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Propagation of Waves in Ducts

Exhaust noise of internal combustion engines is known to be the biggest pollutant of the present-day urban environment. Fortunately, however, this noise can be reduced sufficiently (to the level of the noise from other automotive sources, or even lower) by means of a well-designed muffler (also called a silencer). Mufflers are conventionally classified as dissipative or reflective, depending on whether the acoustic energy is dissipated into heat or is reflected back by area discontinuities.

However, no practical muffler or silencer is completely reactive or completely dissipative. Every muffler contains some elements with impedance mismatch and some with acoustic dissipation. In fact, combination mufflers are getting increasingly popular with designers.

Dissipative mufflers consist of ducts lined on the inside with an acoustically absorptive material. When used on an engine, such mufflers lose their performance with time because the acoustic lining gets clogged with unburnt carbon particles or undergoes thermal cracking. Recently, however, better fibrous materials such as sintered metal composites have been developed that resist clogging and thermal cracking and are not so costly. Besides, long strand unglued glass fibers can stand high temperatures. Nevertheless, no such problems are encountered in ventilation ducts, which conduct clean and cool air. The fan noise that would propagate through these ducts can well be reduced during propagation if the walls of the conducting duct are acoustically treated. For these reasons the use of dissipative mufflers is much more common in air-conditioning systems.

Reflective mufflers, being nondissipative, are also called reactive mufflers. A reflective muffler consists of a number of tubular elements of different transverse dimensions joined together so as to cause, at every junction, impedance mismatch and hence reflection of a substantial part of the incident acoustic energy back to the source. Most of the mufflers currently used on internal combustion engines, where the exhaust mass flux varies strongly, though periodically, with time, are of the reflective or reactive type. In fact, even the muffler of an air-conditioning system is generally provided with a couple of reflective elements at one or both ends of the acoustically dissipative duct.

Clearly, a tube or pipe or duct is the most basic and essential element of either type of muffler. A study of the propagation of waves in ducts is therefore central to the analysis of a muffler for its acoustic performance (transmission characteristics). This chapter is devoted to

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the derivation and solution of equations for plane waves and three-dimensional waves along rectangular ducts, circular tubes and elliptical shells without and with mean flow, without and with viscous friction, with rigid unlined walls and compliant or acoustically lined walls. We start with the simplest case and move gradually to the more general and involved cases.

1.1 Plane Waves in an Inviscid Stationary Medium

In the ideal case of a rigid-walled tube with sufficiently small cross dimensions* filled with a stationary ideal (nonviscous) fluid, small-amplitude waves travel as plane waves. The acoustic pressure perturbation (on the ambient static pressure) p and particle velocity u at all points of a cross-section are the same. The wave front or phase surface, defined as a surface at all points of which p and u have the same amplitude and phase, is a plane normal to the direction of wave propagation, which in the case of a tube is the longitudinal axis.

The basic linearized equations for the case are:

Mass continuity

$$\rho_o \frac{\partial u}{\partial z} + \frac{\partial \rho}{\partial t} = 0; \tag{1.1}$$

Dynamical equilibrium

$$\rho_o \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0; \tag{1.2}$$

Energy equation (isentropicity)

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma(p_o + p)}{\rho_o + \rho} \cong \frac{\gamma p_o}{\rho_o} = c_o^2 (say) \tag{1.3}$$

where

z is the axial or longitudinal coordinate,
p and ρ are acoustic perturbations on pressure and density,
p_o and ρ_o are ambient pressure and density of the medium,
s is the entropy,
$p/p_o \ll 1, \quad \rho/\rho_o \ll 1$

Equation 1.3 implies that

$$d\rho = \frac{dp}{c_o^2}; \quad \frac{\partial \rho}{\partial t} = \frac{1}{c_o^2}\frac{\partial p}{\partial t}; \quad \frac{\partial \rho}{\partial z} = \frac{1}{c_o^2}\frac{\partial p}{\partial z}. \tag{1.4}$$

The equation of dynamical equilibrium is also referred to as momentum balance equation, or simply, momentum equation. Similarly, the equation for mass continuity is commonly called continuity equation.

*The specific limits on the cross dimensions as a function of wave length are given in the next section.

Substituting Equation 1.4 in Equation 1.1 and eliminating u from Equations 1.1 and 1.2 by differentiating the first with respect to (w.r.t.) t, the second with respect to z, and subtracting, yields

$$\left[\frac{\partial^2}{\partial t^2} - c_o^2 \frac{\partial^2}{\partial z^2}\right] p = 0. \tag{1.5}$$

This linear, one-dimensional (that is, involving one space coordinate), homogeneous partial differential equation with constant coefficients (c_o is independent of z and t) admits a general solution:

$$p(z, t) = C_1 f(z - c_o t) + C_2 g(z + c_o t). \tag{1.6}$$

If the time dependence is assumed to be of the exponential form $e^{j\omega t}$, then the solution (1.6) becomes

$$p(z, t) = C_1 e^{j\omega\left(t - z/c_o\right)} + C_2 e^{j\omega\left(t + z/c_o\right)}. \tag{1.7}$$

The first part of this solution equals C_1 $at\, z = t = 0$ and also at $z = c_o t$. Therefore, it represents a progressive wave moving forward unattenuated and unaugmented with a velocity c_o. Similarly, it can be readily observed that the second part of the solution represents a progressive wave moving in the opposite direction with the same velocity, $c_{o.}$

Thus, c_o is the velocity of wave propagation, Equation 1.5 is a wave equation, and solution (1.7) represents a standing wave defined as superposition of two progressive waves with amplitudes C_1 and C_2 moving in opposite directions.

Equation 1.5 is called the classical one-dimensional wave equation, and the velocity of wave propagation c_o is also called phase velocity or sound speed. As acoustic pressure p is linearly related to particle velocity u or, for that matter, velocity potential ϕ defined by the relations

$$u = \frac{\partial \phi}{\partial z}; \quad p = -\rho_o \frac{\partial \phi}{\partial t}, \tag{1.8}$$

the dependent variable in Equation 1.5 could as well be u or ϕ. In view of this generality, the wave character of Equation 1.5 lies in the differential operator

$$L \equiv \frac{\partial^2}{\partial t^2} - c_o^2 \frac{\partial^2}{\partial z^2}, \tag{1.9}$$

which is called the classical one-dimensional wave operator.

Upon factorizing this wave operator as

$$\frac{\partial^2}{\partial t^2} - c_o^2 \frac{\partial^2}{\partial z^2} = \left(\frac{\partial}{\partial t} + c_o \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} - c_o \frac{\partial}{\partial z}\right), \tag{1.10}$$

one may realize that the forward-moving wave [the first part of solution (1.6) or (1.7)] is the solution of the equation

$$\frac{\partial p}{\partial t} + c_o \frac{\partial p}{\partial z} = 0, \tag{1.11}$$

and the backward-moving wave [the second part of solution (1.6) or (1.7)] is the solution of the equation

$$\frac{\partial p}{\partial t} - c_o \frac{\partial p}{\partial z} = 0, \tag{1.12}$$

Equation 1.7 can be rearranged as

$$p(z,t) = \left[C_1 e^{-jkz} + C_2 e^{+jkz}\right] e^{j\omega t}, \tag{1.13}$$

where

$k = \omega/c_o = 2\pi/\lambda$,
k is called the wave number or propagation constant, and λ is the wavelength.

As particle velocity u also satisfies the same wave equation, one can write

$$u(z,t) = \left[C_3 e^{-jkz} + C_4 e^{+jkz}\right] e^{j\omega t}. \tag{1.14}$$

Substituting Equations 1.13 and 1.14 in the dynamical equilibrium equation (1.2) yields

$$C_3 = C_1/\rho_o c_o, \quad C_4 = -C_2/\rho_o c_o\,,$$

and therefore

$$u(z,t) = \frac{1}{Z_o}\left(C_1 e^{-jkz} - C_2 e^{+jkz}\right) e^{j\omega t}, \tag{1.15}$$

where $Z_o = \rho_o c_o$ is the characteristic impedance of the medium, defined as the ratio of the acoustic pressure and particle velocity of a plane progressive wave.

For a plane wave moving along a tube, one could also define a volume velocity v_v (= Su) and mass velocity

$$v = \rho_o S u, \tag{1.16}$$

where S is the area of cross-section of the tube. The corresponding values of characteristic impedance (defined now as the ratio of the acoustic pressure and the said velocity of a plane progressive wave) would then be as follows:

$$\text{For particle velocity u, characteristic impedance} = p/u = \rho_o c_o; \tag{1.17a}$$

$$\text{For volume velocity } v_v\text{, characteristic impedance} = p/v_v = \rho_o c_o/S; \tag{1.17b}$$

$$\text{For mass velocity } v\text{, characteristic impedance} = p/v = c_o/S. \tag{1.17c}$$

For the latter two cases, the characteristic impedance involves the tube area S. As it is not a property of the medium alone, it would be more appropriate to call it characteristic impedance of the tube. For tubes conducting hot exhaust gases, it is more appropriate to deal with acoustic mass velocity v. The corresponding characteristic impedance is denoted in these pages by the symbol Y for convenience:

$$Y_o = c_o/S. \tag{1.18}$$

Equations 1.15, 1.16 and 1.18 yield the following expression for acoustic mass velocity:

$$v(z,t) = \frac{1}{Y_o}\left(C_1 e^{-jkz} - C_2 e^{+jkz}\right)e^{j\omega t} \tag{1.19}$$

Subscript 0 with Y and k indicates nonviscous conditions. Constants C_1 and C_2 in Equations 1.13 and 1.19 are to be determined by the boundary conditions imposed by the elements that precede and follow the particular tubular element under investigation. This has to be deferred to the next chapter, where we deal with a system of elements or an acoustic filter.

1.2 Three-Dimensional Waves in an Inviscid Stationary Medium

In order to appreciate the limitations of the plane wave theory, it is necessary to consider the general 3D (three-dimensional) wave propagation in tubes. The basic linearized equations corresponding to Equations 1.1 and 1.2 for waves in stationary nonviscous medium are obtained by replacing $\partial/\partial z$ with the 3D gradient operator ∇. Thus,

$$\text{Mass continuity:} \quad \rho_o \nabla \cdot u + \frac{\partial \rho}{\partial t} = 0; \tag{1.20}$$

$$\text{Dynamical equilibrium:} \quad \rho_o \frac{\partial u}{\partial t} + \nabla p = 0. \tag{1.21}$$

The third equation is the same as Equations 1.3 or 1.4. On making use of this equation in Equation 1.20, differentiating Equation 1.20 w.r.t. to t, taking divergence of Equation 1.21 and subtracting, one gets the required 3D wave equation,

$$\left[\frac{\partial^2}{\partial t^2} - c_o^2 \nabla^2\right] p = 0, \tag{1.22}$$

where the Laplacian ∇^2 is given as follows.

Cartesian coordinate system (for rectangular ducts)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}; \tag{1.23}$$

Cylindrical polar coordinate system (for circular tubes)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{1.24}$$

1.2.1 Rectangular Ducts

For harmonic time dependence, making use of separation of variables, the general solution of the 3D wave equation (1.22) with the Laplacian given by Equation 1.23 can be seen to be

$$p(x, y, z, t) = \left(C_1 e^{-jk_z z} + C_2 e^{+jk_z z}\right)\left(e^{-jk_x x} + C_3 e^{+jk_x x}\right)\left(e^{-jk_y y} + C_4 e^{+jk_y y}\right) e^{j\omega t}, \quad (1.25)$$

with the compatibility condition

$$k_x^2 + k_y^2 + k_z^2 = k_o^2. \quad (1.26)$$

Here, k_x, k_y and k_z are wave numbers in the x, y and z direction, respectively. In the limiting case of plane waves, $k_x = k_y = o$. Then, Equation 1.26 yields $k_z = k_o$ and Equation 1.25 reduces to Equation 1.13.

It may be noted from Equation 1.25 that x-dependent factor involves two unknowns k_x and c_3, and the y-dependent factor involves the unknowns k_y and C_4. These may be evaluated from the relevant boundary conditions as follows.

