Chapter 1

Introduction

We begin by describing the philosophy behind our approach to the study of ordinary differential equations. This philosophy has its roots in the way we understand and apply differential equations; it has influenced our teaching and guided the development of this book. This chapter also contains two user's guides, one for students and one for instructors.

1.1 Guiding Philosophy

In scientific inquiry, when we are interested in understanding, describing, or predicting some complex phenomenon, we use the technique of mathematical modeling. In this approach, we describe the state of a physical, biological, economic, or other system by one or more functions of one or several variables. For example:

- The position $s(t)$ of a particle is a function of the time t.
- The temperature $T(x, y, z, t)$ in a body is a function of the position (x, y, z) and the time t.
- The populations $x(t)$ and $y(t)$ of two competing species are functions of time.
- The gravitational or electromagnetic force on an object is a function of its position.
- The money supply is a function of time.

Next, we attempt to formulate, in mathematical terms, the fundamental law governing the phenomenon. Typically, this formulation results in one or more differential equations; *i.e.*, equations involving derivatives of the functions describing the state of the system with respect to the variables they depend on. Frequently, the functions depend on only one variable, and the differential equation is called *ordinary*. To be specific, if $x(t)$ denotes the function describing the state of our system, an *ordinary differential equation* for $x(t)$ might involve $x'(t)$, $x''(t)$, higher derivatives, or other known functions of t. By contrast, in a *partial differential equation*, the functions depend on several variables.

In addition to the fundamental law, we usually describe the initial state of the system. We express this state mathematically by specifying *initial values* $x(0)$, $x'(0)$, *etc*. In this way, we arrive at an *initial value problem*—an ordinary differential equation together with initial conditions. If we can solve the initial value problem, then the solution is a function $x(t)$ that predicts the future state of the system. Using the solution, we can describe qualitative or quantitative properties of the system. At this stage, we compare the values predicted by $x(t)$ against experimental data accumulated by observing the system. If the experimental data and the function values match, we assert that our model accurately models the system. If they do not match, we go back, refine the model, and start again. Even if we are satisfied with the model, new technology, new requirements, or newly discovered features of the system may render our old model obsolete. Again, we respond by reexamining the model.

The subject of differential equations consists in large part of building, solving, and analyzing mathematical models. Many results and methods have been developed for this purpose. These results and methods fall within one or more of the following themes:

1. Existence and uniqueness of solutions,
2. Dependence of solutions on initial values,
3. Derivation of formulas for solutions,
4. Numerical calculation of solutions,
5. Graphical analysis of solutions,
6. Qualitative analysis of differential equations and their solutions.

The basic results on existence and uniqueness (theme 1) and dependence on initial values (theme 2) form the foundation of the subject of differential equations. The eminent French mathematician Jacques Hadamard referred to a problem as *well-posed* if good results are available concerning these two themes, *i.e.*, if one has existence and uniqueness of solutions and if small changes in initial values make only a small change in the solution. The derivation of formula solutions (theme 3) is a rich and important part of the subject; a variety of methods have been developed for finding formula solutions to special classes of equations. Although many equations can be solved exactly, many others cannot. However, any equation, solvable or not, can be analyzed using numerical methods (theme 4), graphical methods (theme 5), and qualitative methods (theme 6). The ability to obtain qualitative and quantitative information without the aid of an explicit formula solution is crucial. That information may suffice to analyze and describe the original phenomenon (which led to the model, which gave rise to the differential equation).

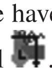
Traditionally, introductory courses in differential equations focused on methods for deriving exact solutions to special types of equations and included some simple numerical and qualitative methods. The human limitations involved in compiling numerical or graphical data were formidable obstacles to implementing more advanced qualitative or quantitative methods. Computer platforms have reduced these obstacles. Sophisticated software and mainframe computers enhanced the use of quantitative and qualitative methods in the theory and applications of differential equations. With the arrival of comprehensive mathematical software systems on personal computers, this modern approach has become accessible to students.

In this book, we use the mathematical software system MATLAB to implement this approach. We use MATLAB's symbolic, numerical, and graphical capabilities to analyze differential equations and their solutions.

Finally, engineers and scientists have to develop not only skills in analyzing problems and interpreting solutions, but also the ability to present coherent conclusions in a logical and convincing style. Students should learn how to submit solutions to the computer assignments in such a style. This is excellent preparation for professional requirements that lie ahead.

1.2 Student's Guide

The chapters of this book can be divided into three classes: general discussion of MATLAB, supplementary material on ordinary differential equations (ODE), and computer problem sets. Here is a brief description of the contents.

