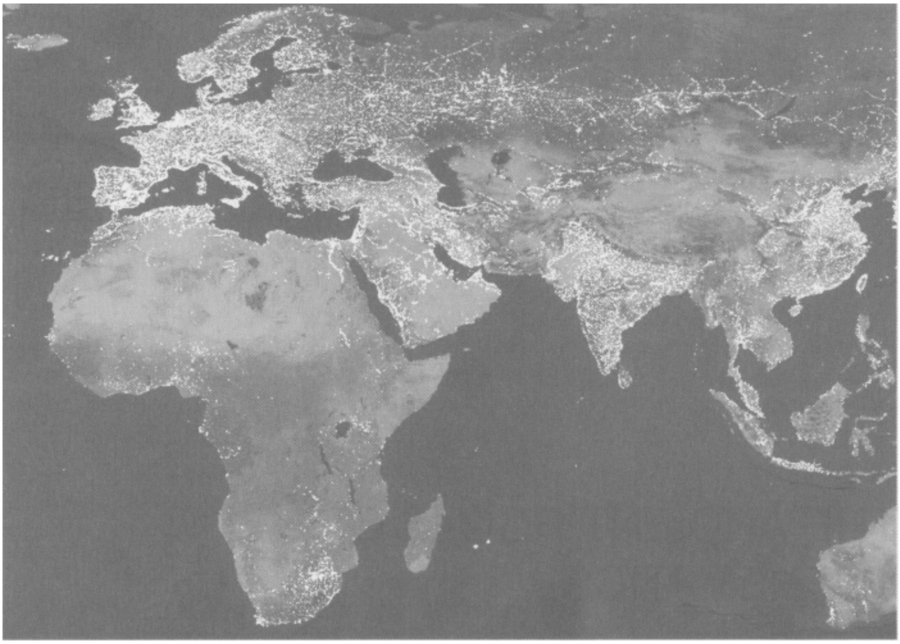
THE ANCIENT ROOTS OF MATHEMATICS

> Mathematics—the unshaken Foundation of Sciences, and the plentiful Fountain of Advantage to human affairs.
>
> ISAAC BARROW (1630–1677)

1.1 Introduction

Mathematics is a human enterprise, which means that it is part of history. It has been shaped by that history, and in turn has helped to shape it. In this chapter we will trace these connections.

Many societies have contributed to mathematics, but a main historical thread is discernible, one that has led directly to today's mathematics. That thread began in the ancient Mediterranean world, swelled mightily in ancient Greece, dwindled at the time of the Roman empire, was kept alive and augmented in the Muslim world, re-entered Western Europe in the Renaissance, developed in Europe for several centuries, then spread throughout the world in the 20th century. We will spend most of our time on this thread, in part because so much is known about it, with a few excursions into other cultures.



Eurasia and Africa.[1]

Fingers, Knots, and Tally Sticks

Experiments have shown that humans, and other animals, are born with innate mathematical abilities. They regularly distinguish between, say, one tree and two trees. The next logical step is counting, that is, establishing a one-to-one correspondence between sets of objects. This is no doubt also an ancient ability.

Once we can count objects, how do we communicate numbers to others? Most of what follows in this chapter is based on the historical, i.e., written record. But writing is a fairly recent invention. Before the written word, people used a variety of methods to represent numbers. Surely one of the first, and still important, methods was the use of various parts of the body. Some quite elaborate systems have been developed. The Torres Strait islanders, an indigenous Australian people, used fingers, toes, elbows, shoulders, knees, hips, wrists, and sternum to represent different numbers. Many languages preserve the remnants of such systems: the word for "five," for example, is "hand" in Persian, Russian, and Sanskrit. And it is no coincidence that our number system is based on ten, the number of fingers.

Perhaps the most popular numbering system used notches on sticks or bones, so-called *tally sticks,* from the French word *tailler*, to cut. These date back at least

[1]From "Earth at Night." C. Mayhew and R. Simmon (NASA/GSFC), NOAA/NGDC, DMSP Digital Archive.

35,000 years, and must rank as one of the most successful technologies ever. As recently as 1826, tally sticks were used in official English tax records.

Another popular counting device was the stone. Our word "calculation" derives from the Latin *calculus*, which is a small stone. Early versions of the abacus were stones on the ground; "abacus" likely derives from the Hebrew *abhaq*, dust.

Knotted strings were a popular accounting tool throughout the world. The most notable examples of these were the amazing Incan *quipu*, which consisted of multiple knotted cords (up to 2000 of them) joined together.

A leading theory of the origin of writing in Mesopotamia, proposed by Denise Schmandt-Besserat, relates to a different method of recording numbers. It starts with the use of small clay tokens, found in archaeological sites, beginning circa 8000 BCE[1]. These tokens, in various standard shapes, were used for accounting: one shape might represent one sheep, for example, another one goat, or ten sheep. Imagine you are a merchant, and have hired someone to deliver a herd of 27 sheep to a neighboring city. The buyer needs to have some way to verify that the number of sheep that arrive is the same number sent. The solution was to encase tokens representing 27 sheep in a clay "envelope," a hollow ball. The ball could be broken open at the destination, and the number of sheep verified.

Now imagine that the sheep's journey has two legs; person A delivers them to person B, who in turn delivers them to the buyer. If B breaks open the ball to verify the count, what is the buyer to do? The solution found was to make impressions on the ball, using the tokens, before they were placed inside. After the clay ball hardened, these impressions could serve as a record as well as the tokens. Eventually, it was realized that the tokens were unnecessary. The "writing" on the ball sufficed.

Agriculture and Civilizations

Some time around 10,000 years ago, humans began developing agriculture, inaugurating the Neolithic, the "new stone age." The first important crops were grains—large-seeded grasses—including wheat, sorghum, millet, and rice. Gradually, various animals were domesticated, notably cattle, sheep, oxen, pigs, and goats. This whole set of developments dramatically changed the way people lived. Instead of living in relatively small nomadic bands of "hunter-gatherers," they started settling into villages. This allowed a larger population density.

In some areas of the world, usually in the flood plains of great river valleys, the agricultural settlements developed civilizations. Among these areas were Mesopotamia, the Nile in Egypt, the Yellow River in China, the Indus River in Pakistan, and the Ganges in India. Civilizations were characterized by more central organization, often including irrigation, granaries to store surplus grain, and cities.

The civilizations were based on the existence of agricultural surplus, which freed people to work on other things. This led to the development of many new technolo-

[1]Circa (abbreviation c.), from the Latin, means "around." We will use it for approximate dates. BCE (Before Current Era) is becoming standard for dates before the year 0, what used to be written B.C. CE is used for dates after the year 0, in place of A.D.

gies. Among these were the plow, wheeled vehicles, and, most notably, writing and metallurgy.

Civilizations required different, more sophisticated, types of mathematics. Geometry was needed for surveying land, building canals, dikes, and ditches, and constructing larger buildings like granaries and palaces. Administering the new city-states, apportioning taxes, and paying workers made increasing demands on arithmetic and algebra, as did the expanded commercial activity.

With the rise of civilization came new class structures. Most people were farmers, but some became blacksmiths, leather workers, engineers, architects, merchants, priests, scribes, surveyors, and of course kings. Some of the new, specialized professions (such as surveyors) nurtured their own mathematical techniques, handing them down through the generations. In some societies, small groups inside the new classes turned their collective attention to developing mathematics generally. Society provided practical inspiration for the new mathematics, but some mathematicians pursued knowledge for its own sake.

EXERCISES

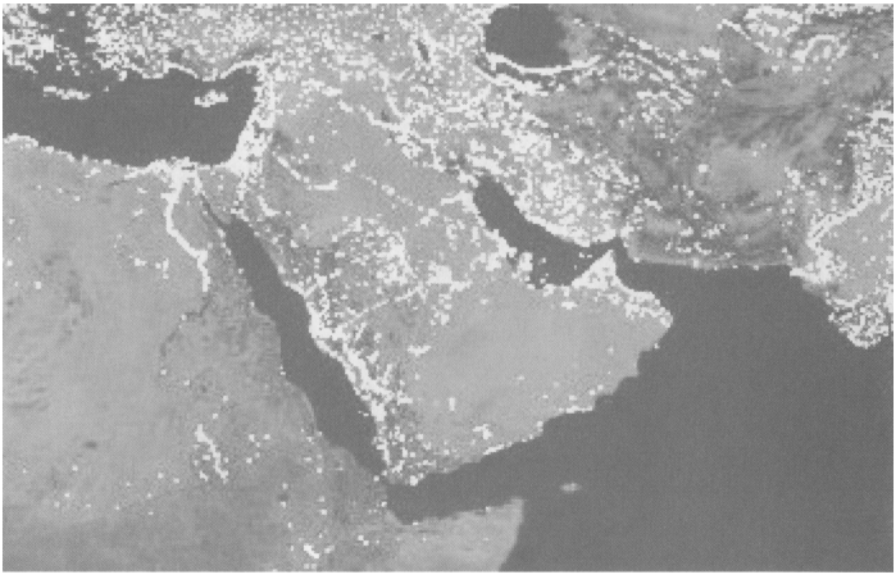
1.1 What mathematics would a pre-agricultural (hunter-gatherer) society need?

1.2 What mathematics would an agricultural village need that a hunter-gatherer society would not?

1.3 What mathematics would a city need that a agricultural village would not?

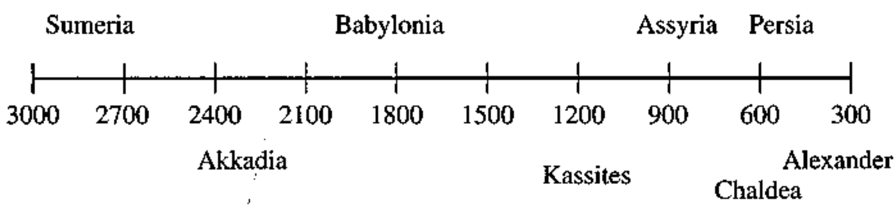
1.4 Look up Incan quipus in your favorite Internet search engine. What did they look like? How were they used?

1.2 Ancient Mesopotamia and Egypt



The Middle East.[1]

Two of the earliest civilizations arose in the Near East.



Ancient Mesopotamian history. All dates are BCE.

Mesopotamia (from the Greek, literally "between the rivers") is the plain between the Tigris and Euphrates rivers, about 600 miles long, in modern-day Iraq. Mesopotamia was home to the earliest known agriculture. The major crops were wheat and barley, but there were also fruits, including dates, grapes, figs, melons, and apples; vegetables, including eggplant, onions, radishes, beans, and lettuce; and sheep, cattle, goats, and pigs.

[1] From "Earth at Night." C. Mayhew and R. Simmon (NASA/GSFC), NOAA/NGDC, DMSP Digital Archive.

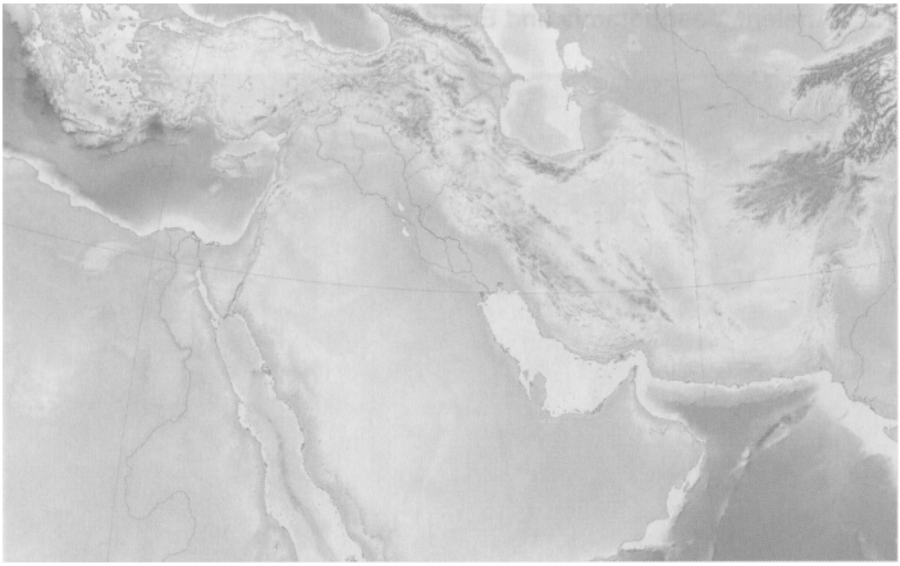


Figure 1.1 The Tigris and Euphrates rivers are above and to the left of center; the Nile is on the left.[1]

Between the Rivers

The farmers relied on the flooding of the Tigris and Euphrates. These floods, which could be violent and destructive, nonetheless left behind very fertile silt. Mesopotamia doesn't get much rain, so the other important ingredient to agriculture was irrigation. One of the critical functions of the government was the construction and maintenance of irrigation systems. The first Mesopotamian civilization was the Sumerian, named after the city-state of Sumer in southern Mesopotamia. It arose circa 3500–3000 BCE. Politically, the early Sumerians did not have an empire; empires came later. Instead, they were organized into city-states, ruled by priest-kings. These city-states built up bureaucracies to manage the irrigation systems and the surplus grains. They even had postal systems. The Sumerians are credited with the invention of plows, the potter's wheel, and wheeled carts. Their greatest invention was an improved writing system. Earlier writing systems had relied principally on *pictograms*, symbols which were meant to look like the thing represented. The Sumerians developed, over many hundreds of years, a system of standardized *ideograms*, symbols that represented ideas.

The Sumerians, and their successors in Mesopotamia, wrote by using a stylus, a reed cut at an angle, to make impressions in wet clay tablets. The tablets were then

[1]Map by Sémhur. Wikipedia Commons.

baked until hard. Their writing is called cuneiform, from the Latin *cuneus* (wedge) and *forma* (shape). It is from these tablets that we learn most of what we know about their history. Ironically, the preservation of this history was often assisted when the buildings housing the tablets were burned. This baked the tablets, making them more durable.

Mesopotamia is a crossroads. This allowed it to be a trading center and to profit from the ideas of other civilizations. It also was subject to regular raids and occasional full-scale invasions from its neighbors. So there was a succession of civilizations and empires through the years. The Sumerians were conquered by the Akkadians, whose most famous ruler was Sargon the Great, who lived around 2250 BCE. The Akkadians were replaced by the Babylonians.

The most important Babylonian king was Hammurabi, who ruled c. 1792–1750 BCE. He is famous for promulgating the first code of laws, a list of 282 short "laws." Here is one: "If a man puts out the eye of an equal, his eye shall be put out." Presumably he can put out the eye of an inferior with impunity. A "tooth for a tooth" is also here. Many of the laws end with "shall be put to death."

If you read about ancient Mesopotamia, you will often find it referred to as Babylon, even during those times when Babylon was not its most important city. Perhaps this reflects Babylonian cultural accomplishments. In particular, the high point of Mesopotamian mathematics was during this time. After the Babylonians, mathematics was mainly stagnant.

The Babylonians in their turn fell to the Kassites, who had a new weapon, horse-drawn chariots, the tanks of their day. In the 9th century BCE, the Assyrians ruled, relying on iron weapons. In the 7th century came the Chaldean empire, when Nebuchadnezzar built the hanging gardens of Babylon and sent many Hebrews into Babylonian exile. The Persians under Cyrus invaded in 538 BCE, and Alexander the Great took over in 330 BCE, bringing Greek culture with him.

Most of the invaders did not displace the local culture. Instead they adopted much of it. In particular, the bureaucracy essential to managing their conquests tended to stay in place. This bureaucracy included the scribes who were at the heart of mathematics. Early on, the Sumerians developed scribal schools, which taught writing and mathematics, among other subjects, to future bureaucrats. Most of these scribes came from wealthy families. They were the ruling elite of their day.

The schools for scribes became centers of culture, including mathematics. Their main emphasis, however, was business and administration. Irrigation systems had to be run, laws administered, lands apportioned, taxes levied. A very important responsibility, and one that had a profound influence on mathematics in Mesopotamia and elsewhere, was maintenance of the calendar, which required accurate measurements of the heavens.

Most of the tablets from which we learn about Mesopotamian mathematics were created at the schools, for the purpose of training scribes. They usually took the form of solving problems. The problems were stated in practical terms: measuring fields, apportioning inheritance, and so on. But the purpose of the tablets was to train students in mathematical methods rather than in practical problem-solving. In some ways, mathematical textbooks haven't changed.

Although the tablets reveal a strong mathematical tradition, they do not reveal a lot of theory, or even general methods. These methods are implied by the results, but apparently were restricted to an oral tradition. Sometimes historians have been able to infer the methods used, sometimes they have to guess them.

Numeration One of the greatest accomplishments of the Mesopotamian culture was the development of the best number system of antiquity.

A problem that any sophisticated number system must address is how to group numbers. Small numbers may be expressed by simple ticks, but if we are to handle larger numbers, they must somehow be grouped together. The Sumerians were the first to establish a consistent grouping system. Unlike our number system, which groups by powers of 10 (1, 10, 100, ...), the Sumerians grouped by powers of 60 (1, 60, 3600, ...). This system is called *sexagesimal* (from the Latin *sexagesimus*, sixtieth), as opposed to our *decimal* system (from *decimus*, tenth).

No one knows exactly why they chose this system, but one of its useful features is that 60 has many divisors: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. By contrast, 10 has only four divisors. More divisors make fractions easier to work with. Consider the multiplication $1/2 \times 3/5$. One way of doing this is to convert to a decimal representation: $.5 \times .6 = .3$. Thus, we can use our regular multiplication and not have to deal with fractions as ratios. It only works, however, because 2 and 5 are divisors of 10. Consider $2/3 \times 11/12$. This problem does not lend itself to an easy decimal representation in base 10. Ten doesn't have enough factors. We will see below how a sexagesimal system can handle this multiplication.

The legacy of the Mesopotamian sexagesimal system survives to this day: we divide hours into 60 minutes, minutes into 60 seconds, and we divide the circle into 360 degrees.

One of the most important advances in representing numbers was the idea of place-value notation, developed by the Babylonians. To understand this, let us look at how we represent numbers in our own place-value system. Consider the number 235:

$$235 = (2 \times 100) + (3 \times 10) + (5 \times 1) = (2 \times 10^2) + (3 \times 10^1) + (5 \times 10^0).$$

The meaning of each digit depends on its *place* in the number. So, for example, 235 is not the same as 253.

