Chapter One

FUNCTIONS AND ALGEBRAIC STRUCTURE

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Contents

1.1 WHAT IS A FUNCTION?

In everyday language, the word *function* expresses the notion of one quantity depending on another. We might say that election results are a function of campaign spending, or that ice cream sales are a function of the weather. In mathematics, the meaning of *function* is more precise, but the idea is the same. If the value of one quantity determines the value of another, we say the second quantity is a function of the first.

> A **function** is a rule that takes numbers as inputs and assigns to each input exactly one number as the output. The output is a function of the input.

Example 1 A 20% tip on a meal is a function of the price, in dollars, of the meal.

(a) What is the input and what is the output of this function?
(b) Write a formula for the function, letting B stand for the price of the meal in dollars and letting T stand for the tip in dollars.

Solution (a) The input is the price in dollars, and the output is the amount of the tip in dollars.
(b) We have $20\% = 0.2$, so 20% of B is

$$T = 0.2B.$$

The letters we use to stand for the input and output of a function, like T and B in Example 1, are called *variables*. The variable standing for the input of a function is called the *independent variable*, and the variable standing for the output is called the *dependent variable* because the output depends on the input. In Example 1 the independent variable is B, and the dependent variable is T.

A formula tells you more than how to calculate the values of a function. The algebraic structure of the formula explains why the function behaves the way it does, as the next example shows.

Example 2 For the function in Example 1, explain how you can see from the formula that the larger the value of the bill, the larger the tip.

Solution As B gets larger, so does $0.2B$, because it is a positive constant times B. So $T = 0.2B$ gets larger.

Function Notation

We use *function notation* to represent the output of a function in terms of its input. The expression $f(20)$, for example, stands for the output of the function f when the input is 20. Here the letter f stands for the function itself, not for a number. If f is the function in Example 1, we have

$$f(20) = 4,$$

since the tip for a \$20 meal is $0.2 \cdot 20 = \$4$. In words, we say "$f$ of 20 equals 4." In general,

$$\text{Output} = f(\text{Input})$$
$$\text{Dependent} = f(\text{Independent}).$$

The **domain** of a function is the set of possible inputs to the function; the **range** is the corresponding set of outputs. What values make sense to substitute into the function $T = f(B) = 0.2B$? Since B is the price of a meal, the inputs must be positive. The output $0.2B$ is the tip, which also must be positive. So the domain of f is all positive values for B, and the range of f is all positive values for T.

Example 3 For the function in Example 1, evaluate $f(9)$ and $f(24)$ and interpret your answer in practical terms.

Solution We have

$$T = f(B) = 0.2B.$$

So

$$f(9) = 0.2 \cdot 9 = 1.80$$

and

$$f(24) = 0.2 \cdot 24 = 4.80.$$

This tells us the tip on a \$9 meal is \$1.80, and the tip on a \$24 meal is \$4.80.

Example 3 illustrates a benefit of function notation: allowing us to refer to a function's value in a compact way. Without function notation, we would have to refer to $f(9)$ as "the output of the tip function when the input is 9," or something similar.

Example 4 Let $n = f(p)$ be the average number of days a house stays on the market before being sold for price p (in \$1000s). What do the following statements mean in terms of the housing market?

(a) $f(250) = 90$ (b) $f(90) = 250$

Solution

(a) This means that a house stays on the market for 90 days on average before being sold for \$250,000.
(b) This means that a house stays on the market for 250 days on average before being sold for \$90,000.

Example 4 illustrates another use of function notation: it allows us to refer to and write statements about a function even when we do not know a formula for it.

Example 5 A headphone company's sales S, in dollars, when they sell each pair of headphones for p dollars is given by

$$S = g(p) = 240p - p^2.$$

(a) Identify the independent and dependent variables.
(b) Evaluate and interpret the meaning of

(i) $g(50)$ (ii) $g(240)$.

Solution

(a) The independent variable is the price p, in dollars, the company charges for a pair of headphones. The dependent variable S is the company's sales, in dollars.
(b) (i) We have

$$g(50) = 240 \cdot 50 - 50^2 = 9500.$$

This tells us the company sells \$9500 worth of headphones when the price for a pair is \$50.

(ii) We have

$$g(240) = 240 \cdot 240 - 240^2 = 0.$$

This tells us the company does not sell any headphones when the price for a pair is \$240. At that price, nobody is willing to buy them.

The Graph of a Function

What if the company in Example 5 wants to find the price it should charge in order maximize its sales? Obviously \$240 is too much, but maybe there is a price higher than \$50 which will bring increased sales. For example, trying $p = 100$ we get

$$g(100) = 13{,}500,$$

which is more sales than when the price is \$50. Is there an even better price? One way to find out is to make a table of values (see Table 1.1). The sales go down from \$14,000 to \$13,500 when the price increases from \$100 to \$150. But maybe there is a better price in between?

Table 1.1 *Values of sales*

p, price, in \$	$g(p)$, sales, in \$
0	0
50	9500
100	14,000
150	13,500
200	8000

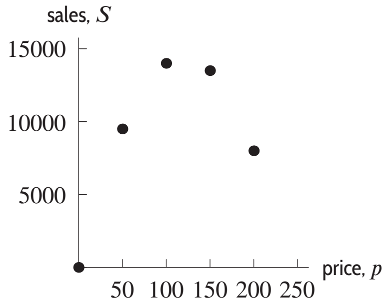


Figure 1.1: Graph of sales

This seems possible from Figure 1.1, which shows the pairs (p, S). If we plot all pairs (p, S) we get the curve in Figure 1.2. This is the *graph* of the sales function.

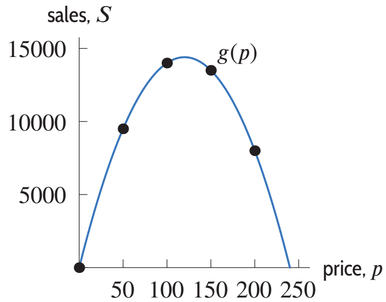


Figure 1.2: Graph of $S = 240p - p^2$

In Figure 1.2, we see that the graph of $S = g(p)$ crosses the p-axis when p is 0 and when p is approximately 240. We say $p = 0$ and $p = 240$ are the **horizontal intercepts**. Similarly, since the graph of $S = g(p)$ crosses the S-axis where S is 0, we say $S = 0$ is the **vertical intercept**.

Example 6 Use Figure 1.2 to estimate what price the company should charge to bring in the greatest sales. Approximately what will the company's sales be at this price?

Solution In Figure 1.2 the point where the graph $S = g(p)$ reaches its maximum value is located at approximately $(125, 15000)$. This tells us that if the company charges a price of about \$125 for each pair of headphones, it will reach its maximum sales of about \$15,000.

In Example 6, the graph shows us at a glance the approximate price the company should charge for its headphones. To find the exact price, we need to use the algebraic structure of the expression defining the function. We do this in the next section.

Variables and Constants

In addition to using letters to stand for the input and output of a function, we sometimes also use letters to stand for other numbers in the formula for the function.

Example 7 A car rental company charges a rental fee of r dollars plus an additional fee of m dollars for each mile a customer drives the rental car. Express the total cost C, in dollars, as a function of the distance d, in miles, the rental car has been driven.

Solution Each customer must pay a flat fee of r dollars. In addition, if the car is driven a total distance of d miles, and the company charges m dollars for each mile driven, a customer would have to pay an additional amount of $m \cdot d$ dollars. Thus, if a customer drives d miles, the total cost of renting the car is

$$C = f(d) = r + m \cdot d.$$

In Example 7 the expression for the function contains the letters r and m in addition to the variable d. No matter how many miles a person drives the rental car, the values of r and m do not change. We call such letters *constants* because for a given function their values do not change.

Exercises for Section 1.1

PROBLEMS

1. The number, N, of napkins used in a restaurant is $N = f(C) = 2C$, where C is the number of customers. What is the dependent variable? The independent variable?

2. A silver mine's profit, P, is $P = g(s) = -300{,}000 + 50{,}000s$ dollars, where s is the price per ounce of silver. What is the dependent variable? The independent variable?

In Problems 3–4, write the relationship using function notation (that is, y is a function of x is written $y = f(x)$).

3. Number of molecules, m, in a gas, is a function of the volume of the gas, v.

4. Weight, w, is a function of caloric intake, c.

5. The cost in dollars of tuition, T, at most colleges is a function $T = f(c)$ of the number of credits taken, c.

   (a) Identify the independent and dependent variables.
   (b) Give the meaning of:

   (i) $f(3) = 3000$ (ii) $f(12) = f(16)$

6. Figure 1.3 shows the graph of the function from Example 3 giving the tip, T, as a function of the bill, B. Which of the points P, Q, and R in Figure 1.3 tells us that $f(20) = 4$?

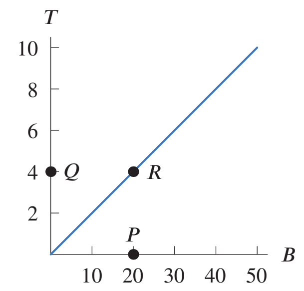


Figure 1.3

7. **(a)** Use the graph of $f(x) = 5 - \sqrt{x}$ in Figure 1.4 to estimate:

   (i) $f(0)$ (ii) $f(10)$ (iii) $f(16)$

   (b) Use the formula for f to evaluate:

   (i) $f(0)$ (ii) $f(10)$ (iii) $f(16)$

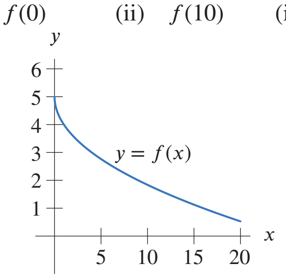


Figure 1.4

In Problems 8–9, evaluate the function for $x = -7$.

8. $f(x) = x/2 - 1$ **9.** $f(x) = x^2 - 3$

10. Let $g(x) = (12 - x)^2 - (x - 1)^3$. Evaluate:

(a) $g(2)$ **(b)** $g(5)$ **(c)** $g(0)$ **(d)** $g(-1)$

11. Let $f(x) = 2x^2 + 7x + 5$. Evaluate:

(a) $f(3)$ **(b)** $f(a)$ **(c)** $f(2a)$ **(d)** $f(-2)$

In Problems 12–17, evaluate the function given that

$$f(x) = \frac{2x+1}{3-5x} \quad \text{and} \quad g(y) = \frac{1}{\sqrt{y^2+1}}.$$

12. $f(0)$ **13.** $g(0)$ **14.** $g(-1)$

15. $f(10)$ **16.** $f(1/2)$ **17.** $g\left(\sqrt{8}\right)$

In Problems 18–23, evaluate the function given that

$$h(t) = 10 - 3t.$$

18. $h(r)$ **19.** $h(2u)$ **20.** $h(k-3)$

21. $h(4-n)$ **22.** $h(5t^2)$ **23.** $h(4-t^3)$

24. If $f(x) = 1 - x^2 - x$, evaluate and simplify $f(1-x)$.

25. Let $f(T)$ be the volume in liters of a balloon at temperature T°C. If $f(40) = 3$,

(a) What are the units of the 40 and the 3?
(b) What is the volume of the balloon at 40°C?

26. The sales tax on an item is 6%. Express the total cost, C, in terms of the price of the item, P.

In Problems 27–29, use the table to fill in the missing value. (There may be more than one answer.)

27. **(a)** $g(0) = ?$ **(b)** $g(?) = 0$
(c) $g(-5) = ?$ **(d)** $g(?) = -5$

y	−10	−5	0	5	10
$g(y)$	−5	0	5	10	−10

28. **(a)** $h(0) = ?$ **(b)** $h(?) = 0$
(c) $h(-2) = ?$ **(d)** $h(?) = -2$

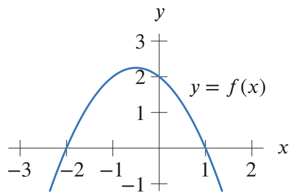
t	−3	−2	−1	0	1	2	3
$h(t)$	−1	0	−3	−2	−1	−2	0

29. **(a)** $h(?) = 2h(0)$ **(b)** $h(?) = 2h(-3) + h(2)$
(c) $h(?) = h(-2)$ **(d)** $h(?) = h(1) + h(2)$

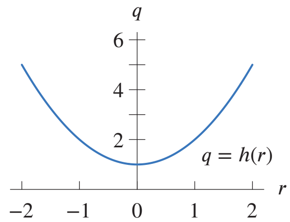
t	−3	−2	−1	0	1	2	3
$h(t)$	−1	0	−4	−2	−1	−2	0

In Problems 30–32, use the graph to fill in the missing value. (There may be more than one answer.)

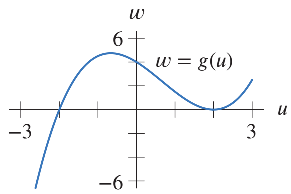
30. **(a)** $f(0) = ?$ **(b)** $f(?) = 0$



31. **(a)** $h(0) = ?$ **(b)** $h(?) = 0$



32. **(a)** $g(0) = ?$ **(b)** $g(?) = 0$



33. Let $S = f(p)$ be the total sales, in dollars, that a company receives when it charges p dollars for a product. Explain the meaning of the following statements.

(a) $f(15) = 112{,}500$ **(b)** $f(a) = 0$
(c) $f(1) = b$ **(d)** $c = f(p)$

34. The braking distance of a car is the distance a car travels from the time the brakes are applied to the time the car comes to a complete stop. Let $d = g(v)$ be the braking distance, in feet, of a car traveling at v miles per hour. Explain the meaning of the following statements.

(a) $g(30) = 111$ **(b)** $g(a) = 10$
(c) $g(10) = b$ **(d)** $s = g(v)$

35. Figure 1.5 shows the graph of a function giving the highway gas mileage of a car (in miles per gallon), H, as a function of the car's speed (in miles per hour), s.

(a) Estimate the highway gas mileage if the car is going 60 miles per hour.
(b) At what speed should you drive in order to maximize the car's fuel efficiency?

