1

Introduction

1.1 Assumption of Small Displacements

University courses often introduce the small displacement assumption in an implicit way, without explaining to the students that it is applicable only in special cases. Take, for example, the structural system shown in Figure 1.1. Point P is attached to the ground using two straight rods that are pin-connected both to the ground and to each other. In a first-year stress analysis course this problem would be solved using the solution procedure shown in Figure 1.2, thus yielding

$$f_1 = f_2 = \frac{1}{2}\frac{f}{\sin\bar{\theta}} = \frac{1}{2}\frac{f}{\sin 45}. \tag{1.1}$$

Note that in Equation (1.1) force f_1 is a function of angle $\bar{\theta}$, where $\bar{\theta}$ is 45°. However, with this approach, one has ignored the fact that the rods can deform (shorten) under load, thus resulting in a downward displacement of point P. This deformation, in turn changes the initial angle $\bar{\theta}$ into $\tilde{\theta}$, Figure 1.3.

In real life, the actual equilibrium of forces occurs not on the initial geometry, but on the deformed geometry. The internal forces (forces between atoms) move with the geometry (changed position of atoms) – the internal forces $\mathbf{f}_1$ and $\mathbf{f}_2$ rotate with the corresponding rods and are always parallel to their corresponding rods. The load $\mathbf{f}$ also moves with point P. Finally the state of equilibrium shown in Figure 1.4 is reached, when

$$f_1 = f_2 = \frac{1}{2}\frac{f}{\sin\tilde{\theta}} > \frac{1}{2}\frac{f}{\sin 45}. \tag{1.2}$$

Large Strain Finite Element Method: A Practical Course, First Edition. Antonio Munjiza, Esteban Rougier and Earl E. Knight.

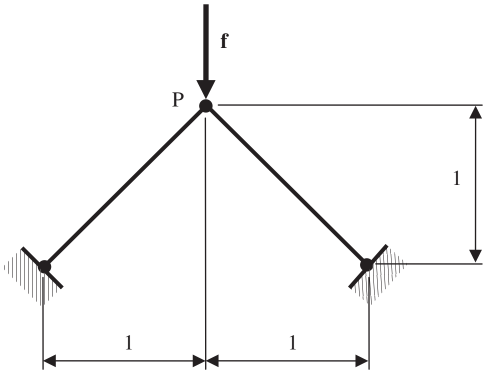

Figure 1.1 A two member truss

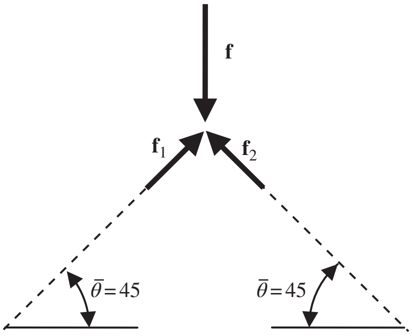

Figure 1.2 Equilibrium of forces at point P

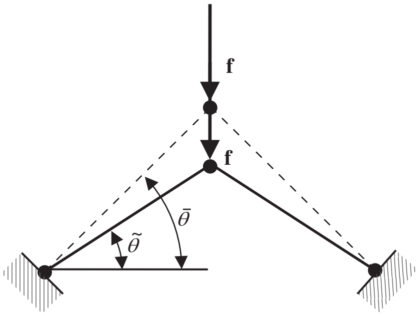

Figure 1.3 The initial (dashed lines) and the current (solid lines) geometries

In other words, the internal forces obtained using Equation (1.1) are wrong and the correct internal forces are obtained using Equation (1.2). The problem with Equation (1.2) is that the geometry of the system is a function of the internal forces, which in turn are a function of the geometry. This yields an implicit equilibrium formulation

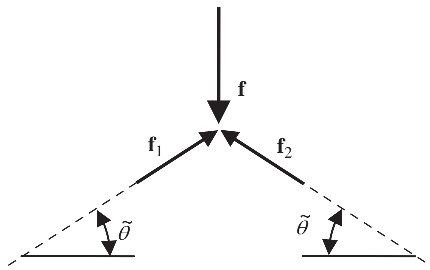

Figure 1.4 Equilibrium of the deformed system

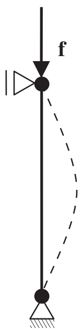

Figure 1.5 Instability of the equilibrium (buckling of struts)

$$\begin{aligned} f_1 &= \frac{1}{2}\frac{f}{\sin\tilde{\theta}(f_1)} \\ f_2 &= \frac{1}{2}\frac{f}{\sin\tilde{\theta}(f_2)}. \end{aligned} \tag{1.3}$$

