Chapter One

Trends and Trades

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1.1 Introduction

High-frequency data in finance is often characterized by fast fluctuations and noise (see, e.g., [7]), a trait that is known to make the volatility of the data very hard to estimate (see, e.g., [13]). Although this characteristic creates many challenges in modeling, it offers itself to the study of distinguishing "signal" from "noise," a topic of interest in the area of *quickest detection* (see [25], [5]). One of the most popular algorithms used in quickest detection is known as the *cumulative sum* (CUSUM) stopping rule first introduced by Page [24]. In this work, we employ a sequence of CUSUM stopping rules to construct an online trading strategy. This strategy takes advantage of the relatively frequent number of alarms CUSUM stopping times may provide when applied to high-frequency data as a result of the fast fluctuations present therein. The trading strategy implemented settles frequently and thus eliminates the risk of large positions. This makes the

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strategy implementable in practice. Prior work has been done by Lam and Yam [20] on drawing connections between CUSUM techniques and the filter trading strategy, yet both the filter trading strategy (see [2, 3]), or its equivalent, the buy and hold strategy (see [12]), run high risks of great losses mainly due to the randomness associated with settling. The well-known trailing stops strategy whose properties have been thoroughly studied in the literature (see, e.g., [15] or [1]) is also related to the filter strategy and thus suffers similar risks.

Although our proposed rule presents clear merits in terms of minimizing the risk of large positions by taking advantage of the high volatility frequently present in high-frequency data, the main purpose of this chapter is to present and illustrate the use of detection techniques (in this case the CUSUM) in high-frequency finance. In particular, the strategy proposed is based on running in parallel two CUSUM stopping rules: one detects an upward (+) change and the other a downward (−) change in the mean of the observations. Once an upward/downward CUSUM alarm (called a "signal") goes off, there is a buy/short sale of one unit of the underlying asset. At that moment, we repeat a CUSUM stopping rule, and for every alarm of the same sign, we continue buying or short selling one unit of the underlying asset until a CUSUM alarm of the opposite sign is set off, at which time we sell off all of what we bought or buy up all of what we short sold. The high frequency of CUSUM alarms in high-frequency tick data permits the implementation of this rule in practice since large exposures on one side, whether on the buy or on the sell side, are settled relatively quickly.

The algorithmic strategy proposed is applied on real tick data of a 30-year asset and a 5-year note sold at auction on various individual days. It is seen that the algorithm is most profitable in the presence of upward or downward trends (which we call "subperiods"), even in the presence of noise, and is less profitable on periods of price stability. The proposed strategy is, in fact, a trend-following algorithm.

To quantify the performance of the proposed algorithmic strategy, we calculate its expected reward in a simple random walk model. Our diagnostic plots indicate that the more biased the random walk is, the more profitable the proposed strategy becomes, which is consistent with the actual findings when the strategy is applied to real data. This is because in the presence of a bias, trends are more likely to form than in the absence of a bias.

We take the analytical approach of discrete data and a linear random walk model, rather than taking the continuous approach via, for example, the geometric Brownian motion model, because we are analyzing the movement of individual ticks of a price, quantized in a linear fashion (e.g., at the level of 1 cent, $\frac{1}{32}$ cent, or $\frac{1}{64}$ cent). Our models focus on tracking the motion of an asset price via these ticks, and so a linear approach is a more realistic setting, when short interest rate effects would be minimal.

We begin our analysis in Section 1.2 by describing a general trading strategy based on following upward or downward trends in a data stream, without specifying the timing mechanism behind such a strategy. We then develop the notion of gain over the time period of an individual trend. In Section 1.3, we build a timing scheme stemming from quickest detection considerations and give a preliminary performance evaluation of the overall strategy on real tick data. Next, in Section 1.4, we analyze the specific case of random walk-based data and calculate the expected value of the gain over a trend in this case. We give an explicit formula for this gain in the special case of simple asymmetric random walk on asset tick changes. Then, in Section 1.5, we give results of Monte Carlo simulations for the asymmetric lazy simple random walk and symmetric lazy random walk on tick changes. In Section 1.6, we discuss the effect of the CUSUM threshold parameter on the trading strategy. We conclude in Section 1.7 by a discussion of ways in which the proposed strategy may be improved with suggestions for further work.

1.2 A trend-based trading strategy

Let $\{S_n\}_{n=0,1,2...}$ be a sequence of data points; for our purposes, they will be samples of the price of an asset. We assume that $S_0 = s$ is a constant, and $S_k = 0$ for some k implies that $S_n = 0$ for all $n > k$. Let $T_0 = 0$, and define T_k, $k = 1, 2, ...$ as an increasing sequence of (stopping) times, called *signals*, noting some trend in the sequence. We call T_k the k-th *signal*.

1.2.1 SIGNALING AND TRENDS

In this subsection, we construct a trading strategy in the case that there are two types of signals: "+ signals" (declaring the detection of an upward trend in the data) and "− signals" (declaring the detection of a downward trend in the data). Let "Property $+(k)$" be the property that causes a + signal to occur as the kth signal, and denote this event by $\{T_k = T_k^+\}$. Likewise, let "Property $-(k)$" be the property that causes a − signal to occur as the k-th signal, and denote this by $\{T_k = T_k^-\}$. Only one type of trend can be detected at a time, so we formally define T_k^+ and T_k^- by

$$T_k^+ := \begin{cases} T_k & \text{if Property } +(k) \text{ occurs} \\ \infty & \text{if Property } -(k) \text{ occurs} \end{cases} \tag{1.1}$$

$$T_k^- := \begin{cases} T_k & \text{if Property } -(k) \text{ occurs} \\ \infty & \text{if Property } +(k) \text{ occurs} \end{cases} \tag{1.2}$$

Thus, $T_k = T_k^+ \wedge T_k^-$ for every $k = 1, 2, \ldots$.

Next, we state what it means for the data to stay in a trend. We define the sequence of signal indices $\alpha(l)$ as follows: let $\alpha(0) = 0$, so $T_{\alpha(0)} = 0$, and for $l \geq 1$, with $k \geq 2$, define the properties

$$\text{"Property } +(l,k)\text{"}: \quad T_j = T_j^- \text{ for every } \alpha(l-1) < j < k \text{ and } T_k = T_k^+$$
$$\text{"Property } -(l,k)\text{"}: \quad T_j = T_j^+ \text{ for every } \alpha(l-1) < j < k \text{ and } T_k = T_k^-.$$

Then, we define the *lth shift point* as, for $l = 1, 2, \ldots$,

$$\alpha(l) := \inf\{k \geq \alpha(l-1) + 2\text{: Property} + (l,k) \text{ or Property} - (l,k) \text{ holds}\}. \tag{1.3}$$

Note that $T_{\alpha(l)}$ is at least two signals after $T_{\alpha(l-1)}$. Definition (1.3) is equivalent to

$$\alpha(l) := \inf\{k \geq \alpha(l-1) + 2\text{:} T_k \text{ has different sign than } T_j, \alpha(l-1) < j < k\}. \tag{1.4}$$

A sequence of the same type of signal will be called a *subperiod* of the sample points. A shift point denotes the *end of a subperiod* of the same type of signal.

Let Δ_n be the number of shares of the asset S held at time n. Set $\Delta_0 = 0$. Note that, for every $n \in (T_{\alpha(l)}, T_{\alpha(l+1)})$, the sign of Δ_n is invariant, that is, either $\Delta_n > 0$ holds for every $n \in (T_{\alpha(l)}, T_{\alpha(l+1)})$ or $\Delta_n < 0$ holds for every $n \in (T_{\alpha(l)}, T_{\alpha(l+1)})$.

Our trading strategy is as follows:

$$\Delta_{n+1} = \begin{cases} \Delta_n & \text{if no signal at time } n, \textit{ i.e. } n \neq T_j \ \forall j \text{ (no change)} \\ \Delta_n + 1 & \text{if } n = T_j = T_j^+ \text{ for some } j,\ \alpha(l) < j < \alpha(l+1) \\ & \quad \text{for some } l \text{ (buy one during a } + \text{ subperiod)} \\ \Delta_n - 1 & \text{if } n = T_j = T_j^- \text{ for some } j,\ \alpha(l) < j < \alpha(l+1) \\ & \quad \text{for some } l \text{ (sell one during a } - \text{ subperiod)} \\ 0 & \text{if } n = T_{\alpha(l)} \text{ for some } l \geq 1 \\ & \quad \text{(buy-up if } T^+_{\alpha(l)};\ \text{sell-off if } T^-_{\alpha(l)}). \end{cases} \tag{1.5}$$

We assume a market in which all market orders are instantly fulfilled. The intent of this strategy is to profit from following subperiods of + or − signals by the old adage “buy low, sell high.” The success of this strategy relies mainly on the length of such subperiods.

1.2.2 GAIN OVER A SUBPERIOD

We wish to analyze the gain G_l, $l = 1, 2, \ldots$, for this trading strategy over the time period $(T_{\alpha(l-1)}, T_{\alpha(l)}]$, called *subperiod* l; this is the amount of cash earned or lost by liquidating the transactions made from signals $T_{\alpha(l-1)+1}, \ldots, T_{\alpha(l)-1}$ at $T_{\alpha(l)}$.

Note that a subperiod is determined by the first signal on that run: if $T_1 = T_1^+$, then the run from signal 1 to signal $\alpha(1) - 1$ is a “bull run” subperiod of individual buy orders followed by a sell-off at time $T_{\alpha(1)} = T^-_{\alpha(1)}$; if $T_1 = T_1^-$, then this run is a “bear run” subperiod of individual short sales followed by a buy-up at $T_{\alpha(1)} = T^+_{\alpha(1)}$. Define G_l to be the gain on subperiod l; thus, G_1 is the gain on the first subperiod, starting at signal $T_{\alpha(0)+1} = T_1$ and ending at signal $T_{\alpha(1)}$. We require, as a condition, the sign of the first signal of the subperiod. Let $c \geq 0$ be the percentage cost per transaction, and define

$$A_l := 1_{\{T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1}\}}, \quad Y_l := \alpha(l) - \alpha(l-1) - 1. \tag{1.6}$$

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The gain on a subperiod is calculated as follows:

$$G_l := \begin{cases} (1-c)\sum_{j=\alpha(l-1)+1}^{\alpha(l)-1} S_{T_j} - (1+c)(\alpha(l)-\alpha(l-1)-1)S_{T_{\alpha(l)}} \\ \quad \text{if } T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1}, \\ (1-c)(\alpha(l)-\alpha(l-1)-1)S_{T_{\alpha(l)}} - (1+c)\sum_{j=\alpha(l-1)+1}^{\alpha(l)-1} S_{T_j} \\ \quad \text{if } T_{\alpha(l-1)+1} = T^+_{\alpha(l-1)+1} \end{cases}$$
$$:= \begin{cases} (1-c)\sum_{j=1}^{Y_l} S_{T_{j+\alpha(l-1)}} - (1+c)(Y_l)S_{T_{\alpha(l)}} \text{ if } T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1}, \\ (1-c)(Y_l)S_{T_{\alpha(l)}} - (1+c)\sum_{j=1}^{Y_l} S_{T_{j+\alpha(l-1)}} \text{ if } T_{\alpha(l-1)+1} = T^+_{\alpha(l-1)+1}. \end{cases} \tag{1.7}$$

For example, if $c = 0.01$, $T_1 = T_1^+$, and $\alpha(1) = 4$, then $T_{\alpha(1)} = T_4 = T_4^-$. Say the prices at the buy-signal times are $S_{T_1} = 5$, $S_{T_2} = 7$, $S_{T_3} = 9$, and we sell everything off at $S_{T_4} = 8$. Then $\Delta_{T_0} = 0$, $\Delta_{T_1} = 1$, $\Delta_{T_2} = 2$, $\Delta_{T_3} = 3$, and we liquidate at time T_4 to $\Delta_{T_4} = 0$. The gain on the first subperiod would then be $G_1 = (0.99)(3)(8) - (1.01)(5 + 7 + 9) = 2.55$.

Combining the $1 - c$ terms and adding on the random variable $2cY_lS_{\alpha(T_l)}$, we have after some algebra a sum of price increments:

$$\begin{aligned} G_l + 2cY_lS_{\alpha(T_l)} &= \left(c + (-1)^{A_l}\right)\left[Y_lS_{T_{\alpha(l)}} - \sum_{j=1}^{Y_l} S_{T_{j+\alpha(l-1)}}\right] \\ &= \left(c + (-1)^{A_l}\right)\sum_{j=1}^{Y_l}(S_{T_{\alpha(l)}} - S_{T_{j+\alpha(l-1)}}). \end{aligned} \tag{1.8}$$

We can rewrite each difference in the sum as a telescoping sum: setting

$$Z_k := S_{T_{k+1}} - S_{T_k}, \quad k = 1, 2, ..., \tag{1.9}$$

as the incremental price change between signals k and $k + 1$, we have

$$S_{T_{\alpha(l)}} - S_{T_{j+\alpha(l-1)}} = \sum_{k=j+\alpha(l-1)}^{\alpha(l)-1}(S_{T_{k+1}} - S_{T_k}) = \sum_{k=j+\alpha(l-1)}^{\alpha(l)-1} Z_k = \sum_{k=j}^{Y_l} Z_{k+\alpha(l-1)}.$$

Substituting this back into (1.8) yields

$$G_l + 2cY_lS_{\alpha(T_l)} = \left(c + (-1)^{A_l}\right)\sum_{j=1}^{Y_l}\left[\sum_{k=j+\alpha(l-1)}^{\alpha(l)-1} Z_k\right] = \left(c + (-1)^{A_l}\right)\sum_{j=1}^{Y_l} jZ_{j+\alpha(l-1)}. \tag{1.10}$$

Therefore, by (1.11), the gain over subperiod l is

$$G_l = \left(c + (-1)^{A_l}\right) \sum_{j=1}^{Y_l} jZ_{j+\alpha(l-1)} - 2cY_l S_{\alpha(T_l)}. \tag{1.11}$$

Note that, in the absence of transaction costs (i.e., $c = 0$), the expected gain G_l is entirely determined by price increments and the sign of the first signal of the subperiod.