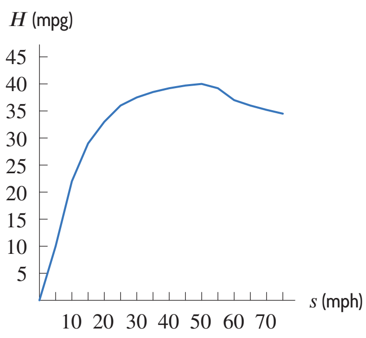


Figure 1.5

36. A corporate bond has a face value of p dollars. The interest each year is 5% of the face value. After t years the total interest is the product of the number of years, t, and the interest received each year. The payout is the sum of the face value and the total interest.

(a) Express the total interest I, in dollars, as a function of the age t, in years, of the bond.
(b) Express the payout P, in dollars, as a function of t.

In Problems 37–40, which letters stand for variables and which for constants?

37. $V(r) = (4/3)\pi r^2$

38. $f(x) = b + mx$

39. $P(t) = A(1 - rt)$

40. $B(r) = A(1 - rt)$

41. Einstein's famous equation $E = mc^2$ expresses energy E as a function of mass m. Which letters in this equation represent variables and which represent constants?

42. A tip of r percent on a bill of B dollars is given in dollars by

$$\text{Tip} = \frac{r}{100}B.$$

Which letters in the expression $(r/100)B$ would you call variables and which would you call constants if you were considering

(a) The tip as a function of the bill amount?
(b) The tip as a function of the rate?

43. The price of apartments near a subway is given by

$$\text{Price} = \frac{1000 \cdot A}{10d} \text{ dollars,}$$

where A is the area of the apartment in square feet and d is the distance in miles from the subway. Which letters are constants and which are variables if

(a) You want an apartment of 1000 square feet?
(b) You want an apartment 1 mile from the subway?
(c) You want an apartment that costs $200,000?

The lower the price for purchasing a song from an online music store, the more times the song is downloaded. Let $d = r(p)$ give the average number of daily downloads of a song as a function of the price of the song (in cents). Let p_0 be the price currently being charged for the song (in cents). What does the statement in Problems 44–51 tell you about number of downloads from the store?

44. $r(99)$

45. $r(p_0)$

46. $r(p_0 - 10)$

47. $r(p_0 - 10) - r(p_0)$

48. $365r(p_0)$

49. $r(0.80p_0)$

50. $\dfrac{r(p_0)}{24}$

51. $p_0 \cdot r(p_0)$

Let $f(n) = \frac{1}{2}n(n+1)$. Evaluate:

52. $f(100)$

53. $f(n+1) - f(n)$

Different strains of a virus survive in the air for different time periods. For a strain that survives t minutes, let $N = h(t)$ be the number of people infected (in thousands). The most common strain survives for t_0 minutes. What does the statement in Problems 54–55 tell you about the number of people infected?

54. $h(t_0 + 3)$

55. $\dfrac{h(2t_0)}{h(t_0)}$

A car with tire pressure P lbs/in^2 gets gas mileage (in mile per gallon) $g = f(P)$. The recommended tire pressure is P_0. What does the statement in Problems 56–57 tell you about the car's tire pressure and gas mileage?

56. $f(0.9P_0)$

57. $f(P_0) - f(P_0 - 5)$

Evaluate and simplify the expressions in Problems 58–60 given that $w(s) = 4 - 7s$.

58. $3w(s-5)$

59. $w\left(s^2 - 2\right) + 2$

60. $4w(7 - 4s) - 7$

1.2 FUNCTIONS AND EXPRESSIONS

An *algebraic expression* is a way of representing a calculation, using letters to stand for numbers. For example, the expression $\frac{1}{2}bh$, which gives the area of a triangle whose base has length b and height h, describes the following calculations:

- multiply the base b by $1/2$,
- and then multiply by the height h.

Notice that the verbal description is longer than the expression, and the algebraic expression enables us to see in compact form all the features of the calculation at once.

What Do We Mean by Algebraic Structure?

Looking at the structure of an algebraic expression can help us understand how the expression behaves. For example, as x increases from 1:

$6 + x$	starts at 7 when $x = 1$ and increases each time x increases.
$\frac{7}{x}$	also starts at 7 when $x = 1$ but decreases each time x increases.
$6 + x^2$	starts at 7 when $x = 1$ and increases faster than $6 + x$ because x is squared.

We can make these observations without performing time-consuming calculations. Throughout this book we pay attention to the *algebraic structure* of expressions. The next examples shows how this can be helpful.

Example 1 A student's grade in a course depends on two test grades, t_1 and t_2, and a final exam grade, f. The average of the two tests counts for 60% of the grade, and the final counts for 40% of the grade, according to the formula

$$\text{Course grade} = 0.6\left(\frac{t_1 + t_2}{2}\right) + 0.4f.$$

(a) Find the course grade given that the student's grades on tests one and two are 80 and 88, respectively, and the final grade is 74.
(b) Which would be better, to get 10 extra points on the second test or to get 10 extra points on the final?
(c) Explain how you could have seen the answer to part (b) without calculating the course grade.

Solution (a) Using the formula, we have

$$\text{Course grade} = 0.6\left(\frac{80 + 88}{2}\right) + 0.4 \cdot 74 = 80.$$

(b) If we add 10 points to the second test grade then it becomes $88 + 10 = 98$, so

$$\text{Course grade} = 0.6\left(\frac{80 + 98}{2}\right) + 0.4 \cdot 74 = 83.$$

If we add 10 points to the final then it becomes $74 + 10 = 84$, so

$$\text{Course grade} = 0.6\left(\frac{80 + 88}{2}\right) + 0.4 \cdot 84 = 84.$$

So it is better to get the extra points on the final.

(c) The part of the formula for the tests can be expanded using the distributive law[1] as

$$0.6\left(\frac{t_1+t_2}{2}\right) = 0.6\cdot\frac{1}{2}(t_1+t_2) = 0.3t_1 + 0.3t_2.$$

So

$$\text{Course grade} = 0.3t_1 + 0.3t_2 + 0.4f.$$

If we ignore all the terms except the one involving the second test grade, $0.3t_2$, we can see that the second test grade is multiplied by 0.3. This means that any extra points on the second test get multiplied by 0.3. On the other hand, the term involving the final is $0.4f$, so any extra points on the final get multiplied by 0.4. Since 0.4 is greater than 0.3. it is better to add the points to the final.

Paying attention to algebraic structure means looking inside an algebraic expression and thinking ahead about what will happen if we perform an operation on it. For instance, in Example 1, we look inside the expression for the course grade and notice that the part of it giving the contribution from tests can be expanded using the distributive law. Then thinking ahead, we see that in the expanded form t_2 will be multiplied by the constant 0.3, which is less than the constant 0.4 multiplying the final grade.

Since functions are often defined by expressions, by paying attention to the algebraic structure of expressions we can learn about key features of functions.

Example 2 Bernardo plans to travel 400 miles over spring break to visit his family. He can choose to fly, drive, take the train, or make the journey as a bicycle road trip. If his average speed is r miles per hour, then the time T, in hours, that the journey takes is given by

$$T = f(r) = \frac{400}{r}.$$

(a) Find $f(80)$ and $f(200)$ and give a practical interpretation of your answers.
(b) Use your answer in part (a) to show that $f(200)$ is less than $f(80)$. What does this mean in practical terms?
(c) Show how you can see from the graph of f that $f(200)$ is less than $f(80)$.
(d) Use the algebraic structure of the expression defining $f(r)$ to explain how you can see that $f(200)$ is less than $f(80)$ without doing any calculations.

Solution (a) We have

$$f(80) = \frac{400}{80} = 5 \quad \text{and} \quad f(200) = \frac{400}{200} = 2.$$

This means that it takes Bernardo 5 hours traveling at 80 miles per hour (e.g. by train) and 2 hours traveling at 200 miles per hour (e.g. by airplane).

(b) Since $f(200) = 2$ and $f(80) = 5$, we see that $f(200)$ is less than $f(80)$. This means that the faster he goes the less time he takes, which makes sense.

(c) The T-values on the graph in Figure 1.6 get smaller as the r-values gets larger. Larger r-values correspond to greater speeds, so again we see that the faster he goes the less time it takes.

[1]See page 349 for a review of the distributive law.

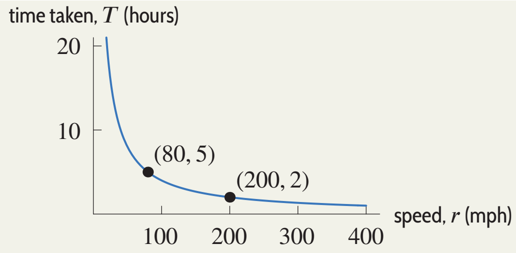


Figure 1.6: Graph of time taken as a function of speed

(d) We have

$$f(r) = \frac{400}{r}.$$

As the value of r becomes larger, the value of $f(r) = 400/r$ becomes smaller. Since 200 is greater than 80, we can see that $f(200)$ will be less than $f(80)$ without evaluating either expression.

Equivalent Expressions

A number can be expressed in many different ways. For example, $1/2$ and 0.5 are two different ways of expressing the same number. Similarly, two different algebraic expressions can define the same function. For instance, in Example 5 in Section 1.1 we considered sales S of headphones as a function of their price p, given by

$$S = g(p) = 240p - p^2.$$

This function can also be expressed in the form

$$S = g(p) = p(240 - p).$$

To see that the two expressions give the same output, we expand using the distributive law:[2]

$$p(240 - p) = p \cdot 240 - p \cdot p = 240p - p^2.$$

We say that $240p - p^2$ and $p(240 - p)$ are *equivalent expressions* because they have the same value for all values of p.

Example 3 Use the algebraic structure of

$$S = g(p) = p(240 - p)$$

to explain, without calculating, how you can see that when the price p of a pair of headphones is \$240, the sales S will be \$0.

Solution The expression $p(240 - p)$ is a product of two factors, p and $240 - p$. The second factor is zero when $p = 240$, because $240 - 240 = 0$, so

$$S = g(240) = \text{number} \cdot \text{zero}.$$

Since any number times zero is zero, $S = 0$.

One way to see that two expressions are equivalent is to graph them. For example, consider the function

$$S = g(p) = 14{,}400 - (p - 120)^2.$$

[2] See page 349 for a review of the distributive law.

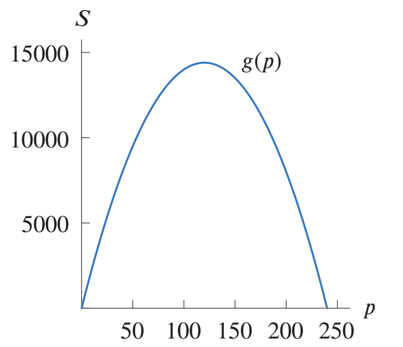


Figure 1.7: Graph of $g(p) = 14{,}400 - (p - 120)^2$

Looking at the graph of g in Figure 1.7, we see that it appears to be the same as the graph of

$$S = g(p) = 240p - p^2$$

in Figure 1.2 in Section 1.1. This is because the expression $14{,}400 - (p - 120)^2$ is equivalent to the expression $240p - p^2$. In later chapters we will study how to show equivalence algebraically for many different types of expressions. Here we are mainly interested in seeing how equivalent expressions are useful.

Example 4 Use the algebraic structure of the expression for sales S given by

$$S = g(p) = 14{,}400 - (p - 120)^2$$

to determine the price p that gives the maximum value of sales S. What are the sales at this price?

Solution The expression $g(p)$ has the form

$$g(p) = 14{,}400 - (p - 120)^2 = 14{,}400 - \text{Squared expression}.$$

Note that the expression $(p - 120)^2$ is always greater or equal to zero since the square of a number is always greater or equal to zero. If $p - 120 = 0$, then the squared expression $(p - 120)^2$ will equal zero.

Subtracting a positive number from 14,400 results in a number that is smaller than 14,400. Subtracting zero from 14,400 leaves it the same. So, the maximum value of sales occurs when $p - 120 = 0$, that is, when $p = 120$. When the price $p = 120$, the headphone company's sales will be $S = 14{,}400$. Notice that the exact answer obtained in this example, $p = 120$, is different from the approximate answer, $p = 125$, obtained graphically in Example 6 in Section 1.1. Although graphs are useful, we need algebra to find exact answers.

What Operations Produce Equivalent Expressions?

Since an algebraic expression is a way of representing a calculation, the rules for producing equivalent expressions come from the rules of arithmetic. For example, we can use regrouping and reordering terms to produce equivalent expressions such as

$$(3x - 5) + (7x + 10) \quad \text{and} \quad 10x + 5.$$

See Appendix A for a review of the rules for producing equivalent expressions by regrouping, reordering, and the distributive law.

We also know adding 0 to any number does not change its value, so we can produce equivalent expressions

$$x \quad \text{and} \quad (x+2)-2 \quad \text{by adding } 2-2=0. \tag{1.1}$$

Similarly, since multiplying a number by 1 does not change its value, we can produce equivalent expressions

$$x \quad \text{and} \quad \frac{9x}{9} \quad \text{multiplying by } 9/9=1. \tag{1.2}$$

Therefore we can express the function $f(x) = x$ in the form $f(x) = (x+2) - 2$, as we can express the function $g(x) = \frac{9x}{9}$ in the form $g(x) = x$. Expressing a function in different forms can reveal different features of the function.

Exercises for Section 1.2

IDENTIFYING ALGEBRAIC STRUCTURE

In Exercises **1–11**, say whether the value of the expression increases, decreases, or does not change as x starts at 1 and increases. Assume A is a positive constant.

1. $A + x$ **2.** $A - x$ **3.** $x - A$

4. Ax **5.** $-Ax$ **6.** $\frac{A}{x}$

7. $\frac{x}{A}$ **8.** $-Ax^2$ **9.** $(-Ax)^2$

10. $\frac{Ax}{x}$ **11.** $(A + x) - x$

In Exercises **12–19**, write an expression for the sequence of operations.