For a rigid-walled duct of breadth b and height h (Figure 1.1), the boundary conditions are

$$\frac{\partial p}{\partial x} = 0 \quad at \quad x = 0 \quad \text{and} \quad x = b \quad (1.27a)$$

and

$$\frac{\partial p}{\partial y} = 0 \quad at \quad y = 0 \quad \text{and} \quad y = h. \quad (1.27b)$$

Substituting these boundary conditions in Equation 1.25 yields, respectively,

$$C_3 = 1; \quad k_x = \frac{m\pi}{b}, \quad m = 0, 1, 2, \ldots\ldots. \quad (1.28a)$$

and

$$C_4 = 1; \quad k_y = \frac{n\pi}{h}, \quad n = 0, 1, 2, \ldots\ldots., \quad (1.28b)$$

and Equation 1.25 then becomes

$$p(x, y, z, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \cos\left(\frac{m\pi x}{b}\right)\cos\left(\frac{n\pi y}{h}\right)\left(C_{1,m,n} e^{-jk_{z,m,n}z} + C_{2,m,n} e^{+jk_{z,m,n}z}\right) e^{j\omega t}, \quad (1.29)$$

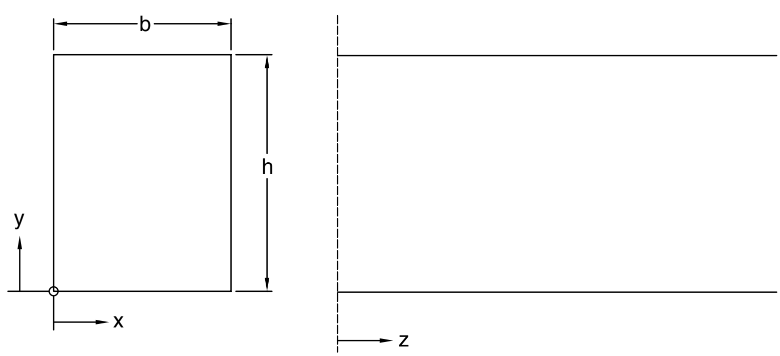

Figure 1.1 A rectangular duct and the Cartesian coordinate system (x, y, z)

where, as per Equation 1.26, the transmission wave number for the (m, n) mode, $k_{z,m,n}$ is given by the relation

$$k_{z,m,n} = \left[k_o^2 - (m\pi/b)^2 - (n\pi/h)^2\right]^{1/2}. \tag{1.30}$$

In order to evaluate axial particle velocity corresponding to the (m, n) mode, we make use of the z-component of the momentum equation (1.21)

$$\rho_o \frac{\partial u_{z,m,n}}{\partial t} + \frac{\partial p}{\partial z} = 0, \tag{1.31}$$

which yields

$$\begin{aligned} u_{z,m,n} &= \frac{-\partial p/\partial z}{j\omega\rho_o} \\ &= \frac{k_{z,m,n}}{k_o\rho_o c_o}\left\{C_{1,m.n}e^{-jk_{z,m,n}z} - C_{2,m,n}e^{+jk_{z,m,n}z}\right\}\cos\left(\frac{m\pi x}{b}\right)\cos\left(\frac{n\pi y}{h}\right)e^{j\omega t}. \end{aligned} \tag{1.32}$$

Now, mass velocity can be evaluated by integration over the area of cross-section in Figure 1.1:

$$\begin{aligned} v_{z,m,n} &= \rho_o \int_o^h \int_o^b u_{z,m,n} dx dy \\ &= \int_o^b \cos\left(\frac{m\pi x}{b}\right)dx \int_o^h \cos\left(\frac{n\pi y}{h}\right)dy \frac{k_{z,m,n}}{\omega}\left\{C_{1,m,n}e^{-jk_{z,m,n}z} - C_{2,m,n}e^{+jk_{z,m,n}z}\right\}e^{j\omega t}, \end{aligned} \tag{1.33}$$

which yields

$$\begin{aligned} v_{z,m,n} &= 0 \quad \text{for} \quad m \neq 0 \quad \text{and/or} \quad n \neq 0 \\ &= \frac{bh}{c_o}\left\{C_1 e^{-jk_o z} - C_2 e^{+jk_o z}\right\}e^{j\omega t} \quad \text{for} \quad m = n = 0. \end{aligned} \tag{1.34}$$

Thus, acoustic mass velocity is nonzero only for the plane wave or (0, 0) mode for which Equation 1.19 is recovered. Incidentally, it shows that the concept of acoustic volume velocity or mass velocity does not have any significance for higher-order modes. Equation 1.32 shows that for the same acoustic pressure, amplitude of the particle velocity for the (m, n) mode is less than $\left(k_{z,m,n}/k_o \text{ times}\right)$ that for the plane wave. It can be noted that for the (0, 0) mode, $k_{z,m,n} = k_o$ and Equation 1.29 reduces to Equation 1.13. Thus, plane wave corresponds to the (0, 0) mode solution in Equation 1.29.

Any particular mode (m, n) would propagate unattenuated if $k_{z,m,n}$ is greater than zero. Then, use of Equation 1.30 yields

$$k_o^2 - \left(\frac{m\pi}{b}\right)^2 - \left(\frac{n\pi}{h}\right)^2 > 0 \tag{1.35a}$$

or

$$\lambda < \frac{2}{\left\{\left(\frac{m}{b}\right)^2 + \left(\frac{n}{h}\right)^2\right\}^{1/2}} \tag{1.35b}$$

Obviously, a plane wave of any wavelength can propagate unattenuated, whereas a higher mode can propagate only insofar as inequality (1.35b) is satisfied. Thus, if $h > b$, the first higher mode (0, 1) would get cut-on (that is, it would start propagating) if

$$\lambda < 2h \quad \text{or} \quad f > \frac{c_o}{2h}. \tag{1.36}$$

In other words, only a plane wave would propagate (all higher modes, even if present, being cut-off, that is, attenuated exponentially) if the frequency is small enough so that

$$\lambda > 2h \quad \text{or} \quad f < \frac{c_o}{2h}, \tag{1.37}$$

Thus, the cut-off frequency of a rectangular duct (Figure 1.1) is given by

$$f_{co} = \frac{c_o}{2h}, \tag{1.38}$$

where h is the larger of the two transverse dimensions of the rectangular duct.

1.2.2 Circular Ducts

The wave equation (1.22), with the Laplacian given by Equation 1.24 governs wave propagation in circular tubes (see Figure 1.2). Upon making use of the method of separation of variables, and writing time dependence as $e^{j\omega t}$ and θ dependence as $e^{jm\theta}$, one gets

$$p(r, \theta, z, t) = \sum\nolimits_m R_m(r) e^{jm\theta} Z(z) e^{j\omega t}. \tag{1.39}$$

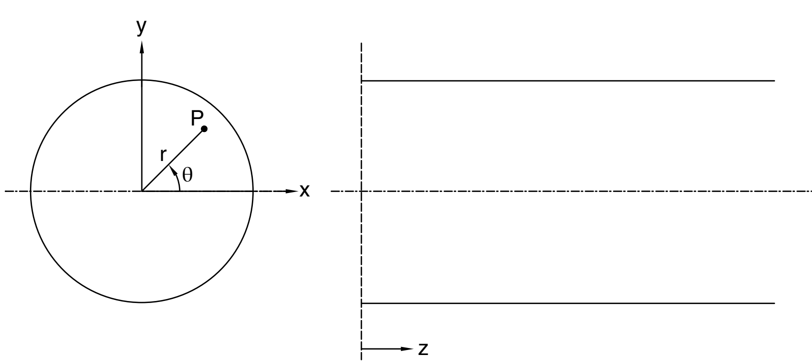

Figure 1.2 A cylindrical duct/tube and the cylindrical polar coordinate system (r, θ, z)

Assuming the z-dependence function $Z(z)$ as in Equation 1.25 with

$$\frac{d^2Z}{dz^2} = -k_z^2 Z \tag{1.40}$$

and substituting Equations 1.39 and 1.40 in the wave equation, one gets a Bessel equation for $R(r)$:

$$\frac{d^2R_m}{dr^2} + \frac{1}{r}\frac{dR_m}{dr} + \left(k_o^2 - k_z^2 - \frac{m^2}{r^2}\right)R_m = 0. \tag{1.41}$$

As indicated in Appendix A, Equation 1.41 has a general solution

$$R_m = C_3 J_m(k_r r) + C_4 N_m(k_r r), \tag{1.42}$$

where the radial wave number k_r is given by

$$k_r^2 = k_o^2 - k_z^2, \tag{1.43}$$

and $J_m(\cdot)$ and $N_m(\cdot)$ are Bessel function and Neumann function, respectively.

$N_m(k_r r)$ tends to infinity at $r = 0$ (the axis). But acoustic pressure everywhere has got to be finite. Therefore, the constant C_4 must be zero.

Again, the radial velocity at the walls $(r = r_o)$ must be zero. Therefore,

$$\frac{dJ_m(k_r r)}{dr} = 0 \text{ at } r = r_o. \tag{1.44}$$

Thus, k_r takes only such discrete values as satisfy the equation

$$J'_m(k_r r_o) = 0. \tag{1.45}$$

Upon denoting the value of k_r corresponding to the nth root of this equation as $k_{r,m,n}$, one gets

$$p(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(k_{r,m,n}r\right)e^{jm\theta}e^{j\omega t} \times \left(C_{1,m,n}e^{-jk_{z,m,n}z} + C_{2,m,n}e^{+jk_{z,m,n}z}\right), \tag{1.46}$$

where

$$k_{z,m,n} = \left(k_o^2 - k_{r,m,n}^2\right)^{1/2}. \qquad \text{(cf. Eq. 1.30)} \tag{1.47}$$

As the first zero of J'_o (or that of J_1) is zero, $k_{r,0,1} = 0$ and $k_{z,0,1} = k_o$. Thus, for the (0, 1) mode, Equation 1.46 reduces to Equation 1.13, the equation for the plane wave propagation. Hence, the plane wave corresponds to the (0, 1) mode of Equation 1.40 and propagates unattenuated.

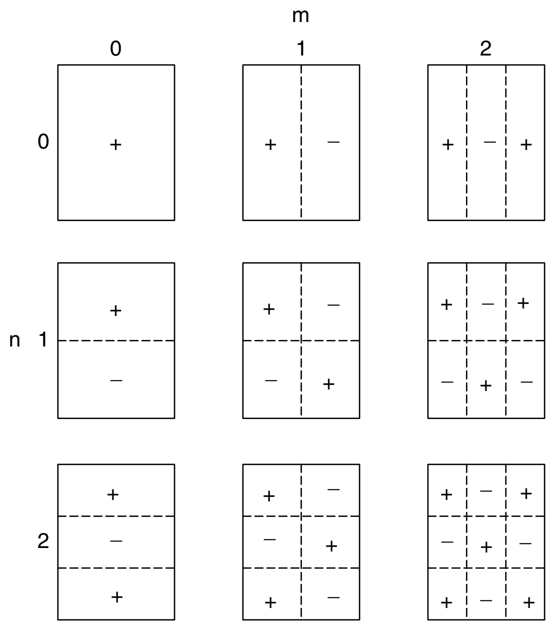

Figure 1.3 Nodal lines for transverse pressure distribution in a rectangular duct up to m = 2, n = 2 (Reproduced with permission from [5])

In most of the literature [1–3], n represents the number of the zero of the derivative $J'_m(k_r r_o)$ as per Equation 1.45. This introduces a dissimilarity between the notation for rectangular ducts and circular ducts. In rectangular ducts, m and n represent the number of nodes in the transverse pressure distribution as shown in Figure 1.3. A similar picture could emerge for circular ducts if n were to denote the number of circular nodes in the transverse pressure distribution. This is shown in Figure 1.4. With this notation [4,5], the plane mode would have the (0, 0) label in circular as well as rectangular ducts, and m and n would have the same connotation, that is, the number of nodes (in respective directions) in the transverse pressure distribution.

This new notation is adopted here henceforth. According to this, $n = 0$ would represent the first root of Equation 1.45 and n would represent the $(n+1)^{\text{st}}$ root thereof. In Equation 1.46, the summation $n = 1$ to ∞ would read $n = 0$ to ∞ as in Equation 1.29 for rectangular ducts.

The first two higher-order modes (1, 0) and (0, 1) will get cut-on if $k_{z,1,0}$ and $k_{z,0,1}$ are real, that is, if $k_o > k_{r,1,0}$ and $k_{r,0,1}$. The first zero of J'_1 occurs at 1.84 and the second zero of J'_o occurs at 3.83. Thus, the cut-on wave numbers would be $1.84/r_o$ and $3.83/r_o$, respectively. In other words, the first azimuthal or diametral mode starts propagating at $k_o r_o = 1.84$ and the first axisymmetric mode at $k_o r_o = 3.83$. If the frequency is small enough (or wave length is large enough) such that

$$k_o r_o < 1.84, \quad \text{or} \quad \lambda > \frac{\pi}{1.84} D, \quad \text{or} \quad f < \frac{1.84}{\pi D} c_o, \tag{1.48}$$

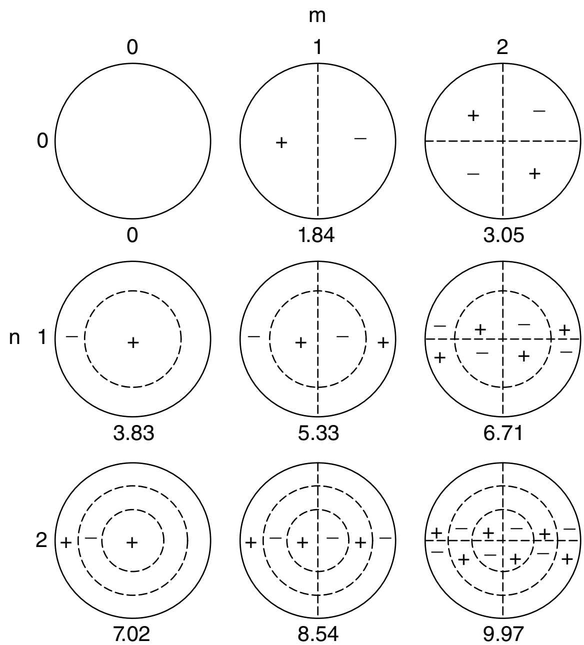

Figure 1.4 Nodal lines for transverse pressure distribution in a circular duct up to m = 2, n = 2 (Reproduced with permission from [5])

where D is the diameter $2r_o$, then only the plane waves would propagate. Thus the cut-off frequency of a circular tube is given by

$$f_{co} = \frac{1.84}{\pi}\frac{c_o}{D} = 0.5857\frac{c_o}{D} \qquad \text{(cf. Eq. 1.38)} \tag{1.49}$$

Fortunately, the frequencies of interest in exhaust noise of internal combustion engines are low enough so that for typical maximum transverse dimensions of exhaust mufflers Equation 1.49 is generally satisfied. Therefore, plane wave analysis has proved generally adequate. In the following pages, as indeed in most of the current literature on exhaust mufflers, one-dimensional wave propagation has been used throughout, with only a passing reference to the existence of higher modes or three-dimensional effects. In practice, muffler configurations are designed making use of the 1D analysis, and 3D analysis is used for a final check.