The formulation shown in Equation (1.3) is obviously nonlinear. As such, it can be difficult to resolve. For this reason, the formulation given by Equation (1.1) is often used instead, which is fine only when:

a. the initial and the deformed geometry are nearly identical – this case occurs only when the displacements are infinitesimally small or tolerably small in practical applications.
b. the displacements do not change progressively with the applied load – infinitesimally small perturbations (inaccuracies) in the geometry will not lead to disproportionally large changes in the internal forces, Figure 1.5.

The situation shown in Figure 1.5 is called instability of the equilibrium. Buckling of struts is only one example of an unstable equilibrium. There exists an entire field of applied science dedicated to the analysis of structural stability. It is often formulated in terms of modal analysis, which students usually find difficult to understand, although the concept is relatively simple:

> *For a given load, there may exist a particular deformed shape in which the structure has in a sense "escaped" from under the load: for example, in Figure* 1.5 *the load has stayed vertical, but the strut has moved (escaped) sideways and, as such, it does not support the load any longer.*

In order to solve the above problems, the second order formulation was developed. It is not a general formulation, but rather a patchwork of application-specific formulations that address either the problem of large displacements or structural stability. Some classical examples are the deformation of a rope, the deformation of membranes, and the deformation of slender structures used in civil engineering, aerospace engineering and naval architecture.

As an alternative, the theoretically exact generalized large displacements approach was introduced in the late 1990s and early years of the 21st century and it has gained significant popularity. The idea is relatively simple:

Always consider equilibrium using the deformed geometry of the solid.

The resulting formulation is called the large displacement formulation, for it represents the exact equilibrium of internal and external forces regardless of the size of the displacements. As such, it captures: (a) equilibrium of systems with small displacements, (b) any instability of equilibrium, and (c) equilibrium of systems with large displacements. In contrast to the 2nd order theory, this is the exact theoretical formulation. It is, by default, nonlinear.

1.2 Assumption of Small Strains

In order to simplify how one solves solid deformation problems, the assumption of small strains through the engineering strain is often introduced. Many times this approach is utilized without any thought of explaining that it is only valid if the strains are infinitesimally small. For this purpose, the strain is often defined as engineering strain

$$\varepsilon = \frac{\Delta L}{\bar{L}} = \frac{\tilde{L} - \bar{L}}{\bar{L}}, \tag{1.4}$$

where ΔL is the elongation of a rod of initial length $\bar{L}$ and deformed length $\tilde{L}$. The assumption of small strains is only valid in exceptional circumstances such as deformation of glass at room temperature and similar materials.

When it comes to plastics, rubber, metals, clay, gels, granular materials, glass fibers, carbon fibers, biological tissues, mechanics of cells (such as red blood cells), bitumen, kerogen, and many other materials of modern technology, modern industry, modern science and modern engineering, the assumption of small strains is simply not valid.

In order to rectify the problem for specific applications, various second order formulations have been developed. These parallel the second order formulations for large displacements and are in general applicable only to a specific narrowly defined problem.

In contrast, in this book the theoretically exact generalized large strain formulation is explained.

1.3 Geometric Nonlinearity

The large strain formulation combines naturally with the large displacement formulation. The result is a formulation that reproduces a theoretically exact solution (as opposed to the second order formulation) for both large displacements and large strains.

As such, it addresses (in an exact manner) geometric nonlinearity. Geometric nonlinearity by definition includes nonlinear aspects of deformation that arise from large displacements and/or large strains.