12. Subtract x from 1.

13. Subtract 1 from x.

14. Add 3 to x and double the result.

15. Double x and add 3 to the result.

16. Subtract x from 1, double, and add 3.

17. Subtract 1 from x, double, and add 3.

18. Add 3 to x, subtract the result from 1, and double.

19. Add 3 to x, double, and subtract 1 from the result.

PROBLEMS

20. The number of gallons left in a gas tank after driving d miles is given by $G(d) = 17 - 0.05d$.

(a) Which is larger, $G(50)$ or $G(100)$?
(b) Explain your answer in terms of the expression for $G(d)$, and give a practical interpretation.

21. If you drive to work at v miles per hour, the time available for breakfast is $B(v) = 30 - 480/v$ minutes.

(a) Which is greater, $B(35)$ or $B(45)$?
(b) Explain your answer in terms of the expression for $B(v)$ and give a practical interpretation.

22. In Example 2 on page 9, Bernardo's trip time T, in hours, is a function of his average speed r, in miles per hour, which is given by

$$T = f(r) = \frac{400}{r}.$$

(a) Make a table of values for $r = 10, 25, 80, 100$, and 200, and graph the function.
(b) Sketch a graph of $T = f(r)$. Determine the behavior of T values as r gets larger.
(c) Use the expression defining $T = f(r)$ to explain its behavior as r gets larger.

23. In Example 3 on page 3 the tip T, in dollars, for a meal with a bill of B dollars is given by the function

$$T = f(B) = 0.2B.$$

Pares says she has an easy way to figure out the tip: she moves the decimal point in the bill one place to the left, then doubles the answer.

(a) Check that Pares' method gives the same answer on bill amounts of \$8.95 and \$23.70 as evaluating the expression $f(B) = 0.2B$ at $B = 8.95$ and 23.70.
(b) Write an expression for Pares' method. Does her method define the same function as f? Explain your answer using algebraic structure.

24. To convert kilograms to pounds, Abby halves the number of kilograms, n, then subtracts 10% from the result of that calculation, whereas Renato subtracts 10% first and then halves the result.

   (a) Write an algebraic expression for each method.
   (b) Do the methods give the same answer?

25. Let $f(x) = 2x^2$ and $g(x) = (2x)^2$.

   (a) Sketch the graphs of the functions on the same coordinate axes. Do the graphs appear to be the same or different?
   (b) Using the expression defining the functions and algebraic operations, verify your answer in part (a).

26. **(a)** Say in words how to compute an output value of $f(x) = (x-2)^2 + 3$ corresponding to an input.
   (b) Use your answer in part (a) to determine whether the function f has a maximum and/or minimum value based on the algebraic structure of the expression of f. If so, what are the maximum and minimum values and at what x-values do they occur?

In Problems 27–33, are the expressions equivalent?[3]

27. $a + (2 - d)$ and $(a + 2) - d$

28. $3(z + w)$ and $3z + 3w$

29. $(a - b)^2$ and $a^2 - b^2$

30. $\sqrt{a^2 + b^2}$ and $a + b$

31. $-a + 2$ and $-(a + 2)$

32. $bc - cd$ and $c(b - d)$

33. $x^2 + 4x^2$, $5x^2$, and $4x^4$

In Problems 34–37, describe the sequence of operations that produces the expression.

34. $2(u + 1)$

35. $2u + 1$

36. $1 - 3(B/2 + 4)$

37. $3 - 2(s + 5)$

38. Abby and Leah go on a 5-hour drive for 325 miles at 65 mph. After t hours, Abby calculates the distance remaining by subtracting $65t$ from 325, whereas Leah subtracts t from 5 then multiplies by 65.

   (a) Write expressions for each calculation.
   (b) Do the expressions in (a) define the same function?

39. To calculate the balance after investing P dollars for two years at 5% interest, Sharif adds 5% of P to P, and then adds 5% of the result of this calculation to itself. Donald multiplies P by 1.05, and then multiplies the result of this by 1.05 again.

   (a) Write expressions for each calculation.
   (b) Do the expressions in (a) define the same function?

40. On Figure 1.8, indicate intervals of length

   (a) $x + 1$ **(b)** $2(x + 1)$
   (c) $2x$ **(d)** $2x + 1$

   What does your answer tell you about whether $2x + 1$ and $2(x + 1)$ are equivalent?

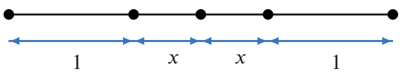


Figure 1.8

41. Apples are 99 cents a pound, and pears are \$1.25 a pound. Write an expression for the total cost, in dollars, of a pounds of apples and p pounds of pears.

42. **(a)** Write an expression for the total cost of buying 8 apples at \$$a$ each and 5 pears at \$$p$ each.
   (b) Find the total cost if apples cost \$0.40 each and pears cost \$0.75 each.

In Problems 43–46, write an expression for the sales tax on a car.

43. Tax rate is 7%, price is \$$p$.

44. Tax rate is r, price is \$20,000.

45. Tax rate is 6%, price is \$1000 off the sticker price, \$$p$.

46. Tax rate is r, price is 10% off the sticker price, \$$p$.

47. A teacher calculates the course grade by adding the four semester grades g_1, g_2, g_3, and g_4, then adding twice the final grade, f, then dividing the total by 6. Write an expression for the course grade.

48. A student's grade in a course depends on the three test grades, t_1, t_2, and t_3, a homework grade, h, and a final exam grade, f. The homework counts for 10% of the grade, the average of the three tests counts for 60% of the grade, and the final counts for 30% of the grade, according to the formula

$$\text{Course grade} = 0.1h + 0.6\left(\frac{t_1 + t_2 + t_3}{3}\right) + 0.3f.$$

   (a) Find the course grade given that the student's homework grade is 67, the grades on tests one, two and three are 80, 96, and 82, respectively, and the final grade is 74.
   (b) Which would be better, to get 10 extra points on the third test, or to get 10 extra points on the final?
   (c) Explain how you could have seen the answer to part (b) without calculating the course grade.

[3] See Appendices A, D, E and G for further review and practice with expressions.

49. A car travels from Tucson to San Francisco, a distance of 870 miles. It has rest stops totaling 5 hours. While driving, it maintains a speed of v mph. Give an expression for the time it takes. What is the difference in time taken between a car that travels 65 mph and a car that travels 75 mph?

50. The volume of a cone with radius r and height h is given by the expression $(1/3)\pi r^2 h$. Write an expression for the volume of a cone in terms of the height h if the radius is equal to half of the height.

51. The number of people who attend a concert is $160 - p$ when the price of a ticket is $\$p$.

(a) What is the practical interpretation of the 160?
(b) Why is it reasonable that the p term has a negative sign?
(c) The number of people who attend a movie at ticket price $\$p$ is $175 - p$. If tickets are the same price, does the concert or the movie draw the larger audience?
(d) The number of people who attend a dance performance at ticket price $\$p$ is $160 - 2p$. If tickets are the same price, does the concert or the dance performance draw the larger audience?

52. A rectangle with base b and height h has area bh. A triangle with the same base and height has area $(1/2)bh$. Write a brief explanation of where the $1/2$ in this expression comes from by comparing the area of the triangle to the area of the rectangle.

Suppose P and Q give the sizes of two different animal populations, where $Q > P$. In Problems 53–56, which of the expressions is larger? Briefly explain your reasoning in terms of the two populations.

53. $P + Q$ and $2P$

54. $\dfrac{P}{P+Q}$ and $\dfrac{P+Q}{2}$

55. $(Q - P)/2$ and $Q - P/2$

56. $P + 50t$ and $Q + 50t$

One algebra class has a total of A students, while a second class has B students, where A is larger than B. What does the expression in Problems 57–60 represent in practical terms?

57. $A + B$

58. $A - B$

59. $\dfrac{A+B}{2}$

60. $\dfrac{A}{A+B}$

One month into an epidemic, I people are infected, R people have recovered from infection, and S people remain susceptible to infection. Initially, the entire population is susceptible, and no one is infected or has recovered. Assume that once someone is infected or has recovered, he or she cannot be reinfected and that the total number of people remains constant. Write simplified expressions representing the quantities described in Problems 61–64.

61. The total population.

62. The fraction of the total population that has recovered.

63. The percentage of the population that remains susceptible after one month, assuming there are twice as many infected people as recovered people, and ten times as many susceptible people as infected people.

64. The fraction of the total population that has recovered after one month, assuming that one in four initially susceptible people have been infected, and of these one in five has recovered.

In Problems 65–69, both a and x are positive. What is the effect of increasing a on the value of the expression? Does the value increase? Decrease? Remain unchanged?

65. $ax + 1$

66. $\dfrac{x}{a} + 1$

67. $x + \dfrac{1}{a}$

68. $ax - \dfrac{1}{a}$

69. $a + x - (2 + a)$

In Problems 70–73, which expression represents the larger number?

70. $10 + t^2$ and $9 - t^2$

71. $\dfrac{6}{k^2+3}$ and $k^2 + 3$

72. $(s^2 + 2)(s^2 + 3)$ and $(s^2 + 1)(s^2 + 2)$

73. $\dfrac{12}{z^2+4}$ and $\dfrac{12}{z^2+3}$

In Problems 74–77, decide whether the expressions can be put in the form

$$\frac{ax}{a+x}.$$

For those that are of this form, identify a and x. Assume that a and b are constants.

74. $\dfrac{3y}{y+3}$

75. $\dfrac{b^2\theta^2}{b^2+\theta^2}$

76. $\dfrac{8y}{4y+2}$

77. $\dfrac{5(y^2+3)}{y^2+8}$

1.3 FUNCTIONS AND EQUATIONS

In Example 2 in Section 1.2 we found the time taken for Bernardo to travel 400 miles at an average speed of r miles per hour by evaluating the expression for the function

$$T = f(r) = \frac{400}{r}.$$

We found the time by substituting an input into the expression for T. Sometimes we might want to go in the other direction and find the input that results in a specific output. To do this we solve an equation.

Solving Equations

Example 1 What average speed must Bernardo maintain to make the trip in 10 hours?

Solution We want

$$T = 10$$

so we want the value of r that makes

$$\frac{400}{r} = 10.$$

Assuming $r \neq 0$, we multiply both sides by r to get

$$400 = 10r,$$

and then we divide both sides by 10 to get

$$40 = r.$$

So Bernardo must travel at 40 miles per hour. Figure 1.9 shows this fact graphically.

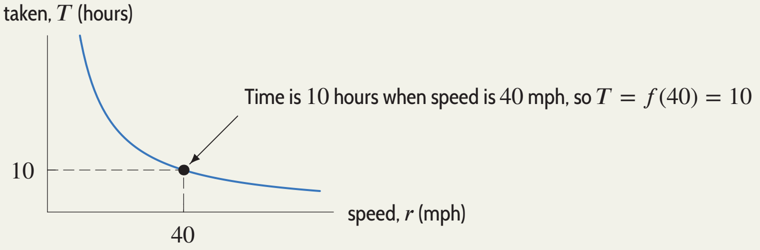


Figure 1.9: Solution to $T = 10$ is $r = 40$

The statement

$$\frac{400}{r} = 10$$

is an example of an *equation*. In general, an equation is a statement that two expressions are equal. A value of the variable that makes the statement true is a *solution* of the equation. So $r = 40$ is a solution to the equation in Example 1. Finding all the solutions to an equation is called *solving the equation*.[4]

[4] See Appendix B for a review of solving equations.

What Do We Mean by Algebraic Structure of an Equation?

In Example 1 we solved the equation step by step, at each step performing the same operation on both sides. Sometimes looking at the *algebraic structure* of an equation can help us find a quicker strategy for finding a solution. For example:

$6 + x = 10$ has solution $x = 4$ since adding 4 to 6 gives 10.

$\frac{7}{x} = 1$ has solution $x = 7$ since dividing 7 by 7 gives 1.

$6 + x^2 = 6$ has solution $x = 0$ since adding 0 to 6 gives 6.

By paying attention to the structure of an equation, we can sometimes solve an equation almost immediately, without manipulating the equation.

Example 2 Explain how you can see the solution to the equation in Example 1 using algebraic structure.

Solution We want to solve the equation

$$\frac{400}{r} = 10.$$

The expression on the left-hand side is a fraction representing 400 divided by r, so we want to answer the question

what number do we need to divide 400 by in order to get 10?

Since 400 divided by 40 is 10, the answer is $r = 40$, which is the same answer we found algebraically in Example 1.

Example 3 A hotel charges its guests for using its Wi-Fi network. The cost, in dollars, is given by the function $C = 10 + 6t$, where t is the number of hours a guest uses the Wi-Fi network. If a guest has a $40 charge for Wi-Fi use, for how many hours did the guest use the hotel's Wi-Fi network?

Solution We want to know when $C = 40$. So we want to solve the equation

$$10 + 6t = 40.$$

Looking at the algebraic structure of the equation, notice $6t$ is the number we need to add to 10 in order to get 40. Since $10 + 30 = 40$, we have

$$6t = 30.$$

Next, we need to answer the question what value multiplied by 6 gives 30? Since $6 \cdot 5 = 30$, we see

$$t = 5$$

is a solution. This means the guest used the hotel's Wi-Fi for a total of 5 hours.

Equivalently, we can find the solution by performing the following steps:

$$\begin{aligned} 10 + 6t &= 40 \\ 6t &= 30 && \text{subtract 10 from by sides} \\ t &= 5. && \text{divide both sides by 6} \end{aligned}$$

The Difference Between Equations and Expressions

An equation must have an equal sign separating the expressions on either side. An expression never has an equal sign. For example, $3(x-5)+6-x$ is an expression, but $3(x-5)+6-x=0$ is an equation. Although they look similar, they mean quite different things. One way to avoid confusion is to interpret the meaning of an expression or an equation.