Since the Babylonians had a sexagesimal system, they would represent the number 235 in powers of 60. Thus

$$235 = (3 \times 60) + (55 \times 1) = (3 \times 60^1) + (55 \times 60^0),$$

so we might write this as 3,55, using a comma to separate powers of 60.

What number would 3,40,6 represent?

$$\begin{aligned} 3{,}40{,}6 &= (3 \times 60^2) + (40 \times 60^1) + (6 \times 60^0) \\ &= (3 \times 3600) + (40 \times 60) + (6 \times 1) \\ &= 10800 + 2400 + 6 \\ &= 13206 \end{aligned}$$

One of the advantages of a place-value system is the ability it gives us to express arbitrarily large numbers with a small set of symbols, ten symbols in the case of our decimal system. The number symbols used in Mesopotamia changed dramatically through the years. The Sumerians used hundreds of symbols, both pictorial and phonetic. Their successors, the Akkadians, developed a standardized system of number ideograms. These ideograms represented the *idea* of a number, divorcing it from concrete notions such as a "hand," for five.

The Babylonians reduced the number of symbols to two, one for 1 and one for 10. They repeated these symbols as necessary to get the numbers from 1 to 59, as in Figure 1.2. For numbers greater than 59, they used their place value system, as we do for numbers greater than 9. Figure 1.3 shows how they would write the number $2{,}34 = 2 \times 60 + 34$.

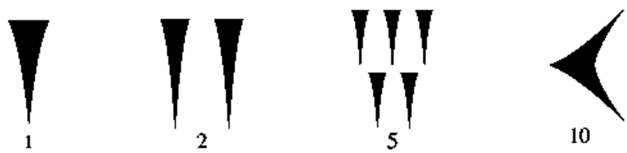


Figure 1.2 Babylonian symbols.

Figure 1.3 Babylonian 2,34.

This system could also handle numbers less than one, in the same way as our decimal system. We write $2/5$ as .4. If we use a semi-colon, instead of a decimal point, the Babylonians could use ; 24 for two-fifths, since 24 is two-fifths of 60. Since 60 has so many divisors, this was a convenient way of writing fractions. Another example: since 20 is one-third of 60, we would write $1/3 = ;20$. Here is an example of a mixed fraction.

$$70\frac{2}{15} = 1 \times 60 + 10 + 2 \times \frac{1}{15} = (1{,}10) + 2 \times (;4) = 1{,}10;8$$

The Babylonians did not use a semi-colon, or any indicator of where fractions started, so there was an ambiguity to their numerals. For example, they wrote all these numbers the same way:

$$2{,}5{,}0 = 2 \times 60^2 + 5 \times 60 = 7500$$

$$2,5 = 2 \times 60 + 5 = 125$$

$$2;5 = 2 + 5 \times 60^{-1} = 2\frac{1}{12}$$

$$;2,5 = 2 \times 60^{-1} + 5 \times 60^{-2} = \frac{5}{144}.$$

They would determine which number was meant by the context.

The other missing element in this system was the notion of zero. The Babylonians did not have a zero number, and would never write a number such as 2,5,0. They wrote 2,5 and interpreted it as 2,5,0 from context. They did have to develop some way to indicate skipped digits, such as what we mean when we write 0 in the middle of a number, as in 205. This gap was indicated in different ways, often with just a space.

Computation and Algebra The Babylonian number system allowed for a sophisticated arithmetic. Like us, the Babylonians wrote down multiplication tables. They also had tables of squares and cubes of numbers. For division, they used tables of reciprocals. For example, consider the problem $32 \div 25$. If we had a table giving us .04 as the reciprocal of 25, we could translate the division problem into its equivalent multiplication problem, $32 \times .04$. This the Babylonians regularly did, aided by the fact that 60 has many divisors. Of course, this works well only when the reciprocal has a nice form; think of trying it with $32 \div 7$. The Babylonian reciprocal tables were usually restricted to the nicer reciprocals. For other divisions, approximation techniques were used.

The Babylonians were very good at calculating square roots. A tablet from around 1600 BCE gives the approximation $\sqrt{2} = 1;24,51,10$. In decimal terms, this is about 1.414213, while the correct value starts $1.414214\ldots$. The approximation is within one-millionth of the correct value.

The method they used to obtain such accuracy is not known with certainty, but may be what later was called Heron's method, since Heron was the first to write it down, over 1500 years later. This method can be used to find the square root of any number N. You start with any guess for $\sqrt{N}$, say, x_1. You then generate a sequence of numbers $x_2, x_3, x_4, \ldots$, as follows.

$$x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right)$$
$$x_3 = \frac{1}{2}\left(x_2 + \frac{N}{x_2}\right)$$
$$x_4 = \frac{1}{2}\left(x_3 + \frac{N}{x_3}\right)$$

You can continue this pattern as long as you like. The numbers $x_1, x_2, x_3, x_4, \ldots$ get closer and closer to $\sqrt{N}$.

As an example, we will approximate $\sqrt{2}$. We first guess $x_1 = 1.5$. (The initial guess doesn't have to be too accurate. Just pick something reasonable.) Then

$$x_2 = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.41666666666667$$

$$x_3 = \frac{1}{2}\left(1.41666666666667 + \frac{2}{1.41666666666667}\right) \approx 1.41421568627451$$

$$x_4 = \frac{1}{2}\left(1.41421568627451 + \frac{2}{1.41421568627451}\right) \approx 1.41421356237309.$$

Already, all the digits given for x_4 are accurate.

Babylonian mathematicians had a good understanding of linear equations in one variable, $ax + b = c$, even though the scribes had no general notion of a variable. They could also solve systems of two linear equations in two unknowns.

Many of the tablets concern problems that involve solving quadratic equations, equations we would write as $ax^2 + bx + c = 0$. Here is an example: "I summed the areas of my two square-sides so that it was 0;21,40. A square-side exceeds the (other) square-side by 0;10."

Let us translate that into modern notation. The number 0;21,40 refers to $21/60 + 40/3600 = (21 \times 60 + 40)/3600 = 13/36$. Also, 0;10 is $10/60$, or $1/6$. So we have two squares, one of side x, say, and the other of side $x - 1/6$. Since the sum of their areas is $13/36$, the problem is to solve the equation

$$x^2 + \left(x - \frac{1}{6}\right)^2 = \frac{13}{36}.$$

The solution is then given step-by-step. It starts like this: "You break off half of 0;21,40 and you write down 0;10,50."

Note that the problem itself is stated in geometric form. This was typical; there was no clear distinction between algebra and geometry. The Babylonians did not have an algebraic notation. They also had no symbols for arithmetic operations, like $+, -, \times, \div$. The problems were stated in words.

The solution of the problem is given by a set of specific instructions on how to proceed. This also was typical. There was no notion of a general solution like our quadratic equation, even though the scribes clearly had methods for solving many such equations. The idea of a general theory had yet to be developed.

Given these restrictions, their accomplishments in algebra were impressive.

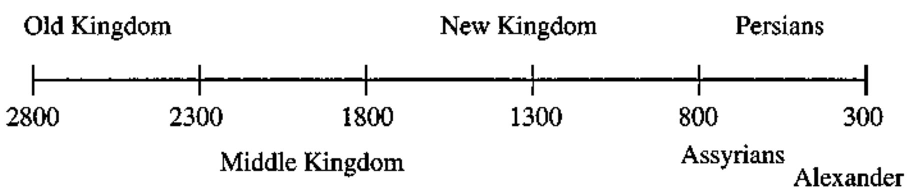
Geometry The people of Mesopotamia dealt with many practical problems requiring geometric knowledge for their solutions. Surveyors had to measure distance and compute areas. Builders of large structures needed knowledge of distance, area, and volume.

Surviving tablets give us some insight into their geometry. The scribes had rules for computing the areas of triangles, rectangles, and various other plane figures such as pentagons, hexagons, and trapezoids. Rules for computing with circles usually have $\pi = 3$, although the better approximation of $3\frac{1}{8}$ was also used. The Babylonians had the correct formula for the volume of a truncated pyramid, which is a pyramid with its top cut off.

What we now call the Pythagorean Theorem, $a^2 + b^2 = c^2$, where a and b are the legs of a right triangle and c its hypotenuse, was known in Mesopotamia at least 1000 years before Pythagoras. A famous clay tablet from the Old Babylonian Period lists a number of Pythagorean triples, which are sets of three numbers obeying the Pythagorean equation, for example, 3–4–5 (since $3^2 + 4^2 = 5^2$). See Section 5.11 for more on these triples.

We must remind ourselves when dealing with cultures several thousand years in the past, that our knowledge is spotty. In the matter of geometry, it has been suggested that clay tablets were not the best medium for drawings. Perhaps the scribes did their best work drawing in sand, or some other medium which has been lost. Having said that, it is certain that their geometry never approached anything like the sophistication attained in ancient Greece. The existing tablets address particular problems, not general theory. There is no notion of proof.

Egypt before Alexander



Ancient Egyptian history. All dates are BCE.

Gift of the Nile Egypt consists of a desert cut by the Nile River. The upper Nile, about 600 miles from Aswan to Memphis, is a narrow valley, not more than 15 miles wide, bordered by cliffs. The lower Nile, about 150 miles from Memphis to the Mediterranean, is a fan-shaped marshy delta. Outside the river valley is only desert.

In ancient times, before modern flood control systems, the Nile flooded every year. The flood left behind rich soil. About 7000 years ago, the Egyptians began to farm this soil. As in Mesopotamia, there was little rain, so the agriculture that developed was reliant on irrigation. Egyptians raised wheat and barley and a variety of vegetables for food, and flax for clothing. The most important domesticated animals were cattle, but they also kept sheep, goats, and pigs. It was one of the most productive agricultural areas in the world, producing in good years a large surplus that could support a sophisticated civilization.

The Egyptians had a written language by about 3200 BCE, almost as early as in Mesopotamia. The *hieroglyphs*, from the Greek for "sacred carvings," were pictographs. A little later, the Egyptians developed the cursive *hieratic* script, and, in the first millennium BCE, an alphabetic system called *demotic*.

Hieroglyphs were painted or carved on monuments, while the hieratic and demotic systems were written using ink on papyrus. *Papyrus* was a type of paper made from a reed, *Cyperus papyrus*, found in the Nile Delta. Our word "paper" derives from papyrus. Papyrus sheets were cheaper than the clay tablets used in Mesopotamia, but they don't last as long. Thus much of what we know of Egyptian mathematics, with some exceptions noted below, comes from inscriptions on monuments.

One meaning of hieroglyphic in English is "hard to decipher," and Egyptian hieroglyphs were unreadable by modern scholars until the decipherment of the famous Rosetta Stone. This stele (inscribed stone slab) was found by Napoleon's armies in Egypt in 1799. It had a message written in Greek, hieroglyphic, and demotic, which allowed Jean Champollion, after much work, to decipher it in 1821.

Egypt was first brought together under a single ruler about 3100 BCE. In the Old Kingdom (c. 3000–2200 BCE), the Egyptians adopted much from the Sumerians, including irrigation systems, the plow, and metallurgy. They too had a class of scribes to assist in administration, and again it was from this class that most of their mathematics originated. It was during the Old Kingdom that the biggest pyramids were built, including the famous Great Pyramid of Giza (c. 2500 BCE).

Egypt was not organized into city-states like Mesopotamia, but instead was centrally ruled by the pharaoh, who was considered a god. Only under such a centralized system could monuments such as the pyramids be constructed. According to the Greek historian Herodotus (c. 484–425 BCE), who first called Egypt the "gift of the Nile," the Great Pyramid required the labor of 400,000 men at a time, for three months of the year, over twenty years. That was after the effort of ten years building the road needed to transport the materials. (Herodotus wrote 2000 years after the fact, so the precise numbers shouldn't be taken too seriously. Nonetheless, they are not far from modern estimates.)

A period of political unrest followed the Old Kingdom, until the arrival of the Middle Kingdom (c. 2100–1800 BCE). Egyptian culture flourished in this period. In fact, it was the high point of ancient Egyptian mathematics. The Middle Kingdom ended with the invasion of the Hyksos, from Syria-Palestine.

The Hyksos were expelled, starting the New Kingdom (c. 1600–1100 BCE). In this period, Egypt expanded its power into the Middle East (Palestine and Syria) and to the south (Nubia and the Sudan). The New Kingdom was followed by a period of weak kings and a number of invasions, including conquests by the Assyrians in 671 BCE, the Persians in 525 BCE, and finally, in 332 BCE, Alexander the Great.

Mathematics in ancient Egypt was applied to many of the same uses as in Mesopotamia: building irrigation systems and granaries, levying taxes, paying workers, and apportioning the surplus grain. Of particular note is surveying. Farms had to be marked off again after each yearly flood destroyed the previous year's boundaries. For this, surveyors needed *geometry*—from the Greek *geo*, Earth, and *metria*, measure.

As mentioned above, most of what we know about ancient Egypt is from the thousands of inscriptions on monuments they left behind. The most important sources for the later mathematics are, however, a dozen or so papyri. Two stand out. The *Moscow Mathematical Papyrus* (which is in the Moscow Museum of Fine Arts), dating from about 1850 BCE, contains a list of twenty-five problems. Eleven of these twenty-five concern ways of making different beers and breads. The Rhind Mathematical Papyrus (bought in the 19th century in Luxor, Egypt by a man named Rhind) is a scroll 13 inches wide and 18 feet long, which contains eighty-seven problems and tables. It dates from around 1650 BCE, but its author writes that he copied it from a work written 200 years before that.

Numeration and Arithmetic The Egyptians had one system of numeration for each of their three writing systems: hieroglyphic, hieratic, and demotic. All of them were decimal, that is, they grouped numbers by powers of ten. The hieroglyphic system had symbols for 1, 10, 100, etc. Multiples of these were represented by repeating symbols. The hieratic and demotic systems added symbols for 2, 3, …, 9 and 20, 30, …, 90, and so on, which made writing large numbers much easier. Unlike the Mesopotamians, the Egyptians never developed a fully positional system.

Addition and subtraction in these systems was rather like in our own. For multiplication, they used a doubling system called *duplation*. It is similar to how modern computers multiply. Here is an example, computing $11 \cdot 17$. First we write powers of 2, with their corresponding multiples of 17. Each line is obtained by multiplying the previous line by 2.

1	17
2	34
4	68
8	136

Why only four lines? Because we can write $11 = 1 + 2 + 8$. Therefore $11 \cdot 17 = (1 + 2 + 8) \cdot 17 = 1 \cdot 17 + 2 \cdot 17 + 8 \cdot 17$. To complete the multiplication, we need only add the entries in the right-hand column corresponding to 1, 2, and 8, to get $11 \cdot 17 = 17 + 34 + 136 = 187$. Division was handled using the same idea, although it was a bit more complicated due to remainders.

Egyptians didn't use fractions as we do. They only used *unit fractions* of the form $1/n$, for example, $1/2$, $1/3$, or $1/4$. The sole exception is their use of $2/3$. Other fractions were expressed as sums of unit fractions. A famous problem from the Rhind Papyrus asks how to divide six loaves of bread among ten men. The answer given was $1/2 + 1/10$. You can check that this equals $6/10$. In fact, dividing the loaves is easy using $1/2 + 1/10$. Cut five of the loaves in half, giving one-half to each man. Then cut the last loaf into tenths, giving each man one tenth. Each man ends up with one-half plus one-tenth.

The Egyptians did not prove that any fraction can be written as the sum of distinct unit fractions (they didn't have the notion of proof), but it can be proven. To do the actual computations, they used extensive tables. The Rhind Papyrus, for example, includes a table decomposing fractions of the form $2/n$ into sums of unit fractions.

Geometry One area where Egyptian mathematics excelled was geometry. They knew how to compute areas of rectangles, triangles, and trapezoids, as well as volumes of rectangular boxes and various cylinders. They also used similar triangles.

Two triangles are similar if they have the same three interior angles. One triangle is a blown up version of the other. They are useful because the ratios of their corresponding sides are the same. As for circles, again we refer to the Rhind Papyrus, where the area of a circle is given as $(8d/9)^2$, where d is the diameter of the circle. This was equivalent to approximating π by $256/81$, about 3.16.

Given the importance of pyramids to them, it is no surprise that the Egyptians knew how to compute the volume of a pyramid. The *Moscow Papyrus* has a method for calculating the volume of a truncated pyramid. (You can see a drawing of a truncated pyramid on the back of a U.S. dollar bill, underneath an eye.) The method is equivalent to the formula

$$V = \frac{1}{3}h(a^2 + ab + b^2),$$

where a is the length of the lower base, b is the length of the upper base, and h is the height (see Figure 1.4).

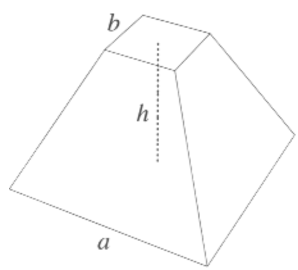


Figure 1.4 A truncated pyramid.

The Egyptians did not pursue theoretical geometry the way the Greeks later did. They were, however, unsurpassed in practical geometry. This can be seen in their monuments. The base of the Great Pyramid is an almost perfect square about 756 feet per side. The length of the sides differ by less than one foot—and this was done using 2.5 ton stone blocks (more than 2,300,000 of them). Later Egyptian construction never reached this level of precision.

Algebra The Egyptians were able to solve linear equations, and some quadratic equations. They, as all ancients, were hampered by the lack of a good notation. For them, all problems were word problems.

One method the Egyptians scribes used is called the method of *false position.* This method was adopted by many other peoples, and used as late as the Middle Ages in Europe. As an example, the Rhind Papyrus offers this in Problem 26: "A quantity whose fourth part is added to it becomes 15. What is the quantity?" In modern notation, we want to solve the equation $x + x/4 = 15$ for x. The method of false position involves guessing a (probably incorrect) solution, then adjusting it using proportionality. In our problem, the scribe guessed that $x = 4$, to make the fraction $x/4$ easier. If $x = 4$, then $x + x/4 = 5$. Since we want $x + x/4 = 15$, we multiply by 3, because $15/5 = 3$. If we multiply $x = 4$ by 3, we get $x = 12$. You can check that $12 + 12/4 = 15$, so we have solved the problem.

Astronomy and the Calendar In ancient Egypt, as in many places at many times, an important use of mathematics was in astronomy. The astronomers of Egypt were priests, which suggests that astronomy was not only a practical science. Astronomers kept track of the Sun, Moon, planets, and stars. One notion they used to track the seasons was that of a helical rising of a star, which meant that the star rose just before the Sun. The most important helical rising was that of Sothus (which we know as Sirius), the brightest star in the sky. This rising occurred in July, shortly before the onset of the Nile floods. Sothus was known as the Dog Star, and we still refer to this time of year as the dog days of summer.

