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Introduction

1.1 Introduction

The finite element method (FEM) is one of the most common numerical tools for obtaining the approximate solutions of partial differential equations. It has been applied successfully in many areas of engineering sciences to study, model, and predict the behavior of structures. The area ranges across aeronautical and aerospace engineering, the automobile industry, mechanical engineering, civil engineering, biomechanics, geomechanics, material sciences, and many more. The FEM does not operate on differential equations; instead, continuous boundary and initial value problems are reformulated into equivalent variational forms. The FEM requires the domain to be subdivided into non-overlapping regions, called the *elements*. In the FEM, individual elements are connected together by a topological map, called a *mesh*, and local polynomial representation is used for the fields within the element. The solution obtained is a function of the quality of mesh and the fundamental requirement is that the mesh has to conform to the geometry. The main advantage of the FEM is that it can handle complex boundaries without much difficulty. Despite its popularity, the FEM suffers from certain drawbacks. There are number of instances where the FEM poses restrictions to an efficient application of the method. The FEM relies on the approximation properties of polynomials; hence, they often require smooth solutions in order to obtain optimal accuracy. However, if the solution contains a non-smooth behavior, like high gradients or singularities in the stress and strain fields, or strong discontinuities in the displacement field as in the case of cracked bodies, then the FEM becomes computationally expensive to get optimal convergence.

One of the most significant interests in solid mechanics problems is the simulation of fracture and damage phenomena (Figure 1.1). Engineering structures, when subjected to high loading, may result in stresses in the body exceeding the material strength and thus, in progressive failure. These material failure processes manifest themselves in various failure mechanisms such as the fracture process zone (FPZ) in rocks and concrete, the shear band localization in ductile metals, or the discrete crack discontinuity in brittle materials. The accurate modeling and evolution of smeared and discrete discontinuities have been a topic of growing interest over the past few decades, with quite a few notable developments in computational techniques over the past few years. Early numerical techniques for modeling discontinuities in finite elements can be seen in the work of Ortiz, Leroy, and Needleman (1987) and Belytschko, Fish, and Englemann (1988). They modeled the shear band localization as a “weak” (strain) discontinuity that could pass through the finite element mesh using a multi-field variational principle. Dvorkin, Cuitiño, and Gioia (1990) considered a “strong” (displacement) discontinuity by modifying the principle of virtual work statement. A unified framework for modeling the strong discontinuity by taking into account the softening constitutive law and the interface traction–displacement relation was proposed by Simo, Oliver, and

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Figure 1.1 Building destroyed by a 8.8 magnitude earthquake on Saturday, February 27, 2010, with intense shaking lasting for about 3 minutes, which occurred off the coast of central Chile. (Source: Vladimir Platonow (Agência Brasil) [CC-BY-3.0-br (http://creativecommons.org/licenses/by/3.0/br/deed.en)], via Wikimedia Commons; http://commons.wikimedia.org/wiki/File:Terremoto_no_Chile_2010.JPG)

Armero (1993). In the strong discontinuity approach, the displacement consists of regular and enhanced components, where the enhanced component yields a jump across the discontinuity surface. An assumed enhanced strain variational formulation is used, and the enriched degrees of freedom (DOF) are statically condensed on an element level to obtain the tangent stiffness matrix for the element. An alternative approach for modeling fracture phenomena was introduced by Xu and Needleman (1994) based on the cohesive surface formulation, which was used later by Camacho and Ortiz (1996) to model the damage in brittle materials. The cohesive surface formulation is a phenomenological framework in which the fracture characteristics of the material are embedded in a cohesive surface traction–displacement relation. Based on this approach, an inherent length scale is introduced into the model, and in addition, no fracture criterion is required so the crack growth and the crack path are outcomes of the analysis.

In the FEM, the non-smooth displacement near the crack tip is basically captured by refining the mesh locally. The number of DOF may drastically increase, especially in three-dimensional applications. Moreover, the incremental computation of a crack growth needs frequent remeshings. Reprojecting the solution on the updated mesh is not only a costly operation but also it may have a troublesome impact on the quality of results. The classical FEM has achieved its limited ability for solving fracture mechanics problems. To avoid these computational difficulties, a new approach to the problem consists in taking into account the *a priori* knowledge of the exact solution. Applying the asymptotic crack tip displacement solution to the finite element basis seems to have been a somewhat early idea. A significant improvement in crack modeling was presented with the development of a partition of unity (PU) based enrichment method for discontinuous fields in the PhD dissertation by Dolbow (1999), which was referred to as the extended FEM (X-FEM). In the X-FEM, special functions are added to the finite element approximation using the framework of PU. For crack modeling, a discontinuous function such as the Heaviside step function and the two-dimensional linear elastic asymptotic crack tip displacement fields, are used to account for the crack. This enables the domain to be modeled by finite elements without explicitly meshing the crack surfaces. The location of the crack discontinuity can be arbitrary with respect to the underlying finite element mesh, and the crack propagation simulation can be performed without the need to remesh as the crack advances. A particularly appealing feature is that the finite element framework and its properties, such as the sparsity and symmetry, are retained and a single-field (displacement) variational principle is used to obtain the discrete equations. This technique provides an accurate and robust numerical method to model strong (displacement) discontinuities.

The original research articles on the X-FEM were presented by Belytschko and Black (1999) and Moës, Dolbow, and Belytschko (1999) for elastic fracture propagation on the topic of "A FEM for crack growth without remeshing". They presented a minimal remeshing FEM for crack growth by including the discontinuous enrichment functions to the finite element approximation in order to account for the presence of the crack. The essential idea was based on adding enrichment functions to the approximation space that contains a discontinuous displacement field. Hence, the method allows the crack to be arbitrarily aligned within the mesh. The same span of functions was earlier developed by Fleming *et al.* (1997) for the enrichment of the element-free Galerkin method. The method exploits the PU property of finite elements that was noted by Melenk and Babuška (1996), namely that the sum of the shape functions must be unity. This property has long been known, since it corresponds to the ability of the shape functions to reproduce a constant that represents translation, which is crucial for convergence.

The X-FEM provides a powerful tool for enriching solution spaces with information from asymptotic solutions and other knowledge of the physics of the problem. This has proved very useful for cracks and dislocations where near-field solutions can be embedded by the PU method to tremendously increase the accuracy of relatively coarse meshes. The technique offers possibilities in treating phenomena such as surface effects in nano-mechanics, void growth, subscale models of interface behavior, and so on. Thus, the X-FEM method has greatly enhanced the power of the FEM for many of the problems of interest in mechanics of materials. The aim of this chapter is to provide an overview of the X-FEM with an emphasis on various applications of the technique to materials modeling problems, including linear elastic fracture mechanics (LEFM); cohesive fracture mechanics; composite materials and material inhomogeneities; plasticity, damage and fatigue problems; shear band localization; fluid–structure interaction; fluid flow in fractured porous media; fluid flow and fluid mechanics problems; phase transition and solidification; thermal and thermo-mechanical problems; plates and shells; contact problems; topology optimization; piezoelectric and magneto-electroelastic problems; and multi-scale modeling.