Example 4 You have \$10.00 to spend on n bottles of soda, costing \$1.50 each. Are the following expressions? Equations? Give an interpretation of each expression or equation in terms of the context.

(a) $1.50n$ (b) $1.50n = 6.00$
(c) $10 - 1.50n$ (d) $10 - 1.50n = 2.50$

Solution

(a) This is an expression representing the cost of n bottles of soda.
(b) This is an equation whose solution is the number of bottles that can be purchased for \$6.00.
(c) This is an expression representing the amount of money left after buying n bottles of soda.
(d) This is an equation whose solution is the number of bottles you bought if the change you received was \$2.50.

Equivalent Equations

Each step in Example 3 resulted in a different equation that had the same solutions as the original. For instance, in Example 3,

$$6t = 30 \quad \text{is equivalent to} \quad 10 + 6t = 40.$$

In general, we say two equations are *equivalent* if they have exactly the same solutions. The step-by-step method for solving equations is to keep using operations that produce equivalent equations until you get an equation like $t = 5$ that shows the solution directly. For a review of solving equations, see Appendix B.

What Operations Produce Equivalent Equations?

We can think of an equation as a scale on which things are weighed. When the two sides are equal, the scale balances. In order to transform an equation into one that has the same solutions, we must perform an operation on both sides of the equal sign that keeps the relationship between the two sides the same. In this way we can be sure that the new equation has the same solutions—the same values that make the scale balance—as the original one. Figure 1.10 illustrates this idea.

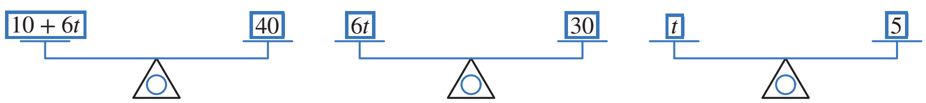


Figure 1.10: Subtracting 10 kilograms from each side of a balanced scale leaves it in balance, as does dividing each side by 6.

In general,

> We can transform an equation into an equivalent equation using any operation that does not change the balance between the two sides. This includes:
> - Adding or subtracting the same number to both sides
> - Multiplying or dividing both sides by the same number, provided it is not zero
> - Replacing an expression in an equation by an equivalent expression.

These operations ensure that the original equation has the same solutions as the new equation, even though the expressions on each side might change. We explore other operations in later chapters.

Finding Solutions Using Graphs

A solution to an equation is a value of the variable that makes both sides have the same value. We can look for solutions by graphing each side of the equation and finding where the two graphs intersect.

Example 5 Solve the equation in Example 3 by graphing each side of the equation

$$10 + 6t = 40.$$

Solution Figure 1.11 shows the graphs of

$$C = 10 + 6t$$

and

$$C = 40.$$

The value of t at the point where these two graphs intersect is the value of t such that

$$10 + 6t = 40.$$

Since the graphs intersect when $t = 5$, the guest used the hotel's Wi-Fi for 5 hours, which agrees with the algebraic solution in Example 3.

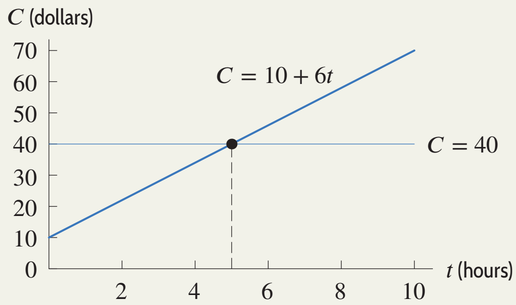


Figure 1.11: Cost for using hotel's Wi-Fi

Using Algebraic Structure

In Example 2 we used the algebraic structure of the expression on the left side of the equation $400/r = 10$ to solve the equation quickly. By recognizing r is the value we need to divide 400 by to get 10, we can see that $t = 40$ is a solution without performing any operations. Often we can see a solution to an equation by comparing the structure of both sides, even when the equation may seem complicated.

Example 6 Without any manipulation, give a solution to the equation:

(a) $\dfrac{x+1}{5} = 1$ (b) $\dfrac{x-7}{3} = 0$ (c) $9 - z = z - 9$ (d) $2^{t+1} = 4$

Solution

(a) The only way a fraction can equal one is for the numerator to equal the denominator. Therefore, $x + 1$ must be 5, so $x = 4$.
(b) The only way a fraction can equal zero is for the numerator to be zero. Therefore $x = 7$.
(c) The left side is the negative of the right side, so the equation requires a number to equal its own opposite. The only such number is zero. So $9 - z$ and $z - 9$ are both zero, and $z = 9$.
(d) Since $2^2 = 4$, the power $t + 1$ on the left side must equal 2. Therefore $t = 1$.

With some equations, it is possible to see from their structure there are no solutions.

Example 7 Without performing any manipulation, explain why the equation has no solutions:

(a) $x^2 = -4$ (b) $t = t + 1$ (c) $\frac{3x+1}{3x+2} = 1$ (d) $\sqrt{w+4} = -3$

Solution

(a) Since the square of any number is positive or zero, this equation has no solutions.
(b) No number can equal one more than itself, so there are no solutions.
(c) A fraction can equal one only when its numerator and denominator are equal. Since the denominator is one larger than the numerator, the numerator and denominator can never be equal. Therefore, there are no solutions.
(d) The square root of a number cannot be negative. Therefore, this equation has no solutions.

Exercises for Section 1.3

IDENTIFYING ALGEBRAIC STRUCTURE

In Exercises **1–6**, without performing any manipulation, give a solution to the equation.

1. $3x = 15$ **2.** $10 - y = 2$ **3.** $\frac{w}{4} = \frac{3}{4}$

4. $\sqrt{p+1} = 7$ **5.** $3 + a^2 = 3$ **6.** $\frac{7}{q+1} = -1$

In Exercises **7–12**, without performing any manipulation, explain why the equation has no solutions.

7. $\sqrt{2x+9} = -5$ **8.** $-2x^2 = 8$

9. $\frac{17}{x-1} = 0$ **10.** $2q + 1 = 2q + 7$

11. $10 + y^2 = 5$ **12.** $5^x = -5$

In Exercises **13–16**, what operation on both sides of the equation isolates the variable on one side? Give the solution of the equation.

13. $13 = -2z$ **14.** $0.5x = 3$

15. $7x = 6x - 6$ **16.** $\frac{3}{7}M = \frac{4}{3}$

PROBLEMS

In Problems **17–20**, is the value of the variable a solution to the equation?

17. $t + 3 = t^2 + 9,\ t = 3$ **18.** $x + 3 = x^2 - 9,\ x = -3$

19. $\frac{a+3}{a-3} = 1,\ a = 0$ **20.** $\frac{3+a}{3-a} = 1,\ a = 0$

In Problems **21–22**, use the graph of $y = v(x)$ in Figure 1.12.

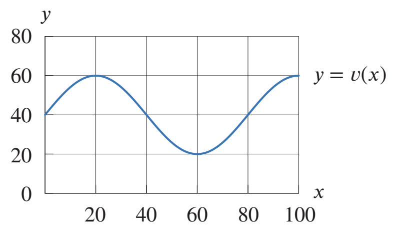


Figure 1.12

21. Solve $v(x) = 60$. **22.** Evaluate $v(60)$.

23. The tuition C, in dollars, for a semester at a small public university t years from now is given by

$$C = 3000 + 100t.$$

(a) Using the graph of C shown in Figure 1.13, estimate how many years it will take for tuition to reach \$3700.
(b) Check your answer to part (a) by substituting it into the equation

$$3000 + 100t = 3700.$$

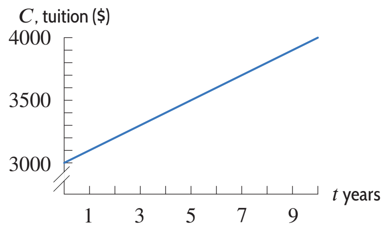


Figure 1.13

24. If a company sells s software packages, its profit per package P, in dollars, is given by

$$P = 10{,}000 - \frac{100{,}000}{s}.$$

(a) Using the graph of P shown in Figure 1.14, estimate the number of packages sold when profits per package are \$8000.

(b) Check your answer to part (a) by substituting it into the equation

$$10{,}000 - \frac{100{,}000}{s} = 8000.$$

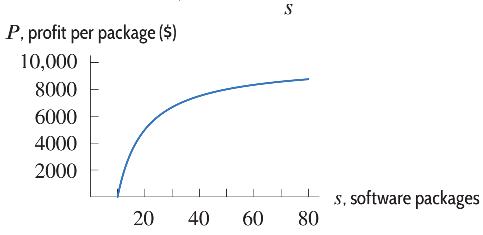


Figure 1.14

Solve the equations in Problems 25–30.[5]

25. $3z = 22$

26. $5x + 12 = 90$

27. $10 - 2x = 60$

28. $3(x - 5) = 12$

29. $\dfrac{x+2}{5} = 10$

30. $2x + 5 = 4x - 9$

31. Scott developed the following solution to the equation $2(x + 3) = 8$.

$$\begin{aligned} 2(x+3) &= 8 \\ 2x + 6 &= 8 \\ 2x &= 2 \\ x &= 1. \end{aligned}$$

Describe an alternate first step that could have been used to arrive at the same solution.

32. The number of gallons of gas, g, in a car's tank, d miles after stopping for gas, is given by

$$g = 15 - d/20.$$

(a) Write an equation whose solution is the number of miles it takes for the amount of gas in the tank to reach 10 gallons.

(b) Make a plot of the gallons left for $d = 40, 60, 80, 100, 120, 140$, and indicate the solution $m = 100$ to the equation in part (a).

33. A town's population P, in thousands, t years after its incorporation is given by the function $P = 30 + 2t$.

(a) Write an equation whose solution is when the town's population reaches 50,000.

(b) Solve the equation in part (a) by graphing both sides on the same axes.

(c) Check your answer by solving the equation algebraically.

34. In 2013 many towns along the Red River in North Dakota suffered serious flooding. Some responded by building levees, banks of sandbags to hold back the river. Suppose that the height h, in inches, of a river along a levee that is 50 inches high t hours after it begins to rain is given by

$$h = 27 + 2.1t.$$

Emergency supplies of sandbags to increase the height of the levee are on the way. When must they arrive to be there before the river reaches the top of the levee?

35. The populations, in year t, of two towns are given by the functions

$$\text{Town } A: \quad P(t) = 600 + 100(t - 2010)$$
$$\text{Town } B: \quad Q(t) = 200 + 300(t - 2010).$$

(a) Write an equation whose solution is the year in which the two towns have the same population.

(b) Make a table of values of the populations for the years 2010–2014 and find the solution to the equation in part (a).

Table 1.2 shows values of three expressions at different values of x. Based on the values in Table 1.2, give solutions to the equation in Problems 36–38.

Table 1.2

x	-1	0	1	2
$f(x)$	1	2	-1	0
$g(x)$	1	0	-1	0
$h(x)$	0	2	-1	-1

36. $f(x) = g(x)$ **37.** $f(x) = h(x)$ **38.** $g(x) = h(x)$

39. Eric plans to spend \$20 on ice cream cones at \$1.25 each. Write an equation whose solution is the number of ice cream cones he can buy, and find the number.

40. Dennis is on a diet of 1800 Calories per day. His dinner is 1200 Calories and he splits the remaining Calories equally between breakfast and lunch. Write an equation whose solution is the number of Calories he can eat at breakfast, and find the number.

[5] See Appendix B for more practice with solving equations.

41. A town's total allocation for firemen's wage and benefits is \$600,000. If wages are calculated at \$40,000 per fireman and benefits at \$20,000 per fireman, write an equation whose solution is the maximum number of firemen the town can employ, and solve the equation.

42. Antonio and Lucia are both driving through the desert from Tucson to San Diego, which takes each of them 7 hours of driving time. Antonio's car starts out full with 14 gallons of gas and uses 2 gallons per hour. Lucia's SUV starts out full with 30 gallons of gas and uses 6 gallons per hour.

(a) Construct a table showing how much gas is in each of their tanks at the end of each hour into the trip. Assume each stops for gas just as the tank is empty, and then the tank is filled instantaneously.
(b) Use your table to determine when they have the same amount of gas.
(c) If they drive at the same speed while driving and only stop for gas, which of them gets to San Diego first? (Assume filling up takes time.)
(d) Now suppose that between 1 hour and 6.5 hours outside of Tucson, all of the gas stations are closed unexpectedly. Does Antonio arrive in San Diego? Does Lucia?
(e) The amount of gas in Antonio's tank after t hours is $14 - 2t$ gallons, and the amount in Lucia's tank is $30 - 6t$ gallons. When does
(i) $14 - 2t = 30 - 6t$?
(ii) $14 - 2t = 0$?
(iii) $30 - 6t = 0$?

43. You have a \$10 discount certificate for a pair of pants. When you go to the store you discover that there is a 25% discount on all pants, but no further discounts apply.

(a) Make a table comparing the two discount methods. For what tag price do you end up paying the same amount with each discount method?
(b) Check your answer by solving an equation algebraically.

44. A drought in Central America causes the price of coffee to rise 25% from last year's price. You have a \$3 discount coupon and spend \$17.00 for two pounds of coffee. What did coffee cost last year?

In Problems 45–52, are the two equations equivalent? If they are, what operation transforms the first into the second?

45. $2(x+3) = 10$
$x + 3 = 10$

46. $2x + 5 = 22$
$2x = 27$

47. $5 - 3x = 10$
$5 = 10 + 3x$

48. $x^2 = 5x$
$x = 5$

49. $3x - 6 = 10$
$3x = 16$

50. $2x = 5x + 8$
$3x = 8$

51. $x = a$
$x^2 = xa$

52. $\frac{x}{3} = 12$
$x = 36$

53. Which of the following are equations?

(a) $3(x+5) = 6 - 2(x-5)$
(b) $ax^2 + bx + c = 0$
(c) $5(2x-1) + (5-x)(x+3)$
(d) $t = 7(t+2) - 1$

54. **(a)** Does $x/3 + 1/2 = 4x$ have the same solution as $2x + 3 = 24x$?
(b) Is $x/3 + 1/2$ equivalent to $2x + 3$?