Chapter 2 explains how to start and run MATLAB on your computer. Chapter 3 introduces basic MATLAB commands. Unless you have previous experience with MATLAB, you should work through Chapter 3 while sitting at your computer. Then you should read Chapter 4, which contains detailed instructions for using MATLAB scripts and printing or "publishing" your work. After that, work the problems in Problem Set A to practice the skills you've learned in Chapters 3 and 4. These steps will bring you to a basic level of competence in the use of MATLAB, sufficient for the first three of the differential equations chapters (Chapters 5, 7, and 8) and for Problem Set B. Chapter 6 on more advanced aspects of MATLAB's Symbolic Math Toolbox is *optional*. It is not needed for most of the rest of the book. However, this chapter may help you to understand why the output to MATLAB's **`dsolve`** command sometimes appears the way it does. Some more advanced aspects of MATLAB, needed for some of the problems starting in Problem Set C, are discussed in Chapter 9. Chapter 10 introduces the MATLAB companion program known as Simulink, which provides a useful graphical tool for solving initial value problems. This chapter is also optional, but you may find it useful if you intend later to work with computer simulations. Since Chapter 10 is optional, we have marked the chapters, sections, and problems using Simulink with the special symbol .

Since the primary purpose of this book is to study differential equations, we have not attempted to describe all the major aspects of MATLAB, the Symbolic Math Toolbox, or Simulink. You can explore MATLAB in more depth using its demos, tutorials, and online help. We suggest starting with the links at "Getting Started with MATLAB" in the MATLAB section of the Documentation Home in the Help Browser. After that you might want to consult more comprehensive books such as the following:

1. S. Attaway, *MATLAB: A Practical Introduction to Programming and Problem Solving*, 5th edition, Butterworth-Heinemann, 2019.

2. W. Gander, *Learning MATLAB: A Problem Solving Approach*, Springer, 2015.

3. A. Gilat, *MATLAB: An Introduction with Applications*, 6th edition, Wiley, 2017.

4. B. Hahn and D. Valentine, *Essential MATLAB for Engineers and Scientists*, 6th edition, Academic Press, 2017.

5. B. Hunt, R. Lipsman, and J. Rosenberg, *A Guide to MATLAB: for Beginners and Experienced Users*, 3rd edition, Cambridge Univ. Press, 2014.

6. C. Moler, *Numerical Computing with MATLAB*, e-book at `http://www.mathworks.com/moler/index_ncm.html`, 2004.

7. C. Moler, *Experiments with MATLAB*, e-book at `http://www.mathworks.com/moler/exm.html`, 2011.

8. H. Moore, *MATLAB for Engineers*, 5th edition, Pearson, 2018.

9. R. Pratap, *Getting Started with MATLAB*, 7th edition, Oxford Univ. Press, 2017.

The eight ODE chapters (5, 7–8, 11–15) are intended to supplement the material in your text. The emphasis in this book differs from that found in a traditional ODE text. The main difference is less emphasis on the search for exact formula solutions, and greater emphasis on qualitative, graphical, and numerical analysis of the equations and their solutions. Furthermore, the commands for analyzing differential equations with MATLAB appear in these chapters.

The six computer problem sets form an integral part of the book. Solving these problems will expose you to the qualitative, graphical, and numerical features of the subject. Each set contains about twenty problems.

You can most profitably attack the problem sets if you plan to do them in two sessions. Begin by reading the problems and thinking about the issues involved. Then go to the computer and start solving the problems. If you get stuck, save your work and go on to the next problem. If there are things you'd like to discuss with your instructor, print out the relevant parts of your input and output. Talk to your instructor or your peers about anything you don't understand. This first session should be attempted well before the assignment is due. After you have reviewed your output and obtained answers to your questions, you are ready for your second session. At this point, you should fill the gaps, correct your mistakes, and polish your solution. Although you may find yourself spending extra time on the first few problems, if you read the MATLAB chapters carefully, and follow the suggestions above, you should steadily increase your level of competence in using MATLAB.

The end of this book contains a collection of Sample Solutions. In addition, on our website

`schol.math.umd.edu/ode/DewMtlb.html`

is a Glossary containing a brief summary of relevant MATLAB commands, built-in functions, and programming constructs. The Sample Solutions show how we solved several problems from this book. These samples can serve as guides when you prepare your own solutions. Emulate them. Strive to prepare coherent, organized solutions. Combine MATLAB's input, output, and graphics with your own textual commentary and analysis of the problem. Edit the final version of your solution to remove syntax errors and false starts. You will soon take pride in submitting complete, polished solutions to the problems.