1.2 An Enriched Finite Element Method

The FEM is widely used in industrial design applications, and many different software packages based on FEM techniques have been developed. It has undoubtedly become the most popular and powerful analytical tool for studying the behavior of a wide range of engineering and physical problems. Its applications have been developed from basic mechanical problems to fracture mechanics, fluid dynamics, nano-structures, electricity, chemistry, civil engineering, and material science (Figure 1.2). The FEM has proved to be very well suited to the study of fracture mechanics. However, modeling the propagation of a crack through a finite element mesh turns out to be difficult because of the modification of mesh topology. To accurately model discontinuities with FEMs, it is necessary to conform the discretization to the discontinuity. This becomes a major difficulty when treating problems with evolving discontinuities where the mesh must be regenerated at each step. Reprojecting the solution on the updated mesh is not only a costly operation but also it may have a troublesome impact on the quality of results.

Modeling moving discontinuities within the classical finite element is quite cumbersome due to the necessity of the mesh to conform to discontinuity surfaces. Mesh generation of complex geometries can be very time consuming with a classical finite element analysis. The main difficulty arises from the necessity of the mesh to conform to physical surfaces. Discontinuities such as holes, cracks, and material interfaces may not cross mesh elements. Moreover, local refinements close to discontinuities and mesh modification to track the geometrical and topological changes in crack propagation problems for example,

Figure 1.2 Bridge damage in Shaharah, Yemen, August 1986. The failure of bridges is of special concern to structural engineers in trying to learn lessons vital to bridge design, construction, and maintenance. (Source: Bernard Gagnon [CC-BY-3.0-br (http://creativecommons.org/licenses/by/3.0.en)], via Wikimedia Commons; http://en.wikipedia.org/wiki/File:Shehara_02.jpg)

can be difficult. Also, when geometries evolve and history dependent models are used, robust methods to transfer the solution to the new mesh are needed. This issue is particularly significant, since computed fields defined on these discontinuities are often the most important ones. In order to overcome these mesh-dependent difficulties, the generalized finite element method (G-FEM) and the X-FEM have been developed to facilitate the modeling of arbitrary moving discontinuities through the partition of unity enrichment of finite elements (PUM), in which the main idea is to extend a classical approximate solution basis by a set of locally supported enrichment functions that carry information about the character of the solution, for example, singularity, discontinuity, and boundary layer. As it permits arbitrary functions to be locally incorporated in the FEM or the meshfree approximation, the PUM gives flexibility in modeling moving discontinuities without changing the underlying mesh, while the set of enrichment functions evolve (and/or their supports) with the interface geometry. In addition to facilitating the modeling of moving discontinuities, enrichment also increases the local approximation power of the solution space by allowing the introduction of arbitrary functions within the basis. This is particularly useful for problems with singularities or boundary layers.

Basically, the G-FEM and the X-FEM are versatile tools for the analysis of problems characterized by discontinuities, singularities, localized deformations, and complex geometries. These methods can dramatically simplify the solution of many problems in material modeling, such as the propagation of cracks, the evolution of dislocations, the modeling of grain boundaries, and the evolution of phase boundaries. The advantage of these methods is that the finite element mesh can be completely independent of the morphology of these entities. The G-FEM and the X-FEM incorporate the analytically known or numerically computed handbook functions within some range of their applicability into the traditional FE (finite element) approximation with the PU (partition of unity) method to enhance the local and global accuracy of the computed solution. Both the X-FEM and G-FEM meshes need not conform to the boundaries of the problem. The FEM is used as the building block in the X-FEM and the G-FEM; hence, much of the theoretical and numerical developments in FEs can be readily extended and applied. Moreover, the X-FEM and G-FEM make possible an accurate solution of engineering problems in complex domains that may be practically impossible to solve using the FE method. The X-FEM and G-FEM are basically identical methods; the X-FEM was originally developed for discontinuities, such as cracks, and used local enrichments while the G-FEM was first involved with global enrichments of the approximation space. The X-FEM and G-FEM can be used with both structured and unstructured meshes. The structured meshes are appealing for many studies in materials science, where the objective is to determine the properties of a unit cell of the material. However, the unstructured meshes tend to be widely used for the analysis of engineering structures and components since it is often desirable to conform the mesh to the external boundaries of the component, although some methods under development today are able to treat even complicated geometries with structured meshes (Belytschko, Gracie, and Ventura 2009). The G-FEM allows for *p*–adaptivity and provides accurate numerical solutions with coarse or practically acceptable meshes by augmenting the FE space with the analytical or numerically generated solution of a given boundary value problem. The X-FEM on the other hand pays most attention to the requisite enrichment of nodes to model the internal boundary (crack or inclusion) of interest. Hence, the X-FEM is less dependent on known closed form solutions and affords greater flexibility.

1.3 A Review on X-FEM: Development and Applications

The X-FEM has gained a lot of attention in the last decade for its advantages in replicating the discontinuity of the displacement field across the crack surface and singularity at the crack front without the need for remeshing. The X-FEM enables the accurate approximation of fields that involve jumps, kinks, singularities, and other non-smooth features within elements (Karihaloo and Xiao, 2003). This is achieved by adding additional terms, that is, the enrichments, to classical FE approximations. These terms enable the approximation to capture the non-smooth features independently of the mesh. The X-FEM has shown its full potential for application in fracture mechanics (Fries and Belytschko, 2010). Applications with cracks involve discontinuities across the crack surface and singularities, or general steep gradients, at the crack front. In the classical FEM, a suitable mesh that accounts for these features has to be provided and maintained; this is particularly difficult for crack propagation in three dimensions. The X-FEM, however, can treat these types of problems on fixed meshes and considers crack propagation by a dynamic enrichment of the approximation.

Crack propagation using an enriched FEM technique was first introduced by Belytschko and Black (1999) that encompasses three major topics; the crack description, the discretized formulation, and the criteria for the crack update. In this method, the meshing task is reduced by enriching the elements near the crack tip and along the crack faces with the leading singular crack tip asymptotic displacement fields using the PUM to account for the presence of the crack. In the case that multiple crack segments need to be enriched using the near-tip fields; a mapping algorithm is used to align the discontinuity with the crack geometry. It was also shown that the use of discontinuous displacements along the crack produces a solution with zero traction along the crack faces. Moës, Dolbow, and Belytschko (1999) introduced a far more elegant and straightforward procedure to introduce a discontinuous field across the crack faces away from the crack-tip by adapting the generalized Heaviside function, and developed simple rules for the introduction of discontinuous and crack tip enrichments. Daux *et al.* (2000) introduced the junction function concept to account for multiple branched cracks and called their method the *extended finite element method* (X-FEM). They have employed this method for modeling complicated geometries such as multiple branched cracks, voids, and cracks emanating from holes without the need for the geometric entities to be meshed. The X-FEM is promising since it avoids using a mesh that conforms to the cracks, voids, or inhomogeneities as is the case with the traditional FEM. In X-FEM, a standard FE mesh for the problem is first created without accounting for the geometric entity. The presence of cracks, voids, or inhomogeneities is then represented independently of the mesh by enriching the standard displacement approximation with additional functions. For crack modeling, both discontinuous displacement fields along the crack faces and the leading singular crack tip asymptotic displacement fields are added to the displacement based FE approximation through the PUM. The additional coefficients at each enriched node are independent. Moreover, the X-FEM provides a seamless means to use higher order elements or special

FEs without significant changes in the formulation. The X-FEM improves the accuracy in problems where some aspects of the functional behavior of the solution field is known *a priori* and relevant enrichment functions can be used.