The Egyptians developed the calendar on which ours is based. Their civil calendar, used for official record keeping (as opposed to the everyday lunar calendar), had 365 days, divided into twelve 30-day months, plus an extra five days at the end of the year. Actually, they knew that the year was about $365\frac{1}{4}$ days, but they never adjusted their calendar with leap years, as we do.

EXERCISES

1.5 Each of the numbers below is given in sexagesimal form. Translate each into our decimal form.

a) 2
b) 3,1,2
c) 1,2;6
d) ;1,40

1.6 Translate each of the decimal numbers into sexagesimal form.

a) 2
b) 122
c) 7265
d) .2
e) $1\frac{1}{3}$

1.7 a) Write the fractions $2/5$ and $11/12$ in sexagesimal form.

b) Use the results of (a) to write $2/5 + 11/12$ in sexagesimal form. (Hint: think of how adding decimals works.)

c) Add $2/5 + 11/12$ in our usual way, and confirm that you get the same answer.

1.8 a) Write the fraction 1/15 in sexagesimal form.

b) Use (a) to divide 7 by 15, expressing the result in in sexagesimal form.

1.9 a) Write the fraction 1/30 in sexagesimal form.

b) Use (a) to divide 43 by 30, expressing the result in in sexagesimal form.

1.10 Use Heron's method to approximate $\sqrt{3}$, accurate to eight decimal places. (A calculator may be necessary.) Check your answer by squaring it.

1.11 Solve the Babylonian problem given in the text: "I summed the areas of my two square-sides so that it was 0;21,40. A square-side exceeds the (other) square-side by 0;10."

1.12 a) Here is another problem from a Babylonian tablet, written around 2000 BCE: "I have added the area and two-thirds the side of my square and it is $0;35$. What is the side of my square?" Translate this into modern notation. The result should be a quadratic equation.

b) Solve the equation. Do you get the same answer as on the tablet? "You take 1. Two-thirds of 1 is $0;40$. Half of this, $0;20$, you multiply by $0;20$ and it, $0;6,40$, you add to $0;35$ and the result $0;41,40$ has $0;50$ as its square root. The $0;20$ which you have multiplied by itself, you subtract from $0;50$, and $0;30$ is the side of the square."

1.13 Use duplation to calculate 13 times 15.

1.14 Use duplation to calculate 15 times 22.

1.15 Show that the formula $(8d/9)^2$ given in the Rhind papyrus for the area of a circle is equivalent to approximating π by $256/81$.

1.16 a) Find the volume of a truncated pyramid with lower base 100 feet, upper base 30 feet, and height 69 feet.

b) No existing papyrus gives the volume of a whole (as opposed to truncated) pyramid, but this can easily be derived from the formula above. What is the formula for the volume of a whole pyramid? (Hint: what happens to b as a truncated pyramid gets closer to a whole pyramid?)

1.17 Around the year 1200 CE, the famous mathematician Fibonacci described a method for expressing any fraction as the sum of distinct unit fractions. The method was simple: find the largest unit fraction less than your number. Then subtract it from your number and repeat. For example, consider $41/42$. The largest unit fraction less than this is $1/2$. Subtracting, we get $41/42 = 1/2 + 20/42$. The largest unit fraction less that $20/42$ is $1/3$. Subtracting, $20/42 = 1/3 + 6/42$, so $41/42 = 1/2 + 1/3 + 6/42 = 1/2 + 1/3 + 1/7$, and we are done.

a) Express $7/10$ as the sum of distinct unit fractions.

b) Express $8/15$ are the sum of distinct unit fractions in two different ways.

1.18 Solve by the method of false position: "A quantity whose seventh part is added to it becomes 32. What is the quantity?"

1.19 Solve by the method of false position: "A quantity whose fifth part is subtracted from it becomes 6. What is the quantity?"

1.20 Suppose that a civilization had a system similar to the Babylonian, but based on 5 instead of 60. If they write the first five numbers as A, B, C, D, and E, how would they write these numbers?

a) 23
b) 72
c) .2
d) .24

1.3 Early Greek Mathematics: The First Theorists

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry.
>
> BERTRAND RUSSELL (1872–1970)

Modern mathematics is distinguished not only by its techniques and results, but by its logical structure. A mathematician does not merely discover formulas, but proves theorems, starting from well-understood assumptions and definitions. This logical structure is the invention of the Greeks, surely one of the greatest inventions in human history. They also applied this new invention, especially in geometry, to produce some very sophisticated mathematics.

More than only mathematics, much of Western intellectual tradition dates to classical Greece. Mathematics was part of a larger philosophical movement in which the Greeks attempted to understand the world in rational, not mythical or religious, ways. This movement extended to the political and social spheres as well. The idea of democracy is usually dated to 5th century BCE Athens.

To understand the enormity of the Greek accomplishment, and appreciate that this advance was not inevitable, consider that most of recorded history occurred *before* classical Greek civilization.

Historians have learned a great deal about Greek mathematics. However, unlike the case in Egypt and especially Mesopotamia, none of this knowledge is first-hand. Papyrus did not last long in the moist climate of the Greek world, so what we have is copies of copies of Greek texts.

Greece before 600 BCE

The Middle East and the Mediterranean Sea.[1]

Unlike Mesopotamia and Egypt, Greece was not a great agricultural center. The land was too mountainous. Greece did have the sea, however. Very little of the mainland is far from the water, and there are many Greek islands in the Aegean Sea. Hence the Greeks were a seafaring people. By the 11th century BCE (1100–1000), they had spread across the Aegean to Ionia, on the shores of what is now Turkey.

By the middle of the 8th century BCE, the Greeks entered a period of expansion, physically and culturally. This was the time of Homer, author of the *Iliad* and the *Odyssey*. In 750 BCE the Greeks established a colony near modern-day Naples. Over the next three centuries, they set up many more colonies around the Mediterranean, in southern Italy, Sicily, Spain, and northern Africa.

The Greeks were borrowers. During this period, they adopted papyrus from Egypt, and adapted the Phoenician alphabet. (Our word "alphabet" comes from the first two Greek letters, *alpha* and *beta*.) As we will see, the early mathematicians were also travelers, and learned much from Mesopotamia and Egypt.

The early Greeks did not have great empires. The basic political unit was the city-state, the *polis*, the origin of our word "politics." There were many forms, from democracies to monarchies, but they were all distinguished by a respect for law. Another notable feature of Greek public life was debate and argumentation. But public life was not shared by all; slavery was common.

[1]From "The Blue Marble." R. Stöckli, R. Simmon. (NASA/GSFC) `http://visibleearth.nasa.gov/`.

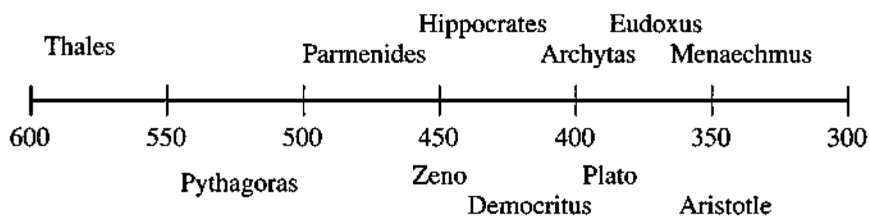
Numeration

The Greeks had a variety of numeral systems. The best known, which appeared in the 6th century BCE and was standard by the 3rd century BCE, was the Ionic system. It had 27 symbols: the 24 letters of the alphabet plus three others. These symbols represented the numbers 1, 2, 3, ..., 9, 10, 20, 30, ..., 90, 100, 200, 300, ..., 900. For example, γ was 3 and μ was 40, so 43 would be written $\mu\gamma$.

There were various ways of writing larger numbers, usually involving adding an extra symbol on top of, or next to, the existing symbols. A similar system handled fractions. The fraction $1/3$, for example, would be written $\acute{\alpha}\acute{\gamma}$ (since $\alpha = 1$).

As you can see, this system was closer to the Egyptian system than to the superior Mesopotamian system. The development of Greek mathematics was not held back by the limitations of this system, since Greek mathematics did not rely heavily on numerical calculations.

Ionia, Miletus



From Thales to the death of Alexander the Great. All dates are BCE.

Classical Greek civilization began in the 6th century BCE, in Ionia, the Greek colony located on the western shores of modern Anatolia in Turkey and some nearby islands. Unlike the Greek mainland, Ionia enjoyed good land for agriculture. Miletus, its greatest city, was an important trading center on the Meander River (which gave us our word "meander"). It was connected to Mesopotamia via overland trading routes. It was also a seaport whose ships traded with Egypt. Thus Miletus had access to the knowledge of these great civilizations.

Some time in the 6th century BCE, Western philosophy was born, and with it theoretical mathematics. In ancient times, philosophy was not distinct from science. The goal was to understand the world. The explanations that the new philosophers gave were natural, as opposed to supernatural. Religion was not abandoned; rather it was no longer considered adequate to explain the natural world by means of the actions of capricious gods in myths. In mathematics, it was not enough to empirically demonstrate results. One should argue why they were true. This was the beginning of formal deductive reasoning.

No one knows why this major intellectual development started in this place at this time. Certainly, it made a difference that Miletus had access to the major intellectual traditions of the Near East. Perhaps the new philosophy was an attempt to explain the contradictions in these traditions. It has also been suggested that the Greek habit of public debate fostered the notion that all assertions should be justified by careful argument.

Thales of Miletus (c. 625–547 BCE)

Thales was credited with beginning the new philosophy. Very little is known about his life. He was from Miletus, probably born into an aristocratic family. He was said to be a merchant, and to have traveled to both Egypt and Mesopotamia. He was famous as a statesman, astronomer, and engineer, as well as mathematician and philosopher.

Many stories have been told of Thales, but all of them date from well after his death, and most are no doubt apocryphal. There is a famous story, perhaps the first absent-minded professor story, of how, intent on studying the heavens, he fell into a ditch. On the other hand, there is another tale that when he was criticized for being impractical, he shrewdly cornered the market on olive presses, thereby making a fortune. Nothing he wrote has been preserved, so we have mostly the legends from later times.

It is said of Thales that, on a trip to Egypt, he impressed his hosts by demonstrating a method to determine the height of a pyramid by measuring shadows. Here is how he did it. Let A be the top of the pyramid, C the tip of the pyramid's shadow, and angle ABC be a right angle. (See Figure 1.5.) Suppose that a staff is held perpendicular to the ground. Let D be its tip, E its base, and F the tip of its shadow. Then the triangles ABC and DEF are similar, which means that the ratios of corresponding sides are equal. In particular, if the height of pyramid and staff are h_1 and h_2, respectively,

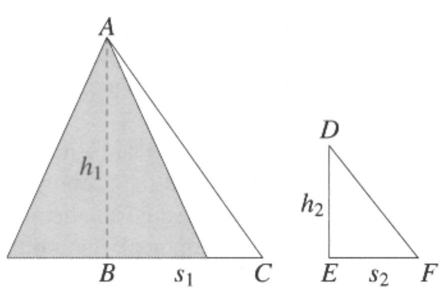


Figure 1.5 Measuring a pyramid's height by its shadow.

and the lengths of the shadows are s_1 and s_2, then $h_1/h_2 = s_1/s_2$. Since h_2, s_1, and s_2 can easily be measured, h_1 can be computed.

For example, if the staff is 6 feet, its shadow is 8 feet, and the pyramid's shadow is 640 feet, then $h_1/6 = 640/8$. Solving this, we get $h_1 = 480$ feet.

Eudemus of Rhodes wrote a book on early Greek geometers in the 4th century BCE. No copies of it remain, but a short bit of it was included in a book of the commentator Proclus (411–485 CE). Thales is credited with five theorems. They are fairly basic; one of them is that a circle is bisected by its diameter. Such a result would surely be known by anyone who had a practical need for it. Thales, however, was said to have been the first to seek a logical, not merely practical, basis for such theorems.

The most famous student of Thales was Anaximander. As with Thales, we know little of Anaximander. He was credited with introducing into Greece, from Mesopotamia, the gnomon, the center of the sundial, which casts the shadow. Anaximander also contributed to geography, drawing a circular map of the world.

In the last half of the 6th century BCE, due largely to the expansion of the Persian empire, Ionia declined as a cultural center. The center of the new Greek philosophy shifted west to Magna Graecia ("greater Greece" in Latin), the Greek colonies in southern Italy.

The Pythagoreans

In the 6th century BCE, an important group of thinkers emerged in Magna Graecia, centered around Pythagoras.

Pythagoras of Samos (c. 572–497 BCE)

Pythagoras was from the Ionian island of Samos. As with Thales, our knowledge of Pythagoras has been pieced together from reports written long after his death. He was said to have studied in Miletus, perhaps with Anaximander. He also traveled to Egypt, and reportedly spent seven years in Babylon, after which he returned to Samos. He was forced to leave Samos around 530 BCE, and settled in Crotona, a Greek seaport in southern Italy. It was there that he founded his society, known as the Pythagoreans, which also spread to neighboring cities, and was for a time very influential. Around 500 he was forced to move again, to the neighboring town of Metapontum, where he died.

Pythagoras left no writings, and his followers had a habit of attributing all of the group's discoveries to him. This practice, and the secrecy surrounding his organization, make it difficult to sort out his individual mathematical accomplishments. He certainly was a leading religious, philosophical, and political figure of his time, but perhaps his greatest accomplishment was founding the society that left such an important mark on our intellectual history.

The Pythagoreans were a society of a few hundred aristocrats. It is sometimes called a brotherhood, but there is one story that its original members included dozens of women. It was certainly selective and hierarchical. Members were divided into the *akousmatikoi*—listeners—who were expected to learn the master's teachings, and the *mathematikoi*, who could develop the teachings. The *mathematikoi* were among the first pure mathematicians. Our words "mathematics" and "mathematician" derive from *mathematikoi* (which in turn was based on *mathesis*, learning).

Religiously, the Pythagoreans practiced an asceticism, were probably vegetarian, and eschewed wine. They believed in the transmigration and reincarnation of souls, where the soul is reborn in another body after death.

Politically, the Pythagoreans were anti-democratic. In fact, democratic forces attacked them and burned their buildings in about 450. Their political influence waned thereafter, although the sect continued beyond that time and continued to produce important mathematics.

What distinguished the Pythagoreans from other mystery cults of the time was their philosophy that numbers, by which they meant counting numbers, were the foundation of the universe. An example of this was their discovery of the connection between musical harmonies and simple ratios. If one measures two strings on a lyre tuned an octave apart, their lengths are in the ratio $2 : 1$. The musical interval of a fifth is associated with the ratio $3 : 2$, a fourth with $4 : 3$, and so on.

The Pythagoreans thought that these musical ratios were reflected in the heavens as well. They theorized that the planets, including the Sun and Moon and a couple of extra ones necessary to produce the number ten, traveled around on invisible spheres. These spheres were spaced according to the harmonic ratios they discovered, and produced sounds, the "music of the spheres."

The Pythagoreans dealt in numerology as well as number theory. The number 1 is associated with reason, 3 with harmony, odd numbers are masculine, even numbers are feminine. Ten, the sum of the first four numbers, was magical. They had the notion of "perfect" numbers, numbers with the property that the sum of their proper factors equals the number itself. The smallest such number is 6, as $6 = 1 + 2 + 3$. This may seem an odd notion to us, endowing numbers with human characteristics, but it inspired some lovely mathematics. Perfect numbers turn out to be associated with a family of prime numbers that is still studied. (See Section 5.7.)

Another area of interest to the Pythagoreans was the study of figurate, or polygonal, numbers, obtained from drawing regular figures with dots. Figure 1.6 shows the first few triangular (bowling pin) numbers. Note that the nth triangular number is the sum of the first n counting numbers. They attached special significance to the *tetractys*, the figure with $10 = 1 + 2 + 3 + 4$ dots. (See the prayer on p. 30.)

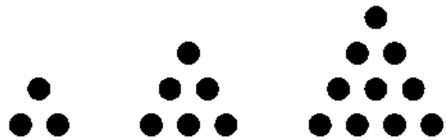


Figure 1.6 Triangular numbers.

A type of figurate numbers with which you may be familiar are the square numbers, which are numbers of the form n^2. As the nth triangular number is the sum of the first n counting numbers, the nth square is the sum of the first n odd numbers. For example, $4^2 = 1 + 3 + 5 + 7$. If you study Figure 1.7, you can see a geometric demonstration of this. The Pythagoreans also studied rectangular and pentagonal numbers. Figurate numbers have continued to fascinate mathematicians into modern times.

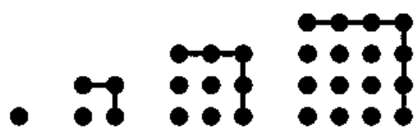


Figure 1.7 Square numbers.

Pythagoras is best known for the Pythagorean Theorem, that the area of the square on the hypotenuse of a right triangle is equal to the sum of the areas of the squares on the legs.

With the angles of the triangle labeled A, B, and C (the right angle), and the side lengths labeled a, b, and c, as in Figure 1.8, the Pythagorean Theorem says

$$c^2 = a^2 + b^2.$$

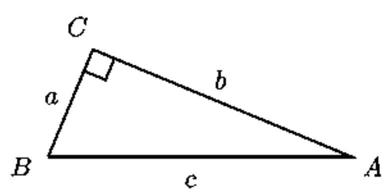


Figure 1.8 A labeled right triangle.

Figure 1.9 illustrates the Pythagorean Theorem. The area of the darker shaded square is equal to the sum of the areas of the two lighter shaded squares.

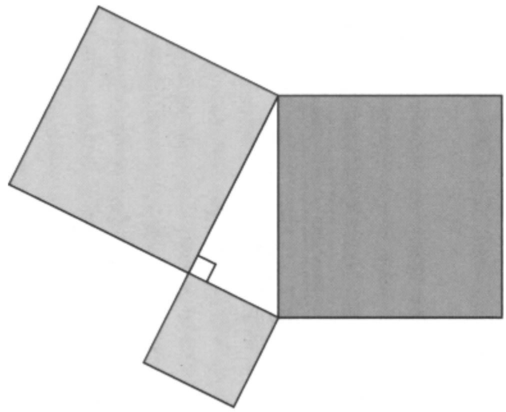


Figure 1.9 The Pythagorean Theorem (the area of the darker square equals the sum of the areas of the two lighter squares).

We have remarked that this theorem was known at least a thousand years before Pythagoras. But Pythagoras gets the credit for the theorem because his school was the first to provide a proof that covers all possible right triangles.

