Chapter 1

Logic

There are several kinds of logic in mathematics. The one based in the construction of Truth tables is called *formal logic*. This is the logic used in computer science to design and construct the guts of your computer. And then there is Aristotle's logic. This is the logic used to make arguments in court or when arguing informally with another person. This is the logic used to prove that something is, or to prove that something is not. This is the logic used to examine combinations of any of the mathematical ideas encountered in this text. While we will examine formal logic and the logic of sets and functions, we will be most interested in Aristotle's *logic of the argument* in this chapter and throughout the rest of the text.

Oh, and there will be no need for a calculator in this book. I have made an effort to emphasize the important mathematical content in this book, not the superfluous, tedious practice of arithmetic. Arithmetic is important when you work with money, but in more challenging mathematical problems it only gets in the way. So cradle your electronic toy if you need to, but there will be almost no use for it as we do our counting.

1.1 Formal Logic

Formal logic is just a series of tables describing how the words *and, or, not* are defined. There is nothing illuminating with this approach, but it does match the operations of the inner workings of your computer. We will minimally justify the tables used here. We will just write them down and show how they agree with your use of the words in your language.

These tables define logic. Not just in English, the language that this book is being written in, but they describe logic in *every* language on earth. If you are reading a Mandarin Chinese translation of this book, then the logic presented here will still be the logic of your language. It is also the binary language in which the software in your computer is written. Take time to savor that thought. Logic as it is applied to languages and computers is universal. Logic is thus common to all forms of communication, analogue or digital.

To begin with we need to know what the logical operations are and what they operate on. *Logic operates on statements*, and ordinarily we will use the letters P, Q, and R to denote the statements that we we are working on. These statements can take on the *logical states* T (for True) and F (for False).

You already have an intuitive understanding of what it means for a statement to be True or False. You know that *The sky is blue* is True on earth, and you know that *You and I are human* is a True statement. *You have five dollars* might be True right now, but it might be False come late Friday evening. Of course *It is raining* is a False statement on a sunny day over my home, but it might be a True statement for you where you live. So let us assume that we know what T and F mean in this context.

The first logical operation that we will investigate is the operation *not*. The *not* operation takes a statement P and changes or negates its logical states. It changes T to F and F to T. Its *Truth table*, the table that lists the logical states of the *not* operation, follows.

P	not P
T	F
F	T

This is just a tabular way of defining what *not* is. Notice that according to the table, if P is T then not P is F, and if P is F then not P is T. As we said, *not* changes a statement's logical state to the complementary logical state.

EXAMPLE 1.1.1 1. If P is the statement *The sky is blue on earth*, then not P is the statement *The sky is not blue on earth.* We have negated P and changed its logical state from T to F.

2. If P is $1 + 2 = 3$ then not P is the statement $1 + 2 \neq 3$. Again the logical state of P has been changed by an application of *not* from T to F.

Because of the nature of the word *not*, two consecutive applications of the operation *not* to P will leave the logical states of P unchanged. For lingual reasons we let not not P = not(not P). In tabular form the compound operation *not not* is written as follows.

P	not P	not(not P)
T	F	T
F	T	F

Notice that if P is T then not P is F, and then not(not P) is T, giving not(not P) the logical states of P. You know this as a *double negative* from your English class.

EXAMPLE 1.1.2 1. If P is *The sky is blue on earth*, then the double negative not(not P) is the awkward sentence *It is False that the sky is not blue on earth.* Your language skills compel you to avoid the double negative and just write *The sky is blue on earth.*

2. Suppose P is *I think this is wrong.* Then not P is *I think this is not wrong*, and not(not P) is the very awkward *I don't think that this is not wrong.* You would be advised by your language teacher to avoid the double negative and just say *I think this is wrong.* The statements P and not(not P) are written with different words, but logically they express the same meaning.

Thus, by applying the logic of the operator *not* to a lingual double negative, we can avoid the double *not.*

Throughout this discussion, suppose that we are given statements P, Q. Several logical operations allow us to compare the logical states of P, Q by combining them.

For instance, we can combine statements P, Q using the *and* operation. This is the *and* that you use all of the time when you write. When applied to P, Q the *and* operation yields the statement "P and Q". This is just the compound statement formed by combing P, Q with the conjunction *and* from English.

EXAMPLE 1.1.3 1. If P is *The sky is blue on Earth* and if Q is *You are a man* then "P and Q" is the statement *The sky is blue on Earth and you are a man.*

2. If P is *This is wrong* and if Q is *These are red* then "P and Q" is *This is wrong and these are red.*

The logical states of P and Q are closely related to the way that the word *and* behaves in language. Thus the logical state of P and Q is T (True) exactly when both P and Q are T. In every other instance, "P and Q" is F (False). Put another way, if one or more of the logical states of P, Q are F (False) then the statement "P and Q" is a Falsehood, its logical value is F.