55. **(a)** Does $8x - 4 = 12$ have the same solution as $2x - 1 = 3$?
(b) Is $8x - 4$ equivalent to $2x - 1$?

56. Which of the equations in parts (a)–(d) are equivalent to

$$2x - (x+3) = 4 + \frac{x-3}{10}?$$

Give reasons for your answers.

(a) $2x - x + 3 = 4 + \dfrac{x-3}{10}$
(b) $2x - (x+3) = 4 + 0.1x - 0.3$
(c) $x - 3 = 3.7 + 0.1x$
(d) $20x - 10(x+3) = 4 + x - 3$

In Problems 57–60, write an equation representing the situation if p is the price of a meal in dollars.

57. The cost of two meals is \$18.

58. The cost of three meals plus a \$5 tip is \$32.

59. The cost of a meal plus a 20% tip is \$10.80.

60. The cost of two meals plus a 20% tip is \$21.60.

In Problems 61–68, does the equation have a solution? Explain how you know without solving it.

61. $2x - 3 = 7$

62. $x^2 + 3 = 7$

63. $\dfrac{3x^2}{3x^2 - 1} = 1$

64. $4 = 5 + x^2$

65. $\dfrac{x+3}{2x+5} = 1$

66. $\dfrac{x+3}{5+x} = 1$

67. $\dfrac{x+3}{2x+6} = 1$

68. $\dfrac{a+1}{2a} = \dfrac{1}{2}$

In Problems 69–72, the solution depends on the constant a. Assuming a is positive, what is the effect of increasing a on the solution? Does it increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

69. $x - a = 0$

70. $ax = 1$

71. $ax = a$

72. $\frac{x}{a} = 1$

73. Check that $t = 1, 2, 3$ are solutions to the equation

$$t(t-1)(t-2)(t-3)(t-4) = 0.$$

Can you find any other solutions?

74. The equation

$$-x\sqrt{7+x} = 2$$

has two solutions. Are they positive, zero, or negative? Give an algebraic reason why this must be the case. You need not find the solutions.

75. The equation $x^2 + 1 = 2x + \sqrt{x}$ has two solutions. Are they positive, zero, or negative? Give an algebraic reason why this must be the case. You need not find the solutions.

76. Given that $x = 4$ is a solution to

$$2jx + z = 3,$$

evaluate the expression $16j + 2z$.

1.4 FUNCTIONS AND CHANGE

Functions describe how quantities change. By comparing values of the function for different inputs we can see how fast the output changes. For example, Table 1.3 shows the population $P = f(t)$ of a colony of termites t months after it was started. Using the Greek letter Δ (pronounced "delta") to represent change, we have

$$\text{Change in first 6 months} = \Delta P = f(6) - f(0) = 4000 - 1000 = 3000 \text{ termites}$$
$$\text{Change in first 3 months} = \Delta P = f(3) - f(0) = 2500 - 1000 = 1500 \text{ termites.}$$

Although the colony grows by twice as much during the first 6 months as during the first 3 months (3000 as compared with 1500), it also had twice as long in which to grow. Sometimes it is more useful to measure not the change in the output, but rather the change in the output divided by the change in the input, or *average rate of change*:

$$\begin{array}{c}\text{Average rate of change}\\ \text{in first 6 months}\end{array} = \frac{\Delta P}{\Delta t} = = \frac{3000 \text{ termites}}{6 \text{ months}} = 500 \text{ termites/month}$$
$$\begin{array}{c}\text{Average rate of change}\\ \text{in first 3 months}\end{array} = \frac{\Delta P}{\Delta t} = = \frac{1500 \text{ termites}}{3 \text{ months}} = 500 \text{ termites/month.}$$

Thus, on average, the colony adds 500 termites per month during the first 6 months, and also during first 3 months. Even though the population change over the first 3 months is less than over the first 6 months (1500 versus 3000), the average rate of change is the same (500 termites/month).

Table 1.3 *Population of a colony of termites*

t (months)	0	3	6	9	12
$P = f(t)$	1000	2500	4000	7000	2800

An Expression for the Average Rate of Change

If the input of the function $y = f(x)$ changes from $x = a$ to $x = b$, we write

$$\text{Change in input} = \text{New } x\text{-value} - \text{Old } x\text{-value} = b - a$$
$$\text{Change in output} = \text{New } y\text{-value} - \text{Old } y\text{-value} = f(b) - f(a).$$

The *average rate of change* of $y = f(x)$ between $x = a$ and $x = b$ is given by

$$\text{Average rate of change} = \frac{\text{Change in output}}{\text{Change in input}} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Example 1 Find the average rate of change of $g(x) = (x-2)^2 + 3$ on the following intervals:

(a) Between 0 and 3 (b) Between -1 and 4.

Solution (a) We have $g(0) = (0-2)^2 + 3 = 7$ and $g(3) = (3-2)^2 + 3 = 4$, so

$$\begin{matrix}\text{Average rate} \\ \text{of change}\end{matrix} = \frac{g(3) - g(0)}{3 - 0} = \frac{4 - 7}{3} = -1.$$

(b) We have $g(-1) = (-1-2)^2 + 3 = 12$ and $g(4) = (4-2)^2 + 3 = 7$, so

$$\begin{matrix}\text{Average rate} \\ \text{of change}\end{matrix} = \frac{g(4) - g(-1)}{4 - (-1)} = \frac{7 - 12}{5} = -1.$$

Example 2 For the termite colony in Table 1.3,

(a) What is the change in the population during the last 6 months? During months 6 to 9?
(b) What is the average rate of change of the population during the last 6 months? During months 6 to 9?

Solution (a) In the last 6 months, we have

$$\text{Change in last 6 months} = \Delta P = f(12) - f(6) = 2800 - 4000 = -1200 \text{ termites}$$
$$\text{Change from months 6 to 9} = \Delta P = f(9) - f(6) = 7000 - 4000 = 3000 \text{ termites.}$$

The change in the last 6 months is negative because the colony decreases in size, losing 1200 termites. However, for the first 3 of those months, months 6 to 9, it increases by 3000 termites.

(b) For the last 6 months, we see that

$$\begin{matrix}\text{Average rate of change} \\ \text{in last 6 months}\end{matrix} = \frac{\Delta P}{\Delta t} = \frac{f(12) - f(6)}{12 - 6} = \frac{-1200 \text{ termites}}{6 \text{ months}} = -200 \frac{\text{termites}}{\text{month}}$$
$$\begin{matrix}\text{Average rate of change} \\ \text{for months 6 to 9}\end{matrix} = \frac{\Delta P}{\Delta t} = \frac{f(9) - f(6)}{9 - 6} = \frac{3000 \text{ termites}}{3 \text{ months}} = 1000 \frac{\text{termites}}{\text{month}}.$$

The average rate of change over the last 6 months is negative, because the net change is negative, as we saw in part (a).

The population change over months 6 to 9 is the same as the population change over the first 6 months. (Both are 3000 termites.) However, the average rate of change is larger for months 6 to 9 than over the first 6 months. This is because the 3-month period (months 6 to 9) is shorter than the 6-month period at the start. A shorter period leads to a *faster* average rate of change.

Algebraic Structure: Change and Average Rate of Change

A car has traveled $F(t)$ miles t hours after it starts a journey. For example, $F(3)$ is the distance the car has traveled after 3 hours, and $F(5)$ is the distance it has traveled after 5 hours. Then:

$$\text{Change in distance} = \Delta F = F(5) - F(3) = 140$$
$$\text{Average rate of change} = \frac{F(5) - F(3)}{5 - 3} = \frac{140}{2} = 70.$$

Since the expression $F(5) - F(3)$ is a difference, it is measured in miles. The statement $F(5) - F(3) = 140$ tells us that the car travels 140 miles between 3 and 5 hours after starting.

Since the expression $(F(5) - F(3))/(5 - 3)$ is a fraction, with the numerator measured in miles and the denominator in hours, the fraction is measured in miler per hour. The statement $(F(5) - F(3))/(5 - 3) = 70$ tells us that the average speed of the car during the two-hour period is 70 mph.

Example 3 A car has traveled $F(t)$ miles t hours after it starts a journey. Interpret the following statements in terms of the journey of the car. Include units.

(a) $F(a + 5) - F(a) = 315$ (b) $\dfrac{F(a + 5) - F(a)}{5} = 63$

Solution

(a) Since $F(a)$ is the distance the car has traveled after a hours and $F(a + 5)$ is the distance it has traveled after $a + 5$ hours (that is, after 5 more hours), $F(a + 5) - F(a) = 315$ means it travels 315 miles between a and $a + 5$ hours after starting.

(b) The denominator, $(a - 5) - a = 5$, represents the total time (in hours) of the journey. We know $F(a+5) - F(a) = 315$ gives the distance (in miles) the car has traveled over the five-hour period. So the ratio tell us that the average rate of change of F, or average speed of the car, during the five-hour period is 63 mph.

Using Units to Interpret the Average Rate of Change

It is often useful to consider the units of measurement of the input and output quantities when interpreting the average rate of change. For instance, as we saw in Example 3, if the input is time measured in hours and the output is distance traveled measured in miles, then the average rate of change is measured in miles per hour. Similarly, we saw in Example 2 if the input is time measured in months and the output is the number of termites in a colony, then the average rate of change is measured in termites per month.

Example 4 A ball thrown in the air has height $h(t) = 90t - 16t^2$ feet after t seconds.

(a) What are the units of measurement for the average rate of change of h? What does your answer tell you about how to interpret the rate of change in this case?

(b) Find the average rate of change of h between

(i) $t = 0$ and $t = 2$ (ii) $t = 1$ and $t = 2$ (iii) $t = 2$ and $t = 4$

(c) Use a graph of h to interpret your answers to part (b) in terms of the motion of the ball.

Solution

(a) In the expression, the numerator is measured in feet and the denominator is measured in seconds,

$$\frac{h(b) - h(a)}{b - a}.$$

So the ratio is measured in ft/sec and measures a velocity.

(b) (i) Between $t = 0$ and $t = 2$, we have

$$\text{Average rate of change} = \frac{h(2) - h(0)}{2 - 0} = \frac{(90 \cdot 2 - 16 \cdot 4) - (0 - 0)}{2 - 0} = \frac{116}{2} = 58 \text{ ft/sec.}$$

(ii) Between $t = 1$ and $t = 2$, we have

$$\text{Average rate of change} = \frac{h(2) - h(1)}{2 - 1} = \frac{(90 \cdot 2 - 16 \cdot 4) - (90 - 16)}{2 - 1} = \frac{42}{1} = 42 \text{ ft/sec.}$$

(iii) Between $t = 2$ and $t = 4$, we have

$$\text{Average rate of change} = \frac{h(4) - h(2)}{4 - 2} = \frac{(90 \cdot 4 - 16 \cdot 16) - (90 \cdot 2 - 16 \cdot 4)}{4 - 2} = \frac{-12}{2} = -6 \text{ ft/sec.}$$

(c) Figure 1.15 shows the ball's height rising to a peak somewhere between 2 and 3 seconds, and then starting to fall. The average velocity is positive during the first two seconds because the height is increasing during that time period. The height is also increasing between $t = 1$ and $t = 2$, but since the ball is rising more slowly, the velocity is smaller. Between $t = 2$ and $t = 4$ the ball rises and falls, experiencing a net loss in height, so its average velocity is negative.

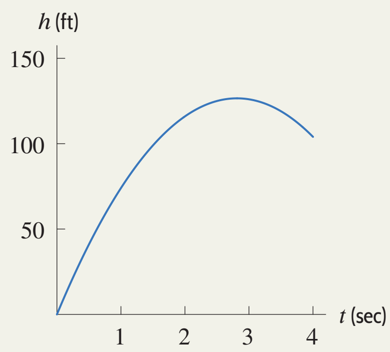


Figure 1.15: Height of a ball

Exercises for Section 1.4

IDENTIFYING ALGEBRAIC STRUCTURE

For the function in Exercises 1–4, give the units for the

$$\text{Average rate of change} = \frac{f(b) - f(a)}{b - a}.$$

1. The population, in people, of a city, $P = f(t)$, is a function of the number of years, t, since 2010.

2. The number of gallons of gas in a car, $g = f(m)$, is a function of the number of miles driven, m.

3. The number of smartphones, $N = f(p)$, purchased is a function of the price p, in dollars, of the smartphone.

4. The cost, $C = f(w)$, in dollars of buying a chemical is a function of the weight bought, w, in pounds.

In Exercises 5–7, let $g(t)$ give the market value (in \$1000s) of a house in year t. What does the statement say about the house?

5. $g(5) - g(0) = 30$

6. $\dfrac{g(10) - g(4)}{10 - 4} = 3$

7. $\dfrac{g(20) - g(12)}{20 - 12} = -1$

In Exercises 8–10, let $s(t)$ give the number of acres of wetlands in a state in year t. Assuming that the area of wetlands goes down over time, what does the statement say about the wetlands?

8. $s(25) - s(0) = -25{,}000$

9. $\dfrac{s(20) - s(10)}{20 - 10} = -520$

10. $s(30) - s(20) < s(20) - s(10)$

PROBLEMS

Find the average rate of change of $f(x) = x^2 + 3x$ on the interval in Problems 11–14.

11. Between 2 and 4.
12. Between −2 and 4.
13. Between −4 and −2.
14. Between 3 and 1.

Find the average rate of change of $g(x) = 2x^3 - 3x^2$ on the interval in Problems 15–18.

15. Between 1 and 3.
16. Between −1 and 4.
17. Between 0 and 10.
18. Between −0.1 and 0.1.

19. The value in dollars of an investment t years after 2010 is given by

$$V = 1000 \cdot 2^{t/6}.$$

Find the average rate of change of the investment's value between 2011 and 2014.