We offer the simplest proof of the Pythagorean Theorem that we know of. Let a right triangle be given. As in Figure 1.10, four copies of the right triangle are arranged inside a square whose side length is the sum of the two legs of the triangle. The shaded area is the area outside the four triangles and inside the large square. In the square on the left, the shaded area is the square on the hypotenuse. In the square on the right, the shaded area is the union of the squares on the legs of the triangle. Since the amount of shaded area doesn't change when we move the four triangles,

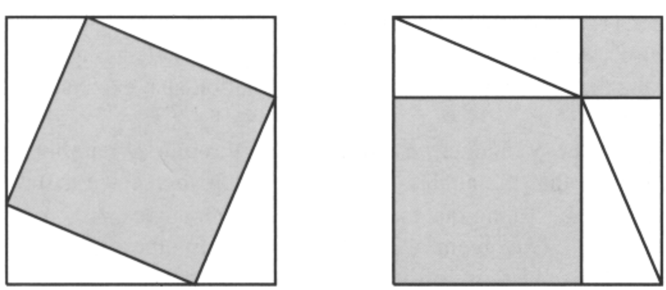


Figure 1.10 Proof of the Pythagorean Theorem.

the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the two legs.

The Pythagoreans also studied Pythagorean triples (as did the Babylonians earlier), which are three positive integers a, b, and c such that $a^2 + b^2 = c^2$. They discovered an infinite family of such triples, namely,

$$\begin{aligned} a &= 2n + 1 \\ b &= 2n^2 + 2n \\ c &= 2n^2 + 2n + 1, \end{aligned}$$

where n is any positive integer. For example, if $n = 2$, we get

$$\begin{aligned} a &= 2 \cdot 2 + 1 = 5 \\ b &= 2 \cdot 2^2 + 2 \cdot 2 = 12 \\ c &= 2 \cdot 2^2 + 2 \cdot 2 + 1 = 13. \end{aligned}$$

You can check that $5^2 + 12^2 = 13^2$. To see that these are always Pythagorean triples:

$$\begin{aligned} a^2 + b^2 &= (2n + 1)^2 + (2n^2 + 2n)^2 \\ &= (4n^2 + 4n + 1) + (4n^4 + 8n^3 + 4n^2) \\ &= 4n^4 + 8n^3 + 8n^2 + 4n + 1 \end{aligned}$$

and

$$\begin{aligned} c^2 &= (2n^2 + 2n + 1)(2n^2 + 2n + 1) \\ &= 4n^4 + 8n^3 + 8n^2 + 4n + 1. \end{aligned}$$

Thus $a^2 + b^2 = c^2$.

The Pythagoreans proved that the sum of the angles in a triangle equals 180 degrees. Their proof is based on the equality of alternate angles, e.g., angles α and α' in Figure 1.11, where the two horizontal lines are parallel. This equality follows from the figure's symmetry; imagine rotating by 180°, exchanging the parallel lines and leaving the diagonal line unchanged. This rotation also exchanges the angles α and α'.

Now let ABC be a triangle, and draw a line through B parallel to AC, as in Figure 1.12. Note that the angles α', β, and γ' together make a straight line, so $\alpha' + \beta + \gamma' = 180°$. Using our theorem on alternate angles, $\alpha = \alpha'$ and $\gamma = \gamma'$. If we substitute these, we obtain $\alpha + \beta + \gamma = 180°$. In other words, the sum of the angles in the triangle ABC is 180°.

One of the most important Pythagorean discoveries was the existence of irrational numbers. A *rational number* is one that can be expressed as a ratio of integers. For example, $2/3$ is rational, as is $5 = 5/1$. It is a natural assumption that all numbers

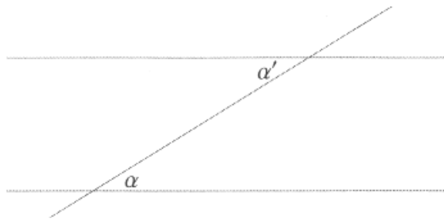


Figure 1.11 Equality of alternate angles.

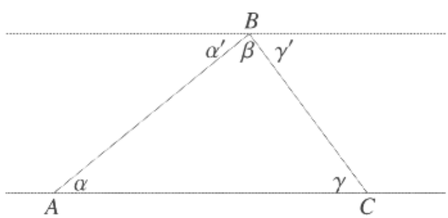


Figure 1.12 The sum of angles in a triangle is 180°.

are rational, but this turns out not to be the case. For example, $\sqrt{2}$, the length of a diagonal from a square of side 1, turns out to be *irrational*, that is, not expressible as the ratio of two integers. This discovery, reportedly made by the Pythagorean Hippasus of Metapontum, certainly complicated theoretical mathematics. One can only guess at its effect on the Pythagoreans, whose philosophy was so dependent on whole numbers and their ratios. Legend has it that Hippasus made the discovery while at sea, and that the others, appalled by the idea, threw him overboard. More on this important theorem can be found in Section 5.3.

The Pythagoreans also had a profound effect on education. They identified four basic areas of education: arithmetic (which essentially meant the theory of numbers), geometry, music, and astronomy. These four were later extolled by Plato and Aristotle, and they loomed large into the Middle Ages, where they were known as the *quadrivium*.

Archytas of Tarentum (c. 438–347 BCE)

The Pythagorean Archytas was a politician and mathematician in southern Italy one hundred years after the death of Pythagoras. He was a number theorist and a geometer, devising a clever mechanical solution to the Delian problem (see "Three

Construction Problems" below). He is also credited with devising the educational system mentioned above, which became the quadrivium.

Archytas had two vastly influential students, Eudoxus and Plato.

Finally, lest we forget that these great mathematicians lived in a very different age, we end with one of their prayers.

> Bless us, divine number, thou who generated gods and men! O holy, holy Tetractys, thou that containest the root and source of the eternally flowing creation! For the divine number begins with the profound, pure unity until it comes to the holy four; then it begets the mother of all, the all-comprising, all-bounding, the first-born, the never-swerving, the never-tiring holy ten, the keyholder of all.
>
> PYTHAGOREAN PRAYER

Elea

Another important center of philosophy in Magna Graecia, not far from Crotona and Metapontum, was the city of Elea.

Parmenides (c. 515–450 BCE)

Parmenides, the founder of the Eleatic school, was born in Elea, into a wealthy family. We know little of his life. Philosophically, he was influenced by the poet Xenophanes, and was perhaps his pupil. Since Xenophanes was from Ionia, Parmenides was aware of the Ionian philosophers. It seems likely, given their proximity, that the Pythagoreans were also known to him. Plato writes that Parmenides visited Athens in 450, when he was an old man, and there met the young Socrates.

The only known work of Parmenides was a philosophical poem titled *On Nature*. Only fragments of it remain.

When Parmenides sang this poem (yes, he sang his philosophy), he entreated his listeners to ignore their senses, and instead follow pure reason. In particular, he insisted that movement and change were illusory; reality is unchanging, eternal.

The Eleatic school was important for its insistence on logic in philosophy. One did not merely assert beliefs, but needed to make formal, logically rigorous arguments in support of them. Members of the school were fond of the type of argument called *reductio ad absurdum*, in which a proposition is proved by demonstrating that its denial leads to a logical absurdity.

In mathematics, a reductio ad absurdum proof is often called a *proof by contradiction*. Here is a simple example, a proof that the number of integers is infinite. We begin by supposing the opposite, that the number of integers is finite. In this case, there must be a largest integer, say n. But then consider $n+1$. It is clearly an integer, and it is larger than n, which gives us a contradiction (logical absurdity). Since this follows logically from assuming that the number of integers is finite, there must be an infinite number of integers.

Zeno (c. 490–425 BCE)

At this point, you will not be surprised to learn that our knowledge of Zeno's life is sketchy. Most of what we know is from Plato's book *Parmenides*. Zeno was born in Elea, and was a student of Parmenides. His importance rests on a book of paradoxes he wrote, in defense of Parmenides' philosophy. We do not even have copies of this book, only commentaries on parts of it written after his death.

Zeno's book was said to contain 40 paradoxes, of which nine have survived, although only as rephrased by other authors. Here are three of the most famous.

The Dichotomy: A runner is running toward a goal. In order to reach this goal, he must first reach the halfway point. He then must go halfway from that point to the goal, and so on. At every point, he must still traverse half the distance to the goal, so can never arrive.

Achilles and the Tortoise: Achilles, the fastest runner in the world, is chasing after a tortoise. In order to catch the tortoise, he must reach the point at which the tortoise started. (We assume that both are running in a straight line.) But when Achilles arrives at that point, the tortoise, slow as he may be, has moved on. So Achilles must then reach the new point at which the tortoise has arrived. This process continues; when Achilles reaches the point at which the tortoise is, the tortoise is no longer there. So Achilles can never catch the tortoise.

The Arrow: Consider a moving arrow at a particular instant of time. At that instant, the arrow occupies a particular place. But the place does not move, therefore the arrow is motionless. Thus motion is impossible.

A *paradox*, in English, can mean an argument or assertion that defies intuition. However, the sense in which Zeno's arguments are paradoxes is more specific. He presented arguments which ended in absurd conclusions, but the important point is that it was not clear exactly where the argument went wrong. The issue is not whether the conclusions are correct, but what exactly is wrong with the reasoning. (The word paradox comes from the Greek *para*, alongside or beyond, and *doxa*, opinion.)

Since we do not have Zeno's original words, we must infer his intent in presenting these paradoxes. One of his goals was probably to support his teacher's assertion that motion was an illusion. Another perhaps was to probe the nature of space and time. Are they infinitely divisible?

Zeno's paradoxes have inspired philosophers and mathematicians for millennia, because they force us to confront the thorny issues of infinity and continuity. These paradoxes are now considered solved, but the solution took the development of a sophisticated theory of limits, and was not completed until late in the 19th century, more than 2300 years after the paradoxes were first posed.

Democritus (c. 460–370 BCE)

Democritus was a native of Abdera, in Greece. He came from a wealthy family, was said to have traveled to many countries, and talked to many scholars before returning to Abdera.

Although not a resident of Elea, he was an academic grandson of Zeno, being a student of Leucippus who was in turn a student of Zeno. He wrote many works on mathematics, but none survive.

Democritus is most famous for developing (with Leucippus) the atomic theory. The word "atom" comes from the Greek *an* (not) and *temnein* (to cut). This is the essence of the atomic theory, that the world consists of atoms that are indivisible. This theory was in part a response to Zeno's paradoxes.

Archimedes also gave Democritus credit for stating, but not proving, that the volume of a cone is one-third that of its related cylinder, and the volume of a pyramid is one-third that of its related prism. (See Figures 1.13 and 1.14.) The first proofs were given by Eudoxus, about 50 years after Democritus.

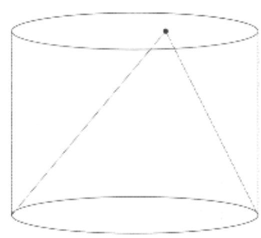


Figure 1.13 A cone and its cylinder.

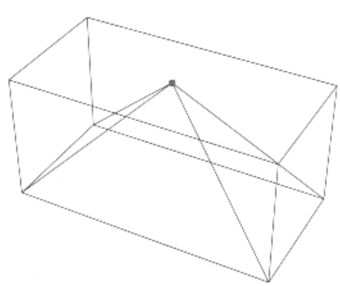


Figure 1.14 A pyramid and its prism.

There is also a tantalizing report by Plutarch (c. 100 CE) that Democritus studied thin sections cut from a cone by planes parallel to its base. This is an idea pursued fruitfully by Archimedes, and is an important part of the integral calculus developed in the 17th century.

Athens

By the middle of the 5th century, and until the late 4th century BCE, Athens was the most important center of Greek mathematics. It was the largest of the Greek city-states; we don't have good data, but estimates place its peak population at 300,000.

Athens was an important Mediterranean trading center. Its ships carried wine and olive oil, marble and silver. It had to import much of its grain. The economy of Athens was heavily dependent on slavery. Some scholars estimate that one-third of the population were slaves.

The political power of Athens waxed and waned in this period, but throughout it remained an important cultural center. It produced some of the most famous art, architecture, theater, science, and philosophy of antiquity. People still go to Athens to admire the Parthenon, and still stage productions of the plays of Aeschylus, Sophocles, Euripides, and Aristophanes. Its most famous philosophers were Socrates, Plato, and Aristotle.

In mathematics, Athenian scholars were responsible for refining the logical structure, overcoming the problem of irrational numbers, and developing theoretical geometry. They also established important institutions, where professional philosophers and mathematicians could flourish.

Three Construction Problems

Among the most important problems in ancient Greece, and beyond, were these three.

1. **Quadrature (or squaring) of the circle:** Construct a square whose area is the same as that of a given circle.

2. **Duplication of the cube (The Delian Problem):** Construct a cube whose volume is twice that of a given cube.

3. **Trisection of an angle:** Divide a given angle into three equal parts.

Obtaining the *quadrature* of a figure means constructing a square with the same area. The quadrature of a circle is in a class of problems that concern studying curved figures using simpler, straight-edged ones. Mathematicians had limited success with these problems until the advent of the calculus in the 17th century.

To duplicate a cube of side a, we need another cube of side $b = \sqrt[3]{2}a$. For then the volume of the cube of side b is $b^3 = (\sqrt[3]{2}a)^3 = 2a^3$, twice the volume of the cube of side a. So the difficulty is geometrically representing $\sqrt[3]{2}$.

What it means to "construct" a figure has been subject to different interpretations, the most famous being the use only of a straightedge and compass. The straightedge allows one to draw a straight line; the compass allows one to draw a circle of given center and radius.

Many mathematicians have worked on these problems, inventing important areas of mathematics in an attempt to solve them. The problems have provided inspiration far beyond ancient Greece. They were finally solved, for straightedge and compass, only in the 19th century, when it was shown that all three are impossible. (We should mention that Pappus, in the 4th century CE, asserted this impossibility, but without proof.)

Hippocrates of Chios (c. 470–410 BCE)

Not to be confused with the physician Hippocrates of Cos (of Hippocratic Oath fame), this Hippocrates was born on the Ionian island of Chios, not far from Pythagoras' birthplace of Samos. Early on, Hippocrates was a merchant, but, after setbacks in his business, he made his way to Athens where he became one of the foremost scholars.

Hippocrates was the leading geometer of his time and wrote an influential text on geometry, *Elements of Geometry*, which has been almost entirely lost. He taught mathematics, being one of the first to make his living that way.

Hippocrates worked on at least the first two of the three construction problems. His advances were typical in the history of difficult mathematical problems, reducing the unsolved problem to another that may be easier to solve. In the case of the quadrature of the circle, he studied a type of intersection of circular arcs called a lune. He showed that if one could always square the lune, then one could square the circle. He further showed how to square a particular type of lune. He was not able to complete this program, however, being unable to square an arbitrary lune.

Similarly, Hippocrates reduced the Delian problem (doubling the cube) to another problem in two dimensions, instead of three. Specifically, he reduced the problem of doubling the cube of side a to that of finding two *mean proportionals* between a and $2a$, that is, numbers x and y such that $a : x = x : y = y : 2a$, where $a : x$, for example, is the ratio of a to x.

Although Hippocrates' text has been lost, historians can make informed guesses about its content based on fragments of his writings included in later works. His work may have been the first to array geometric theorems in a logical sequence, from the simplest to the more advanced. The logic of his surviving proofs is not perfect, but it does demonstrate a sophistication well beyond that of the scholars of a century earlier. One area that had yet to be developed is a system of axioms, or assumptions, upon which to build a geometry.

Plato and His School

Plato (c. 429–347 BCE)

Plato was a son of Athenian aristocrats. When he was young, he became a student of Socrates, the stonemason turned philosopher. After Socrates was executed for irreverence in 399, Plato left Athens, reportedly traveled around Greece, to Egypt, and to Tarentum in Magna Graecia, where he studied with Archytas, the Pythagorean.

In 388 Plato returned to Athens, where he taught, wrote some of the most influential philosophical works ever, and founded his famous Academy. Except for two trips to Syracuse in Sicily, where the Pythagoreans were ensconced, Plato lived the rest of his life in Athens.

Plato was probably not much of a mathematician himself, being more interested in ethics. He had considerable influence on mathematics, however, in two important ways. The first was his philosophy. He believed that the world of the senses was imperfect, that what we experience is but a shadow of what he called "forms" or "ideas." For example, we can draw a circle but it is merely an imperfect representation of the Circle idea. Many mathematicians, though by no means all, subscribe to a form of Platonism that claims that mathematical objects have a real existence, independent of us. All that mathematicians do is study what is already out there. (Others reject this, believing that mathematics is an invention of humans, perhaps only a game we play.)

Plato's other major contribution to mathematics, and to learning in general, was his founding of his school, the Academy. The name, which gave us our word "academic," derived from the name of the site where the school was built. It was founded around 387 BCE.

Above the entrance gate of the Academy was inscribed "Let no man ignorant of geometry enter." The school's curriculum was based on the quadrivium—number theory, plane geometry, music, and astronomy—together with solid (3-dimensional) geometry. After the completion of these studies, the best students went on to study dialectics, a method of critical, persistent questioning, which Plato considered the way to arrive at truth.

Plato, very much the aristocrat, viewed the Academy as an institution to educate the ruling class. Its mathematics was therefore theoretical, not practical, with the exception of military applications. Interestingly, the word "school" derives from the Greek *skhole*, which means "leisure."

The Academy was more than a school; the best scholars from the Greek world came to Athens to study and teach at the school. The closest modern equivalent is the research university. Throughout most of the 4th century BCE, until the rise of the Museum in Alexandria, the Academy was the home of the cream of Greek scholarship. Even after that, it remained an important center of learning. It was finally closed, in 529 CE, by the Christian emperor Justinian, who viewed it as a pagan institution. It thus lasted more than 900 years. This is roughly the age of the oldest current European university, the University of Bologna, founded in 1088.

Eudoxus' Theory of Proportions

Eudoxus (c. 408–355 BCE)

Eudoxus was the most illustrious astronomer and mathematician of his time, and is generally considered to be the second best ancient mathematician, after Archimedes. He was born in the Ionian city of Cnidus. As a young man he traveled to Tarentum, in Sicily, where he studied both mathematics, with Archytas, and medicine. He then spent some months studying with Plato's circle in Athens. He was apparently too poor to live in Athens proper, so he lived in nearby Piraeus, and walked seven miles daily each way to the Academy. After Athens, he returned for a while to Cnidus.