1.3 CUSUM timing

Next, we describe a version of the CUSUM statistic process and its associated CUSUM stopping rule, which we will use to devise a timing scheme based on the quickest detection of trends, and incorporate this scheme to our trading strategy.

1.3.1 CUSUM PROCESS AND STOPPING TIME

In this section, we begin by introducing the measurable space $(\Omega, \mathcal{F})$, where $\Omega = \mathbb{R}^\infty$, $\mathcal{F} = \cup_n \mathcal{F}_n$, and $\mathcal{F}_n = \sigma\{Y_i, i \in \{0, 1, \ldots, n\}\}$. The law of the sequence Y_i, $i = 1, \ldots$, is described by the family of probability measures $\{P_\nu\}$, $\nu \in \mathbb{N}^*$. In other words, the probability measure P_ν for a given $\nu > 0$, playing the role of the *change point*, is the measure generated on Ω by the sequence Y_i, $i = 1, \ldots$, when the distribution of the Y_i's changes at time ν. The probability measures P_0 and P_∞ are the measures generated on Ω by the random variables Y_i when they have an identical distribution. In other words, the system defined by the sequence Y_i undergoes a "regime change" from the distribution P_0 to the distribution P_∞ at the change point time ν.

The *CUSUM statistic* is defined as the maximum of the log-likelihood ratio of the measure P_ν to the measure P_∞ on the σ-algebra $\mathcal{F}_n$. That is,

$$C_n := \max_{0 \le \nu \le n} \log \left. \frac{dP_\nu}{dP_\infty} \right|_{\mathcal{F}_n} \tag{1.12}$$

is the CUSUM statistic on the σ-algebra $\mathcal{F}_n$. The *CUSUM statistic process* is then the collection of the CUSUM statistics $\{C_n\}$ of (1.12) for $n = 1, \ldots$.

The *CUSUM stopping rule* is then

$$T(h) := \inf \left\{ n \geq 0 : \max_{0 \leq \nu \leq n} \log \left. \frac{dP_\nu}{dP_\infty} \right|_{\mathcal{F}_n} \geq h \right\}, \tag{1.13}$$

for some threshold $h > 0$. In the CUSUM stopping rule (1.13), the CUSUM statistic process of (1.12) is initialized at

$$C_0 = 0. \tag{1.14}$$

The CUSUM statistic process was first introduced by Page [24] in the form that it takes when the sequence of random variables Y_i is independent and Gaussian; that is, $Y_i \sim N(\mu, 1)$, $i = 1, 2, \ldots$, with $\mu = \mu_0$ for $i < \nu$ and $\mu = \mu_1$ for $i \geq \nu$. Since its introduction by Page [24], the CUSUM statistic process of (1.12) and its associated CUSUM stopping time of (1.13) have been used in a plethora of applications where it is of interest to perform detection of abrupt changes in the statistical behavior of observations in real time. Examples of such applications are signal processing (see [10]), monitoring the outbreak of an epidemic (see [29]), financial surveillance (see [14] and [9]), and more recently computer vision (see [19] or [30]). The popularity of the CUSUM stopping time (1.13) is mainly due to its low complexity and optimality properties (see, for instance, [21], [22, 23], [6] and [27] or [26]), in both discrete and continuous time models.

As a specific example, we now derive the form in which Page [24] introduced the CUSUM. To this effect, let $Y_i \sim N(\mu_0, \sigma^2)$ that change to $Y_i \sim N(\mu_1, \sigma^2)$ at the change point time ν. We now proceed to derive the form of the CUSUM statistic process (1.12) and its associated CUSUM stopping time (1.13) in the example set forth in this section. To this effect, let us now denote by $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ the Gaussian kernel. For the sequence of random variables Y_i given earlier, we can now compute (see also [28] or [25]):

$$\begin{aligned} C_n &= \max_{0 \leq \nu \leq n} \log \left. \frac{dP_\nu}{dP_\infty} \right|_{\mathcal{F}_n} = \max_{0 \leq \nu \leq n} \log \frac{\prod_{i=1}^{\nu-1} \phi\left(\frac{Y_i - \mu_0}{\sigma}\right) \prod_{i=\nu}^{n} \phi\left(\frac{Y_i - \mu_1}{\sigma}\right)}{\prod_{i=1}^{n} \phi\left(\frac{Y_i - \mu_0}{\sigma}\right)} \\ &= \frac{1}{\sigma^2} \max_{0 \leq \nu \leq n} (\mu_1 - \mu_0) \sum_{i=\nu}^{n} \left[Y_i - \frac{\mu_1 + \mu_0}{2} \right]. \end{aligned} \tag{1.15}$$

In view of (1.14), we initialize the sequence (1.15) at $Y_0 = \frac{\mu_1+\mu_0}{2}$ and proceed to distinguish the following two cases:

- Case 1: $\mu_1 > \mu_0$: divide out $\mu_1 - \mu_0$, multiply by the constant σ^2 in (1.15), and use (1.13) to obtain the CUSUM stopping rule T^+ :

$$T^+(h^+) = \inf \left\{ n \geq 0 : \max_{0 \leq \nu \leq n} \sum_{i=\nu}^{n} \left[Y_i - \frac{\mu_1 + \mu_0}{2} \right] \geq h^+ \right\} \quad (1.16)$$

  for an appropriately scaled threshold $h^+ > 0$.

- Case 2: $\mu_1 < \mu_0$: divide out $\mu_1 - \mu_0$, multiply by the constant σ^2 in (1.15), and use (1.13) to obtain the CUSUM stopping rule T^-:

$$T^-(h^-) = \inf \left\{ n \geq 0 : \max_{0 \leq \nu \leq n} \sum_{i=\nu}^{n} \left[\frac{\mu_1 + \mu_0}{2} - Y_i \right] \geq h^- \right\} \quad (1.17)$$

  for an appropriately scaled threshold $h^- > 0$.

As shown in the study [24] or [11], we can reexpress the stopping times (1.16) and (1.17) in terms of the recurrence relations

$$u_0 = 0; \; u_n := \max \left\{ 0, u_{n-1} + \left(Y_n - \frac{\mu_1 + \mu_0}{2} \right) \right\} \quad (1.18)$$

$$d_0 = 0; \; d_n := \max \left\{ 0, d_{n-1} - \left(Y_n - \frac{\mu_1 + \mu_0}{2} \right) \right\}, \quad (1.19)$$

which lead to

$$T^+(h^+) = \inf\{n > 0 : u_n \geq h^+\}, \quad (1.20)$$

$$T^-(h^-) = \inf\{n > 0 : d_n \geq h^-\}. \quad (1.21)$$

The sequences u_n and d_n of (1.18) and (1.19), respectively, form a CUSUM according to the deviation of the monitored sequential observations Y_n from the average of their pre- and postchange means. The first time that one of these sequences reaches its threshold (in (1.20) or (1.21)), the respective alarm T^+ or T^- fires.

Although the stopping times (1.16) and (1.17) and their respective equivalents (1.20) and (1.21) can be derived by formal CUSUM regime change considerations using the example set forth in this section, they may also be used as general nonparametric stopping rules directly applied to sequential observations as seen in the study by Brodsky and Darkhovsky

[8] or Devore [11]. The former can be used as a general stopping rule to detect an upward change in the mean while the latter a downward one. In many applications, it is of interest to monitor an upward or downward change in the mean of sequential observations simultaneously. This gives rise to the two-sided CUSUM (2-CUSUM), which was first introduced by Barnard [4], and whose optimality properties have been established in Hadjiliadis [17], Hadjiliadis and Moustakides [16], and Hadjiliadis et al. [18]. In the context presented in this section, the 2-CUSUM stopping time takes the form

$$T^{+}(h^{+}) \wedge T^{-}(h^{-}), \tag{1.22}$$

where $T^{+}(h^{+})$ appears in (1.20) and $T^{-}(h^{-})$ in (1.21). The symmetric version of the 2-CUSUM stopping time is that of (1.22) when $h^{+} = h^{-} = h$.

1.3.2 A CUSUM TIMING SCHEME

We now apply the aforementioned CUSUM stopping rule of (1.22) to a stream of data representing the value of the underlying asset without any model assumptions. In other words, the underlying asset is not necessarily assumed to be independent or normally distributed. That is, we apply the forms (1.16) and (1.17) in a nonparametric fashion. Let $M > 0$ denote the "tick size" of the asset being monitored (presuming that S changes in increments of M; we do not know the probability distribution of these changes), and $h > 0$ be a given threshold. Given that $S_0 = s$, recall that $T_0 = 0$. We monitor the progress of upward or downward adjustments in the price S_n of the underlying, by individual ticks.

In view of the previous subsection at time T_k, μ_0 is set to the value of the underlying at time T_k, namely $\mu_0 = S_{T_k}$, and $\mu_1^u = S_{T_k} + M$ and $\mu_1^d = S_{T_k} - M$ are the two "new" mean levels to be monitored against. Thus, as in equations (1.18) and (1.19), which cumulate the deviations of the monitored sequence from the average of their pre- and postchange means, we now monitor the deviations of the underlying sequence S_n, $n = 1, 2 \ldots$, from the quantities

$$\begin{aligned} m_k^u &:= \frac{(S_{T_k} + M) + S_{T_k}}{2} = S_{T_k} + \frac{M}{2}, \\ m_k^d &:= \frac{(S_{T_k} - M) + S_{T_k}}{2} = S_{T_k} - \frac{M}{2}, \end{aligned} \tag{1.23}$$

where $k \geq 0$. To this effect, set $u_0^k = d_0^k = 0$, and for $n \geq 1$, define the CUSUM statistics

$$\begin{aligned} u_n^k &:= \max\{0, u_{n-1}^k + (S_{n+T_k} - m_k^u)\}, \\ d_n^k &:= \max\{0, d_{n-1}^k - (S_{n+T_k} - m_k^d)\}. \end{aligned} \tag{1.24}$$

Thus, for $k \geq 0$, the CUSUM timing scheme for our trend-following trading strategy is defined by using (1.20) and (1.21) (and coming from (1.1) and (1.2)),

$$\begin{aligned} &\text{Property } +(k+1): \ u_n^k \geq h; \quad \text{Property } -(k+1): \ d_n^k \geq h \\ &j_k^* := \min\{n > 0: \ \text{Property } +(k+1) \text{ or } -(k+1) \text{ occurs}\} \\ &T_{k+1} := T_k + j_k^*. \end{aligned} \tag{1.25}$$

In other words, each T_k is the symmetric 2-CUSUM stopping time of (1.22) for cycle k. Finally, at the "end of day," that is, on the final tick, we close out our position, inducing a final shift point to end trading, for algorithmic purposes.

1.3.3 US TREASURY NOTES, CUSUM TIMING

The following figures and chart describe the CUSUM timing scheme (1.25) applied to the trading strategy (1.5) for US Treasury notes sold at auction in 2011. Gains quoted are in increments of \$1000. In Figure 1.1, we show the

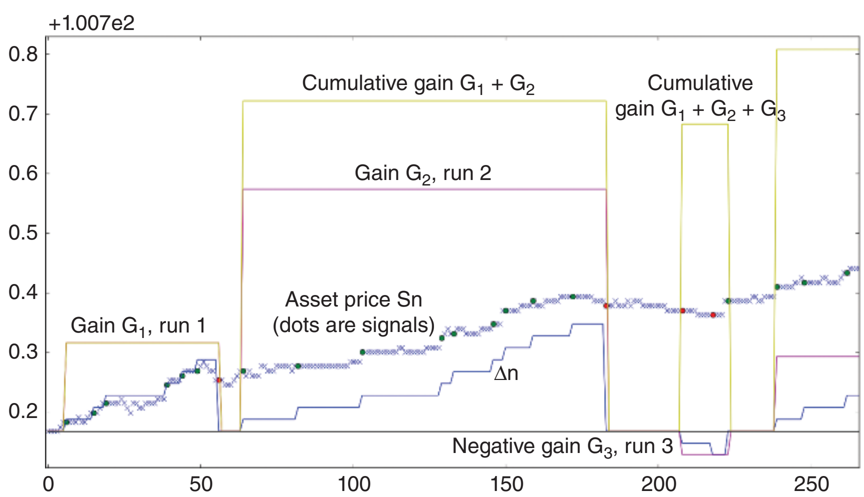

FIGURE 1.1 Plot of the first subperiods, and cumulative gain, for the CUSUM strategy, August 2, 2011, US 5-year treasury note.