Advances in the X-FEM have been led to applications in various fields of computational mechanics and physics. The open source X-FEM codes were released by Bordas and Legay (2005), and numerical implementation and efficiency aspects of X-FEM were studied by Dunant *et al.* (2007). The X-FEM is a robust and popular method that has been used for industrial problems and is implemented by leading computational software companies. These applications have reached a high degree of robustness and are now being incorporated into the general purpose codes such as LS-DYNA and ABAQUS. Many of the techniques that are used in the X-FEM are directly related to techniques previously developed in mesh-free methods. An overview of the X-FEM has been reported by Karihaloo and Xiao (2003), Bordas and Legay (2005), Rabczuk and Wall (2006), Abdelaziz and Hamouine (2008), Belytschko, Gracie, and Ventura (2009), Rabczuk, Bordas, and Zi (2010), and Fries and Belytschko (2010). There are also three recent published books on the X-FEM that have focused on fracture mechanics problems by Mohammadi (2008, 2012) and Pommier *et al.* (2011). In what follows, a comprehensive overview is presented on various achievements of the X-FEM.

1.3.1 Coupling X-FEM with the Level-Set Method

In the context of the X-FEM, the location of non-smooth features is often defined implicitly by means of the level set method (LSM) (Osher and Sethian, 1988). The LSM complements the X-FEM extremely well as it provides the information *where* and *how* to enrich. The extension of the LSM to the description of crack *paths* in two dimensions was proposed by Stolarska *et al.* (2001) and Stolarska and Chopp (2003), and the description of crack *surfaces* in three dimensions was performed by Moës, Gravouil, and Belytschko (2002), Gravouil, Moës, and Belytschko (2002) and Sukumar, Chopp, and Moran (2003a). For crack problems, one enrichment is typically needed at the crack surface and additional enrichments are required at the crack front where both types of information, including the crack surface and the crack front, can be extracted directly from the level set functions. The discontinuous enrichment function that captures the jump in the displacement field across the crack surface depends directly on the level set function that stores the (signed) distance to the surface. The enrichment functions that capture the high gradients at the crack front depend on the level set functions indirectly; there, the level set functions imply a coordinate system in which the enrichment functions are evaluated. Thus, it can be seen that the LSM has important advantages in the context of the X-FEM. On the other hand, the X-FEM is only *one* step in the simulation of crack propagation that leads to an accurate approximation of the displacement, stress, and strain fields. The next step involves a characterization of the situation at the crack tip from which the crack increment is deduced. In fact, on the basis of fracture parameter information, such as stress intensity factors (SIFs), configurational forces, the *J*–integral, local maximum stress and strain measures, and so on, the direction and length of the increment at the crack tip in two dimensions, or at selected points on the crack front in three dimensions, can be modeled. The third and last step involves an update of the crack description such that the increments are considered appropriately (Fries and Baydoun, 2012).

Stolarska *et al.* (2001) presented the first implementation of LSM for modeling of crack propagation within the extended FE framework where the interface evolution was successfully performed by the LSM. Sukumar *et al.* (2001) employed the LSM for modeling holes and inclusions where the level set function was used to represent the local enrichment for material interfaces. Moës, Gravouil, and Belytschko (2002) and Gravouil, Moës, and Belytschko (2002) performed a combined X-FEM and the LSM to construct arbitrary discontinuities in the three-dimensional analysis of crack problems. Ventura, Xu, and Belytschko (2002) introduced the vector LSM for description of propagating crack in the element-free Galerkin method. Ji, Chopp, and Dolbow (2002) presented a hybrid X-FEM–LSM for modeling the evolution of sharp phase interfaces on fixed grids with reference to solidification problems to represent the jump in the temperature gradient that governs the velocity of the phase boundary. Stolarska and Chopp

(2003) presented an algorithm that couples the LSM with the X-FEM to investigate the effects of the proximity of multiple interconnect lines, multiple cracks, interconnect material, and integrated circuit boundaries on the growth of cracks due to fatigue from thermal cycling. Chessa and Belytschko (2003a, b) presented a combined X-FEM and LSM for two-phase flow with surface tension effects, where the velocity was enriched by the signed distance function. They also employed the X-FEM to model arbitrary discontinuities in space–time along a moving hyper-surface using the LSM (Chessa and Belytschko, 2004). Legay, Chessa, and Belytschko (2006) proposed an Eulerian–Lagrangian method for fluid-structure interaction based on the LSM, where the level set description of the interface leads to the formulation of the fluid–structure interaction problem.

An extension of the LSM was introduced by Sethian (1996) based on the *fast marching method*. This technique prevents the need to represent the geometry of the interface during its evolution; the method is computationally attractive for monotonically advancing fronts. Sukumar, Chopp, and Moran (2003a) presented an implementation of the combined X-FEM and fast marching method for modeling planar three-dimensional fatigue crack propagation, where the fast marching method was used to handle its evolution under fatigue growth conditions. Chopp and Sukumar (2003) employed the technique for modeling fatigue crack propagation of multiple coplanar cracks based on coupling the X-FEM with the fast marching method. Sukumar *et al.* (2008) proposed a numerical technique for non-planar three-dimensional linear elastic crack growth simulation based on a coupled X-FEM and the fast marching method.