In the form of a Truth table the *and* operation is diagrammed as follows:

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

The first row states that if both P, Q have logical state T then the conjunction "P and Q" also has logical state T. Once we know that the right hand entry of the first line in the table is T then the rest of the rows follow as F.

EXAMPLE 1.1.4 1. If P is *I am a human being* and if Q is *I am sitting in my chair* then "P and Q" is T exactly when *I am a human being* is T and *I am sitting in my chair* is T. Any other combination of T's and F's for P, Q will produce a logical state F for "P and Q".

2. If P is *The sky is red over me* and if Q is *The ground is dry beneath me* then the logical value of "P and Q" is F if we are

on Earthsince the sky is not red there. If we are on Mars then the logical value of "P and Q" is T because the sky is red and the ground is dry on Mars.

Another way to combine statements is through the use of the conjunction *or*. The use of *or* in logic is denoted by the operation *or*. Thus, statements P, Q are combined to form the conjunctive statement "P or Q", which is read just like the *or* statements that you read and write.

The compound statement "P or Q" has logical state T exactly when one or more of the statements has logical state T. But it might be easier to remember how *or* behaves with False statements. When the logical states of both P and Q are F then "P or Q" has logical state F, and this is the only case in which the logical state of "P or Q" is F.

We will always use the *inclusive or* here so that the statement "P or Q" includes the case where both P, Q have logical state T. That is, we we read "P or Q" as *P, Q, or both P and Q.*

EXAMPLE 1.1.5 1. If P is *The river is wide* and if Q is *The water is cold* then "P or Q" is read as *The river is wide or the water is cold.* Since "P or Q" is T when either P, Q has logical state T, the compound statement *The river is wide or the water is cold* has logical state T if the river is wide.

2. *The river is wide or the water is cold* is T if we are talking about the Missouri River and its waters are cold. *The river is wide or the water is cold* is T if we are talking about the Missouri River and the water we are talking about is in my coffee.

3. Let P be the statement *All is nothing* and let Q be the arithmetical statement $1 + 1 = 3$. Both P and Q have logical state F, so that "P or Q" has logical state F. Since both P, Q have logical state F then "P or Q" has logical state F.

The next logical operations, called *DeMorgan's laws,* show us how the logical operations *and, or, not* combine with each other. Simply put, *DeMorgan's laws* are lingual ways of simplifying a sentence that uses *and* , *or*, and *not* is a more complex manner.

Given statements P, Q then *DeMorgan's laws* are written as

$$\begin{aligned} \text{not}(P \text{ or } Q) &= (\text{not } P) \text{ and } (\text{not } Q) \\ \text{not}(P \text{ and } Q) &= (\text{not } P) \text{ or } (\text{not } Q). \end{aligned}$$

Notice that in our use of DeMorgan's Law, the distribution of the *not* operator changes *or* to *and*, or it changes *and* to *or*. Compare this to the following lingual examples of uses of DeMorgan's laws. When read properly, you will see that the symbolism we use here is the same as our use of *and, or, not* above.

We will use parentheses to emphasize a statement's meaning, so that there is no confusion as to what word modifies what phrase.

EXAMPLE 1.1.6 1. The statement

(The river is not wide) or (the water is not cold)

is equivalent to the statement

It is not True that (The river is wide and the water is cold).

Complex to be sure, but that is the purpose behind DeMorgan's laws. It will take a complicated statement and make it easier to read.

2. The statement

(This is not a king) and (this is not a queen),

is equivalent to the statement

This is not (a king or a queen).

3. The statement

This box does not contain (a red and a yellow crayon),

is equivalent to

(This box does not contain a red crayon) or
(it does not contain a yellow crayon).

EXAMPLE 1.1.7 1. Let P be the statement that *This is a king* and let Q be the statement that *This is a queen.* The statement "not(P or Q)" is also written as

It is False that (this is a king or a queen),

while "(not P) and (not Q)" is written as

(This is not a king) and (this is not a queen).

Which do you prefer? Logically they both mean the same thing.

2. Let P be the statement that *This box contains a red crayon* and let Q be *This box contains a yellow crayon.* Then "not(P and Q)" is written as

It is False that (this box contains a red and yellow crayon),

while its equivalent formulation "(not P) or (not Q)" under De-Morgan's laws is

(This box does not contain a red crayon) or
(this box does not contain a yellow crayon).