20. Atmospheric levels of carbon dioxide (CO_2) have risen from 336 parts per million (ppm) in 1979 to 396 parts per million (ppm) in 2014.[6] Find the average rate of change of CO_2 levels during this time period.

21. Sea levels were at a low point about 22,000 years ago.[7] Since then they have risen approximately 120 meters. Find the average rate of change of the sea level during this time period.

22. Global temperatures may increase by up to 10°F between 1990 and 2100.[8] Find the average rate of change of global temperatures between 1990 and 2100.

23. In 2007, you have 40 songs in your favorite iTunes playlist. In 2010, you have 120 songs. In 2014, you have 40. What is the average rate of change per year in the number of songs in your favorite iTunes playlist between

(a) 2007 and 2010? (b) 2010 and 2014?
(c) 2007 and 2014?

24. The graph of $P = f(t)$ in Figure 1.16 gives the population of a town, in thousands, after t years.

(a) Find the average rate of change of the population of the town during the first 10 years.
(b) Does the population of the town grow more between $t = 5$ and $t = 10$ years, or between $t = 15$ and $t = 30$ years? Explain.
(c) Does the population of the town grow faster between $t = 5$ and $t = 10$ years, or between $t = 15$ and $t = 30$ years? Explain.

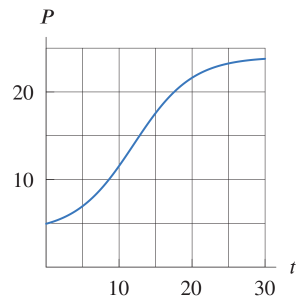


Figure 1.16

25. The most freakish change in temperature ever recorded was from −4°F to 45°F between 7:30 am and 7:32 am on January 22, 1943 at Spearfish, South Dakota.[9] What was the average rate of change of the temperature for this time period?

[6] National Oceanic & Atmospheric Administration, http://www.esrl.noaa.gov/gmd/aggi. Page last accessed September 1, 2014.
[7] See http://en.wikipedia.org/wiki/Sea_level_rise, http://en.wikipedia.org/wiki/Last_glacial_maximum, and related links. Pages last accessed September 1, 2014.
[8] See http://en.wikipedia.org/wiki/Global_warming. Page last accessed September 1, 2014.
[9] *The Guinness Book of Records*. 1995.

In Problems 26–29, use Table 1.4, which gives values of $D = f(t)$, the total US debt (in $ billions) t years after 2005.[10]

Table 1.4

t	D ($ billions)
0	7932.7
1	8507.0
2	9007.7
3	10,024.7
4	11,909.8
5	13,561.6
6	14,790.3
7	16,066.2
8	16,738.2

26. Evaluate
$$\frac{f(5)-f(1)}{5-1}$$
and say what it tells you about the US debt.

27. Which expression has the larger value,
$$\frac{f(5)-f(3)}{5-3} \quad \text{or} \quad \frac{f(3)-f(0)}{3-0}?$$
Say what this tells you about the US debt.

28. Show that
$$\begin{matrix}\text{Average rate of change} \\ \text{from 2012 to 2013}\end{matrix} < \begin{matrix}\text{Average rate of change} \\ \text{from 2011 to 2012.}\end{matrix}$$
Does this mean the US debt is starting to go down? If not, what does it mean?

29. Project the value of $f(15)$ by assuming
$$\frac{f(15)-f(8)}{15-8} = \frac{f(8)-f(0)}{8-0}.$$
Explain the assumption that goes into making your projection and what your answer tells you about the US debt.

In Problems 30–33, use Table 1.5, which gives values of $Q = w(t)$, the atmospheric methane level in parts per billion (ppb) t years after 1980.[11] Methane is a greenhouse gas implicated as a contributor to climate change.

Table 1.5

t	0	5	10	15	20	25	30
Q	1575	1660	1715	1750	1770	1775	1785

30. Evaluate
$$\frac{w(10)-w(5)}{10-5},$$
and say what it tells you about atmospheric methane levels.

31. Which expression is larger,
$$\frac{w(10)-w(0)}{10-0} \quad \text{or} \quad \frac{w(30)-w(10)}{30-10}?$$
Say what this tells you about atmospheric methane levels.

32. Show that
$$\begin{matrix}\text{Average rate of change} \\ \text{from 1995 to 2000}\end{matrix} < \begin{matrix}\text{Average rate of change} \\ \text{from 2000 to 2010.}\end{matrix}$$
Does this mean the average methane level is going down? If not, what does it mean?

33. Project the value of $w(-20)$ by assuming
$$\frac{w(0)-w(-20)}{20} = \frac{w(10)-w(0)}{10}.$$
Explain the assumption that goes into making your projection and what your answer tells you about atmospheric methane levels.

In Problems 34–36, let $f(t)$ be the population of a town that is growing over time. What must be true about a in order for the expression to be positive?

34. $f(a) - f(3)$

35. $\dfrac{f(3)-f(a)}{3-a}$

36. $f(t+a) - f(t+b)$

In Problems 37–39, let $g(t)$ give the amount of electricity (in kWh) used by a manufacturing plant in year t. Assume that, thanks to conservation measures, g is going down over time. What must be true about h in order for the expression to be negative?

37. $g(h) - g(10)$

38. $\dfrac{g(5+h)-g(5)}{h}$

39. $g(h+1) - g(h-1)$

[10] See http://www.treasurydirect.gov/govt/reports/pd/histdebt/histdebt_histo5.htm and www.treasurydirect.gov/govt/reports/pd/mspd/2011/opds092011.prn, and www.treasurydirect.gov/govt/reports/pd/mspd/2013/opds092013.prn. Last accessed September 1, 2014.
[11] National Oceanic & Atmospheric Administration, http://www.esrl.noaa.gov/gmd/aggi. Last accessed September 1, 2014.

1.5 FUNCTIONS, MODELING, AND PROPORTIONALITY

When we want to apply mathematics to a real-world situation, we are not always given a function; sometimes we have to create one. Knowledge about the real world can help us to choose a particular type of function, and we then use information about the situation to select a particular function of this type. This process is called *modeling*. In this section we model proportional relationships.

Direct Proportionality

Suppose a state sales tax rate is 6%. Then the tax, T, on a purchase of price P is given by the function

$$\text{Tax} = 6\% \times \text{Price} \qquad \text{or} \qquad T = 0.06P.$$

We say the tax is proportional to the purchase price: when the price doubles, for example, so does the tax. Notice also that the ratio T/P is constant, $T/P = 0.06$. In general:

> A quantity Q is **directly proportional** to a quantity t if
>
> $$Q = k \cdot t,$$
>
> where k is the *constant of proportionality*. We often omit the word "directly" and simply say Q is proportional to t.

In the tax example, the constant of proportionality is $k = 0.06$, because $T = 0.06P$.

Example 1 A car gets 25 miles to the gallon.

(a) How far does the car travel on 1 gallon of gas? 2 gallons? 10 gallons? 20 gallons?
(b) Express the distance, d miles, traveled as a function of the number of gallons, g, of gas used. Explain why d is proportional to g with constant of proportionality 25.

Solution (a) The car travels 25 miles on each gallon. Thus

$$\begin{aligned}
\text{Distance on 1 gallon} &= 25 \cdot 1 = 25 \text{ miles}\\
\text{Distance on 2 gallons} &= 25 \cdot 2 = 50 \text{ miles}\\
\text{Distance on 10 gallons} &= 25 \cdot 10 = 250 \text{ miles}\\
\text{Distance on 20 gallons} &= 25 \cdot 20 = 500 \text{ miles.}
\end{aligned}$$

(b) Since an additional gallon of gas enables the car to travel an additional 25 miles, we have

$$d = 25g.$$

Thus d is proportional to g, with constant of proportionality 25, the car's mileage per gallon.

Notice that in Example 1, when the number of gallons doubles from 1 to 2, the number of miles doubles from 25 to 50, and when the number of gallons doubles from 10 to 20, the number of miles doubles from 250 to 500. In general:

> ### Behavior of Proportional Quantities
>
> If $Q = kt$, then doubling the value of t doubles the value of Q, tripling the value of t triples the value of Q, and so on. Likewise, halving the value of t halves the value of Q, and so on.

Example 2 Vincent pays five times as much for a car as Dominic. Dominic pays \$300 sales tax. How much sales tax does Vincent pay (assuming they pay the same tax rate)?

Solution Since Vincent's car costs five times as much as Dominic's car, Vincent's sales tax is five times as large as Dominic's, or $5 \cdot 300 = \$1500$.

Example 3 For the same car as in Example 1:

(a) How many gallons of gas are needed for a trip of 5 miles? 10 miles? 100 miles?
(b) Find g, the number of gallons needed as a function of d, the number of miles traveled. Explain why g is proportional to d and how the constant of proportionality here relates to the constant in Example 1.

Solution

(a) Because the car gets 25 miles to a gallon, the amount of gas required to drive 5 miles is $5/25$ of a gallon, or 0.2 gallon. Similarly, the amount of gas required to drive 10 miles is $10/25$ of a gallon, or 0.4 gallon. To drive 100 miles, the gas required is $100/25 = 4$ gallons.
(b) Since each mile takes $1/25$ gallon of gas,

$$g = \frac{d}{25} = \left(\frac{1}{25}\right) d.$$

Thus g is proportional to d and the constant of proportionality is $1/25$, the reciprocal of the constant in Example 1. We can also see that g is proportional to d by solving for g in $d = 25g$.

Sometimes we have to rewrite the expression for a function to see that it represents a direct proportionality.

Example 4 Does the function represent a direct proportionality? If so, give the constant of proportionality, k.

(a) $f(x) = 19x$ (b) $g(x) = x/53$ (c) $F(a) = 2a + 5a$
(d) $u(t) = \left(\sqrt{5}\right) t$ (e) $A(n) = n\pi^2$ (f) $P(t) = 2 + 5t$

Solution

(a) Here $f(x)$ is proportional to x with constant of proportionality $k = 19$.
(b) We rewrite this as $g(x) = (1/53)x$, so $g(x)$ is proportional to x with constant of proportionality $k = 1/53$.
(c) Simplifying the right-hand side, we get $F(a) = 2a + 5a = 7a$, so $F(a)$ is proportional to a with constant $k = 7$.
(d) Here $u(t)$ is proportional to t with constant $k = \sqrt{5}$.
(e) Rewriting this as $A(n) = \pi^2 n$, we see that it represents a direct proportionality with constant $k = \pi^2$.
(f) Here $P(t)$ is not proportional to t, because of the constant 2 on the right-hand side.

Solving for the Constant of Proportionality

If we do not know the constant of proportionality, we can find it using one pair of known values for the quantities that are proportional.

Example 5 A student at a college earns \$80 for 10 hours of work. Express her earnings as a function of the number of hours worked. What is the constant of proportionality, and what is its practical interpretation?

Solution We have

$$E = f(t) = kt,$$

where E is the amount earned and t is the number of hours worked. We are given

$$80 = f(10) = k \cdot 10,$$

and solving for k we have

$$k = \frac{80 \text{ dollars}}{10 \text{ hours}} = 8 \text{ dollars/hour.}$$

Thus, $k = 8$ dollars/hour, which is her hourly wage.

Example 6 A person's heart mass is known to be proportional to his or her body mass.[12]

(a) A person with a body mass of 70 kilograms has a heart mass of 0.42 kilograms. Find the constant of proportionality, k.
(b) Estimate the heart mass of a person with a body mass of 60 kilograms.

Solution (a) With H as heart mass in kg and B as body mas in kg, we have, for some constant k,

$$H = k \cdot B.$$

We substitute $B = 70$ and $H = 0.42$ and solve for k:

$$0.42 = k \cdot 70$$
$$k = \frac{0.42}{70} = 0.006.$$

So the formula for heart mass as a function of body mass, with H and B in kilograms, is

$$H = 0.006B.$$

(b) With $B = 60$, we have

$$H = 0.006 \cdot 60 = 0.36 \text{ kg.}$$

Proportionality and Rates

In general, the average rate of a change of a function depends on the interval over which it is computed. For instance, the average rate of change of the population of the termite colony in Example 2 in Section 1.4 varied over different intervals. In the next example, we look at the average rate of change of a proportional relationship over different intervals.

Example 7 In Example 5, the amount of money earned E, in dollars, by a student who works t hours is given by

$$E = f(t) = 8t.$$

[12] K. Schmidt-Nielsen: *Scaling, Why Is Animal Size so Important?* (Cambridge: CUP, 1984).

(a) What is the average rate of change of the money earned by the student
(i) Between 10 and 20 hours? (ii) Between 6 and 8 hours?

(b) Compare your answers in part (a) to the constant of proportionality. Use the algebraic structure of the formula for $f(t)$ to explain why this happens.

Solution (a) (i) Between 10 and 20 hours, we have

$$\text{Average rate of change of amount earned} = \frac{f(20) - f(10)}{20 - 10} = \frac{\$160 - \$80}{10 \text{ hours}} = \$8 \text{ per hour.}$$

(ii) Between 6 and 8 hours, we have

$$\text{Average rate of change of amount earned} = \frac{f(8) - f(6)}{8 - 6} = \frac{\$64 - \$48}{2 \text{ hours}} = \$8 \text{ per hour.}$$

(b) The average rate of the change is the same over both intervals and equals the constant of proportionality, $k = 8$. Since the formula for $f(t)$ is $8t$, every time t is increased by 1, she earns 8 dollars more. Thus her earnings are changing at an average rate of 8 dollars per hour.

Problem 48 asks you to show the average rate of change of the function $f(t) = kt$ is the same over all intervals and equals the constant of proportionality k. Thus, if two quantities are proportional, the constant of proportionality is the rate of change of one quantity with respect to the other.

Example 8 In Example 1 on page 28, the distance traveled, d miles, when g gallons of gas are used is given by

$$d = 25g.$$

Give a practical interpretation of the constant 25. Include units in your answer.