1.3.2 Linear Elastic Fracture Mechanics (LEFM)

Modeling of crack propagation with the FEM is cumbersome due to the need to update the mesh to conform the geometry of the crack surface. Several FE techniques have been developed to model cracks and crack growth without remeshing. The X-FEM is one of the most powerful techniques developed based on an enrichment strategy for finite elements on the basis of a PU. Belytschko and Black (1999) originally introduced a minimal remeshing FEM for crack growth, where discontinuous enrichment functions were added into the FE approximation to account for the presence of the crack. Moës, Dolbow, and Belytschko (1999) improved the method by incorporating a discontinuous field across the crack faces away from the crack tip for modeling crack growth, where the standard displacement based approximation was enriched near a crack by employing both discontinuous fields and near-tip asymptotic fields through a PUM. Daux *et al.* (2000) extended the X-FEM to model crack problems with multiple branches, multiple holes, and cracks emanating from holes. Sukumar *et al.* (2000) employed the X-FEM in three-dimensional fracture mechanics, where a discontinuous function and the two-dimensional asymptotic crack tip displacement fields were added to the FE approximation to account for the crack using the notion of a PU. Stolarska *et al.* (2001) introduced an algorithm that couples the LSM with the X-FEM to model crack growth, in which the LSM was used to represent the crack location, including the location of crack-tips. Moës, Gravouil, and Belytschko (2002) extended the X-FEM to handle arbitrary non-planar cracks in three dimensions by describing the crack geometry in terms of two signed distance functions that were able to construct a near tip asymptotic field with a discontinuity that conforms to the crack, even when it is curved or kinked near a tip. Ayhan and Nied (2002) proposed an enriched FE approach for obtaining the SIFs for general three-dimensional crack problems. Sukumar and Prevost (2003) presented the X-FEM for two-dimensional crack modeling in isotropic and bimaterial media within the finite element program Dynaflow™, which was later used by Huang, Sukumar, and Prévost (2003b) to demonstrate the numerical modeling of SIFs in crack problems, including crack growth simulation. Stazi *et al.* (2003) presented a method for LEFM using enriched quadratic interpolations, in which the geometry of the crack was represented by a level set function interpolated on the same quadratic FE discretization. Lee *et al.* (2004) presented a combination of the X-FEM and the mesh superposition method (*s*–version FEM) for modeling of stationary and growing cracks, in which the near-tip field was modeled by superimposed quarter point elements on an overlaid mesh and the rest of the discontinuity was implicitly described by a step function on the PU, where the two displacement fields were matched through a transition region.

Budyn *et al.* (2004) developed the X-FEM for multiple crack growth considering the junction of cracks in both homogeneous and inhomogeneous brittle materials, which does not require remeshing as the cracks grow. A similar approach was proposed by Zi *et al.* (2004) for modeling the growth and the coalescence of cracks in a quasi-brittle cell containing multiple cracks.

Advanced issues in LEFM have been conducted by researchers in more recent studies. An application of the X-FEM method to large strain fracture mechanics was presented by Legrain, Moës, and Verron (2005) with a special reference to the fracture of rubber-like materials. Moës, Béchet, and Tourbier (2006) introduced a strategy to impose the Dirichlet boundary conditions within the X-FEM by construction of a corrected Lagrange multiplier space on the boundary that preserves the optimal rate of convergence. Ventura (2006) introduced a method for eliminating the introduction of quadrature subcells when using discontinuous/non-differentiable enrichment functions in the X-FEM by replacing the discontinuous/non-differentiable functions with equivalent polynomials. Asadpoure, Mohammadi, and Vafai (2006) proposed an X-FEM for modeling cracks in orthotropic media based on a discontinuous function and two-dimensional asymptotic crack tip displacement fields. Asadpoure and Mohammadi (2007) modified their previous model by adding new enrichment functions to simulate the orthotropic cracked media, where the required near tip enrichment functions were obtained by extracting basic terms from the complex solutions in the vicinity of the crack tip. Loehnert and Belytschko (2007) employed the X-FEM to investigate the effect of crack shielding and amplification of various arrangements of micro-cracks on the SIFs of a macro-crack, including large numbers of arbitrarily aligned micro-cracks. Sukumar *et al.* (2008) proposed a numerical technique for non-planar three-dimensional elastic crack growth simulations by combining the X-FEM with the fast marching method. Tabarraei and Sukumar (2008) employed the X-FEM on polygonal and quadtree FE meshes for two-dimensional crack growth modeling, where the Laplace interpolant was used to construct basis functions on convex polygonal meshes, and the mean value coordinates were adopted for non-convex elements.

One of the main issues in the X-FEM method is the blending elements, which are constructed between the enriched and standard elements; they are often crucial for a good performance of the local partition of unity enrichments. Chessa, Wang, and Belytschko (2003) employed the enhanced strain method in blending elements to improve the performance of local PU enrichments. Laborde *et al.* (2005) modified the standard X-FEM to circumvent problems in blending elements for the case of crack problems by enriching a whole fixed area around the crack tip. Legay, Wang, and Belytschko (2005) employed the X-FEM within the spectral finite elements for modeling discontinuities in the gradients, where there was no need to implement the blending elements for high-order spectral elements. Fries and Belytschko (2006) developed an intrinsic X-FEM method without blending elements for treating arbitrary discontinuities in a FE context, where no additional unknowns were introduced at the nodes whose supports are crossed by discontinuities. Fries (2008) introduced a corrected X-FEM method without problems in blending elements based on a linearly decreasing weight function for enrichment in the blending elements. Gracie *et al.* (2008b) developed a discontinuous Galerkin formulation without blending elements that decomposes the domain into an enriched and unenriched sub-domains, where the continuity was enforced with an internal penalty method. Benvenuti, Tralli, and Ventura (2008) introduced a regularized X-FEM model for the transition from continuous to discontinuous displacements, where the emerging strain and stress fields were modeled independently using specific constitutive assumptions. Ventura, Gracie, and Belytschko (2009) introduced a weight function blending, where the enrichment function was pre-multiplied by a smooth weight function with a compact support to allow for a completely smooth transition between the enriched and unenriched sub-domains. Tarancon *et al.* (2009) employed the higher-order hierarchical shape functions to reduce unwanted effects of the partial enrichment in the blending elements. Shibanuma and Utsunomiya (2009) presented an alternative formulation for the X-FEM based on the concept of the PU FEM for solving the problem of blending elements, which assures the numerical accuracy in the entire domain. Loehnert, Mueller-Hoeppe, and Wriggers (2011) extended the originally corrected X-FEM method presented by Fries to the three-dimensional case with its extension to finite deformation theory. Chahine, Laborde, and Renard (2011) presented a non-conformal approximation method based on the integral matching X-FEM, in which the transition layer between the singular enrichment area and the rest of the domain was replaced by an interface associated with an integral matching condition of mortar type. Menk and Bordas (2011) presented a procedure to obtain stiffness matrices whose condition number is close to the one of the FE matrices without any enrichment using a domain decomposition technique. Chen *et al.* (2012) presented a strain smoothing procedure within the X-FEM framework for LEFM to outperform the standard X-FEM, where the edge-based smoothing was used to produce a softening effect leading to a close-to-exact stiffness, "super-convergence", and "ultra-accurate" solutions.