Substituting the (m, n) mode component of Equation 1.46 in the equation of dynamical equilibrium for the axial direction, that is,

$$\rho_o \frac{\partial u_z}{\partial t} + \frac{\partial p}{\partial z} = 0, \tag{1.50}$$

yields

$$u_{z,m,n} = -\frac{\partial p/\partial z}{j\omega\rho_o}$$

or

$$u_{z,m,n} = J_m\left(k_{r,m,n}r\right)e^{jm\theta}e^{j\omega t}\frac{k_{z,m,n}}{k_o\rho_o c_o} \times \left\{C_{1,m,n}e^{-jk_{z,m,n}z} - C_{2,m,n}e^{+jk_{z,m,n}z}\right\}. \tag{1.51}$$

Thus, as compared to the plane wave, acoustic particle velocity for the (m, n) mode is $k_{z,m,n}/k_o$ times, for the same acoustic pressure. Of course, as just shown for rectangular ducts, volume or mass velocity does not have a meaning for higher order modes.

1.3 Waves in a Viscous Stationary Medium

The analysis of wave propagation in a real (viscous) fluid with heat conduction from the walls of the tube is originally due to Kirchhoff [6,7]. The presence of viscosity brings into play a coupling between the axial and radial motions of the particle in a circular tube. Even if one were to assume axisymmetry (freedom from θ dependence), the wave propagation in a circular tube would be two-dimensional.

Neglecting heat conduction in the first instance, the basic equations governing axisymmetric wave propagation in stationary medium are [8]:

Mass continuity

$$\frac{\partial\rho}{\partial t} + \rho_0\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right) = 0; \tag{1.52}$$

Dynamical equilibrium (Navier–Stokes equations)

$$\rho_0\frac{\partial u_z}{\partial t} + \frac{\partial p}{\partial z} = \mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}\right) + \frac{\mu}{3}\left(\frac{\partial^2 u_r}{\partial r\partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_z}{\partial z^2}\right); \tag{1.53}$$

$$\rho_0\frac{\partial u_r}{\partial t} + \frac{\partial p}{\partial r} = \mu\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right) + \frac{\mu}{3}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_z}{\partial z\partial r}\right). \tag{1.54}$$

The thermodynamic process being isentropic for small-amplitude waves, Equation 1.3 is the third equation.

Eliminating ρ from Equation 1.52 with the help of Equation 1.3, and using the resulting equation to eliminate p from Equations 1.53 and 1.54 yields

$$\frac{\partial^2 u_z}{\partial t^2} - c_0^2\left(\frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{\partial^2 u_r}{\partial z\partial r}\right) = \frac{\partial}{\partial t}\left[\frac{\mu}{\rho_0}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{3}\frac{\partial^2 u_r}{\partial r\partial z} + \frac{1}{3}\frac{1}{r}\frac{\partial u_r}{\partial z} + \frac{4}{3}\frac{\partial^2 u_z}{\partial z^2}\right)\right]; \tag{1.55}$$

$$\frac{\partial^2 u_r}{\partial t^2} - c_0^2\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_z}{\partial r\partial z}\right) = \frac{\partial}{\partial t}\left[\frac{\mu}{\rho_0}\left(\frac{\partial^2 u_r}{\partial z^2} + \frac{1}{3}\frac{\partial^2 u_z}{\partial z\partial r} + \frac{4}{3}\frac{\partial^2 u_r}{\partial r^2} + \frac{4}{3r}\frac{\partial u_r}{\partial r} - \frac{4}{3}\frac{u_r}{r^2}\right)\right]. \tag{1.56}$$

For a sinusoidal forward progressive wave, if the input is only axial, the steady-state solution would be of the form

$$u_z = U_z(r)e^{j\omega t}e^{-j\beta z}; \tag{1.57}$$

$$u_r = U_r(r)e^{j\omega t}e^{-j\beta z}. \tag{1.58}$$

Upon substituting these in Equations 1.55 and 1.56, decoupling the equations for U_z and U_r, using the order-of-magnitude relation

$$\frac{\mu\omega}{\rho_0 c_0^2} \ll 1, \tag{1.59}$$

which is true for most of the gases (and liquids), and applying the rigid-wall boundary condition, one gets, after considerable algebra [9],

$$U_z(r) = A\{J_0(Cr) - J_0(Cr_0)\}, \tag{1.60}$$

$$U_z(r) = \frac{j\beta A}{C} J_1(Cr), \tag{1.61}$$

where amplitude A is a constant, and

$$C = -\frac{1}{1+j}\left(\frac{2\rho_0\omega}{\mu}\right)^{1/2} = (-1+j)\left(\frac{\rho_0\omega}{2\mu}\right)^{1/2} \tag{1.62}$$

Substituting Equations 1.57, 1.58 and 1.62 in the continuity equation (1.52) gives

$$p = -\frac{\rho_0 c_0^2 \beta}{\omega} A_1 J_0(Cr_o)e^{j\omega t}e^{-j\beta z}, \tag{1.63}$$

which indicates that acoustic pressure p is independent of the radius, where U_r and U_z are not. Figure 1.5 shows typical profiles of the axial velocity v_z, radial velocity u_r and pressure p.

Upon integrating u_z over the cross-section of the tube to calculate volume velocity, multiplying it with ρ_0 to get mass velocity v, dividing p by v, and noting that

$$\frac{J_1(Cr_0)}{J_0(Cr_0)} = -j \quad \text{for} \quad |Cr_0| > 10, \tag{1.64}$$

one gets for characteristic impedance Y:

$$Y = \frac{p}{v} = \pm\frac{c_0}{\pi r_0^2}\left\{1 - \frac{1}{r_0}\left(\frac{\mu}{2\rho_0\omega}\right)^{1/2} + \frac{j}{r_0}\left(\frac{\mu}{2\rho_0\omega}\right)^{1/2}\right\}. \tag{1.65}$$

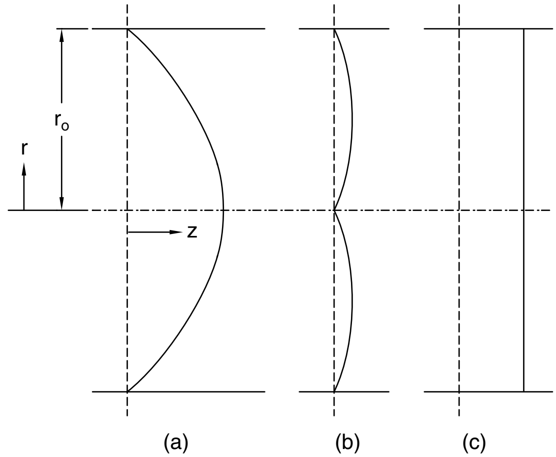

Figure 1.5 Profiles of (a) axial velocity, (b) radial velocity and (c) pressure, at some cross-section of the pipe

Writing Y as c/S [cf. Equation 1.18] gives the velocity of wave propagation in the tube c:

$$c = \pm c_0 \left\{ 1 - \frac{1}{r_0}\left(\frac{\mu}{2\rho_0\omega}\right)^{1/2} + \frac{j}{r_0}\left(\frac{\mu}{2\rho_0\omega}\right)^{1/2} = \pm c_0 \left\{ 1 - \frac{\alpha}{k_0} + j\frac{\alpha}{k_0} \right\} \right\}. \tag{1.66}$$

The corresponding expressions for β become

$$\begin{aligned} \beta &= \pm k_0 \left\{ 1 + \frac{1}{r_0}\left(\frac{\mu}{2\rho_0\omega}\right)^{1/2} - \frac{j}{r_0}\left(\frac{\mu}{2\rho_0\omega}\right)^{1/2} \right\} = \pm\{(k_0+\alpha) - j\alpha\} \\ &= \pm k_0 \left\{ 1 + \frac{\alpha}{k_0} - j\frac{\alpha}{k_0} \right\} \end{aligned} \tag{1.67}$$

where α is the attenuation constant

$$\alpha = \frac{1}{r_0 c_0}\left(\frac{\omega\mu}{2\rho_0}\right)^{1/2}. \tag{1.68}$$

Thus, wave number k for a progressive wave in the tube is

$$k = k_0 + \alpha = k_0\left(1 + \frac{\alpha}{k_0}\right). \tag{1.69}$$

Notably, k is slightly higher than k_0, the wave number in the free medium.

The standing wave solution (1.13) becomes

$$p(z,t) = \left\{ C_1 e^{-\alpha z - jkz} + C_2 e^{\alpha z + jkz} \right\} e^{j\omega t}. \tag{1.70}$$

The acoustic mass velocity v can be got from Equations 1.70 and 1.65:

$$v(z,t) = \frac{1}{Y}\left\{C_1 e^{-\alpha z - jkz} - C_2 e^{\alpha z + jkz}\right\} e^{j\omega t}, \tag{1.71}$$

where Y is the characteristic impedance for the forward wave, corresponding to the positive sign of Equation 1.65; that is,

$$Y = Y_0 \left\{1 - \frac{\alpha}{k_0} + j\frac{\alpha}{k_0}\right\}, \tag{1.72}$$

Y_0 being the characteristic impedance for the inviscid medium, given by Equation 1.18:

$$Y_0 = \frac{c_0}{S}, \quad S = \pi r_0^2.$$

Kirchhoff [6,7] takes into account heat conduction as well. Following a slightly different but more general analysis, he gets expressions that are identical to Equations 1.67 and 1.68 with μ being replaced by μ_e, an effective coefficient of viscothermal friction, given by

$$\mu_e = \mu \left\{1 + \left(\gamma^{1/2} - \frac{1}{\gamma^{1/2}}\right)\left(\frac{K}{\mu C_p}\right)^{1/2}\right\}^2, \tag{1.73}$$

where C_p is the specific heat at constant pressure, and K is the coefficient of thermal conductivity. It may be noted that $\mu C_p/K$ is the Prandtl number. Incidentally, for air at normal temperature and pressure (NTP), Prandtl number is 0.7 and the specific heat ratio γ is 1.4. Thus, for air, Equation 1.73 yields $\mu_e = 1.65\mu$.

Experimental measurements of α by several investigators [2] show disagreement with theoretical values, the discrepancy ranging from 15 to 50%. However, almost all of them confirm the functional dependence of α on $\omega^{1/2}$ and r_0 implied in Equation 1.68. Of course, the attenuation constant α is also a function of surface roughness, flexibility of the tube wall, humidity of the medium, and so on.

In the foregoing analysis, it has been observed that the axial component of acoustic velocity u_z is a function of radius, and its radial dependence remains the same along the axis. This latter property enabled us to define an acoustic mass velocity v, and we got Equation 1.71 to go with Equation 1.70. These two equations correspond to Equations 1.13 and 1.19 for undamped plane waves. This formal similarity of the standing wave solutions suggests strongly that one could perhaps write the basic equation in terms of a mean axial particle velocity u defined as

$$u \equiv \frac{v}{\rho_0 S}, \tag{1.74}$$

taking into account the effect of α in the equation of dynamical equilibrium as an additional pressure-drop term, looking at the velocity of wave propagation c as a real number equal to the real part of Equation 1.66, the corresponding k as in Equation 1.69, and dropping the radial component of acoustic particle velocity altogether. These basic equations would then lead to a one-dimensional damped wave equation with essentially the same solutions as given. Such a representation would make conceptualization as well as analysis considerably easier, and would admit useful generalizations for damped wave propagation in a moving medium, as shown later in Section 1.6.

Thus, two of the basic equations for damped plane waves, the equation of mass continuity and thermodynamic (isentropic) process, are the same as Equations 1.1 and 1.3, whereas the equation for dynamical equilibrium (1.2) becomes

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} + 2\alpha\rho_0 c u = 0, \tag{1.75}$$

where $2\alpha\rho_0 cu$ is the pressure drop per unit length due to viscothermal friction as given by Rschevkin [10].