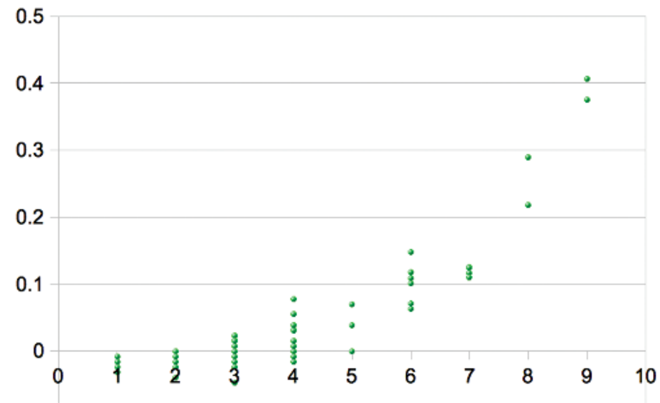

FIGURE 1.2 Lengths of subperiods versus gains, August 2, 2011, US 5-year treasury note.

asset price, along with the number of shares held, per-subperiod gain, and running total gain. Figures 1.2, 1.3, 1.4, 1.5 and 1.6 show the individual subperiod gains, plotted by the number of signals during a subperiod, of the gain for 5-year and 30-year treasury notes, and Figure 1.7 aggregates the data from Figures 1.3, 1.4, 1.5 and 1.6 for 30-year notes.

1.4 Example: Random walk on ticks

We now describe a simple example to model the asset price motions. Assume that $\exists N > 0$ such that the sequence $\{X_j\}_{j\in\mathbb{N}}$ are the steps of a

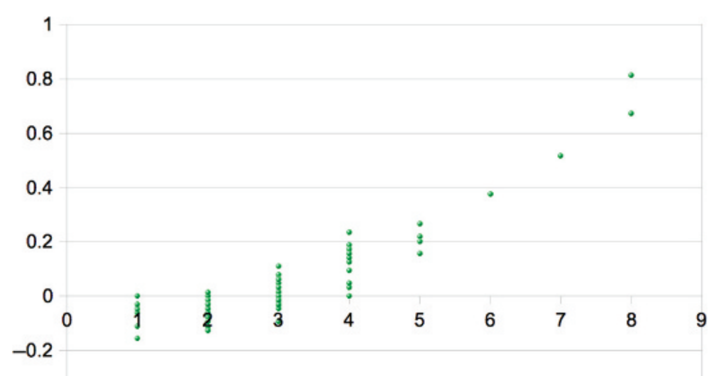

FIGURE 1.3 Subperiod length versus gain, July 29, 2011, US 30-year treasury note.

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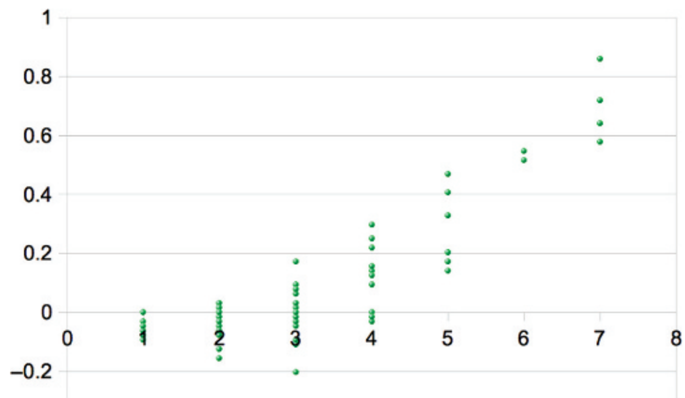

FIGURE 1.4 Subperiod length versus gain, August 1, 2011, US 30-year treasury note.

random walk taking integer values bounded between $-N$ and N, that is, $|X_j| \leq N$ for all $j \in \mathbb{N}$, and that $X_j \in \{-N, -N+1, ..., N-1, N\}$ for every j, with $p_k = P(X_j = k) \geq 0$ and $\sum_{k=-N}^{N} p_k = 1$. Let $S_0 = s$, and for $n \geq 1$, set $S_n = s + \sum_{j=1}^{n} X_j$. We will consider S_n to be a random walk on ticks, rather than price itself, and so normalize tick size to $M = 1$.

Note that, since $\Delta_n = 0 \iff n = \alpha(l)$ for some $l \in \{0, 1, 2, ...\}$, the expected gain over a subperiod is the expected gain over an excursion to zero on Δ_n, and so we can simply consider the first excursion (independent of other excursions) on the time interval $(T_{\alpha(0)} = 0, T_{\alpha(1)}]$. Also, note that

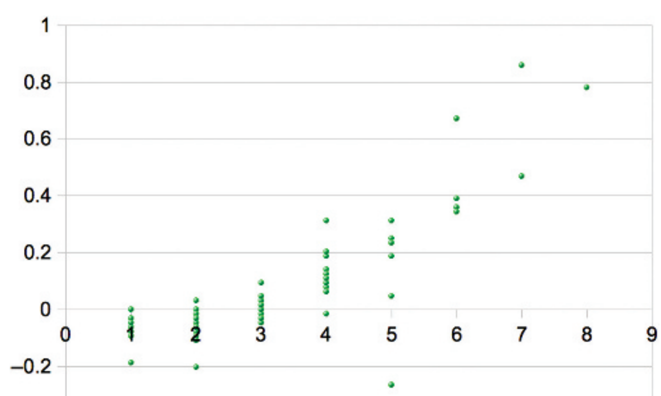

FIGURE 1.5 Subperiod length versus gain, August 2, 2011, US 30-year treasury note.

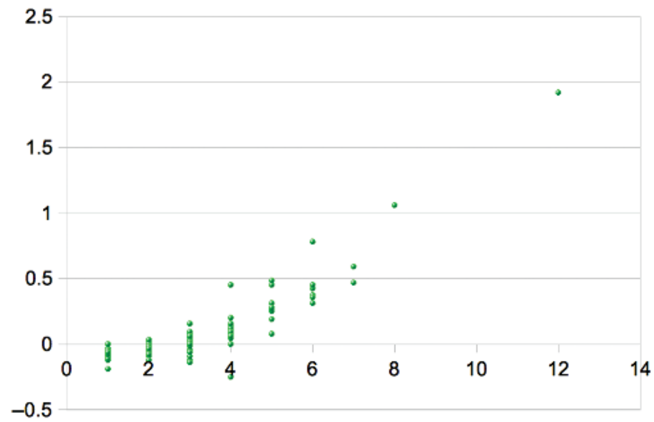

FIGURE 1.6 Subperiod length versus gain, August 3, 2011, US 30-year treasury note.

in this case, if the transaction cost $c = 0$, the G_l of (1.11) are IID random variables.

Set

$$p^+ := P(T_1 = T_1^+), \;\; p^- := 1 - p^+ = P(T_1 = T_1^-), \tag{1.26}$$

and note that signal timing increments are independent. Conditioned on the sign of signal $\alpha(l-1)+1$ at time $T_{\alpha(l-1)+1}$, Y_l is a geometric random

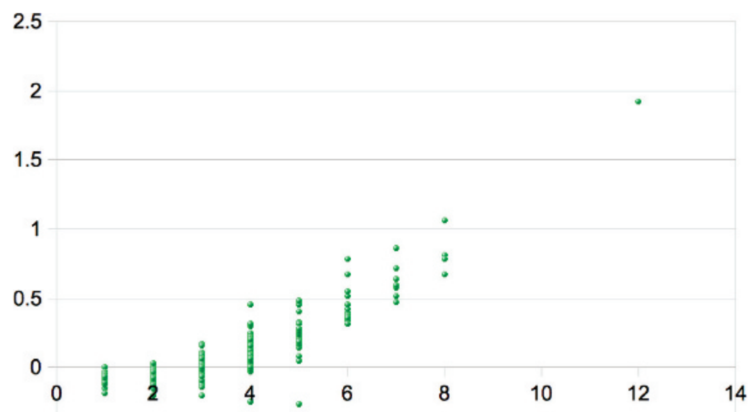

FIGURE 1.7 Figures 1.3, 1.4, 1.5 and 1.6 combined (30-year).

variable (starting at 1) which gives the number of signals of the same sign in subperiod l. The distribution of Y_l, conditioned on $\mathcal{F}_{T_{\alpha(l-1)+1}}$, is

$$Y_l \sim \begin{cases} \text{geom}(p^-) & \text{if } T_{\alpha(l-1)+1} = T^+_{\alpha(l-1)+1}, \\ \text{geom}(p^+) & \text{if } T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1}. \end{cases} \tag{1.27}$$

To explain this, consider the case $T_{\alpha(l-1)+1} = T^+_{\alpha(l-1)+1}$ (the first + signal of a bull run subperiod): a subperiod of + has “failure” probability p^+ (a + signal continues the subperiod with another buy) and “success” probability p^- (a − signal causes a sell-off and ends the subperiod).

A_l is an $\mathcal{F}_{T_{\alpha(l-1)+1}}$-measurable random variable, and every increment in the sum in (1.11) is independent of time $T_{\alpha(l-1)+1}$. Finally, note that the Y_l are independent of the walk up to time $T_{\alpha(l-1)}$, and if $c = 0$, so are the G_l.

1.4.1 RANDOM WALK EXPECTED GAIN OVER A SUBPERIOD

We wish to examine the expected gain $E(G_l)$ over subperiod l. For simplicity in our initial analysis, set $c = 0$. Since the G_l are IID, we will calculate $E(G_1)$. This is, since $\alpha(0) = 0$ and $Y_1 = \alpha(1) - \alpha(0) - 1 = \alpha(1) - 1$, by (1.11),

$$E(G_1) = E\left[(-1)^{A_1} \sum_{j=1}^{Y_1} jZ_j\right]. \tag{1.28}$$

We condition over the possible values of Y_1 and A_1. Note that the sign of T_1 also determines the possibilities of Z_j for $j = 1, 2, ..., Y_1 - 1$. Z_j depends on the type of subperiod it resides on, so by the fact that the event $\{Y_l = n\} \in \mathcal{F}_{T_{\alpha(l)}}$, and by setting, for $j = \alpha(l-1)+1, ..., \alpha(l)$,

$$\begin{aligned} B^+_{j,l,n} &:= E(Z_j \mid T_{\alpha(l-1)+1} = T^+_{\alpha(l-1)+1},\ Y_l = n), \\ B^-_{j,l,n} &:= E(Z_j \mid T_{\alpha(l-1)+1} = T^-_{\alpha(l-1)+1},\ Y_l = n), \end{aligned} \tag{1.29}$$

then, for $n = 1, 2, ...,$ we have

$$\begin{aligned} E\left[\sum_{j=1}^{Y_1} jZ_j \,\middle|\, T_1 = T_1^+, Y_1 = n\right] &= E\left[\sum_{j=1}^{n} jZ_j \,\middle|\, T_1 = T_1^+, Y_1 = n\right] \\ &= \sum_{j=1}^{n} jE\left[Z_j \,\middle|\, T_1 = T_1^+, Y_1 = n\right] = \sum_{j=1}^{n} jB^+_{j,1,n}. \end{aligned} \tag{1.30}$$

Since the conditioning on $B^+_{j,1,n}$ (and, likewise, $B^-_{j,1,n}$) is based only on the walk during the time increments $(T_0, T_1]$ and $(T_{\alpha(1)-1}, T_{\alpha(1)}]$, for $n > 1$, $B^+_{j,1,n}$ and $B^-_{j,1,n}$ are numbers for $j = 1, 2, ..., n-1$. Also, for these j, $B^+_{j,1,n}$ are the same by the strong Markov property at T_{j-1} since the signs on the T_j are all +. However, since the signal $T_{\alpha(1)} = T_{n+1}$ has different sign than T_n, $B^+_{n,1,n}$ has a different distribution. In fact, since this condition implies that $T_{n+1} = T_{\alpha(1)} = T^-_{\alpha(1)}$, $B^+_{n,1,n}$ can be written by the strong Markov property at $T_n = T_{\alpha(1)-1}$ as

$$\begin{aligned} B^+_{n,1,n} &= E\left[Z_n \,\middle|\, T_1 = T_1^+, Y_1 = n\right] = E\left[Z_n \,\middle|\, T_1 = T_1^+, T_{\alpha(1)=n+1} = T^-_{\alpha(1)}\right] \\ &= E\left[Z_n \,\middle|\, T_{\alpha(1)=n+1} = T^-_{\alpha(1)}\right] = B^-_{1,1,n} \end{aligned}$$

To simplify notation, we rewrite $B^+_{1,1,n} = B^+$ and $B^-_{1,1,n} = B^-$, since they do not depend on n. In the case $n = 1$, we simply have $B^+_{1,1,1} = B^-$ and $B^-_{1,1,1} = B^+$, and note that $B^+ \geq 0$ and $B^- \leq 0$. Thus, our sum (1.30) becomes

$$\begin{aligned} E\left[\sum_{j=1}^{Y_1} jZ_j \,\middle|\, T_1 = T_1^+, Y_1 = n\right] &= \sum_{j=1}^{n} jB^+_{j,1,n} = \sum_{j=1}^{n-1} jB^+_{j,1,n} + nB^+_{n,1,n} \\ &= \frac{n(n-1)}{2}B^+ + nB^-. \end{aligned} \tag{1.31}$$

The only thing that needs to change for the analogous argument for $B^-_{j,1,n}$ are the signs; thus, we also have

$$E\left[\sum_{j=1}^{Y_1} jZ_j \,\middle|\, T_1 = T_1^-, Y_1 = n\right] = \frac{n(n-1)}{2}B^- + nB^+. \tag{1.32}$$