1.2 Basic Logical Strategies

We will make exclusive use of logical arguments due to Aristotle some 500 years B.C. They are the basis for every intelligent conversation and every legal argument made since.

The first logical observation is that one statement always has a logical state of F.

The statement "P and (not P)" is a universal Falsehood.

No matter what the logical state of P is, "P and (not P)" is a Falsehood.

To see this, notice that because *not* changes logical states, at any time either P or not P is F. Thus the *and* statement "P and (not P)"

has logical state F. The Truth table for "P and (not P)" is then given as follows:

P	not P	P and (not P)
T	F	F
F	T	F

Observe that the right-hand column of the table is made up of F's. Thus, the statement "P and (not P)" is a Falsehood.

EXAMPLE 1.2.1 1. Let P be *The sky is blue.* Then *(the sky is blue) and (the sky is not blue)* is a Falsehood.

2. Let P be *This statement is True.* Then "P and (not P)" is the statement *This statement is True and this statement is not True*, and this is a Falsehood.

3. Let P be *There is a mountain.* Then "P and (not P)" is *(There is a mountain) and (there is no mountain)*, which is a Falsehood. So is *First there is a mountain, then there is no mountain, then there is.*

We continue our discussion of logical arguments. Given statements P, Q, the statement "P implies Q" is called an *implication*, and it is symbolically written as

$$P \Rightarrow Q.$$

The statement P is called the *premise of the implication* and Q is called its *conclusion.*

The logical states of $P \Rightarrow Q$ are determined by one line of explanation.

> If your argument is correct then Truth leads to Truth.

In other words, if your argument is T and if your premise P is T then your conclusion Q is T. Every other logical state of $P \Rightarrow Q$ follows from this boxed statement.

Note that line one of the following Truth table for "$P \Rightarrow Q$" is logically equivalent to the boxed statement above.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Let us fill in the remaining Truth values for this table. Let P and Q be statements and consider "$P \Rightarrow Q$". We will show how a few simple Truths about argument discovered by Aristotle can be used to fill in the Truth table for the *implication*.

EXAMPLE 1.2.2 We will continually refer to the Truth table for "$P \Rightarrow Q$".

1. Because Truth implies Truth when the argument is correct,

If your argument is correct (T), and if P is T then Q is T.

This is why line 1 is

P	Q	$P \Rightarrow Q$
T	T	T

.

2. Since Truth implies Truth when the argument is correct,

Your argument is False if P is T and Q is F.

This is why line 2 of the Truth table is

P	Q	$P \Rightarrow Q$
T	F	F

.

3. Since any argument begun with a False premise is correct, we can write

Your argument is T if P is F.

This is why lines 3 and 4 of the Truth table are

P	Q	$P \Rightarrow Q$
F	T	T
F	F	T

.

The columnn under Q is the list of all possible logical states for Q in the Truth table for "$P \Rightarrow Q$".

4. Since a False premise leads to either a True or False conclusion,

> Your conclusion is ambiguous if P is F.

This is why lines 3 and 4 of the Truth table are

P	Q	$P \Rightarrow Q$
F	T	T
F	F	T

.

The column under Q completely describes an ambiguous conclusion Q. The T's under "$P \Rightarrow Q$" result from the part 3.

Let us put this implication to work in some elementary arguments.

EXAMPLE 1.2.3 1. Here is a Greek classic. We will use Example 1.2.2(1). Begin with P : *Socrates is a man.* The conclusion will be Q : *Socrates is mortal.* The implication $P \Rightarrow Q$ is *If Socrates is a man then Socrates is mortal.* Since the implication $P \Rightarrow Q$ is correct, and since the Truth of the premise P implies the Truth of the conclusion Q, Socrates is mortal.

2. The premise is P : *I stand on dry land on earth*, and the conclusion is Q : *The sky above me is blue.* The implication is *If I stand on dry land on Earth then the sky above me is blue* is True. Since P is True, and since Truth leads to Truth, Q is True.

3. The premise is P : *Digital technology is like pockets*, and the conclusion is Q : *We have had digital technology for hundreds of years.* The implication is "$P \Rightarrow Q$" *We have had pockets for hundreds of years.* Let us assume that the premise P is True. Since Q is Falsehood, the implication "$P \Rightarrow Q$" has logical state F. But

if we assume that the premise P is False, then Q is still False, but the implication "$P \Rightarrow Q$" is True.

4. Under what conditions will P in part 3 lead us to a True conclusion Q? Have fun with this one.

1.3 The Direct Argument

This formal manipulation of statements is not exactly what we are interested in for this chapter. It is good to know that an argument has logical state T or F, but it is better to know how we can use the implication to correctly deduce a conclusion.