Later, Eudoxus traveled to Egypt, where he worked primarily on astronomy. After Egypt, he returned to Ionia, specifically Cyzicus, where he founded a school and wrote his greatest astronomical works. He returned for a time to Athens, with some of his students, working there with Plato, Aristotle, and others. Finally, he returned to Cnidus, where he helped write a constitution for their new democracy, founded an observatory, taught, and practiced medicine and astronomy. He died at age 53.

Eudoxus' greatest mathematical contribution was his theory of proportions. To understand its importance, recall that the Pythagoreans had proved that the square root of 2 is irrational, that is, not representable as the ratio of two integers. The difficulty this presented to mathematicians was in making precise arguments about irrational numbers, and about geometric figures whose magnitudes might be irrational.

Modern mathematicians get around this difficulty by approximating an irrational number by a sequence of rational numbers, then taking a limit. Eudoxus did something similar, but a bit more complicated.

To start with, Eudoxus' theory did not deal with numbers as such, but with *magnitudes*, e.g., lengths or areas. The key to dealing with magnitudes was knowing how to compare them. Here is how he did it, as it appears in a definition from Euclid's *Elements*.

> Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equal multiples whatever are taken of the first and third, and any equal multiples whatever of the second and fourth, the former multiples alike exceed, are alike equal to, or alike fall short of, the latter multiples respectively taken in corresponding order.

Here "multiple" means integer multiple. Let us translate this into more familiar terms. If we represent the ratio a to b as $a : b$, the question is when we have equality of two ratios, say $a : b = c : d$. The above definition says that we have equality provided the following three conditions are met, for every choice of positive integers m and n.

1. If $ma < nb$ then $mc < nd$.
2. If $ma = nb$ then $mc = nd$.
3. If $ma > nb$ then $mc > nd$.

It is understandable if this definition leaves you underwhelmed. It is difficult to fully appreciate its power unless you see it used in proofs, an exercise that is beyond this text. Mathematicians, however, were not able to replace this treatment of irrationals with anything of equal precision and usefulness until late in the 19th century.

Eudoxus also made rigorous a method called *exhaustion*, earlier invented by Antiphon, which is a limiting procedure to compute areas and volumes. He used the method of exhaustion to prove a result stated by Hippocrates, that the ratio of the areas of two circles is proportional to the square of the ratio of their diameters, or in modern terms, $A = kd^2$, where A is the area of a circle, d its diameter, and k some fixed constant of proportionality.

The method used successive approximations of a circle by polygons (see Figure 1.15). As the number of sides of the polygons grows, they "exhaust" the area of the circle. Combining this with Eudoxus' definition of equality of ratios, and some careful arguments, leads to the desired proof.

Figure 1.15 Approximating the area of a circle.

One effect of Eudoxus' subtle theory was to reinforce the Greek preference for geometry over algebra, a preference that influenced the course of mathematics for the next two thousand years. Greeks did solve some algebraic problems, usually by converting them first to geometric ones.

Finally, we note that Eudoxus introduced the use of spherical geometry in astronomy. His astronomical theory had stars and planets rotating on spheres centered at the Earth. Although this model wasn't accurate, it was sophisticated for its time and was immortalized by being included (in a modified form) in Aristotle's supremely influential works.

Logic

Aristotle (384–322 BCE)

Aristotle was from the Ionian colony of Stagira. His father was physician to the kings of neighboring Macedon. When he was seventeen or eighteen, Aristotle came to Athens to study at the Academy. He stayed on as a scholar until Plato's death in 347 BCE, after which he left Athens.

In 342 Aristotle became the tutor to the young prince Alexander of Macedon, later called Alexander the Great. He stayed as advisor until 335, when Macedonia took control of Athens. In that year, Aristotle returned to Athens and set up his own school, the Lyceum. He remained there until Alexander's death in 323, when he found it prudent to leave. He died the next year in Chalcis.

Aristotle is one of the most influential philosophers of all time. He wrote on a wide variety of topics and was the preeminent authority on the physical sciences for two thousand years. His importance to mathematics lies in his work on logic. He constructed a formal theory of logic, building on the ideas developed over the preceding 250 years or so of Greek philosophy.

Aristotle believed that the only way to certain knowledge was by the use of logic, deducing new knowledge based on old. One has to start somewhere, however. His starting point was a set of axioms and postulates, which were truths that needed no argument. An *axiom*, for Aristotle, was a truth that was not particular to any science, for example, "take equals from equals and equals remain." A *postulate* was a truth that concerned a particular area; for example, "through every two points a straight line may be drawn" is a geometric postulate. The philosopher should start with the minimum number of axioms and postulates needed, and some definitions, then proceed by logical argument to prove things.

This logical structure is still the basis for mathematics, although modern mathematicians do not distinguish between axioms and postulates, usually calling any initial assumption an axiom. We also tend not to use the word "truth" for our axioms, although axioms must be carefully constructed to be of use.

As an example of this aspect of logic, let us reconsider the earlier proof we gave that the number of integers is infinite.

We begin by supposing the opposite, that the number of integers is finite. In this case, there must be a largest integer, say n. But then consider $n + 1$. It is clearly an integer, and it is larger than n, which gives us a contradiction. Since this follows logically from assuming that the number of integers is finite, there must be an infinite number of integers.

This argument assumes some things, for example, that every finite set of integers has a largest element, and that for every integer n, there is an integer $n + 1$. These may seem obvious to you, but they are still logically required. So we may take them as axioms. The argument also assumes that we know what an integer is. Perhaps we might add a definition for integer. (We won't attempt such a definition here; it turns out to be a delicate matter.)

Thus, the revised argument would start with a definition of "integer" and two axioms: (1), that every finite set of integers has a largest element, and (2), that for every integer n, there is an integer $n+1$. We would then proceed with the argument proper.

Perhaps you can think of other axioms or definitions that this argument needs. If so, you have begun to appreciate how difficult and subtle is the task of establishing a logical foundation for mathematics. The Greeks started this process, but as we shall see in later chapters, theirs was not the final word.

At the Lyceum, his school, Aristotle was famed for lecturing while walking the grounds with his students. As a result, teachers and students there were known as Peripatetics, from the Greek *peri* (around) and *patein* (to walk). They didn't spend all of their time walking, however; the Lyceum introduced written examinations into the educational system.

Conic Sections

Menaechmus (c. 380–320 BCE)

Biographical details of Menaechmus' life are sketchy. He and his brother studied at Eudoxus' school at Cyzicus, perhaps with Eudoxus himself. He was a friend of Plato, and perhaps a tutor to Alexander the Great. He later headed the school in Cyzicus, where he died.

Among his many contributions, Menaechmus studied, and probably discovered, conic sections. A *conic section* is a curve obtained by intersecting a double cone with a plane. (See Figure 1.16.) If the plane intersects both parts of the cone, we get a hyperbola. An ellipse, of which a circle is a special case, intersects only one part, and is finite. In between, if the plane is parallel to a side of the cone, we get a parabola, which, unlike the hyperbola, has only one branch and, unlike the ellipse, goes on to infinity.

Menaechmus used conic sections to solve the Delian problem: doubling the cube. Recall that Hippocrates had reduced the problem of doubling the cube of side a to that of finding two mean proportionals between a and $2a$, that is, numbers x and y such that $a : x = x : y = y : 2a$. We would write the equality of the ratios as

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}.$$

These are equivalent to $ay = x^2$ and $2ax = y^2$, the equations of two parabolas. So the problem of finding x and y is equivalent to the problem of finding a point (x, y) on both of the parabolas, i.e., finding an intersection of the parabolas. This is what

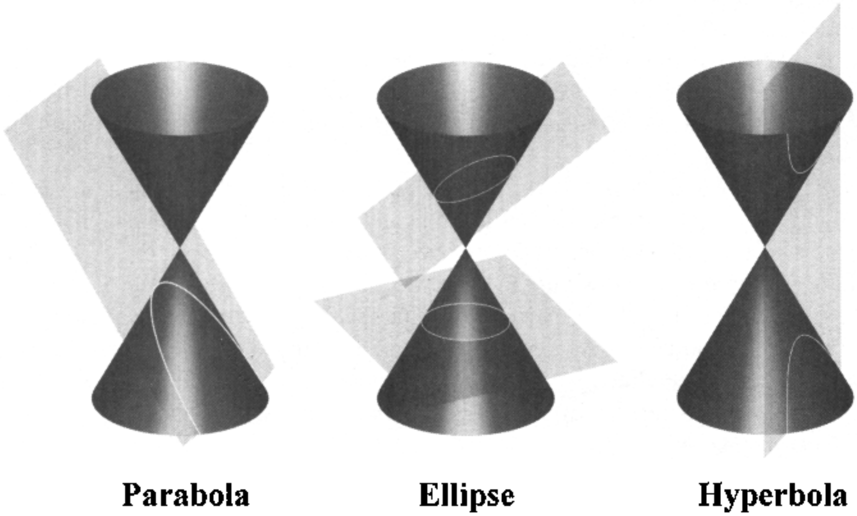


Figure 1.16 Conic sections.[1]

Menaechmus discovered. We don't know how he constructed his conic sections; they cannot be constructed via straightedge and compass, but the Greeks knew other techniques.

Menaechmus' solution to the Delian problem is an example, of which there are many in mathematics, of a discovery that turned out to be more important than its original inspiration would suggest. The Delian problem has faded into obscurity, but conic sections are very important. For instance, comets travel around the Sun in orbits that are conic sections.

EXERCISES

Below is a partial table of Ionic numerals.

Symbol	Number	Symbol	Number	Symbol	Number
α	1	ι	10	ρ	100
β	2	κ	20	σ	200
γ	3	λ	30	τ	300

1.21 What numbers would each of these represent?

[1]Diagram by Pbroks13. Wikipedia Commons.

a) κ
b) $\lambda\alpha$
c) $\sigma\iota\gamma$
d) $\rho\alpha$

1.22 Write each of these numbers in the Ionic system.
a) 21
b) 333
c) 220

1.23 Suppose that the shadow of a building measures 60 meters, at the same time that a 2 meter stick casts a 5 meter shadow. How tall is the building?

1.24 If the two legs of a right triangle have lengths 5 and 12, what is the length of the hypotenuse?

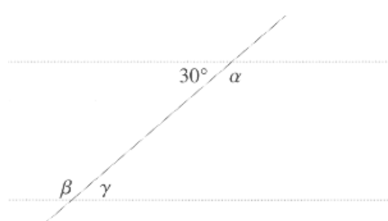
1.25 If the two legs of a right triangle both have length 1, what is the length of the hypotenuse?

1.26 If one of the legs of a right triangle has length 1 and the hypotenuse has length 2, what is the length of the other leg?

1.27 If the two legs of a right triangle have lengths $2n$ and $n^2 - 1$, where n is an integer greater than 1, what is the length of the hypotenuse?

1.28 A triangle with side lengths 8, 15, and 17 is inscribed in a circle. What is the diameter of the circle?

1.29 Find angles α, β, and γ.



1.30 Another figurate number is the *oblong* number, which is a number of the form $n(n+1)$. The first two oblong numbers are $2 = 1 \cdot 2$ and $6 = 2 \cdot 3$.
a) Find the first ten oblong numbers.
b) Recall that triangular numbers are of the form $1 + 2 + 3 + \cdots + n$, and square numbers are of the form $1 + 3 + 5 + \cdots + (2n-1)$. Find a similar pattern for oblong numbers.

1.31 Use the formula for Pythagorean triples to find two other sets of triples, larger than $(5, 12, 13)$.

1.32 The first perfect number is 6. The next one is between 20 and 30. Find it.

1.33 Zeno's dichotomy paradox is related to a famous infinite sum. Find the sum $1/2 + 1/4 + 1/8 + \cdots$.

1.34 Eudoxus proved that a circle of diameter d has an area given by $A = kd^2$, for some constant k. What is k? (Hint: you know the area of a circle in terms of its radius r.)

1.35 How might the atomic theory of Leucippus and Democritus be used to explain Zeno's paradoxes?

1.36 Let $a = 1$, $x = \sqrt[3]{2} = 2^{1/3}$, and $y = 2^{2/3}$.

a) Verify that x and y are mean proportionals between a and $2a$. (Hint: remember that $2^r/2^s = 2^{r-s}$.)

b) Show that this solves the Delian problem for $a = 1$, i.e., that the cube of side x has twice the volume of the cube of side a.

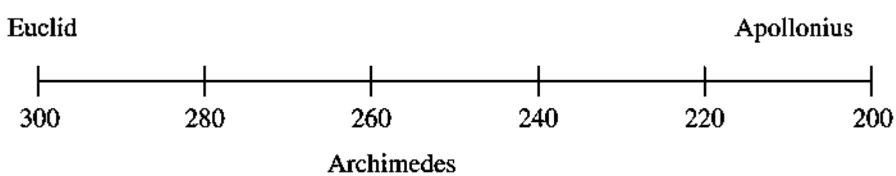
1.37 Consider the proof of the equality of alternate angles given in the text. What axioms and definitions might be required to make this logically rigorous?

1.38 Look up the word *theorem*. What is its origin?

1.4 The Apex: Third Century Hellenistic Mathematics

There is no permanent place in the world for ugly mathematics.

G. H. HARDY (1877–1947)



The Third Century BCE.

Greek mathematics, especially geometry, achieved its highest expression in the 3rd century BCE. The century started with the writing of the most famous math book ever, continued with the work of the greatest of the ancient mathematicians, and finished with the definitive Greek text on conic sections.

Alexander the Great

Macedonia was a small kingdom on the northern boundary of Greece. Under King Philip II, who ruled from 356 to 336 BCE, Macedonia gained effective control of most of Greece, and began a war against the Persian empire to the east. Shortly after the start of this war, Philip was assassinated and his army elected his son Alexander as his successor. Alexander, already a veteran general, was twenty years old.

The war against Persia continued. Alexander won a great battle at Issus, in modern-day Turkey, in 333. He continued his conquests, as far as modern-day Uzbekistan and Pakistan in Asia, and Egypt in Africa, creating the largest empire up to that time. He caught a fever and died in 323 BCE. He was 33 years old.

Among the many stories told of Alexander, perhaps the most famous is that of the Gordian knot. The legend has it that when Alexander entered the city of Gordium, he was shown a sacred knot tied around a pole. Supposedly, the man who could untie the knot was destined to become the king of Asia. Alexander's response was to take out his sword and slice the knot. The phrase "cutting the Gordian knot" is now used to indicate an audacious solution of a complicated problem. Perhaps it can also be considered a metaphor for Alexander's ruling philosophy. Another legend has it that as Alexander lay dying he was asked to whom would he leave his empire. His answer: "to the strongest."

In fact, the empire fell apart after Alexander's death. Major pieces were ruled by several of his generals, including Antigonus in Macedonia, Seleucus in the east, and Ptolemy in Egypt. What followed was a period of empire, much different than the time of the great Greek city-states. The successors of Seleucus, the Seleucids, ruled much of west Asia, gradually declining in power until their last holdings in Syria were conquered by the Romans in 64 BCE. The Ptolemys in Egypt ruled until the death of Cleopatra in 31 BCE, again falling to the Romans.

The civilization of this time, from Alexander until the rise of the Romans, is known as *Hellenistic*, distinguishing it from the earlier *Hellenic* period of Greek culture. Hellenistic culture spread across all of Alexander's empire; a form of Greek became the common language of trade and government. The rulers were educated in classical Greek culture.

Alexandria and Its Museum and Library

The greatest city of this time, both commercially and culturally, was Alexandria in Egypt. The city was founded by Alexander in 331, the first of seventeen cities of that name. It was a major port, located where the Nile empties into the Mediterranean. As such, it was situated to profit from Egypt's large surplus of grain, which was shipped to many Mediterranean ports.

Egypt was ruled by the Macedonian general Ptolemy I Soter from the time of Alexander's death in 323 until 283 BCE. His capital was Alexandria, and it was there that he built the Museum ("temple of the Muses") and Library. The Museum recruited the leading scholars of the Greek world, paying them a salary, providing free board and freedom from taxes. Originally, it was not a school; it has been compared to the modern Institute for Advanced Study in New Jersey, a place for scholars to discuss, and invent, ideas. Over time, students were attracted to the Museum, to learn from the experts.

The Library at Alexandria was the largest in the ancient world, eventually housing over 500,000 manuscripts. Ships sailing from Alexandria were instructed to gather any manuscripts they could, to add to the Library. One story has it that the Library borrowed manuscripts, copied them, then returned the copies, retaining the originals.

Alexandria quickly eclipsed Athens as the chief center of Greek learning. There were other centers, including Syracuse where Archimedes worked, but Alexandria was the greatest. The city and its Museum remained influential even after the rise of Rome.

Euclid's *Elements*

The *Elements* by Euclid is the most successful textbook in history. Written around 300 BCE, it is second only to the Bible in number of editions, certainly over one thousand. It was *the* mathematical textbook into at least the 19th century. It inspired many students from Abraham Lincoln to Albert Einstein. Edna St. Vincent Millay wrote a sonnet entitled *Euclid Alone Has Looked on Beauty Bare.*

The *Elements* is the culmination of the early period of Greek mathematics. In it, Euclid summarized much of the mathematics developed in the preceding three centuries. But he did more than that; he put this mathematics into a rigorous and consistent logical framework, starting with unproved assumptions and carefully, step-by-step, deducing more advanced theorems from them. It is this structure that gives the book its special character, and is no doubt the reason why high school geometry courses to this day often are students' first introduction to formal mathematical reasoning. It is also far from the modern multicolored, image-laden, mathematical textbook. In fact, the modern reader will find it dry as dust. It remains, however, one of the great intellectual milestones in history.

Euclid (c. 330–270 BCE)

Even less is known of Euclid than of many of his predecessors. He worked at the Museum in Alexandria, probably arriving some time around 300 BCE. It is reasonable to think that he had studied in Athens, perhaps at the Academy, because he was certainly familiar with the works of Eudoxus and other Athenians.

Euclid's date and place of birth are unknown, as is the date of his death. He wrote on many mathematical subjects, including astronomy and optics, perhaps a dozen books in all, but only the *Elements* has survived. Of course, tales are told of him, but they are all from many hundreds of years later. One of the most often repeated is that the king Ptolemy asked him if there were a shorter way to learn geometry than through the *Elements*. He was said to reply that there "is no royal road to geometry." He probably did not add that kings at least can afford tutors like Euclid.

The *Elements* is divided into thirteen "books." Book I starts with five postulates and five "common notions," both of which we would call axioms today. They are given below.

Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and radius.
4. That all right angles equal one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Common Notions

1. Things which equal the same thing also equal one another.
2. If equals are added to equals, then the wholes are equal.
3. If equals are subtracted from equals, then the remainders are equal.
4. Things which coincide with one another equal one another.
5. The whole is greater than the part.

Some of this language may need translation. For example, Postulate 1 means that, given any two points, we can draw a straight line joining them. Common Notion 1 can be stated algebraically: If $a = b$ and $c = b$ then $a = c$.

Postulate 5 sticks out like a sore thumb. In modern language, it might be stated thus: if a line intersects lines 1 and 2, as in Figure 1.17, and the angles α and β sum to less than 180 degrees, then lines 1 and 2 must eventually intersect, on the same side of the third line as α and β. Postulate 5 is also called the *parallel postulate*, because Euclid used this postulate to prove theorems about parallel lines.

If you think about this postulate a bit, you may be able to convince yourself of its truth. Postulates and common notions, however, are supposed to be self-evident. This one seems a bit too involved. Over the centuries many people have attempted to show that it could be derived from the other postulates and axioms. In the 19th century, mathematicians developed Non-Euclidean geometries, in which the other axioms hold but the parallel postulate is false. Euclid was vindicated.

The rest of Book I is a careful, step-by-step, argument, culminating in the Pythagorean Theorem.

Book II contains a number of results which we would think of as algebraic, but in geometric form. In general, the Greeks preferred geometry to algebra, and would often cast problems into geometric form that we now would handle algebraically. As an example, consider the following.

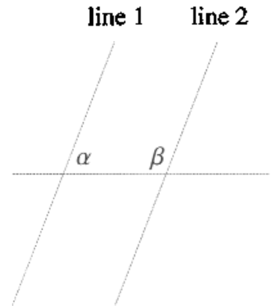


Figure 1.17 Parallel postulate.

Proposition II-4 If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments.

We can picture this as in Figure 1.18, where the line at the left is cut into pieces of lengths a and b. The proposition states that the area of the big square equals the sum of the areas of the two smaller squares and the two rectangles. Of course, drawn as it is in the diagram, this is rather evident, but let's translate this into equations. The area of the larger square is $(a+b)^2$. The smaller squares have areas a^2 and b^2, and each of the rectangles has area ab. So the proposition states that

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

This is our familiar way of squaring a binomial.

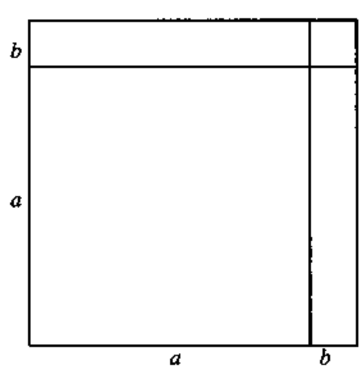


Figure 1.18 Proposition II-4.

Book II contains other results of this nature, including a geometric form of the quadratic formula, which we now write as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

giving solutions of the equation $ax^2 + bx + c = 0$.

Book III contains thirty-seven propositions on circles, starting from basic definitions. Here is one that you may have seen before.

Proposition III-20 In a circle the angle at the center is double the angle at the circumference when the angles have the same circumference as base.

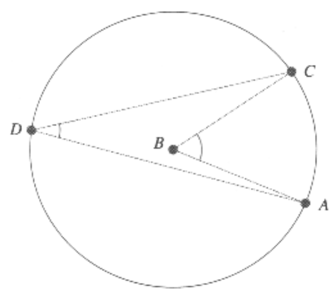


Figure 1.19 Proposition III-20.

Using the notation of Figure 1.19, angle ABC is twice angle ADC.

Book IV has results about inscribing polygons in circles and circles in polygons, and gives constructions of some regular polygons. The regular polygons of three and four sides, i.e., equilateral triangles and squares, are relatively easy to construct. The book ends with the construction of a regular pentagon (5-gon) and 15-gon.

Book V presents some basic results on magnitudes, which we would think of as lengths or areas. Here is an example.

Proposition V-1 If any number of magnitudes are each the same multiple of the same number of other magnitudes, then the sum is that multiple of the sum.

An algebraic version of this is, for a positive integer n and any magnitudes a_1, a_2, $\ldots$, a_k, we have $na_1 + na_2 + \cdots + na_k = n(a_1 + a_2 + \cdots + a_k)$. For us this follows from the associative law of numbers. Euclid didn't consider magnitudes to be the same as numbers, however. Magnitudes and numbers were different things for him.

Book V also presents Eudoxus' theory of proportions, and uses it to prove other results on magnitudes. Many of these results are used in Book VI, which is about similarity, defined as follows (see Figure 1.20).

Definition VI-1 Similar rectilinear figures are such as have their angles severally equal and the sides about the equal angles proportional.

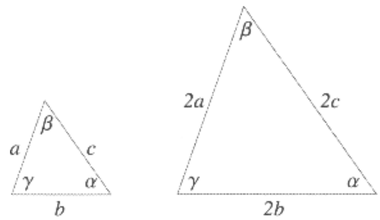


Figure 1.20 An example of similar triangles.

In Proposition VI-19 it is proved that the ratio of the areas of two similar triangles is the square of the ratio of their corresponding sides. As an illustration, the larger triangle in Figure 1.20 has four ($= 2^2$) times the area of the smaller.

Books VII, VIII, and IX cover number theory. These books do not rely on the results of the first six books, because they are about numbers, not magnitudes, which Euclid viewed as entirely different entities. Thus, Euclid proved again theorems such as the distributive law, even though he had a similar theorem about magnitudes. The number theory chapters are based on Pythagorean work, although this work had been later logically reorganized, perhaps by Theaetetus.

We mention four important results from these three books. The first has to do with greatest common divisors. Recall that the greatest common divisor of two integers is the largest integer dividing them both. Book VII contains the *Euclidean Algorithm*, a method of determining greatest common divisors that is still important. Details on this algorithm can be found in Section 5.4.

Two fundamental results on prime numbers are in these books. The first is a proof that the number of primes is infinite. We give this proof in Section 5.5. The second is the Fundamental Theorem of Arithmetic, that every positive integer can be expressed as the product of primes, and in only one way. This theorem is the subject of Section 5.8.

The culmination of Euclid's number theory chapters is a study of perfect numbers. Recall that a positive integer n is a perfect number if it is the sum of all its divisors, excluding n itself. Euclid was able to connect perfect numbers to certain types of primes. This topic is explored further in Section 5.7.

Book X undertakes a study of incommensurable magnitudes. This book starts with the following definition.

Definition X-1 Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure.

Here is the idea. Suppose that we have two lines, of lengths a and b. Then a and b (thought of as magnitudes, not numbers) are commensurable if there is another line of length c that fits into both a and b a whole number of times. For example, 6 and 10 are commensurable because 2 fits into both, i.e., $6 = 3 \cdot 2$ and $10 = 5 \cdot 2$. In general, if a and b are commensurable, say by c, then $a = mc$ and $b = nc$, for some integers m and n. But then $a/b = m/n$. In other words, the ratio of a to b is a *rational* number. Studying incommensurable magnitudes is tantamount to studying irrational numbers. It should therefore come as no surprise that Euclid relies on Eudoxus' theory of proportions in this chapter.

Books XI–XIII concern solid (3-dimensional) geometry. Books XI and XII prove some extensions of earlier theorems of plane geometry, and obtain results on volumes of a number of solids. For example, proofs are included for the theorems that the volume of a cone is one-third that of the related cylinder, and the volume of a pyramid is one-third that of its related prism. The following important result is also proven.

Proposition XII-18 Spheres are to one another in triplicate ratio of their respective diameters.

"Triplicate ratio" means cube, so the proposition is that the volume of a sphere is proportional to the cube of its diameter. This is equivalent to writing $V = kr^3$, where V is the volume, r the radius, and k some constant of proportionality. We now know that $k = 4\pi/3$, but this was not known to Euclid.

Many of the results in Books XI and XII are proved by the method of exhaustion, that is, by exhausting the area or volume of shape being studied by figures of known properties.

Finally, Book XIII is a study of convex regular polyhedra, the *Platonic solids*. These are 3-dimensional equivalents of regular polygons, solids whose faces are all congruent regular polygons, arranged the same way around each vertex. See Figure 1.21. These polyhedra make good dice and are used in various role-playing games.

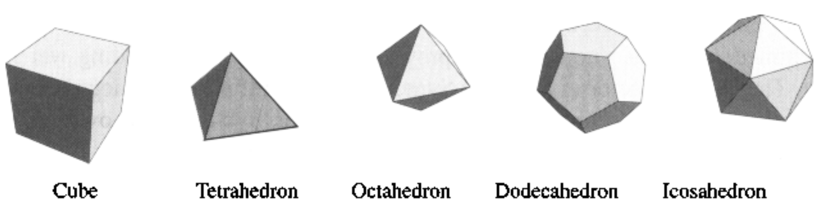


Figure 1.21 The Platonic solids.

Book XIII shows how to construct the Platonic solids, and gives some of their properties. It culminates in the proof that, unlike the regular polygons, there are only a finite number of convex regular polyhedra. In fact, the five in Figure 1.21 are the only ones.

The *Elements* contains some mistakes, and Euclid made some implicit assumptions without stating them. Still, Euclid skillfully surveyed the whole of Greek mathematics of his time. Most of the content of his book was originally due to others, but he put it together so well that previous summaries have not even been preserved.

Archimedes and Higher Geometry

Euclid was not the last word in Greek geometry. The 3rd century BCE was the high water mark for all of ancient geometry. The two most notable figures were Archimedes and Apollonius.

Archimedes (c. 287–212 BCE)

Widely considered to be the greatest mathematician of antiquity, Archimedes was born and raised in the Greek settlement of Syracuse in Sicily. His father was an astronomer, and no doubt his first mathematics teacher. Later, Archimedes traveled to Egypt, probably studying at the Alexandria Museum with students of Euclid. He then returned to Syracuse to work.

In addition to mathematics, Archimedes made important discoveries in physics, especially hydrostatics (the physics of water) and was an engineer of note. The most famous, probably apocryphal, story told of Archimedes concerns his solution of a problem presented to him by King Hiero. The king had recently acquired a golden crown, and wanted to verify that it was indeed made entirely of gold, without in any way modifying the shape. Archimedes was pondering the problem while in his bath, and it occurred to him that he could use the displacement of water when the crown was lowered into a bath to determine its density, hence its composition. He was so excited that he jumped out of the bath and ran naked through the streets yelling "Eureka! Eureka!" Eureka means "I have found it."

Among his many mechanical inventions was the Archimedean screw, a mechanical device for lifting water. Although it has now been mostly replaced for that purpose by other sorts of pumps, it is still used in a variety of applications from sewage treatment plants to fish hatcheries.

Archimedes was also noted for his prowess in designing and building war machines. They were a factor in allowing Syracuse to hold off a Roman siege for three years during the second Punic War. In the end, however, Syracuse was overrun, and Archimedes was slain by a Roman soldier.

Archimedes wrote many works on a wide variety of topics. Unlike Euclid, he did not write textbooks, but rather original research monographs. As with the other ancients, none of the originals have survived, and many of his works have been lost.

He was a pioneer of mathematical physics, the creation of mathematical models for physical situations. He proved the law of the lever, a result that had been previously known but not rigorously studied. This law states that the two sides of the lever are in balance when the product of the weight and the distance from the fulcrum on one side of the lever equals the similar product from the other side. Archimedes used this law in many of his mechanical inventions. He is said to have claimed: "Give me a lever long enough and a fulcrum on which to place it, and I shall move the world."

Archimedes also studied centers of gravity of various shapes, a topic that combined his interest in physics and geometry. The center of gravity of a body is the point from which it can (at least conceptually) be suspended and be at rest.

Hydrostatics is yet another area of science where Archimedes excelled. He discovered the *Archimedes principle*, that a body immersed in water will displace a volume of fluid that weighs as much as the body would weigh in air. (This was the theoretical principle behind the solution of the golden crown problem.)

Among his many mathematical works, we will visit three. The first is a short treatise, called *Measurement of the Circle*. In the tradition of studying the squaring of the circle, Archimedes proves that the area of the circle is equal to one-half the radius times the circumference. If we define π as the ratio of the circumference of circle to its diameter, $\pi = C/2r$, this gives

$$A = \frac{1}{2}rC = \frac{1}{2}r(2\pi r) = \pi r^2.$$

It was previously known that the area was proportional to r^2, i.e., $A = kr^2$. Archimedes showed that constant of proportionality is π.

He then goes on to numerically approximate π. He bounds the area of the circle above and below by inscribing polygons in the circle, and inscribing the circle in other polygons, as in the method of exhaustion. By considering polygons with an increasing number of sides (ultimately 96 sides), he arrives at the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$. For many years after this, the estimate of $3\frac{1}{7}$ was commonly used for π; it is called the Archimedean value of π.

Archimedes' masterpiece was *On the Sphere and Cylinder*. In it he studied a sphere and its circumscribed cylinder (Figure 1.22). He proved that the surface area of the sphere is $2/3$ that of the cylinder (including the two caps), and the volume of the sphere is $2/3$ that of the cylinder.

Let us see how we can use Archimedes' results to find formulas for the surface area and volume of a sphere. First, we figure the surface area of the cylinder. Here is the approach:

$$A = A(\text{top}) + A(\text{bottom}) + A(\text{sides}).$$

If the radius of the sphere is r, each of the caps (top and bottom) is a circle of radius r, so has area πr^2. To figure the area of the sides of the cylinder, mentally unroll it. You will get a rectangle whose height is the height of the cylinder, $2r$, and width is the circumference of the circle, $2\pi r$. So the area of the side is the product of the width and height, or $(2\pi r)(2r)$. Putting this together,

$$A = \pi r^2 + \pi r^2 + (2\pi r)(2r) = 6\pi r^2.$$

Thus the surface area of the sphere is $2/3(6\pi r^2) = 4\pi r^2$.

The volume of the cylinder is the area of the base times the height, or $\pi r^2(2r) = 2\pi r^3$. Hence, Archimedes' theorem tells us that the volume of the sphere is $\frac{4}{3}\pi r^3$. As with the circle, before this theorem, the Greeks knew that the volume of the sphere was proportional to r^3 but did not know the constant of proportionality.

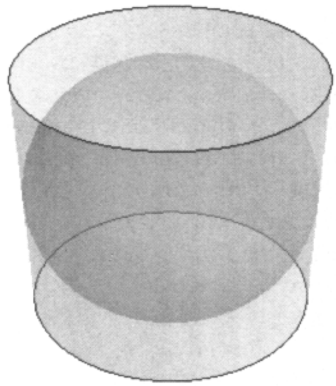


Figure 1.22 A sphere and its circumscribed cylinder.

Archimedes was so proud of these results that he requested the diagram of the sphere and cylinder be carved on his gravestone.

Archimedes proved the above results in the Greek fashion, in a way that did not reveal how he discovered them. In particular, he perfected the method of exhaustion. This method is useful, however, only after you know what the answer is. His method of discovery was mysterious until about 100 years ago. In 1899 a treatise containing a copy of his work called *The Method* was discovered in Constantinople, in the library of a Greek monastery. It had been written on a parchment in the 10th century, but then overwritten by a religious work in the 13th century. The practice of reusing parchments was not uncommon, since parchment was quite valuable. Such a reused parchment even has a name, palimpsest.

In *The Method* Archimedes gives an ingenious technique for computing areas and volumes, one that combines his work in geometry and physics. He mentally slices the shape he is studying into thin cross sections, then balances these against cross sections of a known shape, using the law of the lever. Some modern mathematicians have seen hints of the integral calculus (developed in the 17th century) in this process. It was not, however, rigorous, so after he used this method to discover results, he proved them in the usual deductive way.

Marcellus, the general who commanded the army that overran Syracuse, was upset to find that Archimedes had been killed by one of his soldiers. He erected a small column to mark Archimedes' grave, and had the figure of the sphere inscribed in the cylinder carved on it, as Archimedes had wanted. Cicero, the Roman philosopher and statesman, upon being appointed governor of Sicily in 75 BCE, 137 years later, found the grave neglected and overgrown. He cleaned it up and restored the marker. But it was again neglected, to be found only in 1965 during excavation for the construction of a hotel.

Conic Sections

Recall that conic sections were originally studied by Menaechmus to solve the Delian problem, doubling the cube. From the algebraic point of view, they are a natural topic of study, after straight lines. Specifically, a straight line is the set of points solving an equation of the form

$$ax + by + c = 0.$$

If we allow second powers of the variables, we get conic sections, which are solutions of equations of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

The Greeks did not have this algebra, so they studied conic sections geometrically.

Apollonius (c. 240–174 BCE)

Apollonius was reportedly born in the Greek city of Perga, in modern-day Turkey, and lived as a young man in Pergamum. In the late 3rd century, Pergamum, also in Turkey, became a major intellectual center, modeled after Alexandria. It housed the second largest library in the Greek world.

After Pergamum, Apollonius moved to Alexandria, where he spent most of his career. He published a number of works on astronomy, geometry, and arithmetic, most of which have been lost.

Apollonius' masterwork was the *Conics*. It was the first known mathematical text to systematically and exhaustively treat a single topic—conic sections. It contains eight books and 487 propositions. The first part covers what was known to previous mathematicians, and the second part consists of original contributions.

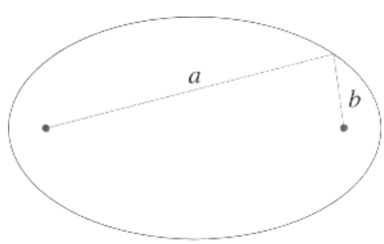


Figure 1.23 An ellipse and its foci.

Apollonius gave us the terms for the three conics: ellipse, hyperbola, and parabola. Among the many results in the *Conics*, the ellipse is shown to be the set of points the sum of whose distances from two fixed points is constant. The two points are called the *foci* (singular *focus*) of the ellipse. An ellipse and its foci are shown in Figure 1.23; the sum $a + b$ is the same for any point on the ellipse. As the two foci get closer together, the ellipse becomes less elongated, until, when the foci are the same point, the ellipse is a circle.

You can use the idea of the foci to construct an ellipse physically. Take a pad of paper, and stick two thumb tacks in where the foci are to be, as in Figure 1.24. Then take a length of string, and tie the two ends to the tacks. If you stretch the string taut, the two parts of the string will give the distances to the two tacks. The sum of these distances is the length of the string. So, if you take a pencil and trace out the curve you get by moving the pencil, always keeping the string taut, the resulting figure will be an ellipse.