These three basic equations lead to the one-dimensional damped wave equation [cf. Equation 1.5]

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial z^2} + 2c\alpha \frac{\partial}{\partial t}\right] p = 0. \tag{1.76}$$

Looking for a propagating solution of the type

$$p = Ce^{j\omega t} e^{\beta z} \tag{1.77}$$

one gets, on substituting Equation 1.77 in Equation 1.76,

$$\begin{aligned} \beta &= \pm\left(-k^2 + 2jk\alpha\right)^{1/2} \\ &\cong \pm jk\left(1 - j\frac{\alpha}{k}\right) \\ &= \pm(jk + \alpha), \end{aligned} \tag{1.78}$$

where the following inequality has been assumed:

$$\alpha^2/k^2 \simeq \alpha^2 \quad k_0^2 \ll 1. \tag{1.79}$$

Thus, we recover Equation 1.70 for acoustic pressure p and, hence, Equation 1.71 for the acoustic mass velocity.

It is important to note here that Equation 1.75 is not an exact equation and therefore should not be used to find the values of c, k, α and Y, which are to be adopted from the foregoing relatively rigorous analysis.

1.4 Plane Waves in an Inviscid Moving Medium

Wave propagation is due to the combined effect of inertia (mass) and elasticity of the medium, and therefore a wave moves relative to the particles of the medium. When the medium itself is moving with a uniform velocity U, the velocity of wave propagation relative to the medium remains c. Therefore, relative to a stationary frame of reference (that is, as seen by a stationary observer), the forward wave would move at an absolute velocity of $U + c$ and the backward moving wave at $U - c$. The waves are said to be convected downstream by mean flow. This is borne out by the following analysis.

Let the medium be moving with a velocity U, the gradients of which in the r direction as well as z direction are negligible. The basic linearized equations for this case are the same as for stationary medium [Equations 1.1–1.3] except that the local time derivative $\partial/\partial t$ is replaced by substantive derivative D/Dt, where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial z}. \tag{1.80}$$

Thus, the mass continuity and momentum equations are

$$\rho_0\frac{\partial u}{\partial z} + \frac{D\rho}{Dt} = 0 \tag{1.81}$$

and

$$\rho_0\frac{Du}{Dt} + \frac{\partial p}{\partial z} = 0, \tag{1.82}$$

respectively. The third equation is, of course, the isentropicity relation (1.3).

Eliminating ρ and u from these three equations yields the convective one-dimensional wave equation

$$\left(\frac{D^2}{Dt^2} - c_0^2\frac{\partial^2}{\partial z^2}\right)p = 0 \tag{1.83}$$

or

$$\frac{\partial^2 p}{\partial t^2} + 2U\frac{\partial^2 p}{\partial z\partial t} + \left(U^2 - c_0^2\right)\frac{\partial^2 p}{\partial z^2} = 0. \tag{1.84}$$

Making use of the separation of variables and assuming again a time-dependence function $e^{j\omega t}$, the wave equation (1.84) may be seen to admit the following general solution:

$$p(z,t) = \left(C_1 e^{-j\omega/(c_0+U)z} + C_2 e^{+j\omega/(c_0-U)z}\right)e^{j\omega t}, \tag{1.85}$$

or

$$p(z,t) = \left(C_1 e^{-jk_0 z/(1+M)} + C_2 e^{+jk_0 z/(1-M)}\right)e^{j\omega t}. \tag{1.86}$$

Writing

$$u(z,t) = \left(C_3 e^{-jk_0 z/(1+M)} + C_4 e^{+jk_0 z/(1-M)}\right)e^{j\omega t}, \tag{1.87}$$

substituting Equations 1.86 and 1.87 in convective wave Equation 1.82 and equating the coefficients of $e^{-jk_0z/(1+M)}$ and $e^{+jk_0z/(1-M)}$ separately to zero yields.

$$C_3 = \frac{C_1}{\rho c_0} \quad \text{and} \quad C_4 = -\frac{C_2}{\rho c_0}$$

Thus, acoustic mass velocity $v(z, t)$ is given by

$$v(z,t) = \rho_0 S u(z,t) = \frac{1}{Y_0}\left(C_1 e^{-jk_0z/(1+M)} - C_2 e^{+jk_0z/(1-M)}\right)e^{j\omega t}, \tag{1.88}$$

where the characteristic impedance Y_0 is the same as for stationary medium–Equation 1.18.

Equation 1.85 indicates (symbolically) the convective effect of mean flow on the two components of the standing waves, as mentioned in the opening paragraph of this section.

1.5 Three-Dimensional Waves in an Inviscid Moving Medium

As indicated earlier in Section 1.2, analysis of three-dimensional waves in a flow duct is needed for understanding the propagation of higher-order modes and for evaluating the limiting frequency below which only the plane wave would propagate unattenuated.

Combining the arguments presented in Sections 1.2 and 1.4 yields the following basic relations:

$$\text{Mass continuity:} \quad \rho_0 \nabla \cdot u + \frac{D\rho}{Dt} = 0; \tag{1.89}$$

$$\text{Dynamical equilibrium:} \quad \rho_0 \frac{Du}{Dt} + \nabla p = 0; \tag{1.90}$$

$$\text{The convected 3D wave equation:} \quad \left(\frac{D^2}{Dt^2} - c_0^2 \nabla^2\right) p = 0. \tag{1.91}$$

Here, the mean-flow velocity is assumed to be constant in space and time, that is, independent of all coordinates.

For a rectangular duct (Figure 1.1), the solution to Equation 1.91 would be

$$p(x,y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \cos\frac{m\pi x}{b}\cos\frac{n\pi y}{h} \times \left\{C_{1,m,n}e^{-jk^+_{z,m,n}z} + C_{2,m,n}e^{+jk^-_{z,m,n}z}\right\}e^{j\omega t}, \tag{1.92}$$

where $k^+_{z,m,n}$ and $k^-_{z,m,n}$ are governed by the equation [cf. Equation 1.30]

$$k^2_{z,m,n} + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{h}\right)^2 = \left(k_0 \pm M k_{z,m,n}\right)^2 \tag{1.93}$$

or

$$k^{\pm}_{z,m,n} = \frac{\mp M k_0 + \left[k_0^2 - \left(1 - M^2\right)\left\{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{h}\right)^2\right\}\right]^{1/2}}{1 - M^2}. \tag{1.94}$$

Thus, the condition for higher-order modes $(m, n > 0)$ to propagate unattenuated is given by the condition that the sum under the radical sign is not negative, or

$$k_0^2 - \left(1 - M^2\right)\left\{\left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{h}\right)^2\right\} \geq 0. \tag{1.95}$$

In other words, only a plane wave would propagate if the frequency is small enough so that

$$\lambda > \frac{2h}{\left(1 - M^2\right)^{1/2}} \quad \text{or} \quad f < \frac{c_0}{2h}\left(1 - M^2\right)^{1/2} \tag{1.96}$$

[cf. inequality (1.37)], where h is the larger of the two transverse dimensions of the rectangular duct.

Clearly, the cut-off frequency for the first higher mode (0, 1) for a flow duct is lower than that of a stationary-medium duct by a factor $\left(1 - M^2\right)^{1/2}$, where M is the average Mach number of the mean flow.

It is worth noting here that the cut-off frequency is the same for downstream as well as upstream propagation.

The same remarks hold for propagation of higher-order modes in a circular duct, the solution for which can readily be seen to be (following the algebra of Section 1.2.2)

$$p(r, \theta, z, t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} J_m\left(k_{r,m,n} r\right) e^{jm\theta} e^{j\omega t} \times \left\{C_{1,m,n} e^{-jk^+_{z,m,n} z} + C_{2,m,n} e^{+jk^-_{z,m,n} z}\right\}, \tag{1.97}$$

where $k^+_{z,m,n}$ and $k^-_{z,m,n}$ are governed by the equation

$$k^2_{z,m,n} + k^2_{r,m,n} = \left(k_0 + Mk_{z,m,n}\right)^2 \tag{1.98}$$

or

$$k^{\pm}_{z,m,n} = \frac{\mp Mk_0 + \left[k_0^2 - \left(1 - M^2\right)k^2_{r,m,n}\right]^{1/2}}{1 - M^2}. \tag{1.99}$$

Thus, the condition for higher-order modes (m and/or $n > 0$) to propagate unattenuated is given by

$$k_0^2 - \left(1 - M^2\right)k^2_{r,m,n} \geq 0. \tag{1.100}$$

In other words, only a plane wave would propagate if the frequency is small enough so that

$$k_0 r_0 < 1.84\left(1 - M^2\right)^{1/2},$$

or

$$\lambda > \frac{\pi D}{1.84\left(1 - M^2\right)^{1/2}},$$

or

$$f < \frac{1.84c_0}{\pi D}\left(1 - M^2\right)^{1/2} = 0.5857\frac{c_0}{D}\left(1 - M^2\right)^{1/2} \tag{1.101}$$

The lowering of the cut-off frequency by mean flow has been demonstrated experimentally by Mason [11,12]. In particular, he has shown that the cut-off frequency for circular tubes with flow is indeed lowered by a factor $\left(1 - M^2\right)^{1/2}$ for low Mach numbers $(M < 0.2)$ that are typical of exhaust mufflers.

Now the particle velocity $u(x, y, z, t)$ can be determined by assuming for it a form similar to that of pressure [i.e. Equation 1.92], with constant $C_{1,m,n}$ and $C_{2,m,n}$ replaced by new constants $C_{3,m,n}$ and $C_{4,m,n}$, and summing $u_{z,m,n}$ so obtained over m and n. Thus,

$$\begin{aligned} u_z(x,y,z,t) &= \frac{1}{\rho_0 c_0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\cos\frac{m\pi x}{b}\cos\frac{n\pi y}{h} \\ &\times\left\{\frac{k_{z,m,n}^{+}}{k_0 - Mk_{z,m,n}^{+}}C_{1,m,n}e^{-jk_{z,m,n}^{+}z} - \frac{k_{z,m,n}^{-}}{k_0 + Mk_{z,m,n}^{-}}C_{2,m,n}e^{+jk_{z,m,n}^{-}z}\right\}. \end{aligned} \tag{1.102}$$

Similarly, the particle velocity $u(r, \theta, z, t)$ for 3D waves in a circular tube with mean flow can be readily proved to be given by the equation

$$\begin{aligned} u_z(r,\theta,z,t) &= \frac{1}{\rho_0 c_0}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}J_m\left(k_{r,m,n}r\right)e^{jm\theta}e^{j\omega t} \\ &\times\left\{\frac{k_{z,m,n}^{+}}{k_0 - Mk_{z,m,n}^{+}}C_{1,m,n}e^{-jk_{z,m,n}^{+}z} - \frac{k_{z,m,n}^{-}z}{k_0 + Mk_{z,m,n}^{-}}C_{2,m,n}e^{+jk_{z,m,n}^{-}z}\right\}. \end{aligned} \tag{1.103}$$

1.6 One-Dimensional Waves in a Viscous Moving Medium

As has been shown in Section 1.3, a wave front in a tube containing a viscous fluid is not plane inasmuch as axial particle velocity is not the same all over the cross-section, although acoustic pressure is constant for most of the common gases for which inequality (1.59) is satisfied. Nevertheless, as shown later in that section, one could write the equivalent one-dimensional equations following Rschevkin [10]. These equations are extended here to account for the additional aeroacoustic losses due to turbulent friction, and also the convective effect of mean flow. They imply use of a quasi-static approach [13] wherein it is assumed that the steady flow relations apply with acoustic perturbations as well. On subtracting one from the other and linearizing in terms of acoustic perturbations ρ and u, we get the required aeroacoustic equation for propagation of one-dimensional waves in a moving medium with friction. This principle or approach is indeed the very basis of aeroacoustics, and is used extensively in Chapter 3.

With subscripts 0 and T denoting mean and total (perturbed) states, we can write

$$\rho_T = \rho_0 + \rho; \qquad p_T = p_0 + p; \qquad u_T = U + u; \tag{1.104}$$

where, for the linear case,

$$\left(\frac{\rho}{\rho_0}\right)^2 \ll 1, \qquad \left(\frac{p}{p_0}\right)^2 \ll 1, \qquad \left(\frac{u}{c_0}\right)^2 \ll 1, \tag{1.105}$$

so that terms involving quadratic terms in the acoustic perturbation variables p, ρ and u can be neglected.