The first line T, T, T of the Truth table for $P \Rightarrow Q$ can be restated as *If our argument is correct then Truth leads to Truth,* or in other words, *If the premise is True and if the argument is correct then the conclusion is True.* This form of argument is called the *direct argument.* It is not new to you since you unconsciously use direct arguments in your everyday life.

EXAMPLE 1.3.1 1. The premise is P : *The sky is not blue* and the conclusion is Q : *We are not on earth.* A correct argument is

If the sky is not blue then we are not on earth.

Conclude that the conclusion Q is True.

2. Something more mathematical begins like this. The premise is $P : 1 + 1 = 2$. Argue correctly as follows:

$1 + 1 = 2$
If we add 1 to both sides of $1 + 1 = 2$ then $1 + 1 + 1 = 2 + 1$.
If $2 + 1 = 3$ then $1 + 1 + 1 = 3$.
The conclusion $Q : 1 + 1 + 1 = 3$ is then True.

A chain-like form of argument shows us the structure inherent in longer arguments called *transitive property.* These longer arguments are what people make when they logically move from one idea to the next. Basically, the *transitive property of implications* is a way to leap from two or more implications to one implication. Hence

If $P \Rightarrow Q$ and if $Q \Rightarrow R$ then $P \Rightarrow R$.

A series of implications and the transitive property provide us with a method for arguing efficiently with many implications. This series of implications is called the *transitive argument.*

> Assume the Truth of the premise P.
> Show that $P \Rightarrow Q$ is True
> Show that $Q \Rightarrow R$ is True
> Conclude the Truth of R.

To justify that this column forms an argument that we can use to deduce R from P, we will argue lingually.

Proof: Assume the Truth of P. If $P \Rightarrow Q$ is True then by the Direct Argument Q is True. If $Q \Rightarrow R$ is True then by the Direct Argument we conclude the Truth of R. Therefore, our transitive argument concludes the Truth of R from the Truth of P.

Let us review what we just argued in terms of True statements. We begin with a True statement P. The assumption is that $P \Rightarrow Q$ and $Q \Rightarrow R$ are True, which allows us to make a correct transitive argument

$$P \Rightarrow Q \text{ and } Q \Rightarrow R \text{ implies } P \Rightarrow R.$$

From the Truth of P and the Truth of $P \Rightarrow R$ we use the Direct Argument to conclude the Truth of R.

In a later section we will argue as we did above and in greater detail, thus producing three more argument forms.

EXAMPLE 1.3.2 This example shows how the above discussion can be applied to longer arguments.

a) The premise is P: $10 < 2^{10}$.
b) $P \Rightarrow Q$: Because $10 < 2^{10} = 1024$ then $11 < 2^{10}$.
c) $Q \Rightarrow R$: Because $11 < 2^{10}$ then $11 < 2 \cdot 2^{10} = 2^{11}$.
d) Conclude R: $11 < 2^{11}$.

Using this iterated form of argument people form longer and more complicated arguments, which allows them to perform more complicated intellectual tasks. These tasks could be just a way of adding numbers, or it could be the design of your computer's software, or it could be that the arguments take the arguer to intellectual places that no one had conceived before. The lesson to learn here is that, while the tabular thinking of logic is good for some tasks, there will always come a time in problem solving when we must use argument and a more enlightened form of thinking if we are to make progress on hard problems.

REMARK 1.3.3 When your computer operates it is working its way through a very long and tedious argument based on the very simple *binary logic* introduced in this section. The steps in the computer's argument are mechanical, a form of arithmetic completed by a machine. The men and women who designed this computer had to think through the *binary logic* during the implementation phase of the software.

However, for the men and women who put the larger internal logical parts of the computer together in the design phase, the problems encountered could not be solved with a simple manipulation of *binary logic*. They had to think creatively through the problems presented to them by the design phase. These solutions would often include a leap of the imagination that could not be anticipated when the design for the computer was initially proposed. The logical problems yet to come will require those leaps of the imagination before we can solve our problems.

1.4 More Argument Forms

Converse Statements

The implication $P \Rightarrow Q$ comes with what is called its *converse*.

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.

Let us write down the Truth table for $Q \Rightarrow P$ and compare it to $P \Rightarrow Q$.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$Q \Rightarrow P$
T	T	T
T	F	T
F	T	F
F	F	T

As you can see, the implication and its converse do not have the same Truth table. The logical state of the implication $P \Rightarrow Q$ in the third row is T, while the logical state of the implication $Q \Rightarrow P$ in the third row is F. Thus the converse implication $Q \Rightarrow P$ can have logical state F even when $P \Rightarrow Q$ has logical state T. For this reason, the converse cannot be used as a True statement even when the original implication is True. Hence *all* are forewarned to avoid the classic error of using the converse of an implication to advance an argument.