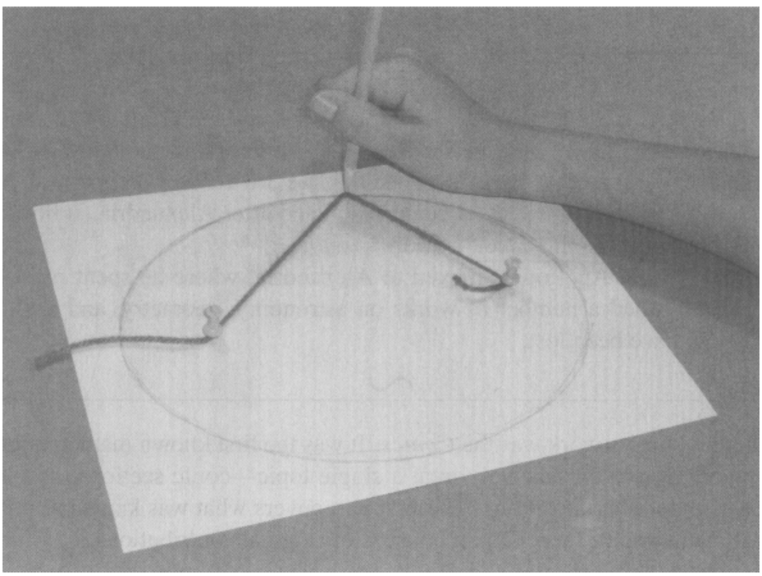


Figure 1.24 Drawing an ellipse.

Apollonius gave a similar result for the hyperbola. In that case, the *difference* of the distances to the foci is a constant. (See Figure 1.25.)

Apollonius did not have a similar result for a parabola, but his contemporary Diocles showed that the parabola is the set of points equidistant from a point (its focus) and a line (its *directrix*). See Figure 1.26.

At the heart of every large, modern telescope is a mirror in the shape of a paraboloid, a solid figure obtained by rotating a parabola. This uses an important reflection property, illustrated in Figure 1.27. In the figure, we have a cross-section of a paraboloid, i.e., a parabola, with the focus marked by a dot. All light that comes from a direction perpendicular to the directrix, after bouncing off the mirror, passes through

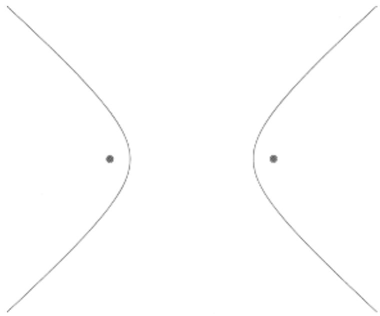


Figure 1.25 A hyperbola and its foci.

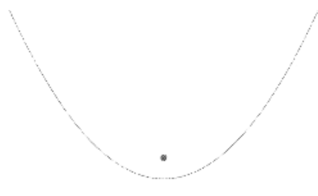


Figure 1.26 A parabola with focus and directrix.

the focus. This property is used to focus the light to form an image of distant objects. The word "focus" is from the Latin, meaning fireplace or hearth.

The *Conics* was very influential for many years. In the 17th century, Kepler drew on it when he discovered that planetary orbits are ellipses, with the Sun located at one focus.

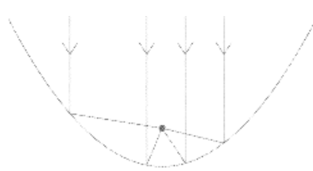


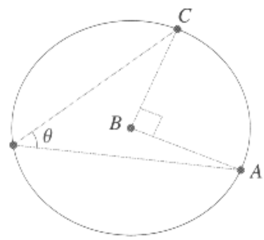
Figure 1.27 Reflection property of a parabola.

EXERCISES

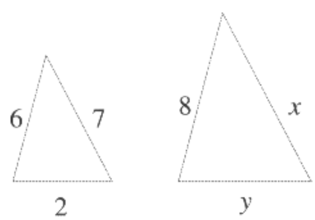
1.39 In the text, we restated Common Notion 1 in algebraic terms. Do the same for Common Notion 2.

1.40 In the text, we restated Common Notion 1 in algebraic terms. Do the same for Common Notion 3.

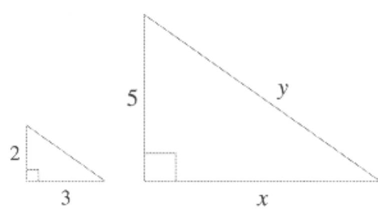
1.41 Given that ABC is a right angle, find θ.



1.42 Given that the two triangles below are similar, find x and y.



1.43 The two right triangles below are similar. Find x and y. (Hint: the Pythagorean Theorem may help.)



1.44 Euclid's Proposition II-4 is a geometric version of the binomial identity

$$(a+b)^2 = a^2 + 2ab + b^2.$$

In this problem, we consider the analogous trinomial identity

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

a) Draw a version of Figure 1.18 to represent the trinomial identity.
b) By considering areas, show that your diagram illustrates the trinomial identity.

1.45 The first proposition of the *Elements* is the construction of an equilateral triangle using straightedge and compass. Show how to carry out such a construction.

1.46 Given three points in the plane, not all on a line, show how to construct, using straightedge and compass, the circle that passes through them.

1.47 The law of the lever, proved by Archimedes, applies to playground teeter totters. Suppose that a 50 pound child sits 9 feet from the fulcrum (center) of the teeter totter. How far from the fulcrum would a 75 pound child sit in order to balance the other child?

1.48 The Earth weighs about 1.3×10^{25} pounds (13 followed by 24 zeroes). Suppose that Archimedes had a fulcrum placed 8000 miles from the Earth (about one diameter), and he can manage 200 pounds of weight on his end of the lever. How long should he be from the fulcrum to "move the world?"

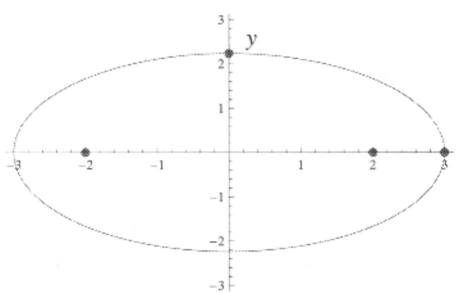
1.49 Suppose that a sphere is inscribed in a cylinder, and that we measure the surface area of the cylinder to be 27.1434 cm^2 and the volume to be 10.8573 cm^3. Using Archimedes' theorem, find the surface area and volume of the sphere.

1.50 Find the surface area and volume of a sphere of radius 2 feet.

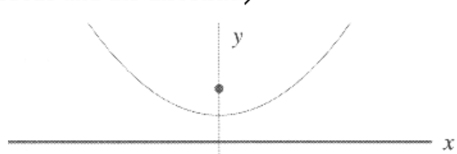
1.51 An ellipse has foci at $(2, 0)$ and $(-2, 0)$, and contains the point $(3, 0)$, as in the figure below.

a) What is the sum of the distances from $(3, 0)$ to the two foci?

b) If the point $(0, y)$ is on the ellipse, what is y?



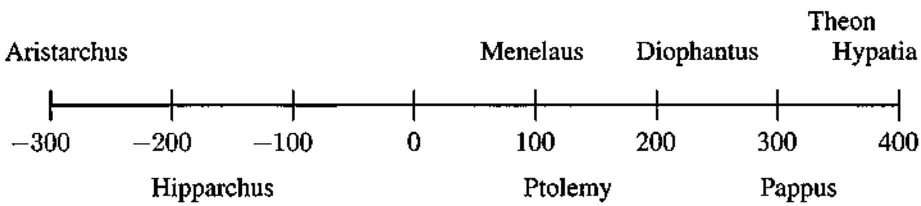
1.52 Below is a parabola with the x-axis as its directrix. Its equation is $y = \frac{1}{8}x^2 + 2$. Find the coordinates of the focus. (Hint: the point on the parabola with $x = 0$ is equidistant from the focus and the directrix.)



1.53 Conic sections can be created by folding papers. Try typing "folding conic sections" in your favorite Internet search engine to find out how to do this.

1.5 The Slow Decline

> The Greeks held the geometer in the highest honor, and, to them, no one came before mathematicians. But we Romans have established as the limit of this art, its usefulness in measuring and reckoning. The Romans have always shown more wisdom than the Greeks in all their inventions, or else improved what they took over from them, such things at least as they thought worthy of serious attention.
>
> CICERO (106–43 BCE)



Late ancient Greek mathematics.

Progress is not inevitable. Mesopotamian mathematics reached a peak in the Old Babylonian period (c. 1900–1550 BCE), then stagnated for more than a thousand years. Egyptian mathematics also stagnated after the early part of the second millennium BCE.

Ancient Greek mathematics achieved its greatest heights in the 3rd century BCE. The brilliance of this period was not matched again until the 17th century. Research did not entirely cease, however, at least for six hundred years or so. There were notable developments in trigonometry and number theory, and some very influential texts appeared in this period.

The Roman Empire

Roman citizens overthrew their Etruscan rulers and established a republic in 509 BCE. In the ensuing centuries, they gradually extended their power, to the rest of Italy, then the western Mediterranean, then the eastern Mediterranean. By the middle of the first century BCE, they ruled the entire Mediterranean, including Asia Minor and Egypt. By that time, they had also lost their republic, in which much of the power was entrusted to an elected senate, to be replaced ultimately by an imperial system. Starting with Julius Caesar (100–44 BCE), the Romans proceeded to conquer what is now France, southern Germany, and Britain.

The height of the Roman empire was in the years of the Pax Romana ("Roman peace") from 27 BCE to 180 CE. In these years, Rome was ruled by a succession of emperors. The "peace" didn't mean no wars. There were still wars of conquest, often quite brutal. The historian Tacitus quotes a barbarian chieftain from a conquered Germanic tribe: "They make a wilderness and call it peace."

The Romans were great builders, constructing a vast system of roads, some of which are still in use, and an impressive network of aqueducts to carry water to a city that eventually housed about a million people. Culturally, they were no match for Greece. They did learn much from the Greeks, but only in some areas. In particular, as indicated in the Cicero quote above, they did not value higher mathematics. Certainly, their architects and engineers mastered practical mathematics, but the Romans produced no notable theoretical mathematics.

Greek culture did not disappear in the Roman era. The Latin language dominated only the western part of the Roman empire, while Greek remained the lingua franca[1] of the eastern Mediterranean. Furthermore, some Greek centers of learning, including Alexandria, continued their intellectual tradition, if at a somewhat reduced level of achievement.

Astronomy and Trigonometry

Astronomy has long been a stimulus for mathematics. In this period, it gave rise to a major area of math—trigonometry. The term comes from the Greek, *trigon*, meaning triangle, and *metron*, measure. The subject has developed beyond that in modern times, but early trigonometry was all about measuring triangles, that is, the sides and angles in a triangle.

Trigonometry is based on similarity of triangles. Consider a right triangle (Figure 1.28), where the angle ACB is a right angle. The ratio of the lengths of two corresponding sides will be the same in any triangle with the same angles. For example, suppose that in the triangle $A'B'C'$ the angle $A'C'B'$ is a right angle, and the angle α' is the same as the angle α. Since the sum of the angles in a triangle is always the same, this means that the angles ABC and $A'B'C'$ are also equal; in other words, the two triangles are similar. Hence the ratios of corresponding sides will be equal.

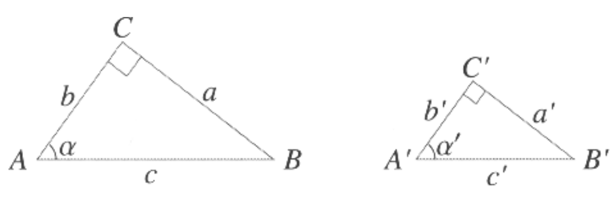


Figure 1.28 Two similar right triangles.

Consider the ratio a/c. Since the triangles are similar, $a/c = a'/c'$. In general, in any right triangle which also shares the angle α, the ratio of its corresponding sides will be the same number a/c. In trigonometry, we give that ratio a name: the sine of

[1] From the Italian, literally "Frankish language," lingua franca refers to a common language used to communicate by people with different native tongues. For example, English is now the lingua franca of the scientific community.

α, or $\sin\alpha$. Similarly, we name other ratios. The most common names are

$$\sin\alpha = \frac{a}{c}, \quad \cos\alpha = \frac{b}{c}, \quad \tan\alpha = \frac{a}{b}.$$

The symbol cos is short for cosine, and tan is short for tangent. A convenient way to remember these is to rename the sides: a is called opp, because it is opposite the angle, b is adj, for adjacent, and c is hyp, for hypotenuse.

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}}, \quad \cos\alpha = \frac{\text{adj}}{\text{hyp}}, \quad \tan\alpha = \frac{\text{opp}}{\text{adj}}$$

Why is this useful? It allows you to learn about an unknown triangle by studying a known similar triangle. Aristarchus used this idea, albeit not with the modern terms, to determine the ratio of the distances from the Earth to the Sun and Moon. First he measured the angle between the Sun and Moon when the Moon is half-full, α in Figure 1.29. (The diagram is not to scale.) He found it to be 87°. He also noted that, because the Moon is half-full, the Earth-Moon-Sun angle was a right angle. Then he used the argument above to deduce that the ratio of the distances to the Moon and the Sun, m/s, is $\cos 87°$ (m is adj, s is hyp). Finally, using various properties of triangles, he estimated that $\frac{1}{20} < \cos 87° < \frac{1}{18}$, so that $\frac{1}{20} < \frac{m}{s} < \frac{1}{18}$. Therefore, he concluded, the Sun is between 18 and 20 times as far away as the Moon.

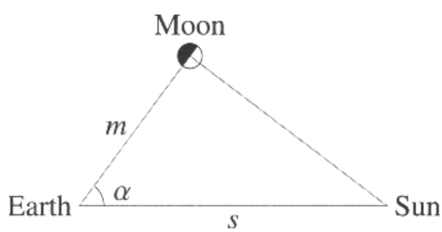


Figure 1.29 Sun and Moon distances.

Aristarchus' argument is perfect, but unfortunately his answer was off by a factor of about 20. The difficulty lay with the measurement; the angle is not 87° but about 89.8°. The small difference in angle makes a big difference in the cosine, because the angle is near 90°.

Aristarchus (c. 310–230 BCE)

Aristarchus was born on the Greek island of Samos. He probably studied and worked in Alexandria, in the early 3rd century BCE. His only surviving work is *On the Sizes and Distances of the Sun and Moon*, from which the previous argument is taken. He also estimated the actual distances, not just their ratio, and from these the sizes of the Sun and Moon.

Aristarchus is most famous for being the first astronomer known to have posited that the Earth revolves about the Sun, and not vice versa. Other scholars of his day had a scientific problem with this, namely, if the Earth moves, why is it that the stars don't appear to change during the year? His answer, that the stars are very far away, was not popular, although it has proven to be correct. Like Galileo 1900 years later, he was accused of impiety.

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The most difficult part of Aristarchus' argument, besides the measurement, was the approximation of $\cos 87°$. Of course, he could have constructed a small triangle with the proper angles, and physically measured its sides. But, being a mathematician, he preferred more mathematical methods which held at the least the promise of greater accuracy. (He was actually quite accurate: $\cos 87° \approx 1/19.1073$.)

If astronomers were to use such arguments regularly, it would be useful to have tables giving values, say, of the cosine of every angle between 1 and 89 degrees. Then the astronomer need merely look up the answer, saving a lot of time. In fact, this is what happened. In order to study the heavens, they found such tables to be a great aid. In order to make these tables, however, they had to develop the mathematics of trigonometry.

The first mathematician known to have computed trigonometric tables was Hipparchus.

Hipparchus (c. 190–120 BCE)

Hipparchus was one of the greatest astronomers in history. He was born in Nicaea, Bythnia (now Iznik, Turkey), where he made his first astronomical observations. Later on he moved to the Greek island of Rhodes in the Aegean Sea. Little else is known of his life.

Hipparchus wrote at least a dozen works, of which only one minor commentary survives. Most of what we know of his work comes from references in the texts of others, most notably Ptolemy.

~

Hipparchus' work in astronomy built on earlier work by the Babylonians and Greeks, notably Eudoxus and Apollonius. Hipparchus made careful observations, and compiled a catalog of 850 stars. He estimated the distances to the Sun and Moon.

Much of mathematical astronomy in this period was dedicated to predicting the motions of the Sun, Moon, and planets. The word "planet," which originally included the Sun and Moon, comes from the Greek *planasthai*, to wander, because planets wander against the unchanging background of the stars. It was this wandering that the astronomers wanted to model mathematically.

The details of the mathematical models used by Hipparchus are too involved to go into here, but basic to the mathematics was the understanding of triangles, both in the plane and on spheres. To assist in the computations, Hipparchus reportedly constructed a table of *chords* subtended by arcs of a circle of standard radius r. For example, the chord $\operatorname{crd}\alpha$ is the length l in Figure 1.30. Hipparchus computed approximations of $\operatorname{crd}\alpha$, for α a multiple of $7\frac{1}{2}^\circ$, up to 360°, a total of 48 ($= 360/7.5$) numbers. (He used a standard radius of 3438.)

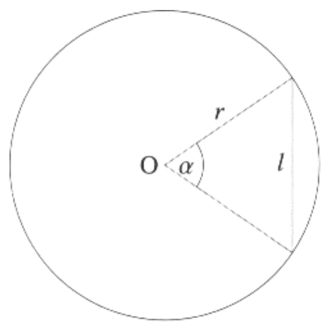


Figure 1.30 Chord subtended by arc.

What, you might ask, has this to do with trigonometry? The answer can be found in Figure 1.31, which includes the triangle from the previous diagram, cut in half. Notice that $\dfrac{l/2}{r} = \sin(\alpha/2)$. Since $\operatorname{crd}\alpha = l$, we have

$$\operatorname{crd}\alpha = 2r\sin(\alpha/2).$$

So if we know the chord l, we know the sine, and vice versa.

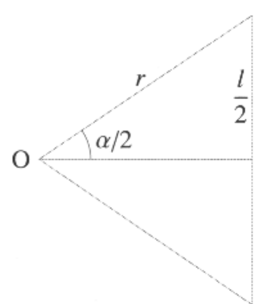


Figure 1.31 Relation of chord to sine.