The theoretically exact formulation is based on the multiplicative decomposition of deformation. The concept is relatively simple:

Write the current coordinates of the material points of the solid as

$$\tilde{x} = \tilde{x}(\xi, \eta, \zeta), \tag{1.5}$$

where (ξ, η, ζ) somehow uniquely define a given material point and as such do not change with deformation. Now, Equation (1.5) can be written as

$$\tilde{x} = \tilde{x}_S(\tilde{x}_R(\tilde{x}_T(\xi, \eta, \zeta))), \tag{1.6}$$

where $\tilde{x}_T$ represents the material points' translation. It is followed by the rotation $\tilde{x}_R$ and the stretch $\tilde{x}_S$. In other words, the function $\tilde{x}$ is a composition of three functions

$$\tilde{x} = \tilde{x}_S \circ \tilde{x}_R \circ \tilde{x}_T, \tag{1.7}$$

where $\tilde{x}_S$ stretches the solid, $\tilde{x}_R$ rotates the solid and $\tilde{x}_T$ translates the solid. It is like one person first comes and translates material point P. The second person comes and rotates the solid. Finally the third person stretches the solid. Only the third stage causes internal forces in the material and can, for example, break the material.

The function $\tilde{x}$ describes the deformation of the solid body and is therefore called the deformation function or simply, deformation. The deformation $\tilde{x}$ is made from the composition of translation, rotation and stretch in any order. Translation and rotation move the solid as though it was rigid. As such, they do not stretch the solid. In contrast, stretch changes the shape (the geometry) of the solid. In an infinitesimally close vicinity of a given material point P, all these functions are de-facto linear functions of coordinates x, y and z, as shown in Figure 1.6.

This leads to the multiplicative decomposition of rotation and stretch. First, the translation is removed from the deformation and what is left is decomposed into a product of stretch and rotation. In addition, stretch is expressed as a product of different types of stretches, such as volumetric stretch, shear stretch, elastic stretch, plastic stretch, etc.

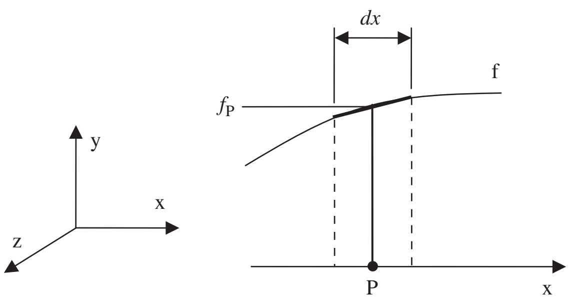

Figure 1.6 Linearity of deformation in the infinitesimal vicinity of a given point P: Note that for infinitesimally small dx any function f(x) reduces to $f(x) = f_P + \alpha x$

1.4 Stretches

In large strain large displacements deformation, it is convenient to formulate the problem not in terms of strains, but in terms of stretches. The reasons for this are as follows:

a. Stretch is well represented by a second order tensor.
b. Stretch is easily calculated from the deformation function.
c. Stretch is easily separated from rotation.
d. Stretch can be further decomposed into an elastic part, a plastic part, a volumetric part, etc.
e. Multiplicative decomposition of stretches comes naturally.
f. Any type of strains (strain measures) can be calculated from the stretches.
g. Stretches are applicable to nonlinear material formulations including nonlinear anisotropic materials.

1.5 Some Examples of Large Displacement Large Strain Finite Element Formulation

Biological Tissue. In Figure 1.7 a 3D finite element based simulation of blood plasma containing red blood cells is shown. It is evident that individual red blood cells stretch significantly and consequently their shape is changing.

Membrane Structures. In Figure 1.8 a 2.5D finite element simulation of a membrane structure subject to an initial velocity is shown (such as a flag on a mast). In this case, the material of the structure does not undergo large strains, i.e., it does not stretch a lot. However, despite this, the displacements are extremely large. This problem can be categorized as a large displacement, small strain type of problem. Nevertheless, the simulation results shown are obtained by using the large displacement, large strains finite element formulation.

2D Solid Structures. In Figure 1.9 a problem similar to a rubber cylinder hitting the ground is shown. This is a 2D solid problem consisting of large displacements and large strains. The results shown are obtained using the finite element formulation described in detail in this book.