The implementation of the X-FEM in dynamic fracture has been mostly focused on simulation of the dynamic crack propagation and estimation of the dynamic SIFs for arbitrary two- and three-dimensional cracks. Réthoré, Gravouil, and Combescure (2005b) proposed an energy-conserving scheme in the framework of the X-FEM to model the dynamic fracture and time-dependent problems that give proof of stability of the numerical scheme in linear fracture mechanics. Menouillard *et al.* (2006, 2008) presented an explicit time stepping method based on a mass matrix lumping technique for enriched elements, and demonstrated that the critical time step of an enriched element is of the same order as that of the corresponding element without extended DOF. Elguedj, Gravouil, and Maigre (2009) presented a generalized mass lumping technique for explicit dynamics simulation using the X-FEM with arbitrary enrichment functions that was based on an exact representation of the kinetic energy of rigid body modes and enrichment modes. Gravouil, Elguedj, and Maigre (2009) presented a general explicit time integration technique for X-FEM dynamics simulations with a standard critical time step by developing a classical element-by-element strategy that couples the standard central difference scheme with the unconditionally stable-explicit scheme. Fries and Zilian (2009) studied the convergence properties of different time integration methods in the framework of X-FEM for moving interfaces, including one-step time-stepping schemes, the implicit Euler method, the trapezoidal rule, and the implicit midpoint rule. Menouillard and Belytschko (2010a) presented a method to enrich the X-FEM using the meshless approximation for dynamic fracture problems, where the mesh-free approximation was used to smooth the stress state near the crack tip during the propagation, and decreasing unphysical oscillations in the stress due to the propagation of the discontinuity. In a later work, Menouillard and Belytschko (2010b) proposed a method based on enforcing the continuity of forces corresponding to the enriched DOF to smoothly release the tip element while the crack tip travels through the element. Menouillard *et al*. (2010) proposed a new enrichment method with a time-dependent enrichment function for dynamic crack propagation in the context of the X-FEM and studied the effect of different directional criteria on the crack path. Motamedi and Mohammadi (2010a, b) presented a dynamic crack analysis for composites based on the orthotropic enrichment functions within the X-FEM framework by evaluating the dynamic SIFs using the domain separation integral method. Esna Ashari and Mohammadi (2012) proposed the X-FEM for fracture analysis of delamination problems in fiber-reinforced polymer reinforced beams, where the stress singularities near the debonding crack-tip were modeled by orthotropic bimaterial enrichment functions. Liu, Menouillard, and Belytschko (2011) developed a higher-order X-FEM method based on the spectral element method for the simulation of dynamic fracture, where the numerical oscillations were effectively suppressed and the accuracy of computed SIFs and crack path were appropriately improved. Motamedi and Mohammadi (2012) introduced the time-independent orthotropic enrichment functions for dynamic crack propagation of moving cracks in composites based on the X-FEM, where the enrichment functions were derived from the analytical solutions of a moving/propagating crack in orthotropic media.

The importance of error estimation in the X-FEM numerical analysis has been investigated by various researchers. Chahine, Laborde, and Renard (2006) performed a convergence study for a variant of the X-FEM on cracked domains by using a cut-off function to localize the singular enrichment area, and illustrated that the convergence error of the proposed variant is of order h for a linear FEM. Ródenas *et al.* (2008) presented a stress recovery procedure that provides accurate estimations of the discretization error for LEFM problems based on the superconvergent patch recovery (SPR) technique for the X-FEM framework. Panetier, Ladeveze, and Chamoin (2010) presented a method to obtain the local error bounds in the context of fracture mechanics by evaluating the discretization error for quantities of interest computed in the X-FEM using the concept of constitutive relation error. Ródenas *et al*. (2010) introduced a recovery-type error estimator yielding upper bounds of the error in energy norm for LEFM problems using the X-FEM that yields equilibrium at a local level. Shen and Lew (2010a, b) introduced an optimally convergent discontinuous Galerkin-based X-FEM for fracture mechanics problems, in which an optimal order of convergence was obtained in comparison with other variants of X-FEM technique. Nicaise, Renard, and Chahine (2011) performed an *a priori* error estimate on the standard X-FEM with a fixed enrichment area and the X-FEM with a cut-off function by estimating the error on the SIFs. Prange, Loehnert, and Wriggers (2012) presented a simple recovery based error estimator for the discretization error in the X-FEM analysis of crack problems, where enhanced smoothed stresses were recovered to enable the error estimation for arbitrary distributed cracks. Byfut and Schröder (2012) presented a higher-order X-FEM method by combining the standard X-FEM with higher-order FEM method based on the Lagrange-type and hierarchical tensor product shape functions, and demonstrated the methodological aspects that are necessary in the *hp*-adaptivity of X-FEM for obtaining the exponential convergence rate. González-Albuixech *et al*. (2013a) investigated the convergence rate of solution obtained from the domain energy integral for computation of the SIFs in the solution of two-dimensional curved crack problems using the X-FEM. Ródenas *et al*. (2013) presented a technique to obtain an accurate estimate of the error in energy norm using a moving least squares (MLS) recovery-based procedure for X-FEM problems. Rüter, Gerasimov, and Stein (2013) proposed a goal-oriented *a posteriori* error estimator for X-FEM approximations in LEFM problems to compute upper bounds on the error of the *J*–integral.

More advanced concepts have been studied by researchers in the X-FEM analysis of elastic linear fracture mechanics. Park *et al*. (2009) developed a mapping method to integrate weak singularities that result from enrichment functions in the G-FEM/X-FEM and is applicable to two- and three-dimensional problems including arbitrarily shaped triangles and tetrahedra. Mousavi and Sukumar (2010) presented an alternative Gaussian integration scheme to construct the Gauss quadrature rule over arbitrarily-shaped elements in two dimensions without the need for partitioning that was efficient and accurate in evaluation of weak form integrals. Bordas *et al*. (2010, 2011) investigated the accuracy and convergence of enriched finite element approximations by employing the strain smoothing to higher order elements, and highlighted that the strain smoothing in enriched approximations are beneficial when the enrichment functions are polynomial. Mousavi, Grinspun, and Sukumar (2011a, b) presented a higher order X-FEM with harmonic enrichment functions for complex crack problems, in which the numerically computed enrichment functions for the crack were obtained via the solution of the Laplace equation with Dirichlet and vanishing Neumann boundary conditions. Legrain, Allais, and Cartraud (2011) employed the X-FEM in the context of quadtree/octree meshes, where particular attention was paid to the enrichment of hanging nodes that inevitably arise with these meshes, and an approach was proposed for enforcing displacement continuity along hanging edges and faces. Richardson *et al*. (2011) presented a method for simulating quasi-static crack propagation that combines the X-FEM with a general algorithm for cutting triangulated domains, and introduced a simple and flexible quadrature rule based on the same geometric algorithm. Shibanuma and Utsunomiya (2011) studied the reproductions of *a priori* knowledge in the original X-FEM and the PU-FEM based X-FEM for the crack analysis, and showed that there is a serious lack of the reproduction of *a priori* knowledge in the local enrichment area close to the crack tip in case of the original X-FEM; however, *a priori* knowledge can be accurately reproduced over the entire enrichment in the PUFEM based X-FEM. Fries and Baydoun (2012) and Baydoun and Fries (2012) presented a method for two- and three-dimensional crack propagation that combines the advantages of explicit and implicit crack descriptions, and described a propagation criterion for three-dimensional fracture mechanics using the proposed hybrid explicit–implicit approach. Minnebo (2012) introduced a three-dimensional integral strategy for numerical integration of singular functions in the computation of stiffness matrix and SIFs using the interaction integral method produced by the X-FEM in LEFM. Benvenuti (2012a) proposed the Gauss quadrature of integrals of discontinuous and singular functions in the three-dimensional X-FEM analysis of regularized interfaces. González-Albuixech *et al*. (2013b) introduced a curvilinear gradient correction based on the level set information used for the crack description within the X-FEM framework to compute the SIFs in curved and non-planar cracks. Amiri *et al*. (2014) presented a method based on local maximum entropy shape functions together with enrichment functions used in

PUMs to discretize problems in LEFM. Pathak *et al*. (2013) presented a simple and efficient X-FEM approach for modeling three-dimensional crack problems, in which the crack front was divided into a number of piecewise curve segments and the level set functions were approximated using the higher order shape functions.