The greatest astronomical discovery of Hipparchus was the precession of the equinoxes. As the Earth revolves around the Sun, it also rotates on its axis, like a spinning top. The axis of this spin always points at the north star, Polaris, or so it seems. In fact, the Earth wobbles, so the axis points in different directions, completing a circle in about 26,000 years. So, in about 12,000 years, the north star will be Vega, not Polaris. (Actually, the axis will point about 5° from Vega, but close enough.) Since this change is so slow, it is no surprise that it took many centuries of observations to notice it. The way Hipparchus spotted it was by noticing that the location of the Sun against the background stars at the Spring and Fall equinoxes had changed, hence the name precession of the equinoxes.

The next major developments in trigonometry, over two hundred years after Hipparchus, were due to Menelaus.

Menelaus (c. 70–130 CE)

Virtually nothing is known of the life of Menelaus. He worked in Alexandria and in Rome. Only one of his works, *Sphaerica*, survives, in an Arabic translation.

Menelaus also made a table of chords. Although it has been lost, it was probably more extensive than that of Hipparchus, for it was in six books.

In the *Sphaerica*, Menelaus treats trigonometry as a science separate from astronomy. Most importantly, he deals with spherical trigonometry, triangles on a sphere, proving a number of results that were important to mathematical astronomy. Astronomers were interested in spherical trigonometry because they studied the celestial sphere. The mathematics of spherical triangles is different from that of flat triangles. For example, the sum of the angles of a spherical triangle is not always 180 degrees. In fact, it is not hard to construct spherical triangles where each of the three angles is 90 degrees, so that the sum is 270 degrees. (See the exercises.)

The apex of ancient trigonometric studies, as well as ancient astronomy, was in the work of Claudius Ptolemy.

Claudius Ptolemy (c. 100–178)

Little is known of the life of Ptolemy, the greatest of the ancient astronomers. He may have been born in Egypt; he certainly worked at the Museum in Alexandria. Some of the astronomical observations he recorded can be dated to the period 124–142 in Alexandria.

Ptolemy wrote many works on science, mathematics, and astrology (yes), including two that were the standards in their fields for many centuries: the *Geography*, and his astronomical masterpiece, the *Almagest*.

Ptolemy's *Almagest* is the astronomical equivalent of Euclid's *Elements*. It was the culmination of Greek astronomy, not to be superseded until the work of Copernicus, Kepler, Galileo, and Newton. The book was originally called *Mathematiki Syntaxis—Mathematical Collection*. Later, it became known as *Megisti Syntaxis*—the greatest collection, and then in Arabic *al-magisti*, which morphed into the *Almagest*, by which it is commonly known today.

In the *Almagest*, Ptolemy gives a table of chords for all angles from one-half of a degree to 180 degrees, in intervals of one-half of a degree. The numbers, following the Greek astronomical tradition, are in the Babylonian sexagesimal notation.

In order to approximate these numbers, Ptolemy had to develop a variety of mathematical techniques. For example, using symmetry, it was easy to find the chords of 45° and 30°. Then Ptolemy demonstrated how, given chords for two angles, he could find the chord for the difference of those two angles. This allowed him to compute the chord of 15°. The formula Ptolemy discovered is equivalent to the following modern version.

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

The computations were not easy; it has been suggested that he employed (human) calculators to finish his table.

Given his table, Ptolemy could solve any triangles needed. He also applied these results to spherical trigonometry, using theorems of Menelaus and others, as well as his own. Finally, he presented a detailed mathematical model for each of the planets. The goal was to be able to predict their movements, e.g., to predict the time that Mars would rise on any given date, at any place on the Earth. In fact, he computed a number of such predictions, then compared them against actual observations to confirm the theory.

Ptolemy also produced another influential book, the *Geography*. In this work, Ptolemy listed 8000 locations, and drew a large map of the known world and twenty-six regional maps. There was a mathematical issue here: how does one represent the spherical Earth on a flat piece of paper? (Scholars were aware long before Columbus that the Earth was round. In fact, Eratosthenes had quite a good estimate of its radius in the 3rd century BCE.) Ptolemy came up with two solutions to this, two *projections* from a sphere to a plane. These were not improved upon for more than a thousand years.

Even though Ptolemy was aware of Eratosthenes' work, he used an inferior (too small) estimate of the Earth's size. Columbus used this size in the 15th century, one reason he thought he had reached Asia when he landed in the Americas.

The Silver Age of Hellenistic Mathematics

After the 3rd century BCE, the pace of mathematical research slowed. While there were advances, such as in trigonometry, much of the scholarly work was directed at preserving, not enhancing, mathematical knowledge.

There was a spurt of activity, however, from the middle of the 3rd century through the 4th century. In particular, works by Diophantus and Pappus stand out.

Diophantus (c. 210–290 CE)

About all that is known of Diophantus is that he worked in Alexandria in the 3rd century, and that he wrote the *Arithmetica*, in thirteen books.

A famous puzzle about him, written a couple hundred years after his death, asks for the number of years he lived.

> "Here lies Diophantus," the wonder behold. Through art algebraic, the stone tells how old: "God gave him his boyhood one-sixth of his life, One twelfth more as youth while whiskers grew rife; And then yet one-seventh ere marriage begun; In five years there came a bouncing new son. Alas, the dear child of master and sage after attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life."

ANTHOLOGIA PALATINA

The *Arithmetica* is in thirteen books. We have versions of six books in Greek, from a 13th century copy. In the 1970s four of the other books were discovered in Arabic translations. These books are not arranged in the step-by-step logical fashion of Euclid's *Elements*. Rather, they consist of a series of problems, 290 in the surviving books, more like the style of ancient Babylon or Egypt.

The *Arithmetica* is different from Euclid in another way; it has little geometry. The problems are about algebra and elementary number theory. This was apparently original with Diophantus. For this he has been called the "father of algebra."

Here is one of the more elementary problems.

> To divide a given number into two having a given difference.

Diophantus explained how to solve this for the case where the given number is 100 and the difference is 40. Before giving his solution, let us translate this into our modern notation. We are looking for numbers, say x and y, such that $x + y = 100$ and $y - x = 40$. From the second equation, we can solve for y, getting $y = x + 40$. Substituting this into the first equation, we get $2x + 40 = 100$. This yields $x = 30$ and $y = x + 40 = 70$.

Diophantus solved this in a similar way, starting with: if x is the smaller number, then $2x + 40 = 100$. His style was to state a general problem, then demonstrate a procedure for solving a particular case of the problem. He did not give a general solution, although his method could often be used to solve the general problem.

The problem above is called *determinate*: there are two equations in two unknowns, so there is only one solution. Most of Diophantus' problems are *indeterminate*: there are more unknowns than equations, so there might be many solutions. For example:

> To divide a given number into two squares.

The example he worked with had 16 as the given number. So the problem is to find two squares which sum to 16, that is, to solve $x^2 + y^2 = 16$. In these problems, Diophantus contented himself with finding a single solution, although again his method might serve to find others. He only accepted positive rational solutions. His methods were often ad hoc, and clever. In this case, the two numbers he found were $\frac{12}{5}$ and $\frac{16}{5}$. (Check that they work!)

One of the most important innovations in the *Arithmetica* is the use of notation. Before this, algebraic problems were described entirely in words. This was called rhetorical algebra. Diophantus invented a system of notation that he used to make the solution of a problem easier. For example, he used ς for the variable name, what we would call x, and Δ^{υ} for x^2. Thus, he would write $\Delta^{\upsilon}\beta$ for $2x^2$, since β, the second letter of the alphabet, was standard then for 2. Another example:

$$\Delta^{\upsilon}\beta\varsigma\gamma$$

means the same as our $2x^2 + 3x$, since γ is 3.

His system was not complete, so he needed to use some words still, in a style historians call syncopated algebra. Fully symbolic algebra didn't arrive for another 1300 years.

The *Arithmetica* remained influential for many years, inspiring other mathematics well into the 17th century.

The other influential work from this period is the *Synagoge*, or *Collection*, of Pappus.

Pappus (c. 290–350)

Pappus was the most important geometer since Apollonius, some five hundred years earlier. Of the life of Pappus, we know only that he worked in Alexandria in the first half of the 4th century, and he had a son named Hermodorus. We know when he worked, because a solar eclipse that he observed can be dated to October 18, 320.

In addition to mathematics, Pappus wrote on astronomy, geography, and hydrostatics, but little survives of these works.

Pappus' *Collection* is exactly that, a collection of separate books, written by Pappus, but perhaps put together by a student. The books are on different geometric topics, and of varying quality. Many of the books contain surveys of the work of his predecessors. In fact, the *Collection* is our best source for the lost works of many Greek geometers.

Book 5 concerns *isoperimetric* figures, those with different shapes but equal perimeters or surface areas. It contains a proof that, of all regular solids with the same surface area, the sphere has the largest volume. Concerning a similar problem in the plane, he has praise for honey bees, whose honeycombs are composed of hexagons.

> Bees were endowed with a certain geometrical forethought... There being, then, three figures which of themselves can fill up the space round a point, viz. the triangle, the square and the hexagon, the bees have wisely selected for their structure that which contains the most angles, suspecting indeed that it could hold more honey than either of the other two.

> Bees, then, know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each. But we, claiming a greater share in wisdom than the bees, will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always the greater, and the greatest of them all is the circle having its perimeter equal to them.

Book 7, "On the Domain of Analysis," is the most influential. It discusses the "analytic" method Greek mathematicians used to solve problems or discover proofs, as opposed to the formal "synthetic" method of proof Euclid made famous. The latter is logically more rigorous but hides the actual discovery process. Book 7 also contains a number of theorems that were important in the development of projective geometry in the 17th century.

The *Collection* has been called the requiem for ancient Greek geometry. The next time geometry of this quality was developed was in the 17th century.

The Decline of Rome and the Rise of Christianity

Greek mathematics was dead by 500 CE. One is tempted to ask why it died, just as Edward Gibbon, in his *The History of the Decline and Fall of the Roman Empire*, famously asked why the Roman empire died. Perhaps the better question is why each, the empire and the mathematics, survived for so long. What was distinctive about the Roman empire was not that it ended, but that it was so successful for so long. Similarly, Greek mathematics is characterized by its long run of brilliance. So let us ask what sustained this brilliance.

An intellectual tradition usually requires a continuity of practitioners. It is not easy to pick up and read a copy of Euclid. Students then, as now, relied on an oral tradition to introduce them to the subtleties of mathematics. Hence the tradition might not survive the absence of mathematicians for a generation or two. The rise of larger institutions in which mathematicians could work was of considerable importance. Greek examples include the Pythagorean society, Plato's Academy, and the Alexandrian Museum. Unlike now, however, the community of theoretical mathematicians in this period was never large, so it was susceptible to interruption.

The support of mathematics has, until recent times, always been rather tenuous. The cutting edge of theoretical mathematics is usually not very practical. Even though history has demonstrated many times that what begins as purely theoretical eventually develops practical applications, this connection is not always evident to those in power. Ancient mathematicians were heavily reliant on the patronage of the elite. They enjoyed this patronage in Greece before Alexander, and under the Ptolemies afterwards. As we have already seen, though, the Romans did not value mathematics highly, so their conquest of Egypt in the first century BCE was a blow to mathematics. Also notable in this conquest, in a skirmish between Julius Caesar's army and local troops in 48 BCE, the library at the Alexandrian Museum was mostly destroyed by a fire. The Roman Marc Antony donated the library of Pergamum to Cleopatra as a replacement. It was stored at the nearby temple of Serapis.

Greek mathematics arose as part of a larger philosophical movement, in an attempt to find rational explanations for natural phenomena. The religious and philosophical climate underwent major changes in Roman times, which weakened this motivation. The most important change was the rise of Christianity, which had different philosophical priorities. Its ascendance was also accompanied by religious conflicts. It took some time for Christianity to become a major force, but by the 4th century it had risen to the status of official religion. This was a further blow to the Greek mathematical tradition.

As important as these cultural concerns were social conditions, which took a turn for the worse after 180 CE, when the emperor Marcus Aurelius died. What followed was a period of civil unrest, economic decline, and plague, especially in the western part of the empire. In the period 235–284 occurred what historians call the Crisis of the Third Century. During this time, dozens of men were declared emperor by some part of the Roman army, usually to die shortly thereafter. The constant civil unrest weakened the borders, so raids from the north and east became more frequent. A plague starting in 251 decimated the population in many places.

The empire stabilized again in the early 4th century, but in a quite different form. In particular, the emperor Constantine made two momentous changes: he converted to Christianity, and he moved the capital of the empire to the city of Byzantium, later renamed Constantinople (now called Istanbul). The western empire did not recover, however, continuing to suffer invasions from the north. Rome itself was sacked in 410 by the Visigoths. The traditional end of the western empire is usually dated to 476, when the German Odovacer deposed the western emperor Romulus (who was not recognized in Constantinople).

The eastern, Greek-speaking, part of the empire fared much better, with Constantinople remaining a major cultural and political center for many centuries.

Egypt's Alexandria was not immune to the troubles of this time. Its economy suffered as trade declined. Support for mathematics also declined. In 391 a Christian mob attacked and destroyed the pagan temple of Serapis and much of its library, the one established in Cleopatra's reign.

Theon (c. 335–405) and Hypatia (c. 355–415)

Theon was a mathematician and Neoplatonist who worked in Alexandria. He published work on astronomy and mathematics, as well as astrology. His most distinguished student was his daughter Hypatia, who became a leading scholar and teacher in her own right.

Hypatia is the most famous woman mathematician of antiquity. When she was sixty years old, on her way to a lecture, she was taken from her carriage by a mob, stripped, dragged to a neighboring church, and brutally murdered. She had become involved in a nasty political struggle and was accused of sorcery.

Theon wrote a number of commentaries on earlier works. The commentary was a popular form of scholarly writing for many centuries. The commentaries might include original research, or only exposition. Theon's commentary on Ptolemy's astronomical works *Almagest* and *Handy Tables* are mainly explanatory. Although a leading scholar of his day, it does not appear that he was a very original thinker.

Theon is most famous for his edition of Euclid's *Elements*. This edition added little to the original, but replaced previous editions and became the standard for many years. In Western Europe, this was the source of all subsequent editions of Euclid until late in the 19th century.

Hypatia helped her father with his work and produced commentaries of her own. Our knowledge of her work is spotty, but it appears that she surpassed her father. She was certainly a famous teacher, offering instruction in philosophy and religious literature as well as mathematics and astronomy. Her edition of Archimedes' *Measurement of the Circle* was the source of most subsequent editions.

Work at the Alexandrian Museum continued after the time of Hypatia, but did not produce any distinguished mathematics. In general, in the Greek-speaking eastern part of the empire, scholarship declined. In 529 the emperor Justinian ordered all pagan schools closed. The Academy in Athens was taken over by the state, and folded shortly thereafter. The memory of Greek mathematics did not die, however. We shall see later that Islamic mathematicians were able to learn from scholars trained in this tradition.

The story was different in the west, which suffered catastrophic decline. The population diminished. Rome in its heyday had about one million inhabitants; by the 8th century, no western city had more than about 50,000 people. Trade also declined precipitously. This was the heart of the Dark Ages.

Mathematics was still sometimes taught, usually in schools associated with Benedictine monasteries. There monks, often from Ireland, copied and preserved Greek and Latin manuscripts. The audience for these manuscripts was not large; literacy was not widespread even among the nobility.

The monks probably did not understand much of the mathematics. The most popular mathematics text was *De Institutione Arithmetice* of Boethius (480–524), which was based on the *Introductio Arithmetica* of the first century Alexandrian scholar Nicomachus. Although Nicomachus' work was an elementary text at the time it was written, with no proofs, it was the most advanced arithmetic known in western Europe for many centuries.

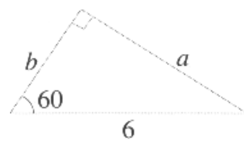
Another influential handbook, by the Italian monk Flavius Magnus Aurelius Cassiodorus (c. 480–575), justified the study of arithmetic by quoting Jesus: "the very hairs of your head are all numbered." In general, mathematical works from the time after Boethius often substituted the citing of authority for proofs. All but practical mathematics was gone from the West.

EXERCISES

The following table can be used in the exercises.

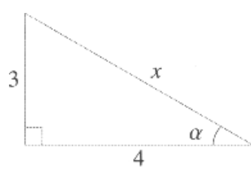
α	$\sin\alpha$	$\cos\alpha$
45°	.707	.707
60°	.866	.5

1.54 Consider the right triangle below.



a) Find lengths a and b.
b) What is $\tan 60°$?

1.55 Consider the right triangle below.



a) Find x.
b) Find $\tan\alpha$.
c) Find $\cos\alpha$.

1.56 Suppose that a right triangle has a hypotenuse of length 6, and the sine of one of its angles (other than the right angle) is .5. Find the lengths of the two legs.

1.57 If $r = 100$ and $\operatorname{crd}\alpha = 75$, what is $\sin(\alpha/2)$?

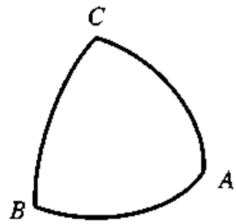
1.58 Suppose that a circle has $r = 100$. Find $\operatorname{crd} 90°$.

1.59 Find $\sin 15°$. (Hint: $15 = 60 - 45$.)

1.60 Find $\sin 75°$.

1.61 Find $\cos 15°$ and $\cos 75°$.

1.62 Suppose that you have three points on the Earth, with A and B on the equator, a quarter of the way around the Earth, and C at the North Pole. What are the angles in the triangle ABC? (Hint: the angles don't add up to 180°.)



1.63 Geometry on the sphere is different from that in the plane. Try this exercise.

> Hang your right arm down by your side. Curl your fingers into a fist, with your thumb pointing forward.
>
> Lift your arm straight out away from your body, keeping your hand as it was, and not twisting your wrist.
>
> Now move your arm so that it points forward. (Your thumb will point to the left.)
>
> Finally, lower your arm straight down again.

Is your thumb now pointing the same direction it was at the start? What does this have to do with a sphere? (Hint: a sphere is the set of points at the same distance, say arm's length, from a center, say a shoulder.)

1.64 According to the puzzle in the text, how long did Diophantus live? (Hint: the age seems to be divisible by 7 and 12.)

1.65 How might Diophantus have written $3x^2 + 2x$?

1.66 Use Diophantus' method to find the general solution of the first Diophantus problem, that is, to find x and y such that $x + y = s$ and $y - x = d$, for any sum s and difference d.

1.67 Another problem Diophantus considered is this: to find a square number between $5/4$ and 2. Find such a number. (Recall that Diophantus only allowed positive rational numbers.)