Next, we give the probability that $Y_1 = n$, conditioned on the sign of T_1. This is easy, since we know that, conditioned on the sign of T_1, Y_1 is a geometric random variable. By (1.27), for $n = 1, 2, ...,$

$$\begin{aligned} P(Y_1 = n \mid T_1 = T_1^+) &= (p^+)^{n-1}(p^-) \\ P(Y_1 = n \mid T_1 = T_1^-) &= (p^-)^{n-1}(p^+). \end{aligned} \tag{1.33}$$

By (1.31), (1.32), and (1.33), and recalling that $p^- = 1 - p^+$, the expected gain on a subperiod, given that the subperiod consists of n signals before a liquidation, is

$$
\begin{aligned}
E(G_1 \mid Y_1 = n) &= p^+ E\left[\sum_{j=1}^{Y_1} jZ_j \,\middle|\, T_1 = T_1^+, Y_1 = n\right] \\
&\quad - p^- E\left[\sum_{j=1}^{Y_1} jZ_j \,\middle|\, T_1 = T_1^-, Y_1 = n\right] \\
&= p^+ \left(\frac{n(n-1)}{2} B^+ + nB^-\right) - p^- \left(\frac{n(n-1)}{2} B^- + nB^+\right) \\
&= \frac{n(n+1)}{2} \left(B^+ p^+ - B^- p^-\right) + n(B^- - B^+). \qquad (1.34)
\end{aligned}
$$

The probability that a subperiod lasts n signals, regardless of its sign, is, by (1.27) and (1.33),

$$
\begin{aligned}
P(Y_1 = n) &= P(Y_1 = n \mid T_1 = T_1^+) P(T_1 = T_1^+) + P(Y_1 = n \mid T_1 = T_1^-) P(T_1 = T_1^-) \\
&= (p^+)^n (p^-) + (p^-)^n (p^+), \qquad (1.35)
\end{aligned}
$$

which also gives the expected number of same-sign signals in a subperiod

$$
E(Y_1) = \sum_{n=1}^{\infty} nP(Y_1 = n) = \frac{p^+}{p^-} + \frac{p^-}{p^+}. \qquad (1.36)
$$

Note that this necessarily matches the calculation via conditioning on T_1's sign; that is, by (1.33),

$$
E(Y_1) = E(Y_1 \mid T_1 = T_1^+) p^+ + E(Y_1 \mid T_1 = T_1^-) p^- = \frac{p^+}{p^-} + \frac{p^-}{p^+}.
$$

We can sum over all possible values n in (1.34), and use (1.35) to get the expected gain of a subperiod in terms of p^+, p^-, B^+, and B^-:

$$\begin{aligned} E(G_1) &= \sum_{n=1}^{\infty} E(G_1 \mid Y_1 = n) P(Y_1 = n) \\ &= \sum_{n=1}^{\infty} \left[\frac{n(n+1)}{2} \left(B^+ p^+ - B^- p^- \right) + n(B^- - B^+) \right] \\ &\quad \times \left((p^+)^n (p^-) + (p^-)^n (p^+) \right) \\ &= B^+ p^+ \left(\left(\frac{p^+}{p^-} \right)^2 - 1 \right) - B^- p^- \left(\left(\frac{p^-}{p^+} \right)^2 - 1 \right). \end{aligned} \tag{1.37}$$

Note that, if $p^+ = \frac{1}{2}$ (which holds for any symmetric random walk), then $E(G_1) = 0$, and as $p^+ \downarrow 0$ or $p^+ \uparrow 1$, $E(G_1) \to \infty$.

1.4.2 SIMPLE RANDOM WALK, CUSUM TIMING

We now calculate the expected return of the first subperiod for a simple random walk asset price, applying CUSUM timing. Set our CUSUM threshold to $h = 1$, and our probability measure to the simple asymmetric random walk on ticks, that is, $N = 1$, with $p_1 = p$, $p_{-1} = 1 - p$ for some $0 < p < 1$. With $M = 1$, we have by (1.23), for every $k \geq 0$,

$$m_k^u = S_{T_k} + \frac{1}{2}, \quad m_k^d = S_{T_k} - \frac{1}{2}.$$

Since $X_j \in \{-1, 1\}$ for every $0 \leq j < T_1$, the possible values of u_j^0 and d_j^0, by (1.24), are $\{0, \frac{1}{2}, 2\}$, where a 2 occurs only with two consecutive ticks of the same type (ending on an even step). T_1 is the first time $2j$ such that $u_{2j}^0 \geq 1$ or $d_{2j}^0 \geq 1$. Hence,

$$\begin{aligned} T_1 = T_1^+ = 2j &\iff X_{k+1} = -X_k \ \forall k, 1 \leq k < 2j-1,\ X_{2j-1} = X_{2j} = 1; \\ T_1 = T_1^- = 2j &\iff X_{k+1} = -X_k \ \forall k, 1 \leq k < 2j-1,\ X_{2j-1} = X_{2j} = -1. \end{aligned}$$

Given $S_0 = s > 0$, S_{T_1} can take only two possible values, from the paths described earlier. For $j = 1, 2, \ldots$, each possibility takes the form of a geometric random variable conditioned on the final two steps $X_{T_1 - 1}, X_{T_1}$,

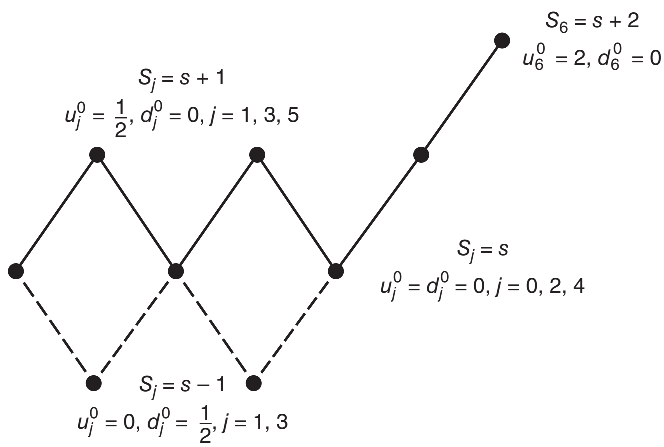

FIGURE 1.8 **The four possible SRW paths for $T_1^+ = 2(3) = 6$.**

where a “failure” is a sequence of two steps of opposite direction; that is, $+1$ then -1, or -1 then $+1$.

$$\begin{aligned}
\{T_1 = T_1^+\} &\iff \{S_{T_1} = s+2\}; \\
&\qquad P(S_{T_1} = s+2,\ T_1^+ = 2j) = [2p(1-p)]^{j-1}p^2 \\
\{T_1 = T_1^-\} &\iff \{S_{T_1} = s-2\}; \\
&\qquad P(S_{T_1} = s-2,\ T_1^- = 2j) = [2p(1-p)]^{j-1}(1-p)^2.
\end{aligned} \tag{1.38}$$

An illustration of the paths leading to a “+” signal T_1^+ is shown in Figure 1.8. The probabilities of each value of S_{T_1} occurring are

$$\begin{aligned}
P(S_{T_1} = s+2) &= \sum_{j=1}^{\infty}[2p(1-p)]^{j-1}p^2 = \frac{p^2}{1-2p(1-p)} \\
P(S_{T_1} = s-2) &= \sum_{j=1}^{\infty}[2p(1-p)]^{j-1}(1-p)^2 = \frac{(1-p)^2}{1-2p(1-p)}.
\end{aligned}$$

Since there is only one possible outcome per signal type, these match the probabilities of each type of signal occurring:

$$\begin{aligned}
p^+ &:= P(T_1 = T_1^+) = P(S_{T_1} = s+2) = \frac{p^2}{1-2p(1-p)}; \\
p^- &:= P(T_1 = T_1^-) = P(S_{T_1} = s-2) = \frac{(1-p)^2}{1-2p(1-p)}.
\end{aligned} \tag{1.39}$$

The increment $Z_k = S_{T_{k+1}} - S_{T_k}$ then takes values in $\{-2, 2\}$ and depends on the sign of the signal of T_{k+1}. Conditioned on this signal sign, and by the strong Markov property at T_k, we get the conditional expectations

$$B^+ = E(Z_k \mid T_{k+1} = T^+_{k+1}) = 2P(S_{T_1} - s = 2 \mid T_1 = T^+_1) = 2, \quad (1.40)$$

$$B^- = E(Z_k \mid T_{k+1} = T^-_{k+1}) = -2P(S_{T_1} - s = -2 \mid T_1 = T^-_1) = -2. \quad (1.41)$$

Thus, by (1.34), (1.40), (1.41), and (1.37), we have the expected gain

$$E(G_1) = \frac{2(p^4 - (1-p)^4)}{1 - 2p(1-p)} \left[\frac{p^2}{(1-p)^4} - \frac{(1-p)^2}{p^4} \right], \quad (1.42)$$

which can be shown to be symmetric about its minimum $p = \frac{1}{2}$ (at $E(G_1) = 0$), with $\lim_{p\downarrow 0} E(G_1) = \lim_{p\uparrow 1} E(G_1) = \infty$.

We also have the expected time until a signal occurs: by (1.38) and (1.39),

$$\begin{aligned}
E(T_1 \mid T_1 = T^+_1) &= \sum_{j=1}^{\infty} (2j) P(T_1 = 2j \mid T_1 = T^+_1) \\
&= 2 \sum_{j=1}^{\infty} j \frac{P(T_1 = 2j,\, T_1 = T^+_1)}{P(T_1 = T^+_1)} = \frac{2p^2}{p^+} \sum_{j=1}^{\infty} j[2p(1-p)]^{j-1} \\
&= \frac{2p^2}{p^+(1 - [2p(1-p)])^2} = \frac{2}{1 - [2p(1-p)]}; \\
E(T_1 \mid T_1 = T^-_1) &= \frac{2}{1 - [2p(1-p)]}; \\
E(T_1) &= E(T_1 \mid T_1 = T^+_1) p^+ + E(T_1 \mid T_1 = T^-_1) p^- \\
&= \frac{2}{1 - [2p(1-p)]}.
\end{aligned} \quad (1.43)$$

Finally, the expected number of same-sign signals in a subperiod is, by (1.36) and (1.39),

$$E(Y_1) = \frac{p^+}{p^-} + \frac{p^-}{p^+} = \frac{p^2}{(1-p)^2} + \frac{(1-p)^2}{p^2} = \frac{p^4 + (1-p)^4}{p^2(1-p)^2}. \quad (1.44)$$

1.4.3 LAZY SIMPLE RANDOM WALK, CUSUM TIMING

Introducing a more complicated random walk distribution, such as a lazy simple random walk, with step distribution

$$X_j = \begin{cases} +1 & \text{with probability } p_1 \\ 0 & \text{with probability } p_0 \\ -1 & \text{with probability } p_{-1}, \end{cases} \tag{1.45}$$

where $p_{-1} + p_0 + p_1 = 1$ increases the complexity of the analysis of the CUSUM timing strategy probabilities, and therefore of calculating the expected gain analytically. We will retain $h = 1$ and $M = 1$.

By introducing a zero tick, we expand the possible cases of "failure" to set off a CUSUM signal. We decompose the lazy random walk path into seven distinct possible components. First, there are three possible patterns that fail to set off a signal, being "up-down" (with probability $p_1 p_{-1}$), "down-up" (with probability $p_{-1} p_1$), and "zero" (a one-step pattern with probability p_0). Note that the first two of these are the two possible failure patterns of (1.38). There are, consequently, four "success" patterns:

- the two from (1.38): "up-up" (with probability p_1^2) and "down-down" (with probability p_{-1}^2);
- and two patterns with zero ticks: "up-zero" (with probability $p_1 p_0$) and "down-zero" (with probability $p_{-1} p_0$).

The number of such patterns that occurs up to a signal time is geometric. Define

$$S^* := p_1^2 + p_{-1}^2 + p_1 p_0 + p_{-1} p_0 \tag{1.46}$$

$$F^* := 1 - S^* = 2p_1 p_{-1} + p_0 \tag{1.47}$$

as the respective signal-pattern success and failure probabilities. Then, we define V_j as the number of failure patterns until signal $j = 1, 2, \ldots$. $V_j \sim \text{geom}(S^*)$ (starting at 0), and, conditioned on V_j, we define W_j as the number of zero-tick patterns that occur during this time frame. Since the W_j zero-ticks can take place at any pattern position of the V_j patterns, $W_j|_{V_j} \sim \text{bin}(V_j, \frac{p_0}{F^*})$. Note that if $p_0 = 0$, this reduces to the case in the previous section.

We can calculate the expected time before a signal: if there are V_j failure patterns (of length 1 or 2 ticks) before signal j, W_j of these are the 1-tick zero-tick failures, and, finally, we have a 2-tick success pattern, then the number of ticks before the first signal is

$$T_1 := W_1 + 2(V_1 - W_1) + 2 = 2V_1 - W_1 + 2. \tag{1.48}$$

The expected time until a signal is, then, by (1.48) and (1.47),

$$\begin{aligned} E(T_1) = 2E(V_1) - E(W_1) + 2 = 2E(V_1) - &\sum_{v=0}^{\infty} E(W_1 \mid V_1 = v)P(V_1 = v) + 2 \\ &= 2E(V_1) - \frac{p_0}{F^*}\sum_{v=0}^{\infty} vP(V_1 = v) + 2 \\ &= \frac{2(1-S^*)}{S^*}\left(1 - \frac{p_0}{F^*}\right) + 2 = \frac{2-p_0}{S^*}. \end{aligned} \tag{1.49}$$

At $p_0 = 0$, (1.49) reduces to (1.43).