EXAMPLE 1.4.1 These examples show that we cannot interchange the implication with its converse. They will have different logical states.

1. Let P be the T statement *The sky is blue*, and let Q be *The world is flat.* Then "$P \Rightarrow Q$" is F.

The converse of "$P \Rightarrow Q$" is the statement "$Q \Rightarrow P$:" *If the world is flat then the sky is blue.* Since its premise Q is F, "$Q \Rightarrow P$" is T. Thus the implication is False while the converse is True, and we cannot exchange them in arguments or conversation.

2. The implication is *If today is Monday then my schedule is clear* and its converse is *If my schedule is clear then today is Monday.* The implication may be True, but the converse is False since my schedule is clear on Sunday.

Contrapositive Statements

Suppose that we consider the implication $P \Rightarrow Q$, assuming that it is T. If Q is F then the Truth table for $P \Rightarrow Q$ shows us that P is also F. Thus, a False premise Q implies a False conclusion P. This is an important implication known as the *contrapositive*.

$$\text{not } Q \Rightarrow \text{not } P.$$

When one writes out the Truth table for the implication and its contrapositive, a curious thing occurs. This Truth table reveals that the two arguments have identical Truth tables.

P	Q	$P \Rightarrow Q$	not $Q \Rightarrow$ not P
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Notice that the rightmost two columns are identical lists of T's and F's. This is completely different from what we found with the converse. The table shows that

The implication and its converse are logically equivalent.
One can be substituted for the other without loss of Truth.

In other words, the statements "$P \Rightarrow Q$" and "not $Q \Rightarrow$ not P" are both True for the same logical values of P and Q.

EXAMPLE 1.4.2 1. The implication *If the sky is not blue then this is not earth* has as contrapositive *If this is earth then the sky is blue.* The implication and its contrapositive are making the same logical statement about the sky.

2. The implication *If my GPS is working then I am not lost* has contrapositive *If I am lost then my GPS is not working.* Notice that both the implication and its contrapositive are making the same logical statement, assuming I always use my GPS.

3. The implication *If my spell-check program is running then I do not misspell all the time* has contrapositive *If I misspell all the time then my spell-check program is not running.* Notice that both the implication and its contrapositive make the same logical statement about a man who cannot spell without technological help.

Counterexamples

The next form of argument does not use T's and F's. It is strictly lingual.

Let P be a statement. A *counterexample to* P is an example that is in logical conflict with the content of P. The existence of a counterexample to P proves that P is False.

The idea behind the proof by counterexample is this. If I claim that P : *All colors are white* is True then you can disprove my claim by *producing some color that is not white.* One non-white color will do. I choose red. With the existence of the color red you have refuted my claim. You have proved that *All colors are white* is a Falsehood.

In the very same manner, we can disprove any statement that asserts that all of the X's in the world are short Y's. All we need do is find a counterexample X that is not a short Y.

The *proof by counterexample* can be summed up as follows:

> The statement *All X's have property Y* is **disproved** by a counterexample of an X that does not have property Y.

These proofs by counterexample all proceed in the same way. Producing just one X that *does not have* quality Y is enough to kill the claim that *All X's have property Y*.

EXAMPLE 1.4.3 1. The claim is P : *All integers are even.* To refute the claim you produce counterexample 3, which is not even. This counterexample refutes the claim that *all integers are even.* You have thus disproved the claim that *All integers are even.* Hence *Some integer is odd.*

2. The claim P : *All people are Truth sayers* claims that *every person tells the Truth at all times.* Your counterexample to refute the claim is the known Falsehood *"1=0"*. Having uttered a False statement, the claim P is disproved.

3. The claim P : *All people are liars* claims that *every person will lie at all times.* Your counterexample to refute the claim is

the known Truth "$1 \neq 0$". Having uttered a Truth, the claim P is disproved.

4. Your claim is P : *All statements are False.* I state that *The number line has no end.* My stated Truth is a counterexample that refutes or disproves the claim.

5. You claim P : *There are no interesting positive integers.* I argue thusly: in that case, there is a least or minimum non-interesting integer, call it x. I find it interesting that there exists such a number, and so I find x interesting. This interest in x is a counterexample to the claim.

6. Someone claims that *Nothing in this world is interesting at all.* I argue that the lack of interesting facts in this world is interesting to me. This shows that something is interesting to someone, which is a counterexample to the claim.