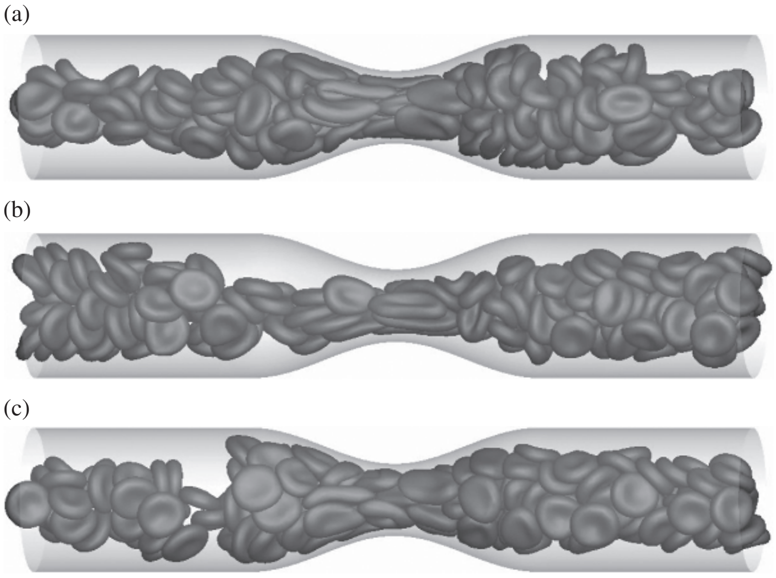

Figure 1.7 The flow of red blood cells accompanied by significant stretching, large translations and large rotations, i.e. complete geometric nonlinearity

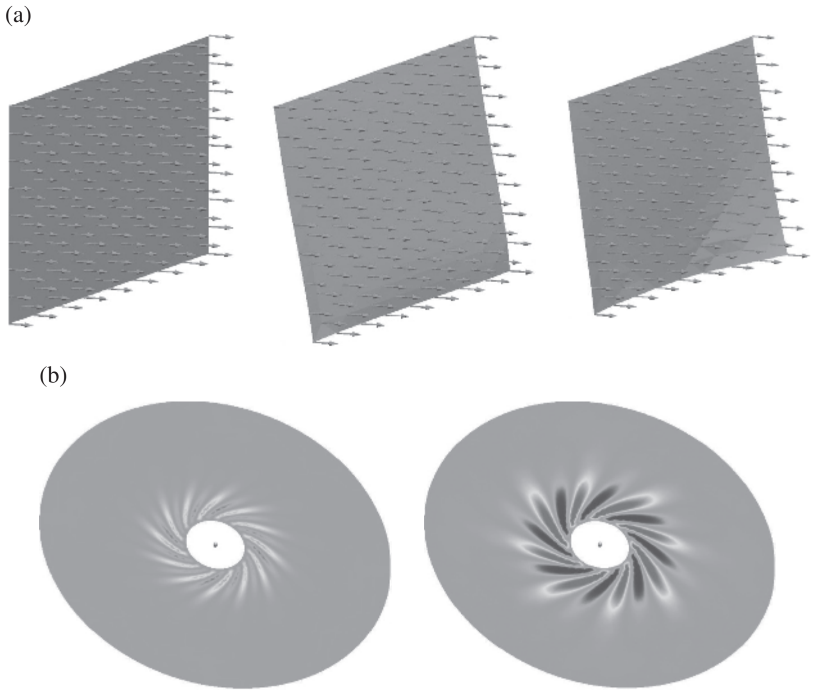

Figure 1.8 (a) A deformation sequence of a flag-like membrane subject to large displacements but small strains; (b) results of geometrically nonlinear analysis of a circular membrane

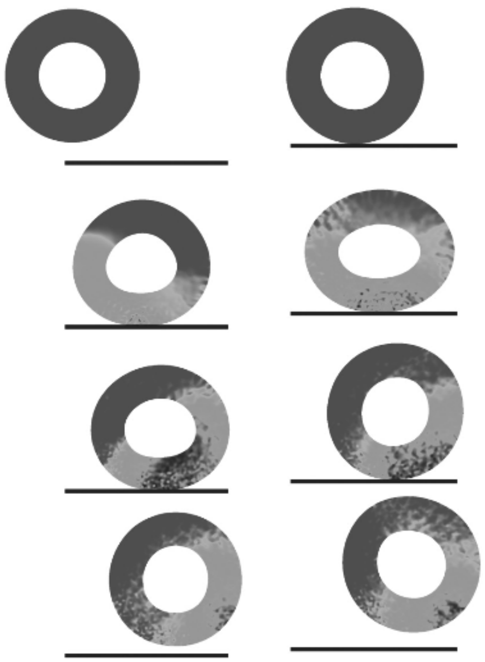

Figure 1.9 A sequence of deformation of a 2D rubber cylinder hitting the ground

Large Displacements Large Strains Shells. Shells are usually termed as 2.5D problems. The finite element formulation for shells, even for small strains and small displacements, is complex. In this book it is demonstrated that the large strain large displacement formulation can be relatively simple to understand, to implement and to use for shells. A typical dynamics problem using shells is shown in Figure 1.10.