1.3 Instructor's Guide

The philosophy that guided the writing of this book is explained at the beginning of this chapter. Here is a capsule summary of that philosophy. We seek:

- To guide students into a more interpretive mode of thinking.
- To use a mathematical software package to enhance students' ability to compute symbolic and numerical solutions, and to perform qualitative and graphical analysis of differential equations.
- To develop course material that reflects the current state of ODE and emphasizes the mathematical modeling of physical problems.
- To minimize the time required to learn to use the software package.

As mentioned in Section 1.2, our material consists of MATLAB discussion, ODE supplements, and computer problem sets. Here are our recommendations for integrating this material into a typical first course in differential equations for scientists or engineers.

1.3.1 MATLAB and Simulink

Our students read Chapters 2, 3, and 4, and work Problem Set A within the first week of the semester. Although Chapter 9 is not essential for Problem Set B, students often find it useful. Attention to these chapters quickly leads students to a basic level of proficiency with MATLAB. The optional Chapter 10 teaches enough about Simulink to make it possible for students to use it for many problems in Problem Sets C–F. Chapter 6 on more advanced aspects of the Symbolic Math Toolbox is also optional, but helps to explain the output from MATLAB's symbolic calculations, which can occasionally puzzle those unfamiliar with the operation of the symbolic engine.

1.3.2 ODE Chapters

These eight chapters (5, 7, 8 and 11–15) supplement the material in a traditional text. We use MATLAB to study differential equations using symbolic, numerical, graphical, and qualitative methods. We emphasize the following topics: direction fields, stability, numerical methods, comparison methods, and phase portraits. These topics are not emphasized to the same degree in traditional texts. We incorporate this new emphasis into our class discussions, devoting some class time to each chapter. Specific guidelines are difficult to prescribe, and the required time varies with each chapter, but on average we spend up to an hour per chapter in class discussion.

The structure of this book requires that numerical methods be discussed early in the course, immediately after the discussion of first order equations. The discussion of numerical methods is directed toward the use of **`ode45`**, MATLAB's primary numerical ODE solver.

1.3.3 Computer Problem Sets

There are six computer problem sets. The topics addressed in the problem sets are:

(A) Practice with MATLAB,

(B) First Order Equations,

(C) Numerical Solutions,

(D) Second Order Equations,

(E) Series Solutions and Laplace Transforms,

(F) Systems of Differential Equations.

Problem Set A is a practice set designed to acquaint students with the basic symbolic and graphical capabilities of MATLAB, and to reacclimate them to calculus. We assign all problems in Problem Set A, and have it turned in rather quickly. We generally assign 3–5 problems from each of the remaining problem sets.

In addition to analyzing problems critically, it is important that students present their analyses in coherent English and mathematics, displayed appropriately on their printouts. The simplest way to accomplish this is to get used to editing MATLAB live scripts and either to submit the scripts electronically for grading or else to export them to a PDF and print them out. An alternative is to master the "publish" feature of MATLAB scripts. These tools facilitate presentations with integrated input, output, graphics, and formatted text. Chapter 4 contains detailed instructions on live scripts and on the "publish" feature. Engineers and scientists do not just solve problems; they must also present their ideas in a cogent and convincing fashion. We expect students to do the same in our course. To encourage students to submit high-quality solutions to the homework problems, we have provided *Sample Solutions* to selected problems at the end of this book. These solutions were prepared by exporting MATLAB live scripts.

Many users of this book will also use the text **Elementary Differential Equations** by William E. Boyce, Richard C. DiPrima, and Douglas B. Meade, John Wiley & Sons, Inc., now in its eleventh edition. The references to Boyce & DiPrima in our book are all to the tenth edition. We have found that our book is easily integrated with this text, or with the text **Differential Equations: An Introduction to Modern Methods and Applications**, third edition, by James R. Brannan & William E. Boyce, John Wiley & Sons, Inc., 2015, which is also referenced several times in the text. We believe our book can likewise be conveniently integrated with any other text for a first course in differential equations for scientists or engineers. Some suggestions on how to accomplish this may be found on our web site:

`schol.math.umd.edu/ode/`

1.4 A Word About Software Versions

New releases of MATLAB appear twice a year (in spring and in fall), and are indexed by the year followed either by "a" or by "b"; this book has been prepared based on Release 2019a. In most new releases of MATLAB, the core MATLAB language has remained largely the same, except for bug fixes and additions of a few new features or commands. Changes to some other MATLAB products, such as Simulink and the Symbolic Math Toolbox, have been more dramatic, and in particular, we have found that the output from **`dsolve`** applied to nonlinear differential equations can vary quite drastically from one release to the next. Thus, depending on what version of MATLAB you are using, you may find that the output to certain commands (especially for symbolic mathematics) is not exactly what you see in this book. Nonetheless, in most cases, we believe that any differences you will notice can be explained using the basic principles of MATLAB and Simulink that you will learn from this text.