The zero-tick success patterns increase the possible asset values at a signal. In (1.40) and (1.41), the only possible values for the price change increment Z_k of (1.9) are $\{-2, 2\}$. Here, the possible values of Z_k are $\{-2, -1, 1, 2\}$, and so, by the Markov property at the times $j-2$, and defining $P_j^T := P(T_1 = j)/S^*$ for $j \geq 2$, we have the probabilities

$$\begin{aligned} P(S_{T_1} = s+2) &= \sum_{j=2}^{\infty} P(S_{T_1} = s+2\,,\, T_1^+ = j) \\ &= p_1^2 \sum_{j=2}^{\infty} P(S_{j-2} = s\,,\, T_1 > j-2) = p_1^2 \sum_{j=2}^{\infty} P_j^T \\ P(S_{T_1} = s+1) &= p_1 p_0 \sum_{j=2}^{\infty} P_j^T, \quad P(S_{T_1} = s-1) = p_{-1} p_0 \sum_{j=2}^{\infty} P_j^T, \\ P(S_{T_1} = s-2) &= p_{-1}^2 \sum_{j=2}^{\infty} P_j^T, \end{aligned} \tag{1.50}$$

which all sum to 1, by the fact that $\sum_{j=2}^{\infty} P_j^T = \frac{1}{S^*}$. The equations in (1.50) also yield the conditional probabilities

$$\begin{aligned} P(S_{T_1} = s + 2 \mid T_1 = T_1^+) &= \frac{p_1^2}{p_1^2 + p_1 p_0}, \\ P(S_{T_1} = s + 1 \mid T_1 = T_1^+) &= \frac{p_1 p_0}{p_1^2 + p_1 p_0}, \\ P(S_{T_1} = s - 1 \mid T_1 = T_1^-) &= \frac{p_{-1} p_0}{p_{-1}^2 + p_{-1} p_0}, \\ P(S_{T_1} = s - 2 \mid T_1 = T_1^-) &= \frac{p_{-1}^2}{p_{-1}^2 + p_{-1} p_0}. \end{aligned} \tag{1.51}$$

An illustration of possible paths leading to a "+" signal can be found in Figure 1.9.

Retaining the definitions of p^+ and p^- from (1.26), we get

$$p^+ = \frac{p_1 p_0 + p_1^2}{S^*}; \;\; p^- = \frac{p_{-1} p_0 + p_{-1}^2}{S^*}, \tag{1.52}$$

which allows us to calculate the expected number of signals on a subperiod. By (1.52) and (1.36),

$$E(Y_1) = \frac{p^+}{p^-} + \frac{p^-}{p^+} = \frac{p_1 p_0 + p_1^2}{p_{-1} p_0 + p_{-1}^2} + \frac{p_{-1} p_0 + p_{-1}^2}{p_1 p_0 + p_1^2}, \tag{1.53}$$

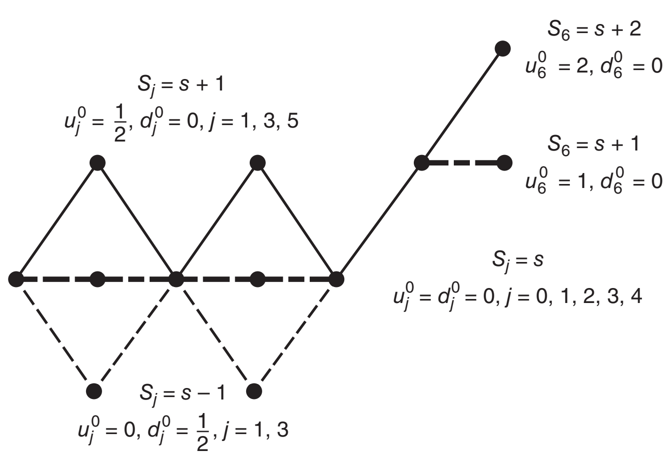

FIGURE 1.9 **The 12 possible LSRW paths for $T_1^+ = 6$.**

which reduces, if $p_0 = 0$, to (1.44). Also, if the walk is symmetric, that is, $p_1 = p_{-1}$, then $E(Y_1) = 2$.

Next, we find B^+ and B^-, the expected size of the incremental changes Z_k, conditioned on the type of subperiod. Generalizing (1.40) and (1.41) (where $p_0 = 0$), we have by (1.51)

$$\begin{aligned} B^+ &= 2P(S_{T_1} - s = 2 \,|\, T_1 = T_1^+) + P(S_{T_1} - s = 1 \,|\, T_1 = T_1^+) \\ &= 1 + \frac{p_1^2}{p_1 p_0 + p_1^2}, \end{aligned} \tag{1.54}$$

$$\begin{aligned} B^- &= -2P(S_{T_1} - s = -2 \,|\, T_1 = T_1^-) - P(S_{T_1} - s = -1 \,|\, T_1 = T_1^-) \\ &= -1 - \frac{p_{-1}^2}{p_{-1} p_0 + p_{-1}^2}. \end{aligned} \tag{1.55}$$

Finally, the expected gain $E(G_1)$ at the end of a subperiod can be found by combining (1.37) with (1.52), (1.54), and (1.55), generalizing the $p_0 = 0$ case (1.42).

1.5 CUSUM strategy Monte Carlo

Here we provide Monte Carlo simulations of the collection of random walks on ticks given in the previous section to numerically analyze the behavior of our strategy against such walks as asset prices.

The two classes of random walks for our simulations are special subclasses of (1.45): they are the lazy symmetric simple random walk

$$X_j = \begin{cases} +1 & \text{with probability } p_1 = \frac{1-p_0}{2} \\ 0 & \text{with probability } p_0 \in \{0, 0.05, 0.1, ..., 0.35\} \\ -1 & \text{with probability } p_{-1} = \frac{1-p_0}{2}, \end{cases} \tag{1.56}$$

and the lazy asymmetric simple random walk with upward drift

$$X_j = \begin{cases} +1 & \text{with probability } p_1 = 0.5 - \frac{p_0}{2} + 0.05j, j \in \{0, 1, ..., 6\} \\ 0 & \text{with probability } p_0 \in \{0, 0.1, 0.2, 0.3, 0.4\} \\ -1 & \text{with probability } p_{-1} = 1 - p_1 - p_0, \end{cases} \tag{1.57}$$

where j allows $p_{-1} > 0$. Each class of walks was run for 200 simulated trading days, with $N = 5000$ ticks for 1 day's trading, and starting price $s = 10,000$ ticks each day (to guarantee that 1 day's trading does not bottom out the asset).

Define the *idle time* of a trading strategy during a day as the (random) set of tick times between subperiods, *that is*, when our algorithm declares that our portfolio be empty. If the day consists of N ticks, then the idle time for the day is defined as

$$\text{idle time } := \{n \in \{1, 2, ..., N\} : \Delta_n = 0\}.$$

The % idle time in a day is simply $\frac{|\text{idle time}|}{N}$. If there are R subperiods in a day, this is

$$|\text{idle time}| = \sum_{l=0}^{R-1} (T_{\alpha(l)+1} - T_{\alpha(l)}) + (N - T_{\alpha(R)}),$$

where $T_{\alpha(R)} = N$ if the final subperiod's end is induced by the end-of-day settling the algorithm requires. We can estimate the average number of subperiods per day by $\frac{N}{E(T)[E(Y)+1]}$, and so, since there is the length of one signal between each subperiod, we can naively estimate the average amount of idle time in a day as the average number of subperiods per day multiplied by the average time to a signal, that is, $\frac{N}{E(T)[E(Y)+1]} \cdot E(T_1) = \frac{N}{E(Y)+1}$. Then, the % idle time in a day is naively estimated by this value divided by the number of ticks per day, or, simply, $\frac{1}{E(Y)+1}$.

Tables containing the results of simulations can be found in the Appendix, Section 1.7. Table 1.A.1 contains experimental averages of the following values for the lazy symmetric random walks represented by (1.56):

- average gain per subperiod (1.37), which can be seen to be close to $E(G_1) = 0$ in all cases due to symmetry;
- average subperiod length, which approximates (1.49) and (1.53)'s $E(T_1)E(Y_1)$: for example, $p_0 = 0.1$ has $7.670 \approx \left(\frac{2-0.1}{1-2(0.45)^2-0.1}\right)(2) = 3.\overline{83}(2) = 7.\overline{67}$;
- average number of signals per subperiod, which approximates (1.53)'s $E(Y_1) + 1$: for example, $p_0 = 0.1$ has $2.998 \approx 2 + 1 = 3$;

- average number of subperiods per day, which approximates R above (which is itself approximated above by $\frac{N}{E(T)[E(Y)+1]}$); for $p_0 = 0.1$, this is $435.185 \approx \frac{5000}{3.8\overline{3}(3)} \approx 434.21$;
- and the average % idle time; for $p_0 = 0.1$, this is $33.2\% \approx \frac{1}{E(Y)+1} = \frac{1}{3}$.

The remaining tables contain similar experimental data for various lazy simple random walks from Section 1.4.3. Results of simulations using frequencies derived from the real data from the 5-year and 30-year bonds are shown in Tables 1.6, 1.7, and 1.8.

Table 1.A.2 contains detail on the subperiods of these walks:

- the average number of subperiods with a specific number of signals; for example, $p_0 = 0.1$, subperiod length $n = 4$ has 27.31, which, when divided by the average total number of subperiods 435.185 from Table 1.A.1, gives $\frac{27.31}{435.185} \approx 0.06275 \approx P(Y_1 = 4) = 0.0625$ from (1.35) using (1.52);
- and the average gain on such a subperiod of length $n = 4$, which is $3.64 \approx E(G_1 \mid Y_1 = 4) = 3.478$ from (1.34) using (1.52), (1.54), and (1.55).

Tables 1.A.3 and 1.A.4 contain the same experimental values as Tables 1.A.1 and 1.A.2, this time from the simple random walk of Section 1.4.2. For example, in Table 1.A.3, examining $p_1 = 0.65$, we have

- average gain per subperiod $16.378 \approx E(G_1) = 16.481$ from (1.42);
- average subperiod length $13.749 \approx E(Y_1) \cdot E(T_1) \approx 3.7389 \cdot 3.6697 = 13.7208$ from (1.44) and (1.43);
- average number of signals per subperiod $4.746 \approx E(Y_1) + 1 = 4.7389$ from (1.44);
- average number of subperiods $288.120 \approx \frac{N}{E(T)[E(Y)+1]} = \frac{5000}{3.6697(4.7389)} = 287.516$; and
- average % idle time $20.8\% \approx \frac{1}{E(Y)+1} = \frac{1}{4.7389} = 21.10\%$.

Note that, for the simple random walk without a "lazy" probability p_0, the average amount of idle time per simulation (the percentage of ticks between subperiods) drops as the walk becomes more asymmetric, as the expected amount of time to get a signal (1.43) (and so be in a subperiod) drops. In Table 1.A.4, the first row of each block approximates (1.35) multiplied by the average number of subperiods from Table 1.A.3 for that $p = p_1$, and the second row approximates (1.34), which is $n^2 - 3n$ for n same-sign signals.

1.6 The effect of the threshold parameter

In this section, we discuss the effect of varying the threshold parameter h on the proposed trading strategy. We first examine this effect on the real data in Section 1.3.3. In particular, Figure 1.10 summarizes the effect of varying thresholds on the gain in all 5 US Treasury bonds of Section 1.3.3. In this figure, it is shown that varying the threshold does not change the sign of the gain. In fact, varying the threshold in the 5-year note leaves the daily gain almost unchanged, while in the 30-year bonds, although a more random variation is observed, no apparent pattern of an increasing or decreasing effect on the gain is observed. This demonstrates a level of robustness of the proposed strategy's gain as a function of the threshold. A closer examination shows that the number of signals per subperiod is almost constant, regardless of the threshold size, as shown in the column "average # of signals per subperiod" in Tables 1.1–1.5. Yet, the number of subperiods per trading day decreases as the threshold increases. This is shown in Figure 1.11, where we note that the number of signals per trading day decreases at the rate of the square root of the threshold. The decrease in the number of subperiods on a given trading day as a result of an increase in the threshold is to be expected since the quantity that varies when the threshold varies is the number of ticks, or equivalently, the amount of time as measured by ticks, required before the completion of a given subperiod. This is true because a smaller threshold gives rise to a more sensitive CUSUM stopping time. In fact, the expected time to a signal (CUSUM alarm) increases as the threshold increases in the order of the square root of the threshold, that is,

$$E[T_1(h)] \approx E[T_1(1)]\sqrt{h}. \tag{1.58}$$

To justify (1.58), note that, on a trend, one of the CUSUM statistics from (1.24) increases quadratically, regardless of the threshold h. For example,