Nonlinear Materials. This book covers geometric (large strains large displacements) nonlinearities in combination with arbitrary anisotropic material nonlinearity. As such, the solid can undergo plastic deformation or damage based nonlinearity leading to localized failure and fracture. For example, in Figure 1.11, the fracture of a glass panel is obtained using the 2.5D large displacement, large strain, nonlinear material based finite element formulation.

In a similar way, the damage based failure sequence of a 2.5D shell is shown in Figure 1.12.

3D Solids. In Figure 1.13 a problem similar to a tennis ball hitting a circular plate (shell) is shown. This is a 3D solid problem consisting of large displacements and large strains. The results shown are obtained using the finite element formulation described in detail in this book. A cross-sectional view of the same problem is shown in Figure 1.14.

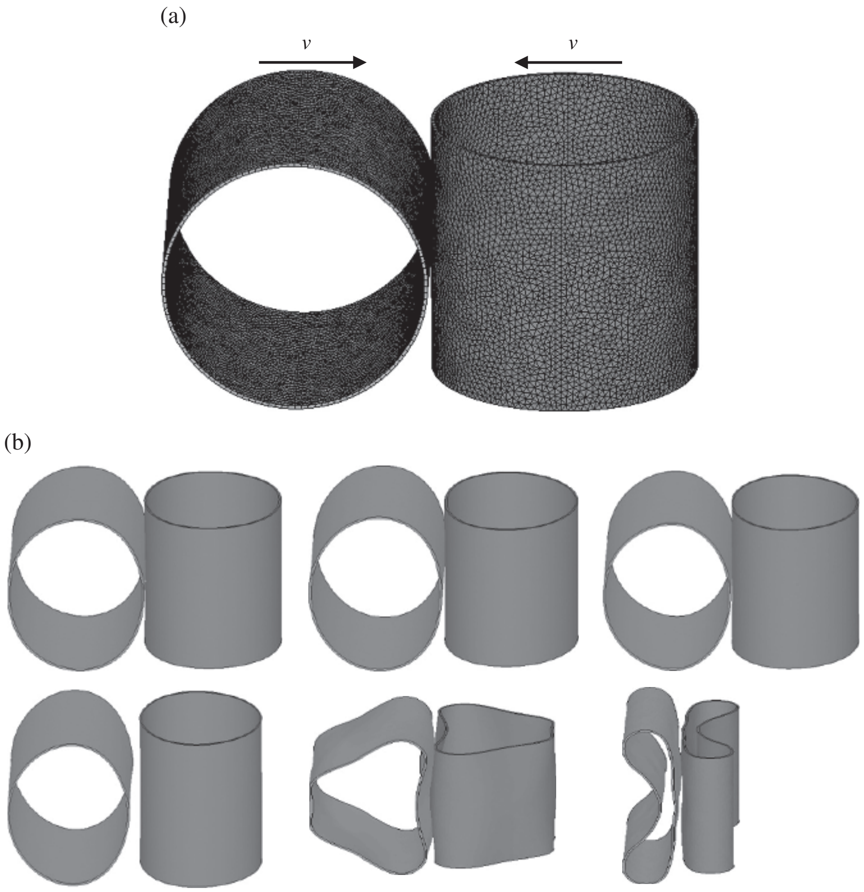

Figure 1.10 (a) Impact of two shells. (b) Deformation sequence obtained using the shell formulation described in detail in this book