Solution The units of g are gallons and the units of distance, d, are miles. Thus the units of k are miles per gallon. The constant tells us that the car gets 25 miles for each gallon of gas.

Families of Functions

Because the functions $y = f(t) = kt$ all have the same algebraic form, we think of them as a family of functions, one function for each value of the constant k. Figure 1.17 shows graphs of various functions in this family. All the functions in the family share common features: their graphs are all lines, and they all pass through the origin. There are also differences between different functions in the family, corresponding to different values of k. Although k is constant for any given member of the family, it changes from one member to the next. We call k a *parameter* for the family.

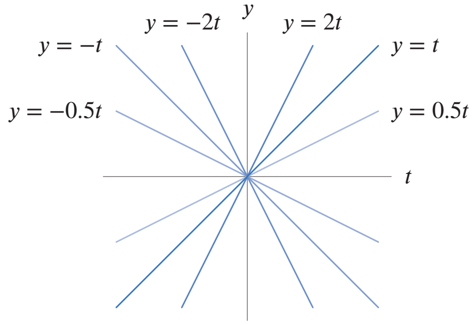


Figure 1.17: The family of functions $y = kt$

Exercises for Section 1.5

IDENTIFYING ALGEBRAIC STRUCTURE

For the formula in Exercises 1–14, is y directly proportional to x? If so, give the constant of proportionality.

1. $y = 5x$
2. $y = x \cdot 7$
3. $y = -x$
4. $y = x \cdot x$
5. $y = \sqrt{5} \cdot x$
6. $y = \sqrt{5x}$
7. $y = x/9$
8. $y = 9/x$
9. $y = \dfrac{3x}{11}$
10. $y = x + 2$
11. $y = 3(x + 2)$
12. $y = 9x - 5x$
13. $y = (3x)^2$
14. $y - 5 = 2x$

PROBLEMS

In Problems 15–18 is the first quantity proportional to the second quantity? If so, what is the constant of proportionality?

15. d is the distance traveled in miles and t is the time traveled in hours at a speed of 50 mph.
16. P is the price paid in dollars for b barrels of oil at a price of \$70.
17. p is the sale price of an item whose original price is p_0 in a 30% off sale.
18. C is the cost of having n drinks at a club, where each drink is \$5 and there is a cover charge of \$20.

In Problems 19–22, assume the two quantities are directly proportional to each other.

19. If $r = 36$ when $s = 4$, find r when s is 5.
20. If $p = 24$ when $q = 6$, find q when p is 32.
21. If $y = 16$ when $x = 12$, find y when x is 9.
22. If $s = 35$ when $t = 25$, find t when s is 14.
23. The interest paid by a savings account in one year is proportional to the starting balance, with constant of proportionality 0.06. Write a formula for I, the amount of interest earned, in terms of B, the starting balance. Find the interest earned if the starting balance is

    (a) \$500 **(b)** \$1000 **(c)** \$5000.

In Problems 24–32, determine if the graph defines a direct proportion. If it does, estimate the constant of proportionality.

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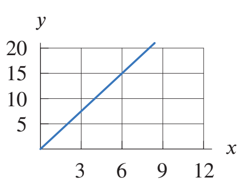

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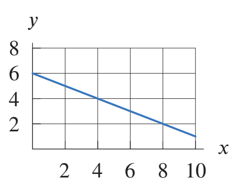

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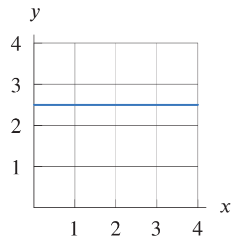

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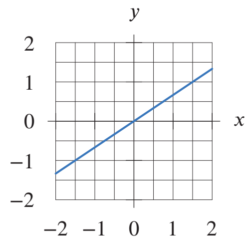

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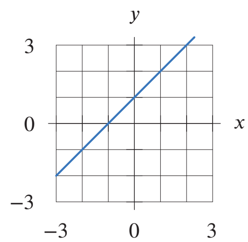

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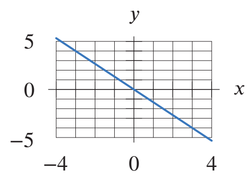

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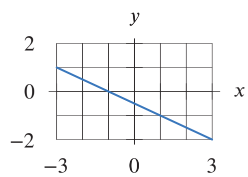

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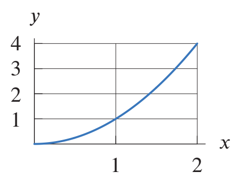

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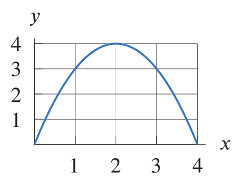


33. A factory makes 50 vehicles a week. Is t, the number of weeks, proportional to v, the number of vehicles made? If so, what is the constant of proportionality?
34. One store sells 70 pounds of apples a week, and a second store sells 50 pounds of apples a week. Is the total number of pounds of apples sold, a, proportional to the number of weeks, w? If so, what is the constant of proportionality?

35. The perimeter of a square is proportional to the length of any side. What is the constant of proportionality?

36. A bike shop's revenue is directly proportional to the number of bicycles sold. When 50 bicycles are sold, the revenue is \$20,000.

(a) What is the constant of proportionality, and what are its units?
(b) What is the revenue if 75 are sold?

37. The total cost of purchasing gasoline for your car is directly proportional to the the number of gallons pumped, and 11 gallons cost \$36.63.

(a) What is the constant of proportionality, and what are its units?
(b) How much do 15 gallons cost?

38. The required cooling capacity, in BTUs, for a room air conditioner is proportional to the area of the room being cooled. A room of 280 square feet requires an air conditioner whose cooling capacity is 5600 BTUs.

(a) What is the constant of proportionality, and what are its units?
(b) If an air conditioner has a cooling capacity of 10,000 BTUs, how large a room can it cool?

39. You deposit \$P into a bank where it earns 2% interest per year for 10 years. Use the formula $B = P(1 + r)^t$ for the balance \$B, where r is the interest rate (written as a decimal) and t is time in years.

(a) Explain why B is proportional to P. What is the constant of proportionality?
(b) Is P proportional to B? If so, what is the constant of proportionality?

40. The length m, in inches, of a model train is proportional to the length r, in inches, of the corresponding real train.

(a) Write a formula expressing m as a function of r.
(b) An HO train is $1/87^{\text{th}}$ the size of a real train.[13] What is the constant of proportionality? What is the length in feet of a real TGV train if the HO train is 90.62 inches long?
(c) A Z scale train is $1/220^{\text{th}}$ the size of a real train. What is the constant of proportionality? What is the length, in inches, of a Z scale Eurostar train if the real Eurostar is 1293 feet long?[14]

41. The cost of denim fabric is directly proportional to the amount that you buy. Let C be the cost, in dollars, of x yards of denim fabric.

(a) Write a formula expressing C as a function of x.
(b) One type of denim costs \$28.50 for 3 yards. Find the constant of proportionality and give its units.
(c) How much does 5.5 yards of this type of denim cost?

42. The distance M, in inches, between two points on a map is proportional to the actual distance d, in miles, between the two corresponding locations.

(a) If $1/2$ inch represents 5 miles, find the constant of proportionality and give its units.
(b) Write a formula expression M as a function of d.
(c) How far apart are two towns if the distance between them on the map is 3.25 inches?

43. The blood mass of a mammal is proportional to its body mass. A rhinoceros with body mass 3000 kilograms has blood mass of 150 kilograms. Find a formula for the blood mass of a mammal in terms of the body mass and estimate the blood mass of a human with body mass 70 kilograms.

44. A bond pays 5% interest per year. If you buy \$1000 of the bond, you receive \$50 per year. For \$1500 of the bond, you receive \$75/yr, for \$5000 of the bond, \$250/yr. What is the average rate of change of interest per dollar change of bond purchased:

(a) Between \$1000 and \$1500?
(b) Between \$1500 and \$5000?

45. The *data rate* of an Internet connection is the rate in bytes per second that data, such as a web page, image, or music file, can be transmitted across the connection.[15] Suppose the data rate is 300 kilobytes per second (KBps). How long does it take to download a file that is 60,000 kilobytes (KB)?

46. The distance a car travels on the highway is proportional to the quantity of gas consumed. A car travels 225 miles on 5 gallons of gas. Find the constant of proportionality, give units for it, and explain its meaning.

47. If x is directly proportional to y, explain what happens to y if x is

(a) doubled.
(b) halved.
(c) multiplied by a factor k.
(d) multiplied by a factor $\frac{1}{k}$.

[13] www.internettrains.com, accessed September 1, 2014.
[14] Train sizes from: en.wikipedia.org/wiki/TGV#Rolling_stock, accessed September 1, 2014.
[15] Sometimes mistakenly called "bandwidth." See the Free Online Dictionary of Computing, http://foldoc.doc.ic.ac.uk/foldoc/contents.html, accessed September 1, 2014.

48. Show algebraically that the average rate of change of the function $f(t) = kt$ between any two different values of t is equal to the constant k.

In Problems **49–54**, put the functions in the form $Q = kt$ and state the value of k.

49. $Q = \frac{t}{4}$ **50.** $Q = t(a+1)$ **51.** $Q = bt + ct$

52. $Q = \frac{1}{2}t\sqrt{3}$ **53.** $Q = \frac{at - bt}{c}$

54. $Q = (t-3)(t+3) - (t+9)(t-1)$

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER 1

EXERCISES

In Exercises **1–6**, are the expressions equivalent?

1. $2x + 6$ and $x + 3$

2. $\frac{1}{3}x$ and $\frac{1}{3x}$

3. $\frac{1}{5}x$ and $\frac{x}{5}$

4. $\frac{x}{3} + \frac{y}{3}$ and $\frac{x+y}{3}$

5. $\frac{x}{2} + \frac{y}{2}$ and $\frac{x+y}{4}$

6. $(x+y)^2$ and $x^2 + y^2$

The equation in Exercises **7–10** can be solved by performing a single operation on both sides. State the operation and solve the equation.

7. $0.1 + t = -0.1$ **8.** $-10 = 3 + r$

9. $5y = 19$ **10.** $\frac{-x}{5} = 4$

Evaluate the expression in Exercises **11–13** given that

$$f(x) = \frac{x+1}{2x+1}.$$

11. $f(0)$ **12.** $f(-1)$ **13.** $f(0.5)$

In Exercises **14–17**, which of the given values of the variable are solutions?

14. $x^2 + 2 = 3x$ for $x = 0, 1, 2$

15. $2x^2 + 3x^3 = 5x^5$ for $x = -1, 0, 1$

16. $2(u-1) + 3(u-2) = 7(u-3)$ for $u = 1, 2, 3$, and 6.5

17. $2(r-6) = 5r + 12$, for $r = 8, -8$

In Exercises **18–23**, is the second equation the result of a valid operation on the first? If so, what is the operation?

18. $3 + 5x = 1 - 2x$
$3 + 7x = 1$

19. $1 - 2x^2 + x = 1$
$x - 2x^2 = 0$

20. $\frac{x}{3} - \frac{3}{4} = 0$
$4x - 9 = 0$

21. $9x - 3x^2 = 5x$
$9 - 3x = 5$

22. $\frac{3}{4} - \frac{x}{3} = 2$
$9 - 3x = 2$

23. $5x^2 - 20x = 90$
$x^2 - 4x = 18$

24. Let $f(r)$ be the weight of an astronaut in pounds at the distance r, in thousands of miles from the earth's center. Explain the meaning of the following statements.

(a) $f(4) = 180$ **(b)** $f(a) = 36$
(c) $f(36) = b$ **(d)** $w = f(r)$

25. The depth, in inches, of the water in a leaking cauldron after t hours is given by

$$H(t) = (-0.08t + 6)^2.$$

(a) Find $H(0)$ and interpret its meaning.
(b) Interpret the meaning of $H(75) = 0$.

26. Census figures for the US population, $P(t)$ (in millions), t years after 1950, are in Table 1.6.

Table 1.6

t (years)	0	10	20	30	40	50
$P(t)$ (millions)	150.7	179.0	205.0	226.5	248.7	281.4

(a) Evaluate $P(20)$ and interpret its meaning.
(b) If $P(t) = 281.4$, find t and interpret its meaning.

In Exercises 27–28, give:

(a) $f(0)$ **(b)** The values of x such that $f(x) = 0$.

Your answers may be approximate.

27.

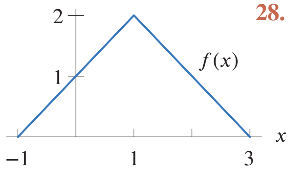
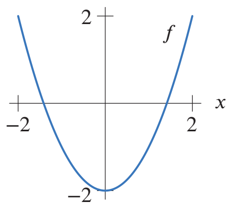


Figure 1.18

28.



29. Using Figure 1.19, find:

(a) $f(4)$
(b) The values of x such that $f(x) = 4$
(c) The values of x such that $f(x) = 1$

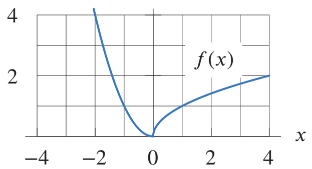


Figure 1.19

30. Use Table 1.7 to evaluate the expression:

(a) $g(10) - 10$ **(b)** $6g(0)$
(c) $g(10) - g(-10)$ **(d)** $g(5) + 7g(5)$

Table 1.7

y	−10	−5	0	5	10
$g(y)$	−5	0	5	10	−10

31. Kari has \$20 and buys m muffins at \$1.25 each.

(a) Write an expression for the amount of money she has left.
(b) Find the amount remaining if she buys 9 muffins.

32. The number of dirty socks on your roommate's floor, t days after the start of exams, is given by

$$s(t) = 10 + 2t.$$

(a) Write an equation whose solution is the number of days it takes for the number of socks to reach 26.
(b) Make a plot of the number of socks for $t = 2, 4, 6, 8, 10, 12$, and indicate the solution to the equation in part (a).