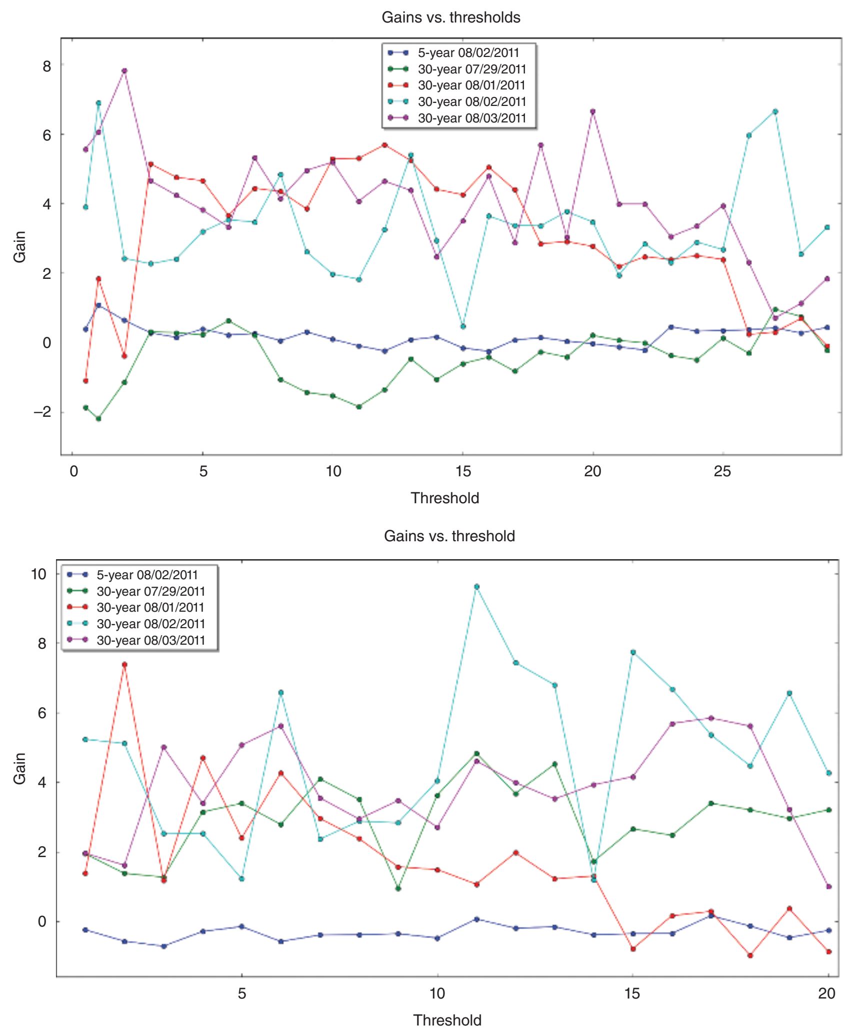

FIGURE 1.10 Total gain versus thresholds for 5-year and 30-year notes. Top graph: Small thresholds that vary as follows: 0.5*tick, tick, 2*tick, 3*tick, etc. (up to 29*tick). Bottom graph: Large thresholds that vary as follows: 50*tick, 2*50*tick, 3*50*tick, etc. (up to 20*50*tick).

on an upward trend, $u_n^k = O(n^2)$, and so the amount of time n it takes to break the threshold, that is, the minimum n to achieve $u_n^k \geq h$, is found by observing $O(n^2) \approx h \Rightarrow n \approx O(\sqrt{h})$. The coefficient can then be found by checking the baseline threshold $h = 1$. We also offer empirical evidence for this from bond data and Monte Carlo simulations.

TABLE 1.1 5-year 08/02/2011 note. Tick size is $M = 0.0078125$, the number of ticks is $N = 17{,}074$. Each threshold h in column 1 is actually hM.

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
0.5	488	1468	0.001	23.340	3.008	33.290	0.385
1	412	1259	0.003	28.488	3.056	31.258	1.074
2	331	1030	0.002	36.332	3.112	29.565	0.637
3	278	864	0.001	43.209	3.108	29.647	0.279
4	241	744	0.001	50.440	3.087	28.804	0.147
5	210	662	0.002	58.895	3.152	27.562	0.389
6	197	609	0.001	60.675	3.091	29.993	0.217
7	187	575	0.001	63.588	3.075	30.356	0.256
8	178	540	0.000	65.157	3.034	32.072	0.054
9	158	500	0.002	74.911	3.165	30.678	0.304
10	155	484	0.001	76.729	3.123	30.344	0.093
50	70	197	−0.004	154.271	2.814	36.752	−0.249
100	44	131	−0.013	245.205	2.977	36.810	−0.577
150	38	111	−0.019	287.921	2.921	35.920	−0.718
200	29	79	−0.010	356.862	2.724	39.387	−0.297
250	27	76	−0.006	397.074	2.815	37.209	−0.157

TABLE 1.2 30-year 07/29/2011 note. Tick size is $M = 0.015625$, the number of ticks is $N = 4588$. Each threshold h in column 1 is actually hM.

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
0.5	467	1334	−0.004	6.503	2.857	33.806	−1.865
1	335	988	−0.007	8.931	2.949	34.786	−2.191
2	233	681	−0.005	13.309	2.923	32.411	−1.142
3	192	565	0.002	15.901	2.943	33.457	0.311
4	174	504	0.002	17.632	2.897	33.130	0.282
5	146	430	0.001	20.651	2.945	34.285	0.218
6	127	375	0.005	24.055	2.953	33.413	0.624
7	123	360	0.002	24.894	2.927	33.261	0.204
8	113	331	−0.009	26.186	2.929	35.506	−1.062
9	104	306	−0.014	28.471	2.942	35.462	−1.436
10	103	302	−0.015	28.573	2.932	35.854	−1.530
50	34	115	0.057	97.676	3.382	27.616	1.937
100	23	75	0.060	145.391	3.261	27.114	1.375
150	18	59	0.070	180.778	3.278	29.076	1.266
200	13	50	0.242	278.077	3.846	21.207	3.140
250	14	47	0.242	229.714	3.357	29.904	3.390

TABLE 1.3 30-year 08/01/2011 bond. Tick size is $M = 0.015625$, the number of ticks is $N = 3244$. Each threshold h in column 1 is actually hM.

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
0.5	358	1036	−0.003	6.095	2.894	32.737	−1.100
1	262	792	0.007	8.531	3.023	31.104	1.839
2	195	582	−0.002	11.031	2.985	33.693	−0.392
3	136	457	0.038	17.669	3.360	25.925	5.140
4	115	393	0.041	19.826	3.417	29.716	4.750
5	103	356	0.045	22.136	3.456	29.716	4.656
6	95	324	0.038	23.579	3.411	30.949	3.655
7	81	291	0.055	28.815	3.593	28.052	4.437
8	76	268	0.057	30.500	3.526	28.545	4.344
9	76	254	0.051	30.039	3.342	29.624	3.844
10	71	251	0.074	32.338	3.535	29.223	5.280
50	33	111	0.042	76.818	3.364	21.856	1.375
100	21	70	0.352	112.333	3.333	27.281	7.390
150	18	57	0.065	114.500	3.167	36.467	1.172
200	16	52	0.294	142.312	3.250	29.809	4.703
250	16	46	0.149	139.750	2.875	31.073	2.391

TABLE 1.4 30-year 08/02/2011 bond. Tick size is $M = 0.015625$, the number of ticks is $N = 4349$. Each threshold h in column 1 is actually hM.

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
0.5	430	1291	0.009	6.802	3.002	32.743	3.901
1	316	982	0.022	9.497	3.108	30.996	6.888
2	218	704	0.011	13.298	3.229	33.341	2.421
3	185	604	0.012	16.411	3.265	30.191	2.264
4	146	509	0.016	20.856	3.486	29.984	2.404
5	126	444	0.025	24.397	3.524	29.317	3.186
6	110	393	0.032	28.391	3.573	28.190	3.531
7	112	382	0.031	26.804	3.411	30.973	3.468
8	104	354	0.046	28.577	3.404	31.662	4.829
9	93	321	0.028	33.269	3.452	28.857	2.609
10	89	298	0.022	32.921	3.348	32.628	1.967
50	38	129	0.138	80.763	3.395	29.432	5.233
100	29	90	0.176	105.207	3.103	29.846	5.110
150	24	75	0.105	128.375	3.125	29.156	2.516
200	19	61	0.133	179.895	3.211	21.407	2.531
250	17	54	0.072	178.706	3.176	30.145	1.219

TABLE 1.5 30-year 08/03/2011 note. Tick size is $M = 0.015625$, the number of ticks is $N = 5153$. Each threshold h in column 1 is actually hM.

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
0.5	549	1674	0.010	6.342	3.049	32.428	5.556
1	425	1311	0.014	8.169	3.085	32.622	6.058
2	276	900	0.028	12.641	3.261	32.292	7.827
3	233	741	0.020	14.670	3.180	33.670	4.654
4	198	634	0.021	17.662	3.202	32.137	4.232
5	180	576	0.021	19.444	3.200	32.078	3.810
6	169	532	0.020	20.604	3.148	32.428	3.325
7	160	507	0.033	22.331	3.169	30.662	5.311
8	139	457	0.030	25.612	3.288	30.914	4.141
9	134	438	0.037	26.560	3.269	30.933	4.953
10	127	414	0.041	28.307	3.260	30.235	5.187
50	52	172	0.038	68.404	3.308	30.972	1.953
100	39	125	0.041	92.282	3.205	30.157	1.609
150	26	92	0.192	150.077	3.538	24.277	5.001
200	21	73	0.161	185.571	3.476	24.374	3.391
250	20	68	0.253	194.600	3.400	24.471	5.063

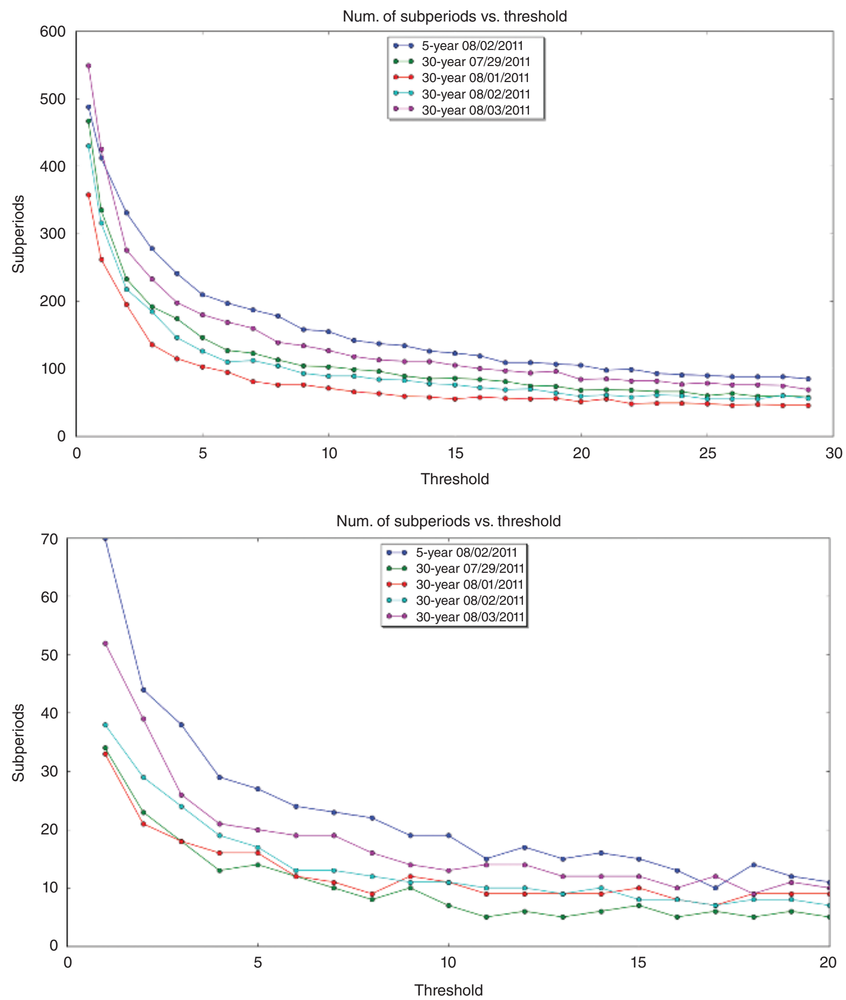

FIGURE 1.11 **Number of subperiods versus thresholds for 5-year and 30-year notes. Top graph: Small thresholds that vary as follows: 0.5*tick, tick, 2*tick, 3*tick, etc. (up to 29*tick). Bottom graph: Large thresholds that vary as follows 50*tick, 2*50*tick, 3*50*tick, etc. (up to 20*50*tick).**

The fifth columns of Tables 1.1–1.5 represent the average subperiod length $E[T_{\alpha(1)}(h)]$; the sixth columns represent the average # of signals per subperiod $E[Y_1] + 1$. Therefore, the expected time to a signal can be found as

$$E\left[T_1(h)\right] = \frac{E[T_{\alpha(1)}(h)]}{E\left[Y_1\right]}. \tag{1.59}$$

TABLE 1.6 Average gains and lengths of subperiods, asymmetric random walk. P(−3) = 0.00012 , P(−2) = 0.00141, P(−1) = 0.05348, P(0) = 0.88619, P(1) = 0.05670, P(2) = 0.00182 , and P(3) = 0.00029, for various thresholds. These probabilities were computed from the 5-year 08/02/2011 bond. Tick size is 1, the number of ticks is 5000 and starting price 10,000. The number of simulations is 1000.