The implementation of X-FEM in FE programs has been performed by various researchers to model the LEFM in practical engineering problems. Rabinovich, Givoli, and Vigdergauz (2007, 2009) presented a computational tool based on a combination of X-FEM and genetic algorithm for the detection and identification of cracks in structures that was used in conjunction with non-destructive testing of specimens. Xiao and Karihaloo (2007) combined the hybrid crack element (HCE) with an X-FEM/G-FEM and incorporated it into a commercial FE package, where the HCE was used for the crack tip region and the X-FEM was employed for modeling crack faces outside the HCE, independent of the mesh with jump functions. Ahyan (2007) presented a three-dimensional enriched FE methodology within the FE program FRAC3D to compute the SIFs for cracks contained in functionally graded materials. Nistor, Pantale, and Caperaa (2008) presented the implementation of X-FEM in their home made explicit dynamic FEM code, DynELA, to simulate the crack propagation in structural problems under dynamic loading. Dhia and Jamond (2010) employed one of the key features of the X-FEM, that is, the Heaviside enrichment function, within a generic numerical method based on the Arlequin framework to reduce the costs of crack propagation simulations. Legrain (2013) proposed a NURBS (non-uniform rational b-spline) enhanced extended FE approach for the unfitted simulation of structures defined by means of CAD (Computer Aided Design) parametric surfaces, in which the geometry of the computational domain was defined using an exact CAD description. Holl *et al*. (2014) presented a multi-scale technique to investigate advancing cracks in three-dimensional spaces with a reference to gas turbine blades, which was able to capture crack growth taking localization effects from the fine scale into account.

1.3.3 Cohesive Fracture Mechanics

LEFM is only applicable when the size of the fracture process zone (FPZ) at the crack tip is small compared to the size of the crack and the size of the specimen (Bazant and Planas, 1998). Hence, alternative models must be chosen to take into account the FPZ. The cohesive crack model is one of the simplest ones that can be represented by a traction–displacement relation across the crack faces near the tip. This model was introduced in the early 1960s for metals by Dugdale (1960) and Barenblatt (1962), and then developed by Hillerborg, Modéer, and Petersson (1976) by introducing the concept of fracture energy into the cohesive crack model and establishing a number of traction–displacement relationships for concrete. The first implementation of an enriched FEM into cohesive fracture mechanics was proposed by Wells and Sluys (2001) by applying the displacement jump into the conventional FEM, in which the path of the discontinuity was completely independent of the mesh structure and the jump function was used as an enrichment function for the whole cohesive crack. Moës and Belytschko (2002a, b) developed the cohesive crack model within the X-FEM framework for modeling the growth of arbitrary cohesive cracks, where the growth of the cohesive zone was governed by requiring the SIFs at the tip of the cohesive zone to vanish. Crisfield and Alfano (2002) presented an enriched FEM for modeling the delamination in fiber-reinforced composite structures with the aid of a decohesive zone model and interface elements, in which the elements around the softening process zone were enriched using the hierarchical polynomial functions. Zi and Belytschko (2003) developed an X-FEM for the cohesive crack model with a new formulation for elements containing crack tips, in which the entire crack was modeled with only one type of enrichment function, that is, the signed distance function, including the elements containing the crack tip so that no blending of the local PU was required. Remmers, de Borst, and Needleman (2003) presented a method for modeling crack nucleation and discontinuous crack growth irrespective of the structure of the finite element mesh, in which the crack was modeled by a collection of cohesive segments with a finite length and the segments were added to finite elements by using the partition of unity property of the finite element shape functions. Mariani and Perego (2003) presented a methodology for the simulation of quasi-static cohesive crack propagation in quasi-brittle materials, where a cubic displacement discontinuity was employed to reproduce the typical cusp-like shape of the process zone at the tip of a cohesive crack. Belytschko *et al*. (2003b) proposed the X-FEM for modeling dynamic crack propagation based on switching from a continuum to a discrete discontinuity, where the loss of hyperbolicity was modeled by a hyperbolicity indicator that enables to determine both the crack speed and crack direction for a given material model. Larsson and Fagerström (2005) presented a theoretical and computational framework for linear and nonlinear fracture behaviors on the basis of the inverse deformation problem with an applied discontinuous deformation separated from the continuous deformation using the X-FEM technique.

Basically, there is a relation between strain softening and fracture mechanics. One motivation for this interest is that strain softening stems from damage and is often a prelude to fracture (Figure 1.3). In fact, it is a manifestation of progressive energy release during microscopic decohesion before a macroscopic crack is apparent. Areias and Belytschko (2005a) presented a numerical procedure for the quasi-static analysis of three-dimensional crack propagation in brittle and quasi-brittle solids, in which a viscosity-regularized continuum damage constitutive model was coupled with the X-FEM formulation resulting in a regularized "crack-band" version of X-FEM. Xiao, Karihaloo, and Liu (2007) proposed an incremental-secant modulus iteration scheme using the X-FEM/G-FEM for simulation of cracking process in quasi-brittle materials described by cohesive crack models whose softening law was composed of linear segments. Asferg, Poulsen, and Nielsen (2007) developed a partly cracked X-FEM element for cohesive crack growth based on additional enrichment of the cracked elements with the capability of modeling variations in the discontinuous displacement field on both sides of the discontinuity to obtain a better stress distribution on crack faces. Benvenuti (2008) and Benvenuti, Tralli, and Ventura (2008) introduced a regularized X-FEM model for the transition from continuous to discontinuous displacements, where the emerging strain and stress fields were modeled independently using specific constitutive assumptions that can address cohesive interfaces with vanishing and finite thickness in a unified way. Mougaard, Poulsen, and Nielsen (2011) developed a cohesive crack tip element together with a coherent fully cracked element within the X-FEM framework based on a double enriched displacement field of *linear strain triangle* type to symmetrize the elements crack opening and reproduce equal stresses at both sides of the crack at the tip. Zamani, Gracie, and Eslami (2012) performed a comprehensive study on the use of higher-order terms of the crack tip asymptotic fields as enriching functions of the X-FEM for both cohesive and traction-free cracks, where two widely used criteria, that is, the SIF criterion and the stress criterion, were used with both linear and nonlinear cohesive laws. Mougaard, Poulsen, and Nielsen (2013) presented a complete tangent stiffness for modeling crack growth in the X-FEM by including the crack growth parameters in an incremental form of the virtual work together with the constitutive conditions in front of the crack tip as direct unknowns in the FEM equations.