33. For accounting purposes, the value of a machine, t years after it is purchased, is given in dollars by

$$V(t) = 100{,}000 - 10{,}000t.$$

(a) Write an equation whose solution is the number of years it takes for the machine's value to reach \$70,000.
(b) Make a plot of the value of the machine at $t = 1, 2, 3, 4, 5, 6$, and indicate, on the graph, the solution to the equation in part (a).

In Exercises 34–39, identify the description as either an expression or equation and write it algebraically using the variables given.

34. Twice n plus three more than n.

35. Twice n plus three more than n is 21.

36. The combined salary of Jason, J, and Steve, S.

37. Twice the combined salary of Jason, J, and Steve, S, is \$140,000.

38. 225 pounds is ten pounds more than Will's weight, w.

39. 50 pounds less than triple Bob's weight, w.

Find the average rate of change of $f(x) = x^3 - x^2$ on the interval in Exercises 40–42.

40. Between 2 and 4. **41.** Between −2 and 4.

42. Between −4 and −2.

In Exercises 43–46, is y directly proportional to x? If so, give the constant of proportionality.

43. $x = 5y$ **44.** $y = 2x - x$

45. $y - 2 = 3x$ **46.** $y/3 = \sqrt{5}x$

In Exercises 47–50,

(a) Evaluate the function at the given input values.
(b) Which gives the greater output value?
(c) Explain the answer to part (b) in terms of the algebraic expression for the function.

47. $f(x) = 9 - x,\ x = 1, 3$

48. $g(a) = a - 2,\ a = -5, -2$

49. $C(p) = \frac{-p}{5}$, $p = 100,\ 200$

50. $h(t) = \frac{t}{5}$, $t = 4, 6$

In Exercises **51–54**, evaluate the expression given that $u = -2$, $v = 3$, $w = 2/3$.

51. $uv - vw$

52. $v + \frac{u}{w}$

53. $u^2 + v^2 - (u - v)^2$

54. $u^v + w^v$

The equation in Exercises **55–60** can be solved by performing two operations on both sides. State the operations in order of use and solve the equation.

55. $\frac{6}{5}t + \frac{2}{5}t = 3$

56. $\frac{A^2}{7} = 3$

57. $(p - 2)^3 = 125$

58. $4 = \frac{1}{y} + 12$

59. $4\sqrt{x} - 3\sqrt{x} = 10$

60. $\left(\frac{q}{4}\right)^3 = -27$

PROBLEMS

61. Professor Priestley calculates your final grade by averaging the number of points, x, that you receive on the midterm with the number of points, y, that you receive on the final. Professor Alvorado takes half the points on each exam and adds them together. Are the two methods the same? Explain your answer algebraically.

62. Your older sister, who has more money than you, proposes that she give you half the difference between the amount of money, $\$q$, that she has and the amount of money, $\$p$, that you have. You propose instead that you give her half your money and she give you half hers. Is there any difference between the methods? Explain your answer algebraically.

63. It costs a contractor $\$p$ to employ a plumber, $\$e$ to employ an electrician, and $\$c$ to employ a carpenter.

- **(a)** Write an expression for the total cost to employ 4 plumbers, 3 electricians, and 9 carpenters.
- **(b)** Write an expression for the fraction of the total cost in part (a) that is due to plumbers.
- **(c)** Suppose the contractor hires P plumbers, E electricians, and C carpenters. Write expressions for the total cost for hiring these workers and the fraction of this cost that is due to plumbers.

In Problems **64–66**, assume that movie tickets cost $\$p$ for adults and $\$q$ for children.

64. Write expressions for the total cost of tickets for:

(a)	2 adults and 3 kids	**(b)**	No adults and 4 kids
(c)	No kids and 5 adults	**(d)**	A adults and C kids

65. Write expressions for the average cost per ticket for:

(a)	2 adults and 3 kids	**(b)**	No adults and 4 kids
(c)	No kids and 5 adults	**(d)**	A adults and C kids

66. A family of two adults and three children has an entertainment budget equal to the cost of 10 adult tickets. An adult ticket costs twice as much as a child ticket. How much money will they have left after seeing two movies? Can they afford to see a third movie?

In Problems **67–69**, you have p pennies, n nickels, d dimes, and q quarters.

67. Write an expression for the total number of coins.

68. Write an expression for the dollar value of these coins.

69. Write an expression for the total number of coins if you change your quarters into nickels and your dimes into pennies.

70. The area, in cm^2, of glass used in a door of width w, in cm, is $A(w) = 4500 + w^2$. (See Figure 1.20.)

- **(a)** From the graph, estimate the width of a door using 7000 cm^2 of glass.
- **(b)** Check your answer to part (a) algebraically.

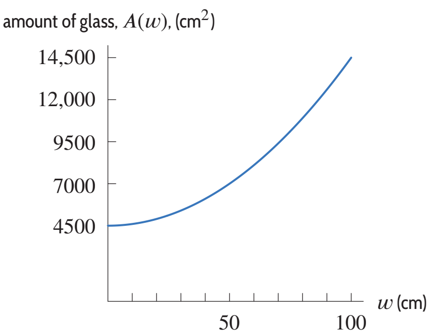


Figure 1.20

71. Hannah has \$100 in a bank account and deposits \$75 more. Write an equation whose solution is the amount she needs to deposit for the balance to be \$300, and find the amount.

In Problems 72–74, assume a person originally owes $\$B$ and has made n payments of $\$p$ each. Assume no interest is charged.

72. What does the expression $B - np$ represent in terms of the person's debt?

73. Write an expression for the number of payments of $\$p$ remaining before this person pays off the debt after he has made n payments.

74. Suppose $B = 5np$. How many payments are required in total to pay it off? How many more payments must be made after n payments have been made?

75. Weight Watchers© assigns points to various foods and limits the number of points you can accumulate in a day. If c is the number of calories, g the grams of fat, and f the grams of dietary fiber, then the number of points for a piece of food is

$$\text{Number of points} = \frac{c}{50} + \frac{g}{12} - \frac{f}{4}.$$

How many grams of fiber can you trade for three grams of fat without changing the number of points?

76. If m is the number of males and f is the number of females in a population, which of the following expresses the fact that 47% of the population is male and 53% of the population is female?

(a) $P = 0.47m + 0.53f$
(b) $P = 0.53m + 0.47f$
(c) $m = 0.47P$ and $f = 0.53P$
(d) $P = 0.47m$ and $P = 0.53f$

77. Here are the line-by-line instructions for calculating the deduction on your federal taxes for medical expenses.

1	Enter medical expenses	
2	Enter adjusted gross income	
3	Multiply line 2 by 7.5% (0.075)	
4	Subtract line 3 from line 1. If line 3 is more than line 1, enter 0.	

Write an expression for your medical deduction (line 4) in terms of your medical expenses, E, and your adjusted gross income, I.

In Problems 78–81, a person's monthly income is $\$I$, her monthly rent is $\$R$, and her monthly food expense is $\$F$.

(a) Interpret each of the expressions in terms of income and expenses.

(b) Do the expressions have the same value? If not, say which is larger, or that there is not enough information to decide.

78. $I - R - F$ and $I - (R + F)$

79. $12(R + F)$ and $12R + 12F$

80. $I - R - F + 100$ and $I - R - (F + 100)$

81. $\dfrac{R+F}{I}$ and $\dfrac{I-R-F}{I}$

The investment portfolio in Problems 82–85 includes stocks and bonds. Let $v(t)$ be the dollar value after t years of the portion held in stocks, and let $w(t)$ be the value held in bonds.

82. Explain what the following expression tells you about the investment:

$$\frac{w(t)}{v(t) + w(t)}.$$

83. The equation $w(t) = 2v(t - 1)$ has a solution at $t = 5$. What does this solution tell you about the investment?

84. Write an expression that gives the difference in value of the stock portion of the investment in year t and the bond portion of the investment the preceding year.

85. Write an equation whose solutions are the years in which the value of the bond portion of the investment exceed the value of the stock portion by exactly \$3000.

In Problems 86–88, explain how you can tell from the form of the equation that it has no solution.

86. $1 + 3a = 3a + 2$

87. $2 + 5x = 6 + 5x$

88. $\dfrac{1}{4z^2} = -3$

In Problems 89–91, decide if the statement is true or false. Justify your answers using algebraic expressions.

89. The sum of three consecutive integers is a multiple of 3.

90. The sum of three consecutive integers is even.

91. The sum of three consecutive integers is three times the middle integer.

The *development time* of an insect is how long it takes the insect to develop from egg to adult. Typically, development time goes down as the ambient temperature rises. In Problems 92–94, use Table 1.8, which gives values of $T = g(H)$, the development time in days at an ambient temperature

H°C for the bluebottle blowfly.[16] What does your answer say about the development time of blowflies?

Table 1.8

H	10	11	12	13	14	15	16	17	18	19	20
T	68	58	50	43	37	32	29	26	24	22	21

92. Evaluate $g(14)$.

93. Estimate the solution to $g(H) = 23$.

94. Evaluate and say which value is larger:

$$\frac{g(12) - g(10)}{12 - 10} \quad \text{and} \quad \frac{g(15) - g(12)}{15 - 12}.$$

In Problems 95–97, use Table 1.9, which gives $V = f(t)$, the value in Canadian dollars[17] (CAD) of \$1 (USD) t days after November 1, 2007. For instance, 1 USD could be traded for 0.9529 CAD on November 1. What does your answer say about the value of the USD?

Table 1.9

t	0	1	2	3	4	5	6
V	0.9529	0.9463	0.9441	0.9350	0.9350	0.9349	0.9295

95. Evaluate $f(3)$.

96. Evaluate $\dfrac{50}{f(5)}$

97. Evaluate and say which value is larger:

$$\frac{f(3) - f(0)}{3 - 0} \quad \text{and} \quad \frac{f(6) - f(3)}{6 - 3}.$$

Give possible values for A and B that make the expression in Problems 98–104 equal to the expression

$$\frac{1 + A + B}{AB}.$$

98. $\dfrac{1 + r + s}{rs}$

99. $\dfrac{1 + 2r + 3s}{6rs}$

100. $\dfrac{1 + n + m + z^2}{(n + m)z^2}$

101. $\dfrac{1 + n + m + z^2}{n(m + z^2)}$

102. $\dfrac{1 + 2x}{x^2}$

103. $\dfrac{13}{35}$

104. 0

The expression in Problems 105–108 can be put in the form

$$ax^2 + x.$$

For each expression, identify a and x.

105. $bz^2 + z$

106. $r(n + 1)^2 + n + 1$

107. $\dfrac{t^2}{7} + t$

108. $12d^2 + 2d$

109. You plan to drive 300 miles at 55 miles per hour, stopping for a two-hour rest. You want to know t, the number of hours the journey is going to take. Which of the following equations would you use?

(A) $55t = 190$ **(B)** $55 + 2t = 300$
(C) $55(t + 2) = 300$ **(D)** $55(t - 2) = 300$

110. **(a)** Does $\dfrac{x}{4} + \dfrac{1}{2} = 3x$ have the same solution as $x + 2 = 12x$?
(b) Is $\dfrac{x}{4} + \dfrac{1}{2}$ equivalent to $x + 2$?

111. **(a)** Does $\dfrac{5x}{3} + 2 = 1$ have the same solution as $5x + 6 = 1$?
(b) Is $\dfrac{5x}{3} + 2$ equivalent to $\dfrac{5x + 6}{3}$?

112. Let $f(N) = 3N$ give the number of glasses a cafe should have if it has an average of N clients per hour.

(a) How many glasses should the cafe have if it expects an average of 50 clients per hour?
(b) What is the relationship between the number of glasses and N? What is the practical meaning of the 3 in the expression $f(N) = 3N$?

113. During the holiday season, a store advertises "Spend \$50, save \$5. Spend \$100, save \$10." Assuming that the savings are directly proportional to the amount spent, what is the constant of proportionality? Interpret this in terms of the sale.

114. The distance D, in miles, traveled by a car going at 30 mph is proportional to the time t, in hours, that it has been traveling.

(a) How far does the car travel in 5 hours?
(b) What is the constant of proportionality? Show that the units on each side of the proportionality relationship agree.

[16] http://www.sciencebuddies.org/science-fair-projects/project_ideas/Zoo_p023.shtml, accessed September 2, 2014.
[17] http://www.oanda.com/convert/fxhistory, accessed November 7, 2007.

115. The formula for the circumference of a circle is given by $C = 2\pi r$, where r is the radius of the circle. Is the circumference proportional to the radius?

116. The number of grams of carbohydrates ingested is proportional to the number of crackers eaten. If 3 crackers cause 36 grams of carbohydrates to be ingested, how many grams of carbohydrates are ingested if 8 crackers are eaten? What are the units of k, the constant of proportionality?

117. Which of the lines (A–E) in Figure 1.21 represent a function that is a direct proportion? For those that are, find k.

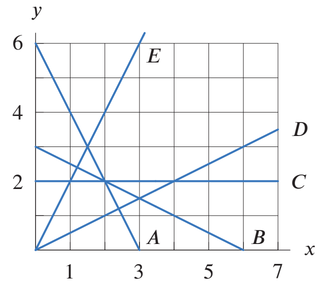


Figure 1.21

118. Observations show that the heart mass H of a mammal is 0.6% of the body mass M, and that the blood mass B is 5% of the body mass.[18]

(a) Write a formula for M in terms of H.
(b) Write a formula for M in terms of B.
(c) Write a formula for B in terms of H. Is this consistent with the statement that the mass of blood in a mammal is about 8 times the mass of the heart?

119. Use the fact that $x = 2$ is a solution to the equation $x^2 - 5x + 6 = 0$ to find a solution to the equation

$$(2w - 10)^2 - 5(2w - 10) + 6 = 0.$$

120. Solve $w(2v + 7) = 3w(v - 1)$ given that $w(v) = 3v + 2$.

[18] K. Schmidt-Nielsen, *Scaling, Why is Animal Size so Important?* (Cambridge: CUP, 1984).