1.5 Proof by Contradiction

In this section we will show that a certain kind of statement is always a Falsehood. These statements are common among amateur mathematicians who do not fully understand the logical ideas that we have been examining in this chapter. We will show that these Falsehoods have similar proofs even though they do not look alike. These proofs are so similar that one proof will be used on one statement by simply replacing certain words in a previous proof. The examples below will make this clear.

We begin with a logical problem that comes from ancient Greece circa 600 B.C.

EPIMENIDES PARADOX 1.5.1 The Cretan Epimenides steps onto a stage in Athenian College and proudly speaks his three lines.

> All Cretans are liars.
> All statements made by Cretans are False.
> I am lying.

The Greek scholars proceed to determine the logical value of *I am lying* in a manner that we will read promptly. They decide that *I am lying* is neither a lie nor the Truth, an intolerable logical situation in any age. We ask for an explanation as to how *I am lying* could lack a logical state.

It is traditional to state that *All Cretans are liars* rather than *All statements made by Cretan's are False.* We will consistently use the longer version for now. To begin our modern approach to the Epimenides Paradox, we will show that the premise for the scholar's discussion is a Falsehood.

THEOREM 1.5.2 All statements made by Cretan's are False *is a Falsehood.*

Proof: This is an obvious proof of the theorem. Because the lying Cretan Epimenides speaks *All statements made by Cretans are False*, the statement is itself a lie. This completes the proof.

One might also disprove *All statements made by Cretans are False* with a counterexample. The first one that comes mind is the Truth *Crete is an island.* Yet another proof that *All statements made by Cretans are False* is False will be used as a template for subsequent proofs in this section.

REMARK 1.5.3 Let R be a statement. Any argument that begins by *assuming something* for the sake of contradiction, and that then concludes both R and its logical negation not R, has concluded a Falsehood called a *contradiction.* Because Truth leads to Truth when the argument is correct, we have proved that the something we assumed initially is a Falsehood. We will make extensive use of this form of proof called *proof by contradiction.*

A *self-referential statement* is a sentence that refers to itself in its lingual content. Statements like *This statement is True*, or *This statement is too long*, or, my favorite, *This statement is self-referential.* A statement labeled with a Q in this section is called a *Q-statement.* Q-statements are examples of *self-referential statements.*

THEOREM 1.5.4 All statements made by Cretans are False *is a Falsehood.*

Proof: Assume for the sake of contradiction that *All statements made by Cretans are False*, and consider the statement

Q: This statement when spoken by a Cretan is not False.

The content of Q states that the statement Q is not False, so we have deduced the statement *Q is not False*. By hypothesis Q, because it is spoken by Epimenides, is False, so *Q is False* is deduced. But R: *Q is False* and its logical negation not R: *Q is not False* form a contradiction. Hence, our premise *All statements made by Cretans are False* is itself a Falsehood, which completes the proof.

Returning to the ancient paradox, we proceed from the False premise *All statements made by Cretans are False.* Thus we can deduce many things, but we have no means of deciding the Truth of those deductions.

Let us quickly review how we argued above. We assumed that the statement P : *All statements made by Cretan's are False* is True. We then deduced the two statements R: *Q is False* and its logical negation not R: *Q is not False*, a contradiction. Since the conclusion is False, we deduce that our premise *All statements made by Cretans are False* is a Falsehood.

EXAMPLE 1.5.5 Let us give a logical analysis of *I am lying* in the context of the Epimenides Paradox above.

We claim that we have deduced that *I am lying* is True, but the Truth is that we cannot identify the logical state of *I am lying.* Beginning with a Falsehood the way we did makes any analysis of the logical state of *I am lying* within the Epimenides Paradox impossible. This illustrates just how badly facts can be distorted when an argument proceeds from a False premise.

That was fun. I hope you derive many hours of pleasure from thinking about the Epimenides Paradox. This logical puzzle demonstrates that if you start with a Falsehood, as we did, then you cannot decide the actual logical state of your conclusion. Now let us consider variations on Epimenides.

EXAMPLE 1.5.6 A Truth sayer is a person who says nothing but the Truth. We will show that P: *All people are Truth sayers* is a Falsehood.

Proof: The method of proof utilized here is the proof by counterexample. One counterexample is produced when I speak the Falsehood $1 = 0$, which is contrary to the statement P. Just for the fun of it, try one of your own counterexamples.

Here is a slight variation on the above example that again shows how we can use a *Q statement.*

EXAMPLE 1.5.7 Assume for the sake of contradiction that *All people are Truth sayers*, and assume that some person speaks the Q-statement

Q: I am lying.

Since we are assuming that *All people are Truth sayers*, *Q is True.* But by its content Q is a lie, or equivalently *Q is not False.* We have thus deduced *Q is True* and its logical negation, a contradiction. Hence *All people are Truth sayers* is a Falsehood.