The X-FEM has been extensively used for crack growth simulation in concrete structures and rock mechanics problems, where the failure is accompanied by the formation of discrete cracks and zones of local damage (Figure 1.4). Unger, Eckardt, and Könke (2007) employed the X-FEM for a discrete crack simulation of concrete using an adaptive crack growth algorithm, in which different criteria were applied to predicting the direction of the extension of a cohesive crack. Deb and Das (2010) proposed the X-FEM for modeling cohesive discontinuities in rock masses, where the displacement discontinuities were modeled using the three- and six-nodded triangular elements. Xu and Yuan (2011) introduced a cohesive zone model with a threshold in combination with the X-FEM to study the effects of fracture criteria in cohesive zone models for mixed-mode cracks. Campilho *et al*. (2011a, b) employed the X-FEM to model crack propagation and to predict the fracture behavior of a thin layer of two structural epoxy adhesives under varying restraining conditions; the stiff and compliant adherends. Benvenuti and Tralli (2012) proposed a regularized X-FEM approach that can tackle in a unified and smooth way the whole process from strain localization to crack inception and propagation and can simulate both the formation of a process zone with finite width and its subsequent collapse into a macro-crack in concrete-like materials. Golewski, Golewski, and Sadowski (2012) employed the X-FEM for three-dimensional numerical modeling of compact shear specimens used for experimental testing of the mode II fracture to estimate the fracture toughness for a mode II fictitious crack. Olesen and Poulsen (2013) presented a simple element for modeling

Figure 1.3 Road damage following a 6.8 magnitude earthquake in Chūetsu on October 23, 2004, which occurred in Niigata Prefecture, Japan. The road and other routes suffered heavy damage due to landslides and faulting that resulted from liquefaction. (Source: Tubbi [CC-BY-3.0-br (http://creativecommons.org/licenses/by-sa/3.0/deed.en)], via Wikimedia Commons; http://commons.wikimedia.org/wiki/File:Chuetsu_earthquake-Yamabe_Bridge.jpg)

Figure 1.4 Teton Dam collapse on June 5, 1976: The collapse of an earthen dam sent a wall of water toward the Idaho Falls. The dam, located in Idaho, USA on the Teton River in the eastern part of the state, between Fremont and Madison counties, suffered a catastrophic failure as it was filling for the first time. (Source: US Government; http://commons.wikimedia.org/wiki/File:Teton_Dam_failure.jpg)

cohesive fracture processes in quasi-brittle materials based on the CST (constant strain triangle) element, where the crack was embedded in the element and a special shape function was introduced for the discontinuous displacements. Zhang, Wang, and Yu (2013) presented a numerical scheme based on the X-FEM for a seismic analysis of crack growth in concrete gravity dams with special reference to the dynamic analysis of the Koyna Dam during the 1967 Koyna earthquake.

1.3.4 Composite Materials and Material Inhomogeneities

Composite structures have been of great concern for possessing advantages of multiple materials resulting in substantial economic benefits. Material inhomogeneities result in a discontinuous displacement gradient that is referred to as *weak discontinuities*. Modeling the deformation and failure mechanisms in composite and polycrystalline materials is critical for improvements to the development and application of advanced structural materials. The material microstructure plays a pivotal role in dictating the modes of fracture and failure, and the macroscopic response of real materials. Sukumar *et al*. (2001) presented the first implementation of the X-FEM for modeling arbitrary holes, inclusions, and material interfaces without meshing the internal boundaries, where the X-FEM was coupled with the LSM to represent the location of holes, inclusions, and material interfaces. Moës *et al*. (2003) presented a modified level set function for problems that involve discontinuities in the gradients of the field to model the material interfaces in micro-structures with complex geometries, and demonstrated the capability of model for the homogenization of periodic basic cells. Sukumar *et al*. (2003b) proposed the X-FEM for crack propagation simulation through a polycrystalline material microstructure with an aim towards the understanding of toughening mechanisms in polycrystalline materials such as ceramics. Sukumar *et al*. (2004) extended the X-FEM to the analysis of cracks that lie at the interface of two elastically homogeneous isotropic materials, where the new crack tip enrichment functions were introduced to span the asymptotic displacement fields for an interfacial crack. Liu, Xiao, and Karihaloo (2004) developed an X-FEM formulation to directly evaluate the mixed mode SIFs without extra post-processing for homogeneous materials and bimaterials. Hettich and Ramm (2006) described the material distribution within a microstructure that consists of different solid phases, including material interfaces by means of the level set and X-FEM methods. Yan and Park (2008) applied the X-FEM combined with the LSM for the simulation of near-interfacial crack propagation in a metal-ceramic layered structure. Yvonnet, Le Quang, and He (2008) presented a numerical approach for modeling the interface effects described by the coherent interface model and determining the size-dependent effective elastic moduli of nano-composites.

There are more recent studies reported in the literature for the X-FEM analysis of composite materials that are commonly used in engineering practice to provide a desired mechanical behavior in fiber-reinforced materials, metal composites, concrete, ceramics, and so on. Huynh and Belytschko (2009) presented a method for treating fracture in composite material by the X-FEM with meshes independent of matrix/fiber interfaces, where the level sets were used to describe the geometry of the interfaces and 12-asymptotic functions were employed for bimaterial crack tips. Zhang and Li (2009) studied the mechanical response of viscoelastic materials with inclusions by using a full integration scheme for the low-Poisson ratio problem and the selective integration scheme treating the volumetric locking for the high-Poisson ratio problem encountered in viscoelastic materials. Dréau, Chevaugeon, and Moës (2010) proposed an approach to improve the geometrical representation of surfaces with the X-FEM, in which the surfaces were represented implicitly using the LSM with higher orders. Aragon, Duarte, and Geubelle (2010) presented a GFEM/XFEM formulation for modeling two-dimensional problems with gradient jumps along discrete lines, such as those of thermal and structural analysis of heterogeneous materials, by introducing new enrichment functions for the solution of multiple intersecting discontinuity lines, such as those of triple junctions in polycrystalline materials. Nouy and Clement (2010) proposed an extended stochastic FEM for numerical simulation of heterogeneous materials with random material interfaces based on coupled X-FEM and spectral stochastic methods, in which the random geometry of material interfaces was described implicitly by using random level set functions. Singh, Mishra, and Bhattacharya (2011) presented a numerical simulation for inhomogeneities/discontinuities, such as cracks, holes, and inclusions, in functionally graded materials based on the X-FEM method. Hiriyur, Waisman, and Deodatis (2011) proposed the X-FEM coupled with a Monte Carlo approach to quantify the uncertainty in the homogenized effective elastic properties of multi-phase materials that allows for an arbitrary number, aspect ratio, location, and orientation of elliptic inclusions within a matrix. Tran *et al*. (2011) developed a modified X-FEM/LSM for numerical simulation of microstructures containing high volume fractions of inclusions to prevent the numerical artifacts in complex microstructures with nearby inclusions. Curiel Sosa and Karapurath (2012) proposed the X-FEM for numerical simulation of delamination in fiber metal laminates, where the orthotropic enrichment functions were used to model the propagation of delamination in composites. Benvenuti, Vitarelli, and Tralli (2012b) presented an approach for modeling of delamination in FRP (fiber reinforced polymer)-reinforced concrete by means of a regularized X-FEM formulation, which takes into consideration the mechanical properties of concrete, adhesive, and FRP. Pathak, Singh, and Singh (2012) performed a numerical analysis of bimaterial interfacial cracks based on the element free Galerkin method and the X-FEM under mixed mode loading conditions that was applicable to dissimilar or layered materials such as ceramic-metal and composite-metal. Wang *et al*. (2012) proposed a numerical simulation of crack growth in brittle matrix of particle reinforced composites. Jiang *et al*. (2013) presented an edge-based smoothing technique combined with the X-FEM method for fracture analysis of composite materials that took the advantages of the X-FEM and ES-FEM methods. Zhao, Bordas, and Qu (2013) presented a hybrid smoothed X-FEM–LSM for modeling nanoscale inhomogeneities with interfacial energy effect, in which the Gurtin–Murdoch surface elasticity model was used to account for the interface stress effect and the Wachspress interpolants were employed to construct the shape functions in the smoothed X-FEM method.