Substituting these relations in the mass continuity equation

$$\frac{D\rho_T}{Dt} + \rho_T \frac{\partial u_T}{\partial z} = \frac{\partial \rho_T}{\partial t} + \frac{\partial}{\partial z}(\rho_T u_T) = 0, \tag{1.106}$$

Subtracting from it the corresponding unperturbed steady flow equation, and noting that both the time derivative as well as space derivative of the mean quantities p_0, ρ_0 and U are zero by definition, yields the equation

$$\frac{\partial \rho}{\partial t} + U \frac{\partial \rho}{\partial z} + \rho_0 \frac{\partial u}{\partial z} = 0 \tag{1.107}$$

which, of course, is identical to Equation 1.81 when one notes that

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_T \frac{\partial}{\partial z} \cong \frac{\partial}{\partial t} + U \frac{\partial}{\partial z}. \tag{1.108}$$

The one-dimensional equation for dynamical equilibrium with viscothermal dissipation and turbulent friction loss can be written as [10,13]

$$\rho_0 \frac{Du_T}{Dt} + \frac{\partial p_T}{\partial z} + 2\alpha \rho_0 c u_T + \xi \rho_0 u_T^2 = 0, \tag{1.109}$$

where $2\alpha\rho_0 c u_T$ is the pressure drop per unit length due to viscothermal friction, $\xi = F/2d$, $F =$ Froude's friction factor, defined as ratio of the pressure drop in an axial length equal to one diameter divided by the dynamic head ${}^1\!/_2\rho_0 u_T^2$, and $d =$ diameter of the tube, or hydraulic diameter (four times the ratio of area and perimeter) if the tube is not circular.

Thus, $\xi\rho_0 u_T^2$ is the pressure drop per unit length due to boundary-layer friction or wall friction. Froude's friction factor F can be obtained as a function of Reynolds number from textbooks on fluid mechanics (see, for example, [14,15]).

For the typical flow velocities in exhaust mufflers, F is given by Lees formula

$$F = 0.0072 + \frac{0.612}{R_e^{0.35}}, \quad R_e < 4 \times 10^5, \tag{1.110}$$

where R_e is the Reynolds number $Ud\rho_0/\mu$ and μ is the coefficient of dynamic viscosity.

Substituting Equation 1.104 in Equation 1.109, subtracting from it the corresponding unperturbed steady flow equation, and making use of the order-of-magnitude relations for the small-amplitude (i.e. linear) waves gives

$$\rho_0 \frac{\partial u}{\partial t} + \rho_0 U \frac{\partial u}{\partial z} + \frac{\partial p}{\partial z} + 2\rho_0 \alpha c u + 2\xi \rho_0 U_0 u = 0$$

or

$$\rho_0 \frac{Du_T}{Dt} + \frac{\partial p}{\partial z} + 2\rho_0(\alpha c + \xi U)u = 0, \tag{1.111}$$

For small-amplitude wave propagation in a moving medium with no transverse gradients, the thermodynamic process is still almost isentropic and therefore Equation 1.3 holds.

Eliminating ρ and u from Equations 1.3, 1.107 and 1.111 yields the desired wave equation

$$\left[\frac{D^2}{Dt^2} - c_0^2 \frac{\partial^2}{\partial z^2} + 2(\xi U + c\alpha)\frac{D}{Dt}\right] p = 0. \tag{1.112}$$

This equation is very similar to Equation 1.76 except that local time derivative operator $\partial/\partial t$ is replaced by the substantive derivative D/Dt, thereby incorporating the convective effect of mean flow, and a flow-acoustic friction term has been added.

On assuming a solution of the form

$$p(z,t) = Ce^{j\omega t}e^{\beta z}, \tag{1.113}$$

substituting it in Equation 1.112, making use of the order-of-magnitude considerations

$$\begin{aligned} &M^2\alpha^2 < \alpha^2 \ll k^2, \\ &\xi^2 M^4 < \xi^2 M^2 \ll k^2, \\ &2\xi M^3 \alpha < 2\xi M\alpha \ll k^2, \end{aligned} \tag{1.114}$$

and some algebraic manipulations [16], one gets two values of β:

$$\beta^{\pm} \cong \mp \left(\frac{\alpha + \xi M + jk}{1 \pm M}\right). \tag{1.115}$$

Thus,

$$p(z,t) = \left[C_1 \exp\left(-\frac{\alpha + \xi M + jk}{1+M} z\right) + C_2 \exp\left(+\frac{\alpha + \xi M + jk}{1-M} z\right)\right] \exp(j\omega t). \tag{1.116}$$

This solution shows clearly that

i. total aeroacoustic attenuation in a moving medium $\alpha(M)$ is a sum of the contributions of the viscothermal effects and turbulent flow friction, and
ii. the factors $1 \pm M$ that represent the convective effect of mean flow apply to the attenuation constants as well as to the wave numbers.

The attenuation constants are

$$\alpha^+ = \frac{\alpha + \xi M}{1+M} = \frac{\alpha(M)}{1+M}, \tag{1.117}$$

$$\alpha^- = \frac{\alpha + \xi M}{1-M} = \frac{\alpha(M)}{1-M}, \tag{1.118}$$

where

$$\alpha(M) = \alpha + \xi M, \quad k = k_0 + \alpha. \tag{1.119}$$

$\alpha(M)$ being the same for waves in both the directions, can be construed to be the 'real' aeroacoustic attenuation constant for a moving medium. The factors $1 \pm M$ in $\alpha^{\pm}$ as well as $\beta^{\pm}$ represent only the Doppler effect due to mean flow convection.

The acoustic mass velocity v can now be written as

$$v(z,t) = \frac{1}{Y}\left[C_1 \exp\left(-\frac{\alpha + \xi M + jk}{1+M}z\right) - C_2 \exp\left(+\frac{\alpha + \xi M + jk}{1-M}z\right)\right]\exp(j\omega t). \tag{1.120}$$

The characteristic impedance Y can be constructed heuristically from Equation 1.72, making use of the foregoing remarks, α being replaced by $\alpha(M)$, and the fact observed in Section 1.4 that mean-flow convection does not alter the characteristic impedance. Thus, for the case on hand,

$$Y = Y_0\left\{1 - \frac{\alpha + \xi M}{k_0} + j\frac{\alpha + \xi M}{k_0}\right\}, \tag{1.121}$$

where, as before, Y_0 is the characteristic impedance for plane waves in an inviscid stationary medium given by Equation 1.18.

Equation 1.121 neglects second-order terms like $M\alpha/k$ and $M^2\xi/k$. These terms would further complicate the algebra inasmuch as Y for the forward direction would not be the same as for the backward direction.

It is worth repeating here that the above analysis is oversimplified for the specific purpose of evaluating the aeroacoustic attenuation constant. In particular, Equations 1.108 and 1.111 are not exact because they are one-dimensional.

Thus, Equations 1.116, 1.120 and 1.121 are approximate. Nevertheless, these equations are very useful from an engineering point of view because of their formal similarity with the corresponding equations for the case of the inviscid moving medium and the viscous stationary medium derived in the foregoing section.

1.7 Waves in Ducts with Compliant Walls (Dissipative Ducts)

In all the foregoing sections, the walls of the duct were assumed to be rigid. However, walls of a finite thickness (typical of the sheet metal from which the exhaust mufflers are fabricated) are in general compliant inasmuch as the transverse impedance is finite. Alternatively, the walls of the duct may be lined with an acoustically absorptive material that would, of course, have a finite normal impedance. This latter application is much more important than the former and is discussed at length in Chapter 6.

The normal impedance of a wall lined with an acoustically absorptive layer can be assumed to be independent of z. In other words, the acoustic layer can be assumed to be homogeneous and 'locally reacting'. The same, however, does not apply to unlined metallic walls of the pipes as are used in exhaust mufflers for internal combustion engines, where the wall impedance would vary with z (increasing near the end plates). Nevertheless, in such mufflers the wall thickness is generally substantial so that the impedance is very large or compliance very small.

Neglecting the viscous friction of the medium as relatively insignificant and assuming the mean flow velocity to be constant all over the cross-section, the propagation of waves in compliant ducts would be governed by a 3D wave equation [Equation 1.22 for a stationary medium and Equation 1.91 for a moving medium]. Here we restrict ourselves to the lowest-mode (corresponding to plane wave) analysis of acoustically lined ducts with stationary medium. The relatively minor effect of moving medium is discussed briefly in Chapter 6.

1.7.1 Rectangular Duct with Locally Reacting Lining

For a stationary medium, the general solution to wave equation (1.22) with the Laplacian ∇^2 in terms of Cartesian coordinates [Equation 1.23] is given by Equation 1.25 with wave numbers k_x, k_y, and k_z being related to k_0 as per Equation 1.26.

Let Z_w be the normal impedance of the walls at their exposed boundary and let b and h be the breadth and height of the free section as shown in Figure 1.6. According to the equation of dynamical equilibrium in the x direction, the x component of acoustic particle velocity u_x is related to acoustic pressure p as

$$\rho_0 \frac{\partial u_x}{\partial t} + \frac{\partial p}{\partial x} = 0 \tag{1.122}$$

or

$$u_x = -\frac{\partial p/\partial x}{j\omega\rho_0}. \tag{1.123}$$

Similarly,

$$u_y = -\frac{\partial p/\partial y}{j\omega\rho_0}. \tag{1.124}$$

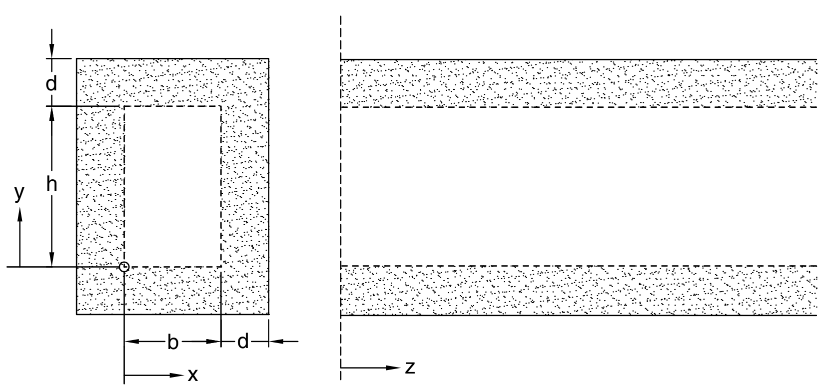

Figure 1.6 Schematic views of an acoustically lined rectangular duct with clear dimensions b and h (cf. Figure 1.1)

Thus, the boundary conditions for a duct with uniform normal wall impedance Z_w would be

$$\frac{p(0,y,z,t)}{-u_x(0,y,z,t)} = \frac{p(b,y,z,t)}{u_x(b,y,z,t)} = Z_{wx}, \tag{1.125}$$

$$\frac{p(x,0,z,t)}{-u_y(x,0,z,t)} = \frac{p(x,h,z,t)}{u_y(x,h,z,t)} = Z_{wy}, \tag{1.126}$$

Substituting solution (1.25) and Equations 1.123 and 1.124 in the four boundary conditions (1.125) and (1.126) yields

$$\frac{\omega\rho_0(1+C_3)}{-k_x(1-C_3)} = Z_{wx}, \tag{1.127}$$

$$\frac{\omega\rho_0}{k_x}\frac{e^{-jk_xb}+C_3e^{+jk_xb}}{e^{-jk_xb}-C_3e^{+jk_xb}} = Z_{wx}, \tag{1.128}$$

$$\frac{\omega\rho_0(1+C_4)}{-k_y(1-C_4)} = Z_{wy}, \tag{1.129}$$

$$\frac{\omega\rho_0}{k_y}\frac{e^{-jk_yh}+C_4e^{+jk_yh}}{e^{-jk_yh}-C_4e^{+jk_yh}} = Z_{wy}. \tag{1.130}$$

Equation 1.127 yields

$$C_3 = \left(\frac{Z_{wx}k_x}{\omega\rho_0}+1\right)\Big/\left(\frac{Z_{wx}k_x}{\omega\rho_0}-1\right), \tag{1.131}$$

Substituting this in Equation 1.128 and rearranging leads to a quadratic in $Z_{wx}k_x/\omega\rho_0$, which in turn yields

$$\frac{Z_{wx}k_x}{\omega\rho_0} = \frac{-\cos k_xb \pm 1}{j\sin k_xb} \tag{1.132}$$

$$= -j\tan\frac{k_xb}{2}, \quad j\cot\frac{k_xb}{2}. \tag{1.133}$$

These two eigen equations can be rewritten in the conventional form

$$\frac{\cot(k_xb/2)}{k_xb/2} = -j\frac{Z_{wx}}{\rho_0c_0}\frac{1}{k_0b/2} \tag{1.134a}$$

and

$$\frac{\tan(k_xb/2)}{k_xb/2} = j\frac{Z_{wx}}{\rho_0c_0}\frac{1}{k_0b/2}. \tag{1.134b}$$

For the limiting case of rigid unlined walls, $Z_{wx} \to \infty$, $C_3 = 1$, and the two equations yield, respectively,

$$k_x = 0, \quad \frac{2\pi}{b}, \quad \frac{4\pi}{b}, \quad \ldots \tag{1.135a}$$

and

$$k_x = \pi/b, \quad 3\pi/b, \quad 5\pi/b, \quad \ldots \tag{1.135b}$$

Thus, the two equations supply alternate values of the series (1.28a), that is,

$$k_x = m\pi/b, \quad m = 0, 1, 2, 3. \tag{1.136}$$

By analogy, the roots or (eigen values) of Equations 1.134a and 1.134b must be alternating with each other. It can readily be checked that, like the series of roots (1.135a), the roots of the transcendental equation (1.134a) belong to symmetric modes, whereas, like the series of roots (1.135b), the roots of Equation 1.134b represent antisymmetric modes, the symmetry here relating to the axis $x = b/2$.