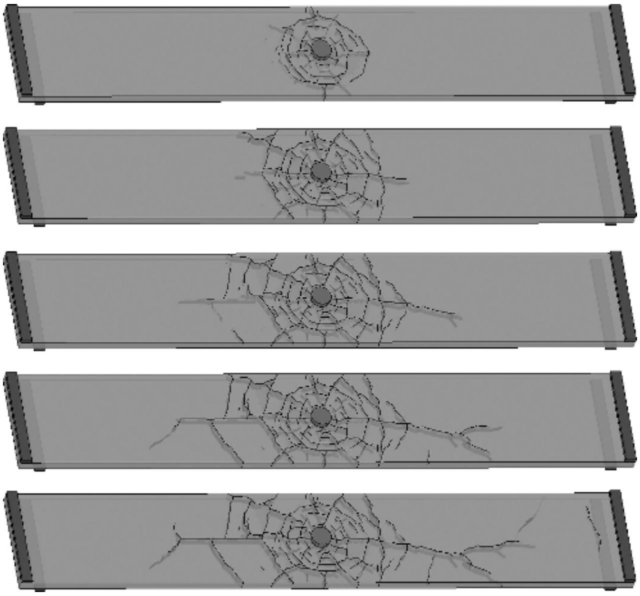

Figure 1.11 Damage based nonlinear material finite element simulation showing fracture of a window screen

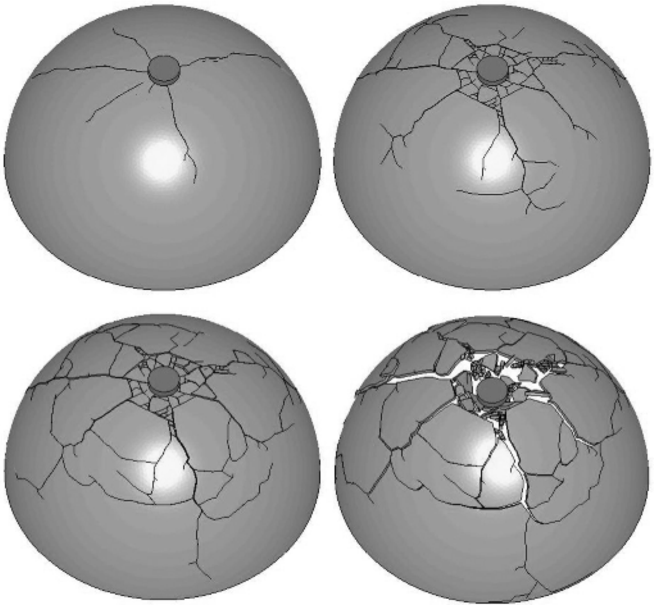

Figure 1.12 Damage based fracture of a spherical shell subject to a penny shaped impactor

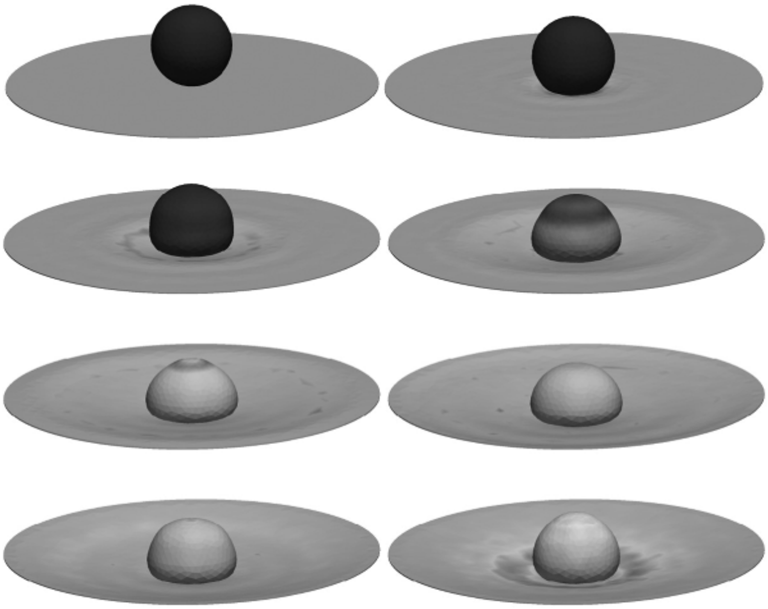

Figure 1.13 The deformation sequence for a tennis ball (3D solid) hitting a circular plate (2.5D shell). Both the shell and the 3D solid deform considerably (large displacements combined with large strains)