EXAMPLE 1.5.8 P: *All statements are True* is shown to be a Falsehood by the use of the counterexample and False statement $1 = 0$.

What follows is an alternative proof that P: *All statements are True* is False, by using the indirect argument and a *Q statement.*

THEOREM 1.5.9 All statements are True *is a Falsehood.*

Proof: For the sake of contradiction assume the statement P : *All statements are True*, and consider the statement

Q: This statement is False.

Since P is True it follows that Q *is True.* But the content of Q-states that Q *is False.* We have deduced Q *is True* and its logical negation Q *is False*, a contradiction. This contradiction proves that our assumed statement P: *All statements are True* is a Falsehood.

Now let us examine several universal statements whose logical state cannot be resolved with a simple counterexample.

THEOREM 1.5.10 All opinions are valid *is a Falsehood.*

Proof: For the sake of contradiction assume *All opinions are valid,* and consider the statement

Q: This opinion is not valid.

Because Q is an opinion, our assumption asserts that Q *is valid.* But the content of Q asserts that Q *is not valid.* We have thus deduced the statement Q *is valid* and its logical negation Q *is not valid,* a contradiction. Hence *All opinions are valid* is a Falsehood. This completes the proof.

You might try to prove that *All opinions are valid* by counterexample, but I do not suggest it. Here is the problem if you try this method of attack in an argument about valid opinions.

EXAMPLE 1.5.11 Suppose you are in a debate about the logical state of *All opinions are valid.* Your correct approach would be to produce an opinion that you claim is not valid. Your debate opponents would then claim that you have produced a valid opinion.

The difficulty with this argument by counterexample is that no one knows a *precise* definition of the term *valid.* No one knows because *valid* is usually given several definitions. Some of these are *having enough vowels, having the right number of words, a professional's conclusion about a scientific argument, an irrational response to the use of opinions.* Therefore, no one knows a precise definition of *valid opinion.*

Without those definitions your debate opponents could claim that every counterexample you put forth is actually a perfectly valid opinion. Opinions claimed by your debate opponents to be *valid* might include statements like $1 = 0$, *you do not exist, and there is no universe.* Since you do not know what *valid* means, anyone arguing with you could legitimately claim that your statements are perfectly valid opinions.

The proof in Theorem 1.5.10 avoids the definition of *valid* by deducing the statement *Q is valid* and its logical negation *Q is not valid,* a contradiction. Therefore, whatever the definition of *valid* is, this contradiction proves that our premise *All opinions are valid* is a Falsehood.

The following example considers the logical state of the statement *All is known.* The statement itself has a problem, as no one has ever written a convincing explanation of what *known* means in this context. Does it mean that we know the logical states of everything, or does it mean that we know the meaning of everything? As yet, no one has answered this question in a lingually professional manner. Nor has anyone realized that this use of the word *All* creates a logical and temporal conflict. If *All is known* then when did you know what *All* means. Does it mean that *All of everything is known* or did it mean that the word *All* is known? At present no one has given a cogent explanation as to why any of these questions can be ignored.

Our next example demonstrates that we do not need to know what *All is known* means.

THEOREM 1.5.12 All is known *is a Falsehood.*

Proof: The proof we use here is exactly the proof used in the previous example where we proved that *All opinions are valid* is False. We will simply replace *opinion* with *statement* and *valid* with *known.* This is surprising physical evidence that *All opinions are valid* and *All is known* are actually the same type of Falsehood.

For the sake of contradiction assume *All is known*, and consider the statement

Q: This statement is not known.

We assumed that all is known, so we deduce Q *is known*. Moreover, the content of Q asserts that Q *is not known*. We have thus deduced the statement Q *is known* and its logical negation Q *is not known*, a contradiction. Therefore, *All is known* is a Falsehood.

The above examples illustrate a general form of statement and argument that can be used to prove that an abstract idea is actually a Falsehood. Let L be a list of *qualities of statements* that contains and that is not restricted to the values *True, known, valid, complicated, assumed, hard to understand.* Fix a quality $Y \in L$ of statements.

Let X be a set of statements that include

Q: This statement in X does not have quality Y.

Evidently, the assertion *All statements in* X *have quality* Y and its abbreviated form

All in X are Y

are logically equivalent. Let us examine the logical state of *All in* X *are* Y. Note that the proof of the following theorem depends on the introduction of a Q-statement in a manner identical to the above proofs.