An identical analysis of Equations 1.129 and 1.130 would show that k_y is given by the transcendental eigen equations

$$\frac{\cot(k_y h/2)}{k_y h/2} = -j\frac{Z_{wy}}{\rho_0 c_0}\frac{1}{k_0 h/2} \tag{1.137a}$$

and

$$\frac{\tan(k_y h/2)}{k_y h/2} = j\frac{Z_{wy}}{\rho_0 c_0}\frac{1}{k_0 h/2}, \tag{1.137b}$$

the roots of which alternate with each other, representing symmetric and antisymmetric modes, respectively, the symmetry being reckoned with respect to the axis $y = h/2$.

Let the infinite roots of Equations 1.134 and 1.137 be

$$k_{x,m}, \quad m = 0, 1, 2, 3, \ldots$$

and

$$k_{y,m}, \quad m = 0, 1, 2, 3, \ldots, \tag{1.138}$$

respectively.

Thus, the general acoustic pressure field equation (1.25) becomes

$$\begin{aligned} p(x,y,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} & \left[e^{-jk_{x,m},x} + \left\{\frac{Z_{w,x}k_{x,m}/\rho_0 c_0 k_0 + 1}{Z_{w,x}k_{x,m}/\rho_0 c_0 k_0 - 1}\right\} e^{+jk_{x,m},x}\right] \\ & \times \left[e^{-jk_{y,n},y} + \left\{\frac{Z_{w,y}k_{y,n}/\rho_0 c_0 k_0 + 1}{Z_{w,y}k_{y,n}/\rho_0 c_0 k_0 - 1}\right\} e^{+jk_{y,n},y}\right] \\ & \left[C_{1,m,n} e^{-jk_{z,m,n}z} + C_{2,m,n} e^{+jk_{z,m,n}z}\right] e^{j\omega t}, \end{aligned} \tag{1.139}$$

where $k_{z,m,n}$ is given by the equation

$$k_{z,m,n} = \left\{k_0^2 - k_{x,m}^2 - k_{y,n}^2\right\}^{1/2}. \tag{1.140}$$

On substituting the $(m,\ n)$ component of Equation 1.139 for acoustic pressure in the momentum equation for the axial direction, evaluating $u_{z,m,n}$, and then summing over m and n, one gets

$$\begin{aligned} u_{z,m,n}(x,y,z,t) = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[e^{-jk_{x,m},x} + \left\{\frac{Z_{w,x}k_{x,m}/\rho_0c_0k_0 + 1}{Z_{w,x}k_{x,m}/\rho_0c_0k_0 - 1}\right\}e^{+jk_{x,m},x}\right] \\ &\times \left[e^{-jk_{y,n},y} + \left\{\frac{Z_{w,y}k_{y,n}/\rho_0c_0k_0 + 1}{Z_{w,y}k_{y,n}/\rho_0c_0k_0 - 1}\right\}e^{+jk_{y,n},y}\right] \\ &\times \frac{k_{z,m,n}}{k_0}\frac{1}{\rho_0c_0}\left[C_{1,m,n}e^{-jk_{z,m,n}z} - C_{2,m,n}e^{+jk_{z,m,n}z}\right]e^{j\omega t}. \end{aligned} \tag{1.141}$$

If all the walls of the duct are not lined with an absorptive material, then the wall impedances $Z_{w,x}$ and $Z_{w,y}$ would be more or less reactive (controlled by mass and elasticity). Then, according to Equations 1.134 and 1.137, k_x and k_y would be real and, as per Equation 1.140, $k_{z,m,n}$ would be real or imaginary, not complex. Thus, the modes that may propagate along an unlined duct with yielding walls would do so without attenuation. In other words, the unlined yielding walls do not introduce axial attenuation.

By the same reasoning it can be seen that ducts lined with acoustically absorptive material (that is, with complex wall impedance) would result in complex values of k_x, k_y, and hence k_z. The imaginary component of k_z would introduce attenuation in the axial direction, and that is the basic principle of dissipative ducts and parallel baffle mufflers discussed at length in Chapter 6.

1.7.2 Circular Duct with Locally Reacting Lining

Waves in a circular duct with stationary medium (see Figure 1.7) are governed by Equation 1.22, with the Laplacian defined in terms of cylindrical polar coordinates according to

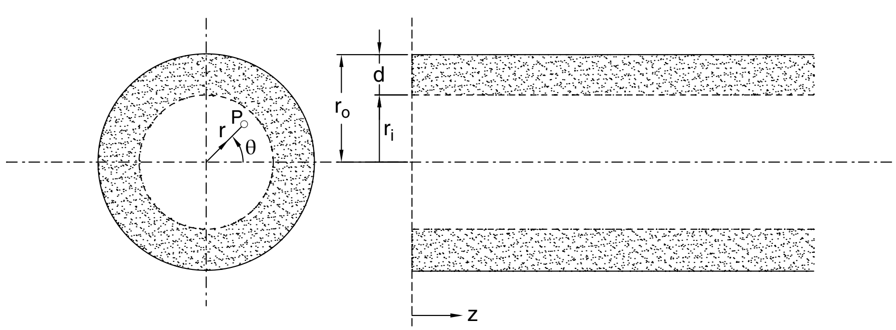

Figure 1.7 Schematic views of an acoustically lined circular duct with clear radius r_i (cf. Figure 1.2)

Equation 1.24; that is,

$$\left[\frac{\partial^2}{\partial t^2} - c_0^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)\right]p = 0. \tag{1.142}$$

Following Section 1.2, the general solution to Equation 1.142 is given by Equation 1.46:

$$p(r,\theta,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} J_m\left(k_{r,m,n}r\right)e^{jm\theta}e^{j\omega t} \times \left\{C_{1,m,n}e^{-jk_{z,m,n}z} + C_{2,m,n}e^{+jk_{z,m,n}z}\right\}, \tag{1.143}$$

with $k_{z,mn}$ being determined from Equation 1.47. The notable difference is that $k_{r,m,n}$ is now determined from the boundary condition that the wall $(r = r_i)$ has a finite impedance Z_w (the rigid walls have infinite impedance).

The momentum equation in the radial direction

$$\rho_0\frac{\partial u_r}{\partial t} + \frac{\partial p}{\partial r} = 0 \tag{1.144}$$

yields

$$u_r = -\frac{\partial p/\partial r}{j\omega\rho_0}. \tag{1.145}$$

Therefore,

$$Z_w \equiv \left(\frac{p}{u_r}\right)_{r=r_i} = \frac{-j\omega\rho_0 p}{\partial p/\partial r} \tag{1.146}$$

$$= \frac{-j\omega\rho_0 J_m\left(k_{r,m,n}r_i\right)}{k_{r,m,n}J'_m\left(k_{r,m,n}r_i\right)}, \tag{1.147}$$

where

$$J'_m\left(k_{r,m,n}r_i\right) = \left[\frac{dJ_m\left(k_{r,m,n}r\right)}{d\left(k_{r,m,n}r\right)}\right]_{r=r_i} \tag{1.148}$$

Thus, $k_{r,m,n}, n = 0, 1, 2\ldots$ are the infinite roots of the transcendental eigen equation

$$\frac{J_m(k_r r_i)}{(k_r r_i)J'_m(k_r r_i)} = j\frac{Z_w}{\rho_0 c_0}\frac{1}{k_0 r_i}. \tag{1.149}$$

It is instructive to compare this equation with Equations 1.134 and 1.137. $J'_m\left(k_{r,m,n}r_i\right)$ of Equation 1.149 corresponds to $\cos(k_x b/2)$ in Equation 1.134a, $\sin(k_x b/2)$ in Equation 1.134b, $\cos\left(k_y h/2\right)$ in Equation 1.137a, and $\sin\left(k_y h/2\right)$ in Equation 1.137b. The correspondence between k_x and k_y, and between r_i, $b/2$ and $h/2$ is of course obvious.

Upon substituting the (m, n) component of Equation 1.143 for acoustic pressure in the momentum equation for the axial direction, evaluating $u_{z,m,n}$, and then summing over m and n, one gets the following equation for acoustic particle velocity:

$$u_z(r,\theta,z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} J_m\left(k_{r,m,n}r\right)e^{jm\theta}e^{j\omega t} \times \frac{k_{z,m,n}}{k_0}\frac{1}{\rho_0 c_0}\left\{C_1 e^{-jk_{z,m,n}z} - C_2 e^{+jk_{z,m,n}z}\right\}, \quad (1.150)$$

The remarks following Equation 1.141 on the attenuation of waves along a rectangular duct with compliant walls apply as well to a circular duct.

It is worth noting that, unlike in the z and r directions, for which the solution in general consists of two terms, we have included only $e^{jm\theta}$ for the azimuthal direction; the $e^{-jm\theta}$ term has been omitted. This is because there are no restrictions or discontinuities in the azimuthal direction that would generate waves going in the opposite direction. The spiralling modes represented by $e^{jm\theta}$ can be excited by nonsymmetries in the system such as area discontinuities. In the exhaust systems of reciprocating machinery, therefore, radial as well as azimuthal modes are excited.

Incidentally, for the hypothetical case of axisymmetry, $m = 0$ and Equations 1.143 and 1.150 have only a single summation (over n); that is, Equation 1.143 reduces to

$$p(r,\theta,z,t) = \sum_{n=0}^{\infty} J_0\left(k_{r,n}r\right)e^{j\omega t} \times \left\{C_{1,m,n}e^{-jk_{z,m,n}z} + C_{2,m,n}e^{+jk_{z,m,n}z}\right\}, \quad (1.151)$$

where $k_{r,n}$ is the $(n+1)$th of the root of the eigen equation

$$-\frac{J_0\left(k_{r,n}r_i\right)}{\left(k_{r,n}r_i\right)J_1\left(k_{r,n}r_i\right)} = j\frac{Z_w}{\rho_0 c_0}\frac{1}{k_0 r_i}. \quad (1.152)$$

Impedance of the lining Z_w at the interface $(r = r_i)$ is evaluated later in Section 1.7.4 as a limiting case of the bulk reacting lining.

1.7.3 Rectangular Duct with Bulk Reacting Lining

A bulk reacting lining allows wave propagation inside the lining along the axis of the duct. Wave number of this wave is equal to the axial wave number inside the duct. In fact, all linings are basically bulk reacting in nature. Local reaction is a limiting or special case of the bulk reaction.

In the bulk reacting model, the lining is assumed to be a homogeneous, highly porous, fibrous or foam type material with open pores, (thermal insulation lining material is characterized by closed pores). Its characteristic impedance $Y_w(f)$ and wave number $k_w(f)$ are often given by complex empirical expressions in terms of flow resistivity, E, as shown in Chapter 6 of this monograph. Subscript w connotes wall lining.

The bulk reaction model consists in writing expressions for acoustic pressure and the axial and transverse particle velocity for a forward progressive wave in the air medium inside as well as the lining materials, and equating pressure and transverse particle velocity across the interface. (The effect of thin protective layer [17] and other practical aspects are discussed later in Chapter 6).

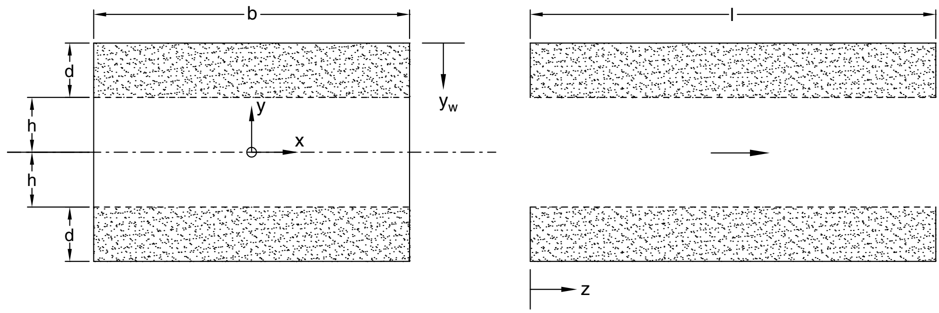

Figure 1.8 Schematic of a bulk reacting rectangular duct lined on two sides

Thus, for a rectangular duct lined on two opposite sides (shown in Figure 1.8), which incidentally represents one unit of the two-unit parallel baffle muffler shown in Figure 1.9, considering the lowest-order mode (corresponding to the plane wave), the field equations are as follows.