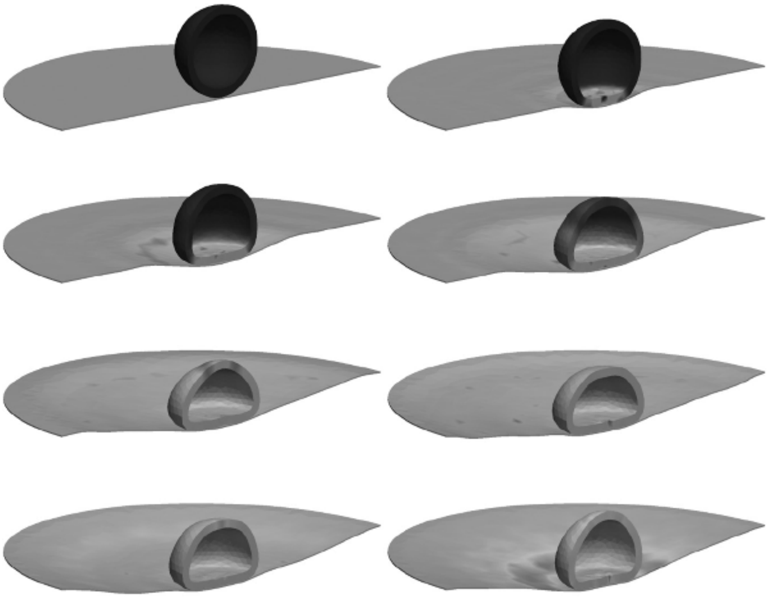

Figure 1.14 A sequence of deformation (a cross section view) showing a tennis ball hitting a circular plate (shell)

1.6 The Scope and Layout of the Book

In this book, the theoretically exact formulation for large strain large displacements based simulations using multiplicative decomposition is presented. This includes: 2D finite element method, 3D finite element method, 2.5D finite element method (plates, shells and membranes), static finite element analysis, transient dynamics finite element analysis, as well as a generalized nonlinear anisotropic material formulation including hyper-elastic, hypo-elastic and unified constitutive formulation. The book is written in such a manner that it is self-contained in the sense that a reader is not required to have any previous knowledge of the subject.

In the first part of the book some essential mathematical tools are covered including a hands-on approach to matrix algebra.

In the second part of the book some necessary physics concepts are introduced including vectors, first order tensors and second order tensors. These are deliberately separated from the first part of the book in order to emphasize that they are not mathematical constructs, but physical realities. As a consequence, relatively easy-to-master approaches for tensorial calculus in 2D, 3D, 4D and *n*D spaces are presented with an aim of familiarizing the reader with the modern notion of tensorial algebra, thus demystifying tensorial calculus itself.

In the third part of the book deformation in 1D is explained (mainly for didactical reasons) in order to help the reader to grasp the subject. Also in this part, 1D deformation kinematics is extended to 2D. This is naturally followed by an extension to 3D. At the end of the third part a unified approach to formulating constitutive laws for general anisotropic materials subject to large displacements is presented.

In the fourth, fifth and sixth parts of the book deformation kinematics is presented using the finite element method in 2D, 3D and 2.5D respectively. In this manner, the finite element formulation is completely separated from the stress calculation (constitutive law), thus enabling material modelers to work completely independent from any finite element developers. This is especially important in modern industry wherein finite element packages are often off-the-shelf commercial packages, while the material models employed may be proprietary and developed completely independently from the finite element package itself.

Internal forces over finite elements are represented using stress tensors (obtained from the material package that uses constitutive laws developed by the material modelers). From these, the equivalent nodal forces are calculated using stress integration over either volume or boundaries of finite elements. These procedures are explained in Chapters 21, 25 and 26.

In this process, a special role is also given to selective stretch sampling, which is in essence a generalization of both reduced and selective integration, Chapter 22.

The resulting nonlinear algebraic equations are solved using suitable algebraic equation solvers. This explicit iterative approach to solving the equations is generally preferred. Some of these are explained in Chapter 3, including the dynamic relaxation and conjugate directions methods.

1.7 Summary

In this chapter, an introduction to geometric and material nonlinearities in general has been provided. This was followed by the scope and layout of the book.

Finally, examples obtained using the finite element formulation described in this book have been shown. The main aim of these examples is to not only make the reader interested in the subject but also to illustrate the power of the large strain large displacement finite element method and its inherent advantages:

a. There are no restraints to the size of displacements.
b. There are no restraints to the size of the strains.

c. Finite element-independent material formulations consisting of anisotropy by default, plasticity, viscosity, etc. are implemented in a format friendly to a material model developer who has no experience with finite elements.

Further Reading

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