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
1	162.401	488.191	0.030	20.648	3.006	32.936	4.9
2	125.799	378.175	0.036	26.665	3.006	32.913	4.5
3	103.358	310.196	0.041	32.508	3.001	32.801	4.3
4	88.207	264.402	0.034	38.121	2.998	32.749	3.0
5	77.574	232.413	0.037	43.390	2.996	32.681	2.9
6	69.419	208.576	0.052	48.530	3.005	32.623	3.6
7	63.469	190.459	0.066	53.123	3.001	32.566	4.2
8	58.775	176.465	0.077	57.408	3.002	32.517	4.5
9	54.911	164.795	0.072	61.455	3.001	32.509	4.0
10	51.568	154.781	0.062	65.533	3.001	32.411	3.2
11	48.860	146.595	0.077	69.171	3.000	32.406	3.8
12	46.511	139.607	0.078	72.754	3.002	32.323	3.6
13	44.562	133.657	0.085	75.995	2.999	32.270	3.8
14	42.796	128.334	0.108	79.217	2.999	32.197	4.6
15	41.267	123.685	0.084	82.236	2.997	32.127	3.5
100	14.414	43.619	0.588	245.445	3.026	29.243	8.5
200	9.628	29.632	1.451	381.445	3.078	26.549	14.0
300	7.636	23.615	2.119	489.755	3.093	25.205	16.2
400	6.477	20.074	2.487	588.740	3.099	23.735	16.1
500	5.672	17.819	3.057	687.449	3.142	22.016	17.3

TABLE 1.7 **Average gains and lengths of subperiods, asymmetric random walk. P(−4) = 0.00494, P(−3) = 0.00997, P(−2) = 0.03732, P(−1) = 0.12561, P(0) = 0.62919, P(1) = 0.12747, P(2) = 0.04575, and P(3) = 0.01279 , P(4) = 0.00697, for various thresholds. These probabilities were computed from the 30-year 07/29/2011 and 08/02/2011 bonds after a best fit that was verified by chi-square test. The *p*-value for 07/29/2011 is 0.87 and for 08/02/2011, 0.815. Tick size is 1, the number of ticks is 5000, and the starting price 10,000. The number of simulations is 1000.**

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
1	443.934	1336.262	0.083	7.535	3.010	33.099	37.0
2	299.194	903.318	0.161	11.208	3.019	32.935	48.3
3	241.959	731.714	0.240	13.879	3.024	32.838	58.0
4	204.308	618.324	0.289	16.452	3.026	32.776	59.1
5	181.312	549.742	0.352	18.563	3.032	32.684	63.8
6	164.086	498.442	0.421	20.548	3.038	32.569	69.1
7	150.539	457.415	0.453	22.410	3.039	32.529	68.2
8	139.256	423.863	0.507	24.255	3.044	32.446	70.6
9	130.517	397.679	0.549	25.909	3.047	32.369	71.6
10	122.775	374.570	0.617	27.547	3.051	32.358	75.8
11	115.983	354.361	0.683	29.171	3.055	32.333	79.2
12	110.395	337.981	0.775	30.676	3.062	32.269	85.6
13	105.205	322.336	0.848	32.226	3.064	32.194	89.2
14	100.997	309.365	0.902	33.602	3.063	32.127	91.1
15	96.980	297.191	0.927	34.964	3.064	32.183	89.9
100	30.065	96.746	5.522	117.792	3.218	29.172	166.0
200	19.133	63.974	10.603	190.220	3.344	27.210	202.9
300	14.566	50.358	16.365	258.656	3.457	24.648	238.4
400	11.936	42.703	22.771	323.669	3.578	22.734	271.8
500	10.217	37.682	28.222	388.293	3.688	20.656	288.3

TABLE 1.8 Average gains and lengths of subperiods, asymmetric random walk. P(−4) = 0.01023, P(−3) = 0.01460, P(−2) = 0.04601, P(−1) = 0.13666, P(0) = 0.58916, P(1) = 0.12914, P(2) = 0.04548, P(3) = 0.01772, and P(4) = 0.01100, for various thresholds. These probabilities were computed from the 30-year 08/01/2011 and 08/03/2011 bonds after a best fit that was verified by chi-square test. The *p*-value for 08/01/2011 is 0.80 and for 08/03/2011, 0.48. Tick size is 1, the number of ticks is 5000 and starting price 10,000. The number of simulations is 1000.

Threshold	# subperiods	# signals	Average gain per subperiod	Average subperiod length	Average # of signals per subperiod	% of idle time	Total gain
1	488.364	1463.672	−0.014	6.834	2.997	33.249	−7.0
2	333.138	998.589	−0.017	10.031	2.998	33.169	−5.6
3	271.557	813.925	−0.015	12.306	2.997	33.162	−4.0
4	229.944	688.463	−0.043	14.532	2.994	33.170	−9.9
5	204.761	613.066	−0.039	16.340	2.994	33.085	−7.9
6	185.382	555.241	−0.034	18.050	2.995	33.077	−6.4
7	170.343	509.950	−0.055	19.636	2.994	33.102	−9.4
8	157.385	471.073	−0.049	21.245	2.993	33.127	−7.7
9	147.463	441.897	−0.020	22.710	2.997	33.022	−3.0
10	138.931	416.129	−0.012	24.127	2.995	32.960	−1.6
11	131.472	393.089	−0.049	25.474	2.990	33.018	−6.5
12	125.161	374.424	−0.053	26.768	2.992	32.993	−6.6
13	119.276	356.992	−0.026	28.096	2.993	32.976	−3.0
14	114.317	341.780	−0.075	29.313	2.990	32.980	−8.5
15	109.788	328.314	−0.068	30.509	2.990	33.009	−7.5
100	35.056	104.634	0.031	96.983	2.985	32.003	1.1
200	22.908	68.150	−0.009	149.200	2.975	31.642	−0.2
300	17.932	53.188	−0.089	192.092	2.966	31.108	−1.6
400	14.932	44.283	−0.198	232.382	2.966	30.602	−3.0
500	13.070	38.673	−0.084	267.170	2.959	30.162	−1.1

To see the square-root effect, let us examine the following rows:

- In rows 2 and 5 of Table 1.2, we can calculate $E[T_1(1)] = \frac{8.931}{1.949} = 4.582$ and $E[T_1(4)] = \frac{17.632}{1.897} = 9.295$, respectively. We now notice that $E[T_1(4)] \approx E[T_1(1)]\sqrt{4}$.
- In row 4 of Table 1.2, we can calculate $E[T_1(3)] = \frac{15.901}{1.943} = 8.184$ and $E[T_1(3)] \approx E[T_1(1)]\sqrt{3}$.
- In rows 2 and 8 of Table 1.3, we can calculate $E[T_1(1)] = \frac{8.531}{2.023} = 4.217$ and $E[T_1(7)] = \frac{28.815}{2.593} = 11.113$, respectively. This leads to $E[T_1(7)] \approx E[T_1(1)]\sqrt{7}$.

We have also generated simulated data for each of the bonds from which once again we can easily decipher the same square-root effect. To be more specific, we have fitted a lazy random walk model to the 30-year bond series data for 07/29/2011 and 08/02/2011 with the appropriate parameters as designated in the caption of Table 1.7. A simple goodness-of-fit test demonstrates the validity of the model selected. The same process is followed in the remaining 30-year bond data. The results of the simulations are summarized in Tables 1.7 and 1.8, respectively. We again demonstrate the square-root effect once again for the same thresholds used in the observed data:

- In rows 1, 4, and 3 of Table 1.7, we can calculate $E[T_1(1)] = \frac{7.535}{2.01} = 3.749$, $E[T_1(4)] = \frac{16.452}{2.026} = 8.12$ and $E[T(3)] = \frac{13.879}{2.024} = 6.857$. Once again we observe the approximations $E[T(4)] \approx E[T(1)]\sqrt{4}$ and $E[T_1(3)] \approx E[T_1(1)]\sqrt{3}$, respectively.
- In rows 1 and 7 of Table 1.8, we can calculate $E[T_1(1)] = \frac{6.834}{1.997} = 3.422$ and $E[T(7)] = \frac{19.636}{1.994} = 9.847$, from which we can extract the approximation $E[T_1(7)] \approx E[T_1(1)]\sqrt{7}$.

The square-root effect suggests that increasing the threshold reduces the number of complete subperiods R on any given trading day, and thus the number of transactions completed therein. Thus, although varying the threshold does not have a systematic effect on the gain in the absence of transaction costs, increasing the threshold would decrease the number of transactions but increase the "riskiness" of the trading strategy. A good measure of performance of the strategy over the course of an entire day of trading is the *total gain* $\sum_{l=1}^{R} G_l$, which, under the zero transaction cost model with IID G_l, has expected value of $E(R)E(G_1)$ by Wald's equation. Examining this product as a function of h under different probabilistic models of the asset price is of interest, especially in terms of maximizing the day's total gain based on the value of h. However, in the presence of transaction costs, Wald's equation fails and analytical derivations are extremely challenging. Besides, transaction costs often vary from firm to firm and thus the appropriate choice of threshold will depend not only on the selection of a measure of "riskiness" but also on the transaction costs related to the specific product or firm.

1.7 Conclusions and future work

In Figures 1.2, 1.3, 1.4, 1.5, and 1.6 of Section 1.3.3, it is shown that the proposed CUSUM trading strategy performs well in subperiods of many signals of one sign before a signal of the opposite sign occurs. This is also evident in Tables 1.3, 1.5, 1.7, 1.A.1, and 1.A.3 related to the results of the simulation in the random walk model of Section 1.4. Such subperiods are characterized by consistent upward or downward trends in prices. On the contrary, the proposed strategy is at a loss in the case of few signals of one sign followed by a signal of the opposite sign. Such subperiods are characterized by stability in prices. This observation suggests that the CUSUM trading strategy can be further improved by an online detection of "regimes of stability" (as contrasted to "regimes of trends"). This suggests the construction of new online algorithm possibly inspired by computer vision (see, for instance, Hadjiliadis and Stamos [19] or Stamos et al. (30)).

Another statistic that is indicative of the contrast between times of stability versus times of instability is known as the *speed of reaction* of the CUSUM, which measures the time between the last reset to 0 of the CUSUM statistic process and the time of the CUSUM alarm (see, for instance, [31]). We intend to examine both of these directions of research in order to improve the performance of the proposed algorithm by limiting trading in times of stability.

A parameter that should be investigated in depth is the transaction cost c. The form of the gain over a subperiod given in (1.11) can be written as

$$G_l = (-1)^{A_l} \sum_{j=1}^{Y_l} jZ_{j+\alpha(l-1)} + c\left(\sum_{j=1}^{Y_l} jZ_{j+\alpha(l-1)} - 2Y_l S_{T_{\alpha(l)}}\right). \quad (1.60)$$

This second term should be analyzed as a fixed percentage, and on a sliding scale (considering, e.g., high-volume rebates). In addition, transaction costs should be investigated via Monte Carlo simulation, as it requires knowledge of the liquidation price of the asset for that subperiod.

A parameter closely related to the transaction cost is the threshold parameter h used in the CUSUM timing. A smaller threshold implies more frequent transactions but decreases the "riskiness" of the strategy on any given trading day. The optimal choice of the threshold should thus be based on the trade-off between an appropriately chosen measure of "risk" of the proposed strategy and the transaction costs in the market where it is applied.

Moreover, it should be noted that the random walk examples included here are not intended as actual asset price models (we do not intend to commit a Bachelierian fallacy); these models are merely used to illustrate the strategy and allow for basic calculations. In future work, it would be of interest to examine the best fit random walk model to actual high-frequency asset data (taking into account such real-world considerations as the bid-ask spread). Furthermore, open problems on this topic include extending analysis of this strategy to other models of asset price motion—primarily, building a binomial model (of which our random walks are the simplest case) and limiting to a continuous geometric Brownian motion. Note that our two sets of random walks investigate different types of "time": the $p_0 = 0$ case investigates "tick time," where the clock moves only when the price moves, and the lazy walk, that is, $p_0 > 0$, considers clock time (since there may be samples where the price does not move). This simple discrepancy induces extra possible paths into the CUSUM timing process. The general binomial model, which may move a price multiple ticks per sample, and still retain the probability of standing still, is certainly, then, of interest.

Finally, we wish to examine the CUSUM strategy with m_k^u and m_k^d set to wait for multiple ticks instead of one (e.g., $m_k^u = S_{T_k} + \frac{bM}{2}$ for some $b > 1$).

Appendix: Tables

In this section are the tables described in Section 1.5.