Air passage (subscript 0):

$$p(z, y, t) = C_1 \left(e^{-jk_{y,0}y} + C_2 e^{jk_{y,0}y}\right) e^{-jk_z z} e^{j\omega t} \tag{1.153}$$

$$u_{z,0}(z, y, t) = \frac{k_z}{k_0 Y_0} C_1 \left(e^{-jk_{y,0}y} + C_2 e^{jk_{y,0}y}\right) e^{-jk_z z} e^{j\omega t} \tag{1.154}$$

$$u_{y,0}(z, y, t) = \frac{k_{y,0}}{k_0 Y_0} C_1 \left(e^{-jk_{y,0}y} - C_2 e^{jk_{y,0}y}\right) e^{-jk_z z} e^{j\omega t} \tag{1.155}$$

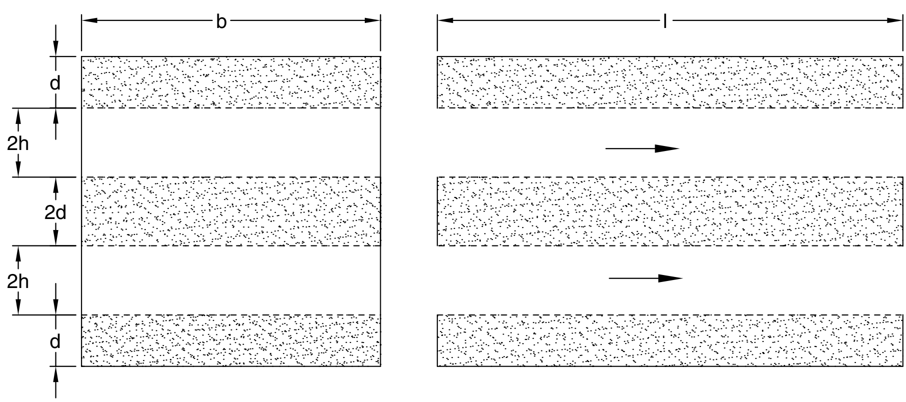

Figure 1.9 Schematic of a parallel baffle muffler consisting of two units of rectangular duct shown in Figure 1.8

Inside the wall lining (subscript w):

$$p_w(z, y, t) = C_1\left(e^{-jk_{y,w}y_w} + C_3 e^{jk_{y,w}y_w}\right)e^{-jk_z z}e^{j\omega t} \tag{1.156}$$

$$u_{z,w}(z, y_w, t) = \frac{k_z}{k_w Y_w} C_1\left(e^{-jk_{y,w}y_w} + C_3 e^{jk_{y,w}y_w}\right)e^{-jk_z z}e^{j\omega t} \tag{1.157}$$

$$u_{y,w}(z, y_w, t) = \frac{k_{y,w}}{k_w Y_w} C_1\left(e^{-jk_{y,w}y_w} - C_3 e^{jk_{y,w}y_w}\right)e^{-jk_z z}e^{j\omega t} \tag{1.158}$$

The overriding compatibility conditions:

$$k_{z,0} = k_{z,w} \equiv k_z \tag{1.159}$$

$$k_{y,0} = \left(k_0^2 - k_z^2\right)^{1/2} \tag{1.160}$$

$$k_{y,w} = \left(k_w^2 - k_z^2\right)^{1/2} \tag{1.161}$$

Boundary conditions:
At the center of the duct,

$$u_{y,0}(z, 0, t) = 0 \Rightarrow C_2 = 1 \tag{1.162}$$

At the rigid wall behind the lining,

$$u_{y,w}(z, 0, t) = 0 \Rightarrow C_3 = 1 \tag{1.163}$$

Across the interface,

$$p(z, h, t) = p_w(z, d, t) \tag{1.164}$$

$$u_{y,0}(z, h, t) = -u_{y,w}(z, d, t) \tag{1.165}$$

Making use of Equations 1.153, 1.156, 1.162 and 1.163, Equation 1.164 yields

$$\cos\left(k_{y,0}h\right) = \cos\left(k_{y,w}d\right) \tag{1.166}$$

Similarly, use of Equations 1.155, 1.158, 1.162 and 1.163 in Equation 1.165 gives

$$\frac{k_{y,0}}{k_0 Y_0}\sin\left(k_{y,0}h\right) = -\frac{k_{y,w}}{k_w Y_w}\sin\left(k_{y,w}d\right) \tag{1.167}$$

Dividing the two sides of Equation 1.66 with the corresponding sides of Equation 1.167 yields

$$\frac{k_0 Y_0}{k_{y,0}}\cot\left(k_{y,0}h\right) = -\frac{k_w Y_w}{k_{y,w}}\cot\left(k_{y,w}d\right) \tag{1.168}$$

Here, $k_{y,0}$ and $k_{y,w}$ are given by Equations 1.160 and 1.161, k_w and Y_w are complex functions of frequency, $k_0 = \omega/c_0$ and $Y_0 = \rho_0 c_0$.

Incidentally, for a locally reacting lining, $k_{z,w} = 0$ and therefore Equation 1.161 yields $k_{y,w} = k_w$. Then, Equation 1.168 would reduce to Equation 1.137a, provided

$$Z_w = -jY_w \cot(k_w d) \tag{1.169}$$

which is impedance of the rigid wall transferred to the interface (distance d), as will be shown in Chapter 3.

Like Equation 1.137a, the transcendental eigen equation (1.168) is solved for the common axial wave number, k_z, by means of the Newton–Raphson iteration scheme as discussed at some length in Chapter 6.

1.7.4 Circular Duct with Bulk Reacting Lining

For the circular duct with bulk reacting lining shown in Figure 1.7, the field equations for the lowest-order mode progressive wave corresponding to Equations 1.153–1.158 for rectangular duct would be as follows [17].

Air passage (subscript 0):

$$p(z, r, t) = C_1 J_0 (k_{r,0} r) e^{-jk_z z} e^{j\omega t} \tag{1.170}$$

$$u_{z,0}(z, r, t) = \frac{k_z}{\omega \rho_0} C_1 J_0 (k_{r,0} r) e^{-jk_z z} e^{j\omega t} \tag{1.171}$$

$$u_{r,0}(z, r, t) = -j \frac{k_{r,0}}{\omega \rho_0} C_1 J_1 (k_{r,0} r) e^{-jk_z z} e^{j\omega t} \tag{1.172}$$

Inside the wall lining (subscript w):

$$p_w(z, r, t) = C_2 \{ J_0 (k_{r,w} r) + C_3 N_0 (k_{r,w} r) \} e^{-jk_z z} e^{j\omega t} \tag{1.173}$$

$$u_{z,w}(z, r, t) = \frac{k_z}{\omega \rho_0} C_2 \{ J_0 (k_{r,w} r) + C_3 N_0 (k_{r,w} r) \} e^{-jk_z z} e^{j\omega t} \tag{1.174}$$

$$u_{r,w}(z, r, t) = \frac{-jk_{r,w}}{\omega \rho_w} C_2 \{ J_1 (k_{r,w} r) + C_3 N_1 (k_{r,w} r) \} e^{-jk_z z} e^{j\omega t} \tag{1.175}$$

where J and N denote the Bessel function and Neumann function, respectively. These are often called Bessel functions of the first kind and second kind, respectively. The Neumann function is often denoted by Y. However, in this monograph, Y has been used for characteristic impedance.

The overriding compatibility conditions:

$$k_{z,0} = k_{z,w} = k_z(say) \tag{1.176}$$

$$k_z^2 + k_{r,0}^2 = k_0^2 \Rightarrow k_{r,0} = \{ k_0^2 - k_z^2 \}^{1/2} \tag{1.177}$$

$$k_z^2 + k_{r,w}^2 = k_w^2 \Rightarrow k_{r,w} = \{ k_w^2 - k_z^2 \}^{1/2} \tag{1.178}$$

Boundary conditions:

At the rigid wall behind the lining, (that is, at $r = r_0$),

$$u_{r,w}(z, r_0, t) = 0 \Rightarrow C_3 = -\frac{J_1\left(k_{r,w} r_0\right)}{N_1\left(k_{r,w} r_0\right)} \tag{1.179}$$

At the interface (that is, at $r = r_i$), in the absence of a thin protective layer or perforated plate,

$$p(z, r_i, t) = p_w(z, r_i, t) \tag{1.180}$$

$$u_{r,0}(z, r_i, t) = u_{r,w}(z, r_i, t) \tag{1.181}$$

which yield the impedance relationship

$$Z_{r,0}(r_i, \omega) = Z_{r,w}(r_i, \omega), \quad Z_r \equiv \frac{p}{u_r} \tag{1.182}$$

On making use of Equations 1.170, 1.172, 1.173, 1.175, 1.180 and 1.181, Equation 1.182 yields

$$j\frac{\omega\rho_0}{k_{r,0}}\frac{J_0\left(k_{r,0} r_i\right)}{J_1\left(k_{r,0} r_i\right)} = j\frac{\omega\rho_w}{k_{r,w}}\frac{J_0\left(k_{r,w} r_i\right) + C_3 N_0\left(k_{r,w} r_i\right)}{J_1\left(k_{r,w} r_i\right) + C_3 N_1\left(k_{r,w} r_i\right)} \tag{1.183}$$

where C_3 is given by Equation 1.179 above.

In Equation 1.183, $j\omega$ has not been cancelled out between the LHS and RHS so as to retain correspondence to Equation 1.182 in terms of the respective impedances $Z_{r,0}$ and $Z_{r,w}$ on the inner and outer side of the interface at $r = r_i$.

It may be noted that

$$\omega\rho_0 = \frac{\omega}{c_0}.\rho_0 c_0 = k_0 Y_0 \tag{1.184}$$

and

$$\omega\rho_w = \frac{\omega}{c_w}.\rho_w c_w = k_w Y_w \tag{1.185}$$

For the limiting case of the locally reacting lining, $k_{z,w} = 0$, and therefore, as per Equation 1.178, $k_{r,w} = k_w$. Then, $Z_{r,w}$ which is given by the right-hand side of Equation 1.183 reduces to the following expression for the locally reacting lining:

$$Z_w = jY_w \frac{J_0(k_w r_i) + C_3' N_0(k_w r_i)}{J_1(k_w r_i) + C_3' N_1(k_w r_i)} \tag{1.186}$$

where C_3' equals C_3 with $\left(k_{r,w}\right)$ replaced with k_w in Equation 1.179. Thus,

$$C_3' = -\frac{J_1(k_w r_0)}{N_1(k_w r_0)} \tag{1.187}$$

Therefore, Z_w in the eigen equation (1.152) for the locally reacting lining is given by Equation 1.186 with C_3 given by Equation 1.187.

Evaluation of the wave number k_w and characteristic impedance Y_w of the lining material in terms of the flow resistivity, and so on is discussed in some detail later in Chapter 6.

1.8 Three-Dimensional Waves along Elliptical Ducts

Automotive exhaust mufflers are often elliptical in cross-section because of the constraint of clearing (space) under a car with low center of gravity. Three-dimensional analysis of elliptical shells is needed not only to evaluate the cut-off frequency for pure plane wave propagation, but also to analyze short flow-reversal end chambers that are often used in automotive mufflers. Here, we assume the medium to be stationary and inviscid.

For sinusoidal time dependence ($e^{j\omega t}$), that is, working in the frequency domain, the 3D wave equation reduces to the Helmholtz equation

$$\left(\nabla^2 + k_0^2\right)p = 0 \tag{1.188}$$

In terms of the elliptical cylindrical coordinates shown in Figure 1.10, the Laplacian ∇^2 is given by [18]

$$\nabla^2 = \frac{2}{h^2\{\cosh(2\xi) - \cos(2\eta)\}}\left[\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right] + \frac{\partial^2}{\partial z^2} \tag{1.189}$$

Here, ξ and η are the radial and angular elliptical coordinates, respectively. Conceptually, these are radial and azimuthal counterparts of the circular polar coordinate system (r, θ) shown in Figure 1.2. Accordingly, the curves of constant ξ define a family of confocal ellipses with semi-major axis. $D_1/2 = h\cosh(\xi)$ and semi-minor axis, $D_2/2 = h\sinh(\xi)$. Each of these ellipses has common foci at $x = \pm h$, where $2h$ is the interfocal distance. Thus, line $F'F$, connecting the two foci, represents the limiting '$\xi = 0$' ellipse. The curves of the constant η denote a family of confocal hyperbolae, as shown in Figure 1.10.