THEOREM 1.5.13 *Let* X *be a set of statements that contains* Q. *Then* All in X are Y *is a Falsehood.*

Proof: By hypothesis, $Q \in X$. For the sake of contradiction assume *All in* X *are* Y. Because $Q \in X$, our assumption implies that *Q has quality Y*. Moreover, the content of Q asserts that *Q does not have quality Y*. We have thus deduced the statement *Q has quality Y* and its logical negation *Q does not have quality Y*, a contradiction. Hence *All in X are Y* is a Falsehood, which completes the proof.

A *perfect logician* is a person who knows all of logic. Let us use our methods to deduce that perfect logicians do not exist.

THEOREM 1.5.14 *There are no perfect logicians.*

Proof: For the sake of contradiction assume that there is a perfect logician, and consider the statement

Q: This statement is not known to some perfect logician.

The self-referential statement Q is a statement of logic, so we deduce the statement *Q is know to every perfect logician.* On the other hand, the content of Q states that *Q is not known to some perfect logician.* We have thus deduced *Q is known to every perfect logician* and its logical negation *Q is not known to some perfect logician,* a contradiction. Therefore, there are no perfect logicians, which completes the proof.

Let us apply our work on *perfect logicians* to a logical puzzle that some consider to be the hardest ever fashioned.

EXAMPLE 1.5.15 At the time of this writing, *The World's Hardest Logic Puzzle* has been a popular stop for those who surf the web. The puzzle begins with 200 perfect logicians on an island, 100 of them are blue eyed, and 100 of them are brown eyed. The problem is that these perfect logicians must determine their eye color through the use of logic alone. When they do, they can leave the island, but not before.

That's it. That is all that we assume in this version of the puzzle. There are some Internet versions of this puzzle that include much more detail than this version, but they and their solutions follow from our solution given below. In other words, once we solve this problem then we can solve any other version of it. Indeed, the manner in which we solved the puzzle makes any further logical investigation unnecessary.

The solution is that the problem begins by assuming that there are 200 perfect logicians, while we have proved that there are *no* perfect logicians. Thus the problem proceeds from a False premise. You can therefore deduce anything you want, but you have no way of knowing which deduction is True. Thus you might deduce using 99 theorems that 200 people leave the island Friday, or you might deduce in a few lines that 200 people leave the island instantly, or you might deduce that seven of them never leave the island.

But we cannot know the logical state of any of these deductions, because we proceed from the False premise that *There are 200 perfect logicians.*

This kind of indirect argument will appear often in the succeeding chapters. The readers should familiarize themselves with it.

One fun example of lingual self-referential behavior is the following story that in the beginning and in the end refers to itself.

A SELF-RECURRING STORY 1.5.16 There once was a girl who liked to travel from town to town, telling this story about herself. One day, while traveling in the dense forest, she entered a small village in a small clearing. She told them that she was hungry and tired, and then asked if she could exchange a telling of her story for some food and a place to sleep. But the villagers knew that only evil came from the dense forest, so they threw garbage at her, and chased her in large numbers. She was so overcome by these people that she stumbled and fell into a great blazing oven just outside the village. There she went up in a black cloud of smoke. This is always how her story ended, though, with her death in a fiery place. It seems that the myth and the miss had this end in common.

Let us end this discussion with a different version of Epimenides.

EXAMPLE 1.5.17 Epimenides, a Cretan, steps into an Athenian party and states that "All Cretans are Truth Sayers. It's a religious thing. We speak only the Truth." A young female student in the room says "Hey, Epimenides. Tell this bartender that I'm old enough to drink." Epimenides cannot tell this to the bartender since he is a Truth Sayer. Not knowing what else to do, he hangs his head and walks out of the party. We respect Epimenides because he did not contradict his first statement by telling the bartender that the lass was old enough for alcohol.

1.6 Exercises

1. Prove that *All cats are bald* is False.

2. Prove that *All birds lack feathers* is False.

3. Prove that *All people are liars* is False. Use a counterexample and a proof by contradiction.

4. Prove that *All statements are False* is False. Use a counterexample and a proof by contradiction.

5. Prove that *Left alone things do not change* is False. Use a counterexample to show that there is something out there that changes when left alone.

6. Prove that *Math is finite* is False by finding a counterexample.

7. Prove that *Nothing is known* is False. Use a counterexample and a proof by contradiction.

8. Prove that *No opinion is valid* is False. Use a proof by contradiction.

9. This is a hard one. Find the logical state of *I am lying* when it exists outside of the Epimenides Paradox.

10. Refer to # 9. See Example 1.5.17 for the definition of *Truth Sayer* If you are a Truth Sayer then can you speak *I am lying*?

11. Let P be a statement that has an unnamed logical state S. Does P have logical state S in every conversation that contains it?