TABLE 1.A.1 Average gains and lengths of subperiods, lazy simple symmetric random walk.

p_1	Average total gain	Average gain per subperiod	Average subperiod length	Average # signals per subperiod	Average # subperiods	Average % idle time
0.000	−0.864	−0.002	7.995	2.998	417.472	33.2
0.050	−5.379	−0.013	7.824	2.999	426.918	33.2
0.100	−3.204	−0.007	7.670	2.998	435.185	33.2
0.150	1.720	0.004	7.579	3.001	440.642	33.2
0.200	4.417	0.010	7.505	3.002	445.020	33.2
0.250	7.205	0.016	7.492	3.003	446.076	33.2
0.300	−3.728	−0.008	7.465	2.996	446.920	33.3
0.350	−0.144	−0.000	7.527	3.000	443.581	33.2

TABLE 1.A.2 Number of subperiods of a specified length, and average gain over those subperiods, lazy simple symmetric random walk.

p_0	0.000	0.050	0.100	0.150	0.200	0.250	0.300	0.350
# 1–subperiods	209.05	213.37	218.10	220.11	222.32	223.14	223.83	222.24
average gain, 1–subperiod	−1.99	−1.90	−1.81	−1.73	−1.66	−1.60	−1.53	−1.48
# 2–subperiods	104.61	106.94	108.50	110.43	111.67	111.18	111.86	110.48
average gain, 2–subperiod	−1.99	−1.90	−1.81	−1.74	−1.66	−1.59	−1.53	−1.48
# 3–subperiods	51.87	53.50	54.30	54.94	55.27	55.98	56.05	55.32
average gain, 3–subperiod	0.01	0.01	0.01	0.01	−0.01	0.01	0.01	0.02
# 4–subperiods	26.12	26.60	27.31	27.88	27.95	27.92	27.73	27.80
average gain, 4–subperiod	4.01	3.82	3.64	3.50	3.36	3.23	3.09	2.97
# 5–subperiods	12.81	13.33	13.50	13.64	13.91	13.80	13.62	13.99
average gain, 5–subperiod	10.02	9.55	9.11	8.80	8.36	7.99	7.74	7.44
# 6–subperiods	6.49	6.66	6.79	6.86	6.97	6.88	6.94	6.87
average gain, 6–subperiod	18.01	17.26	16.47	15.71	14.99	14.44	13.87	13.26
# 7–subperiods	3.30	3.33	3.36	3.38	3.40	3.67	3.48	3.45
average gain, 7–subperiod	28.03	26.55	25.50	24.37	23.35	22.48	21.71	20.72
# 8–subperiods	1.58	1.60	1.67	1.65	1.83	1.77	1.68	1.74
average gain, 8–subperiod	40.04	38.00	36.13	34.79	33.05	31.75	30.97	29.57
# 9–subperiods	0.81	0.76	0.86	0.88	0.85	0.86	0.84	0.83
average gain, 9–subperiod	54.03	51.29	48.82	47.12	45.01	43.09	41.67	39.80
# 10 + -subperiods	0.83	0.82	0.81	0.88	0.87	0.88	0.89	0.85
average gain, 10 + -subperiods	2091.33	1393.27	1852.19	1524.21	1230.49	1715.07	1220.75	1318.50

TABLE 1.A.3 Average gains and lengths of subperiods, asymmetric random walk. $p_0 = 0.0$.

p_1	Average total gain	Average gain per subperiod	Average subperiod length	Average # signals per subperiod	Average # subperiods	Average % idle time
0.500	−1.005	−0.002	8.014	3.001	417.140	33.1
0.550	414.950	1.039	8.572	3.165	399.230	31.6
0.600	1,818.090	5.170	10.413	3.705	351.630	26.8
0.650	4,718.815	16.378	13.749	4.746	288.120	20.8
0.700	10,597.510	48.681	19.612	6.667	217.695	14.6
0.750	22,358.635	144.829	29.404	10.181	154.380	9.2
0.800	47,017.940	468.563	47.264	17.080	100.345	5.1

TABLE 1.A.4 Number of subperiods of a specified length, and average gain over those runs, asymmetric random walk. $p_0 = 0.0$.

p_1	0.500	0.550	0.600	0.650	0.700	0.750	0.800
# 1–subperiods	209.12	192.44	149.00	100.33	56.83	27.99	11.00
average gain, 1–subperiod	−1.99	−1.99	−1.99	−1.99	−1.99	−1.99	−1.98
# 2–subperiods	103.48	95.11	75.41	50.27	29.31	13.47	5.46
average gain, 2–subperiod	−1.99	−2.00	−1.99	−1.99	−1.98	−1.96	−1.94
# 3–subperiods	51.97	49.91	43.08	32.50	20.64	11.08	5.09
average gain, 3–subperiod	0.01	0.01	0.01	0.01	0.03	0.03	0.06
# 4–subperiods	26.20	26.66	27.11	24.35	17.00	10.82	4.56
average gain, 4–run	4.02	4.02	4.02	4.03	4.03	4.08	4.13
# 5–runs	13.76	14.95	17.92	17.76	14.63	9.12	4.30
average gain, 5–run	10.02	10.02	10.02	10.02	10.03	10.08	10.13
# 6–runs	6.18	8.39	11.95	14.38	12.02	8.35	4.28
average gain, 6–run	18.02	18.03	18.04	18.05	18.07	18.13	18.22
# 7–runs	3.29	4.87	8.50	10.80	10.24	7.50	3.98
average gain, 7–run	28.01	28.03	28.00	28.05	28.12	28.10	28.21
# 8–runs	1.70	2.85	5.97	8.47	8.98	6.49	3.79
average gain, 8–run	40.02	40.00	40.03	40.01	40.08	40.08	40.17
# 9–runs	0.69	1.59	3.98	6.70	7.41	5.95	3.31
average gain, 9–run	54.00	54.00	54.05	54.11	54.07	54.14	54.19
# 10 + -runs	0.75	2.44	8.73	22.56	40.63	53.60	54.59
average gain, 10 + -runs	1356.00	2688.00	5096.28	18,888.69	68,970.76	198,829.18	63,3281.20

TABLE 1.A.5 Average gains and lengths of runs, asymmetric random walk. $p_0 = 0.1$.

p_1	Average total gain	Average gain per subperiod	Average subperiod length	Average # signals per subperiod	Average # subperiods	Average % idle time
0.450	7.485	0.017	7.683	3.005	434.700	33.2
0.500	428.825	1.032	8.251	3.173	415.625	31.4
0.550	1,825.045	4.979	9.972	3.705	366.535	26.9
0.600	4,729.605	15.804	13.254	4.767	299.270	20.7
0.650	10,741.020	47.569	18.924	6.727	225.800	14.5
0.700	22,858.345	144.381	28.694	10.364	158.320	9.1
0.750	50,243.475	504.478	47.822	17.994	99.595	4.7

TABLE 1.A.6 Number of subperiods of a specified length, and average gain over those runs, asymmetric random walk. $p_0 = 0.1$.

p_1	0.450	0.500	0.550	0.600	0.650	0.700	0.750
# 1-runs	217.16	199.44	156.22	104.08	58.37	27.67	10.54
avg gain, 1-run	−1.81	−1.81	−1.80	−1.80	−1.78	−1.77	−1.74
# 2-runs	108.08	99.44	78.25	51.39	29.41	14.06	5.47
avg gain, 2-run	−1.81	−1.79	−1.76	−1.69	−1.63	−1.51	−1.34
# 3-runs	54.92	51.84	44.15	34.02	21.74	11.57	4.70
avg gain, 3-run	0.01	0.04	0.14	0.31	0.47	0.64	0.94
# 4-runs	27.02	28.05	28.40	25.00	18.00	10.38	4.34
avg gain, 4-run	3.61	3.73	3.88	4.12	4.29	4.58	4.96
# 5-runs	14.05	15.66	18.58	19.49	15.38	9.06	4.29
avg gain, 5-run	9.02	9.27	9.48	9.82	10.07	10.51	11.06
# 6-runs	6.57	8.90	12.41	14.54	13.19	8.81	4.07
avg gain, 6-run	16.28	16.64	16.96	17.45	17.66	18.25	18.87
# 7-runs	3.58	4.81	8.80	11.25	10.62	7.71	3.69
avg gain, 7-run	25.49	25.62	26.25	26.72	27.38	27.79	28.72
# 8-runs	1.64	2.96	6.07	8.87	8.64	6.85	3.26
avg gain, 8-run	36.23	36.95	37.57	37.91	38.84	39.49	40.15
# 9-runs	0.85	1.78	4.13	6.91	7.64	6.20	3.40
avg gain, 9-run	49.42	50.21	50.84	50.91	51.87	52.39	53.87
# 10 + -runs	0.83	2.73	9.52	23.73	42.81	56.00	55.85
avg gain, 10 + -runs	1119.42	2305.77	6948.10	15,781.54	61,048.80	178,378.58	711,759.83

References

1. Abramov, V., Khan, M. K., and Khan, R. A., A probabilistic analysis of the trading the line strategy, *Quantitative Finance*, Vol. 8, No. 5, pp. 499–512, 2008.
2. Alexander, S. S., Price movements in speculative markets: trends or random walk, *Industrial Management Review*, Vol. 2, No. 2, pp. 7–26, 1961.
3. Alexander, S. S., Price movements in speculative markets: trends or random walk no. 2, *Industrial Management Review*, Vol. 5, No. 2, pp. 338–372, 1964.
4. Barnard, G., Control charts and stochastic processes, *Journal of the Royal Statistical Society B*, Vol. 11, No. 1, pp. 239–271, 1959.
5. Basseville, M., and Nikiforov, I., *Detection of abrupt changes: Theory and applications*, 1993 (Prentice Hall, Englewood Cliffs, NJ).
6. Beibel, M., A note on Ritov's Bayes approach to the minimax property of the CUSUM procedure, *Annals of Statistics*, Vol. 24, No. 2, pp. 1804–1812, 1996.
7. Bozdog, B., Florescu, I., Khashanah K., and Wang, J., A study of persistence in price movement using high-frequency financial data. In *Handbook of modeling high-frequency data in finance*, edited by F. Viens, M. Mariani, and I. Florescu, pp. 27–46, 2012 (John Wiley & Sons, Hoboken, NJ).
8. Brodsky, B. E., and Darkhovsky, B. K., *Nonparametric Methods in Change-Point Problems* (Kluwer, Dordrecht), 1993.
9. Broemling, L. D., and Tsurumi, H., *Econometrics and Structural Change*, 1987 (Marcel Dekker, New York).
10. Cohen, A., *Biomedical Signal Processing*, 1987 (CRC Press, Boca Raton, FL).
11. Devore, J., *Probability and Statistics for Engineering and the Science, 8th edition*, 2012 (Brooks/Cole, Boston, MA).
12. Fama, B., and Blum, M., Filter rules and stock market trading, *Journal of Business*, Vol. 40, pp. 226–241, 1966.
13. Figueroa-Lopez, J. E., Lancette, S., Lee, K., and Yanhun, M., Estimation of NIG and VG models for high-frequency financial data. In *Handbook of modeling high-frequency data in finance*, edited by

F. Viens, M. Mariani, and I. Florescu, pp. 3–26, 2012 (John Wiley & Sons, Hoboken, NJ).

14. Frisen, M., (Ed.), *Financial surveillance*, 2008 (John Wiley and Sons, Hoboken, NJ).
15. Glynn, P. W., and Iglehart, D. L., Trading securities using trailing stops, *Management Science*, Vol. 41, No. 6, pp. 1096–1106, 1995.
16. Hadjiliadis, O., and Moustakides, G. V., Optimal and asymptotically optimal CUSUM rules for change point detection in the Brownian motion model with multiple alternatives, *Theory of Probability and Its Applications*, Vol. 50, No. 1, pp. 131–144, 2006.
17. Hadjiliadis, O., Optimality of the 2-CUSUM drift equalizer rules for detecting two-sided alternatives in the Brownian motion model, *Journal of Applied Probability*, Vol. 42, No. 4, pp. 1183–1193, 2005.
18. Hadjiliadis, O., Hernandez-del-Valle, G., and Stamos, I., A comparison of 2-CUSUM stopping rules for quickest detection of two-sided alternatives in a Brownian motion model, *Journal of Sequential Analysis*, Vol. 28, No. 1, pp. 92–114, 2009.
19. Hadjiliadis, O., and Stamos, I., Sequential classification in point clouds of urban scenes, In *Proceedings of 5th International Symposium on 3D Data Processing Visualization and Transmission*, Paris, France, May 17–20, 2010.
20. Lam, K., and Yam, H. C. CUSUM techniques for technical trading in financial markets, *Financial Engineering and the Japanese Markets*, Vol. 4, pp. 257–274, 1997.
21. Lorden, G., Procedures for reacting to a change in distribution, *Annals of Mathematical Statistics*, Vol. 42, No. 6, pp. 1897–1908, 1971.
22. Moustakides, G. V., Optimal stopping times for detecting changes in distributions, *Annals of Statistics*, Vol. 14, No. 4, pp. 1379–1387, 1986.
23. Moustakides, G. V., Optimality of the CUSUM procedure in continuous time, *Annals of Statistics*, Vol. 32, No. 1, pp. 302–315, 2004.
24. Page, E. S., Continuous inspection schemes, *Biometrika*, Vol. 41, Nos. 12, pp. 100–115, 1954.
25. Poor, H. V., and Hadjiliadis, O., *Quickest Detection*, 2008 (Cambridge University Press, Cambridge, UK).

26. Shiryaev, A. N., On optimum methods in quickest detection problems, *Theory of Probability and Its Applications*, Vol. 8, No. 1, pp. 22–46, 1963.
27. Shiryaev, A. N., Minimax optimality of the method of cumulative sums (CUSUM) in the continuous case, *Russian Mathematical Surveys*, Vol. 51, No. 4, pp. 750–751, 1996.
28. Siegmund, D., *Sequential Analysis*, 1985 (Springer-Verlag, New York).
29. Sonesson, C., and Bock, D., A review and discussion of prospective statistical surveillance in public health, *Journal of the Royal Statistical Society: Series A*, Vol. 166, No. 1, pp. 5–21, 2003.
30. Stamos, I., Hadjiliadis, O., Zhang, H., and Flynn, T., Online algorithms in the classification of urban objects in 3D point clouds, *Proceedings of the 2nd 3D Imaging Processing Visualization and Transmission Conference*, Zurich, Switzerland, October 13–15, 2012.
31. Zhang, H., and Hadjiliadis, O., Drawdowns and the speed of a market crash, *Methodology and Computing in Applied Probability*, Vol. 14, No. 3, pp. 739–752, 2012.