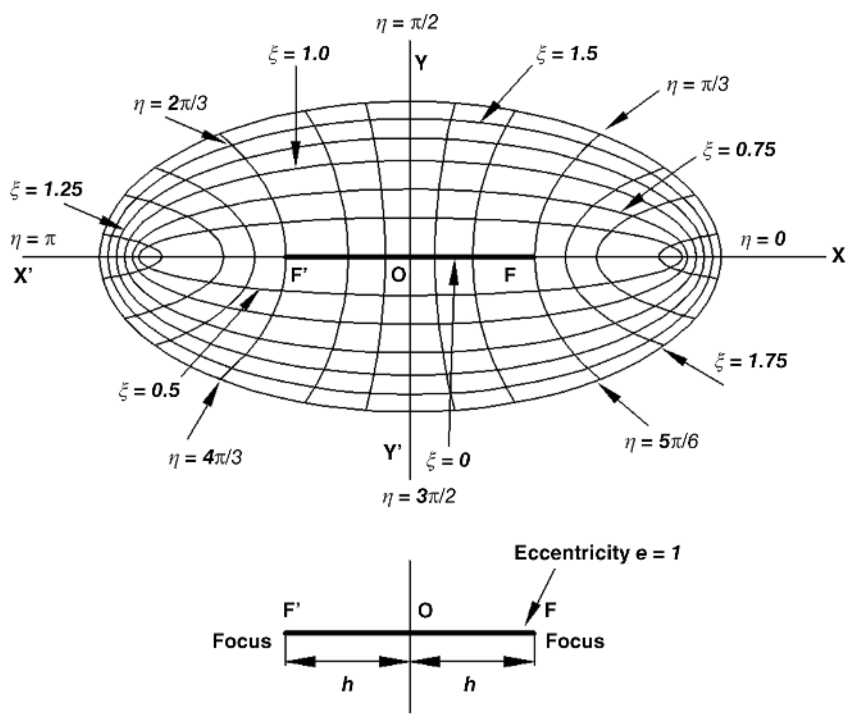

Figure 1.10 An elliptical coordinate system

Incidentally, elliptical cylindrical coordinates are related to the Cartesian coordinates as follows:

$$x = h\cosh(\xi)\cos(\eta,), \quad y = h\sinh(\xi)\sin(\eta), \quad z = z \tag{1.190}$$

where h is semi-interfocal distance. Thus, the $\eta = 0$ and π hyperbolae coincide with X-axis and the $\eta = \pi/2$ and $3\pi/2$ hyperbolae coincide with the Y-axis as shown in Figure 1.10. This behavior is similar to the azimuthal coordinates θ in the cylindrical polar coordinate system (see Figure 1.2).

It may be noted that the tangents at the point of intersection of the family of confocal ellipses and hyperbolae are at right angles, thereby confirming that the elliptical coordinate system is orthogonal and therefore would permit solution by separation of variables. The resulting ordinary differential equations are as follows [18–21]:

$$p(\xi,\eta,z) = p_\xi(\xi)p_\eta(\eta)p_z(z) \tag{1.191}$$

$$\frac{d^2p_\eta}{d\eta^2} + \{a - 2q\cos(2\eta)\}p_\eta = 0 \tag{1.192}$$

$$\frac{d^2p_\xi}{d\xi^2} + \{a - 2q\cosh(2\xi)\}p_\eta = 0 \tag{1.193}$$

$$\frac{d^2p_z}{dz^2} + k_z^2p_z = 0 \tag{1.194}$$

Equation 1.192 is called Mathieu differential equation, and Equation 1.193 is called the modified Mathieu differential equation. In these two equations,

$$q \equiv \frac{\left(k_0^2 - k_z^2\right)h^2}{4} \text{ or } \frac{2\sqrt{q}}{h} = \left(k_0^2 - k_z^2\right)^{1/2} \tag{1.195}$$

Thus, q plays the same role as the radial wave number in a circular duct. The constant ‘a’ is the separation constant which is to be so chosen that solutions are periodic in η, so that $p_\eta(\eta + 2\pi) = p_\eta(\eta)$.

General solutions to Equations 1.192–1.94 are of the type [18]

$$p_\eta(\eta) = C_1ce_m(\eta,q) + C_2se_m(\eta,q) \tag{1.196}$$

$$p_\xi(\xi) = C_3Ce_m(\xi,q) + C_uSe_m(\xi,q) \tag{1.197}$$

$$p_z(z) = C_5e^{-jk_zz} + C_6e^{jk_zz} \tag{1.198}$$

where ce_m and se_m are radial and angular Mathieu functions, respectively, and Ce_m and Se_m are the corresponding Modified Mathieu functions. These are given by [18,22] Mathieu functions:

$$\text{even-even:} \quad ce_{2n}(\eta,q) = \sum_{r=0}^{\infty} A_{2r}^{2n}\cos(2r\eta), \quad m = 2n, \quad n = 0,1,2,\ldots \tag{1.199}$$

$$\text{even-odd:} \quad ce_{2n-1}(\eta, q) = \sum_{r=1}^{\infty} A_{2r-1}^{2n-1} \cos((2r-1)\eta), \quad m = 2n-1, \quad n = 1, 2, 3, \ldots \tag{1.200}$$

$$\text{odd-even:} \quad se_{2n}(\eta, q) = \sum_{r=1}^{\infty} B_{2r}^{2n} \sin(2r\eta), \quad m = 2n, \quad n = 1, 2, 3, \ldots \tag{1.201}$$

$$\text{odd-odd:} \quad se_{2n-1}(\eta, q) = \sum_{r=1}^{\infty} B_{2r-1}^{2n-1} \sin((2r-1)\eta), \quad m = 2n-1, \quad n = 1, 2, 3, \ldots \tag{1.202}$$

Modified Mathieu functions:

$$\text{even-even:} \quad Ce_{2n}(\xi, q) = \sum_{r=0}^{\infty} A_{2r}^{2n} \cosh(2r\eta), \quad m = 2n, \quad n = 0, 1, 2, \ldots \tag{1.203}$$

$$\text{even-odd:} \quad Ce_{2n-1}(\xi, q) = \sum_{r=1}^{\infty} A_{2r-1}^{2n-1} \cosh((2r-1)\eta), \quad m = 2n-1, \quad n = 1, 2, 3, \ldots \tag{1.204}$$

$$\text{odd-even:} \quad Se_{2n}(\xi, q) = \sum_{r=1}^{\infty} B_{2r}^{2n} \sinh(2r\eta), \quad m = 2n, \quad n = 1, 2, 3, \ldots \tag{1.205}$$

$$\text{odd-odd:} \quad Se_{2n-1}(\xi, q) = \sum_{r=1}^{\infty} B_{2r-1}^{2n-1} \sinh((2r-1)\eta), \quad m = 2n-1, \quad n = 1, 2, 3, \ldots \tag{1.206}$$

It may be noted the Modified Mathieu functions are obtained from the corresponding Mathieu functions by replacing circular functions (cos and sine) by the corresponding hyperbolic functions (cosh and sinh). The function names starting with *c* or *C* involve cos and cosh functions and therefore are termed ‘even’, and those starting with *s* or *S* involve sine and sinh functions and therefore are termed ‘odd’. The second adjective denotes the order (2n: even, 2n-1: odd).

The boundary conditions across the interfocal line in Figure 1.10 are:

$$\text{Continuity of acoustic pressure:} \quad p(0, \eta) = p(0, -\eta) \tag{1.207}$$

$$\text{Continuity of pressure gradient:} \quad \frac{\partial p(0, \eta)}{\partial \xi} = -\frac{\partial p(0, -\eta)}{\partial \xi} \tag{1.208}$$

Then, it can be shown that acoustic pressure field inside a hollow elliptical chamber is given by [18]

$$\begin{aligned} p(\xi,\eta,z) \;=\; & \sum_{m=0}^{\infty} Ce_m(\xi,q)ce_m(\eta,q)\left(C_m^1 e^{-jk_z z} + C_m^2 e^{jk_z z}\right) \\ & + \sum_{m=1}^{\infty} Se_m(\xi,q)se_m(\eta,q)\left(S_m^1 e^{-jk_z z} + S_m^2 e^{jk_z z}\right) \end{aligned} \tag{1.209}$$

Constants C_m^1, C_m^2, S_m^1, and S_m^2, denote arbitrary constants to be determined from the boundary conditions in the axial direction.

The mode shapes and their propagation are dependent upon the numerical value of the q-parameter given by Equation 1.195. For evaluation of the cut-on frequencies of axial modes corresponding to a certain q valve, let us use the following relations [23]:

$$\frac{2\sqrt{q}}{h} = k_0, \quad \frac{2\sqrt{q}}{e} = k_0\frac{D_1}{2}, \quad h = \frac{D_1}{2}e, \quad e = \left\{1 - \left(\frac{D_2}{D_1}\right)^2\right\}^{1/2} \tag{1.210}$$

where e is eccentricity of the elliptical section, while D_1 and D_2 are, respectively, the major and minor axes of the elliptical section shown in Figure 1.10.

$q_{m,n}$ and $\overline{q}_{m,n}$, the n^{th} zero of the derivative of the Even and Odd type of the modified Mathieu functions, respectively, of order m, are roots of the following equations characterizing the rigid wall boundary conditions:

$$\left.\frac{dCe_m(\xi,q_{m,n})}{d\xi}\right|_{\xi=\xi_0} = 0 \text{ and } \frac{dSe_m(\xi,\overline{q}_{m,n})}{d\xi} = 0 \tag{1.211}$$

where

$$\xi_0 = \cosh^{-1}(1/e) \tag{1.212}$$

defines the value of the radial elliptical coordinates ξ at the boundary of the elliptical duct or shell shown in Figure 1.10.

By setting $k_z = 0$ in Equation 1.195, the nondimensional cut-on frequencies of the even or odd mode of order m corresponding to the n^{th} parametric zero, that is, $q_{m,n}$ and $\overline{q}_{m,n}$, respectively, are given by

$$k_0(D_1/2)|_{m,n(\text{Even})} = 2\sqrt{q_{m,n}}/e, \quad k_0(D_1/2)|_{m,n(\text{Odd})} = 2\sqrt{\overline{q}_{m,n}}/e. \tag{1.213}$$

Lowson and Bhaskaran [20] tabulated values of the nondimensional frequencies of an elliptical duct in terms of eccentricity e; not in terms of the aspect ratio, D_2/D_1, which would be of greater interest to muffler designers. They did not document nondimensional cut-on frequencies of the purely radial modes and cross modes. Recently, Mimani [23] has produced comprehensive tables of the q-parameters (parametric zeros of the derivative of the modified Mathieu functions). He has also developed interpolating polynomials for any arbitrary value of the aspect ratio, D_2/D_1, and incorporated the convective effect of mean flow.

Table 1.1 The q-parameters and the corresponding nondimensional cut-on frequencies (Extracted from the Mimani Tables [23])

D_2/D_1	Eccentricity e	Even-Odd $m = 0, n = 1$		Even-Even $m = 1, n = 1$	
		$q_{m,n}$	$(k_0 D_1/2)_{m,n}$	$q_{m,n}$	$(k_0 D_1/2)_{m,n}$
0.1	0.995	0.8804	1.8861	3.0042	3.4840
0.2	0.98	0.8523	1.8844	2.8999	3.4761
0.3	0.954	0.8056	1.8818	2.7280	3.4628
0.4	0.917	0.7407	1.8781	2.4909	3.4441
0.5	0.866	0.6582	1.8736	2.1918	3.4190
0.6	0.8	0.5585	1.8682	1.8346	3.3862
0.7	0.714	0.4422	1.8622	1.4240	3.3419
0.8	0.6	0.3099	1.8556	0.9680	3.2795
0.9	0.436	0.1623	1.8486	0.4825	3.1871
0.95	0.312	0.0830	1.8449	0.2382	3.1258
0.99	0.141	0.0169	1.8419	0.0469	3.0693

The Mimani tables [23] contain the parametric zeros $q_{m,n}$ and $\overline{q}_{m,n}$, for even and odd modes along with the corresponding nondimensional cut-on frequencies, for different values of the aspect ratio:

$$D_2/D_1 = 0.1, 0.2, \ldots\ldots\ldots 0.8, 0.9, 0.95, 0.99$$

for a hollow elliptical cross-section rigid-wall duct. These are listed below in Table 1.1 for the lowest two modes.

Incidentally, for the $(m = 0, n = 1)$ mode, $q_{m,n} = (k_0 D_1/2)_{m,n} = 0$. Therefore the $(m = 0, n = 1)$ mode represents the plane wave, corresponding to the (0, 0) mode of circular duct in Figure 1.4. As D_2/D_1 approaches unity, the nondimensional cut-on frequency of the Even-Odd (0, 1) mode approaches that of the (1, 0) mode of the circular duct $(k_0 D/2 = 1.84)$ and that of the Even-Even (1, 1) mode of the elliptical duct tends to that of the (2, 0) mode of the circular duct $(k_0 D/2 = 3.05)$ in Figure 1.4.

Finally, the cut-off frequency, below which all higher-order modes are cut-off (decay exponentially), of an elliptical cross-section duct of major axis D_1 is given by

$$k_0 D_1/2 \cong 1.86 (\text{within} \pm 1\% \text{accuracy}) \tag{1.214}$$

which compares with $k_0 D/2 = 1.84$ for circular duct (Equation 1.48). Thus, Equation 1.48 can also be used to evaluate the cut-off frequency of an elliptical duct, provided D is replaced by D_1, the major axis of the ellipse, not the geometric mean diameter, $(D_1 D_2)^{1/2}$.

In other words, the cut-off frequency of an elliptical section is lower than the corresponding circular duct with the same equivalent diameter by a factor

$$\frac{(D_1 D_2)^{1/2}}{D_1} = \left(\frac{D_2}{D_1}\right)^{1/2} \tag{1.215}$$

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