1.3.5 Plasticity, Damage, and Fatigue Problems

Modeling of fatigue crack growth is a challenging issue in computational fracture mechanics. Fatigue plays an important role in mechanical failure of structures subjected to cyclic loadings. Sukumar, Chopp, and Moran (2003a) and Chopp and Sukumar (2003) proposed a numerical technique for three-dimensional modeling of planar and multiple coplanar fatigue crack growth simulations based on a coupled X-FEM–fast marching method. Stolarska and Chopp (2003) presented an algorithm for modeling the growth of fatigue cracks due to thermal cycling based on a coupled X-FEM–LSM. Ferrié *et al.* (2006) investigated the fatigue crack propagation for a semi-elliptical crack in the bulk of an ultrafine-grained Al–Li alloy based on the synchrotron radiation X-ray microtomography and three-dimensional X-FEM simulation. Giner *et al.* (2008a) employed the X-FEM for analysis of fretting fatigue problems with the FE software ABAQUS™. Xu and Yuan (2009) performed an X-FEM analysis combined with a cyclic cohesive model for the mixed-mode fatigue crack nucleation and propagation in quasi-brittle materials with the FE software ABAQUS™. Shi *et al.* (2010) presented a combined X-FEM–narrow band fast marching method for three-dimensional curvilinear fatigue crack growth and life prediction analysis of metallic structures with the FE software ABAQUS™. Ayhan (2011) employed a three-dimensional X-FEM analysis for fatigue crack propagation of a mode I surface crack and incremented crack front profiles for inclined and deflected surface cracks. Giner *et al.* (2011) used the X-FEM to predict fatigue lives in fretting fatigue problems by means of a combined initiation–propagation model. Singh, Mishra, and Bhattacharya (2011) and Singh *et al.* (2012) investigated the fatigue life of a homogeneous plate containing multiple discontinuities under cyclic loading conditions, where the multiple discontinuities of arbitrary size were randomly distributed in the plate. Ye, Shi, and Cheng (2012) proposed the X-FEM with ABAQUS™ software to simulate crack propagation and to predict the effect of reinforcing particles to the crack propagation behavior of composite materials. Réthoré *et al.* (2012) presented a three-dimensional fatigue crack propagation using advanced experimental, imaging, measurement, and numerical simulation based on the X-FEM technique. Bhattacharya, Singh, and Mishra (2012, 2013) presented an X-FEM numerical simulation for the fatigue crack growth of interfacial cracks in bilayered materials, and investigated the fracture fatigue behavior of center and edge cracked functionally graded materials in the presence of multiple inhomogeneities, such as holes/voids, inclusions, and minor cracks. Baietto *et al.* (2013) proposed a combined experimental–numerical tool based on the X-FEM for modeling two- and three-dimensional fretting fatigue crack growth simulation.

The need for an accurate tool to study crack growth scenarios and assess the damage tolerance is crucial for various industrial structures from the aeronautical industry to the automotive industry. Bordas and Moran (2006) employed the X-FEM together with the level sets for damage tolerance assessment of complex structures for solution of the complex three-dimensional industrial fracture mechanics problems. Xu *et al.* (2010) performed an X-FEM analysis to study the low-speed impact-induced cracking in brittle plates by evaluation of the interaction between the stress field and the initiation and propagation of the radial and circumferential cracks. Chatzi *et al.* (2011) proposed a combined X-FEM–genetic algorithm technique for detection of any type of flaw (cracks or holes) of any shape in structures. Vajragupta *et al.* (2012) presented a micro-mechanical damage simulation of dual phase steels using the X-FEM to study the interaction between failure modes in DP (dual phase) steels. Jung, Jeong, and Taciroglu (2013) performed a dynamic X-FEM analysis for identifying a scatterer embedded in elastic heterogeneous media, such as a crack, a void, or an inclusion with properties that have detectable contrasts from those of the host medium. Feerick, Liu, and McGarry (2013) presented anisotropic damage initiation criteria for the X-FEM prediction of crack initiation and propagation in cortical bone.

Ductile fracture has been the object of numerous works in engineering, which is characterized by the presence of moderate to large plastic deformations prior to the degradation mechanisms (Figure 1.5).

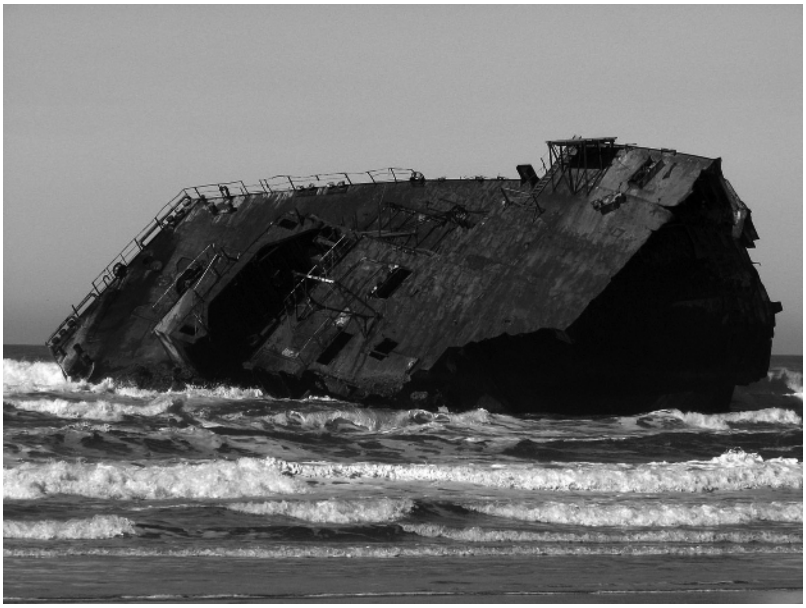

Figure 1.5 The *New Carissa* was a freighter that ran aground on a beach near Coos Bay, Oregon, USA, during a storm on February 4, 1999, and subsequently broke apart. (Source: Erin from Oregon City, OR (Old Carissa) [CC-BY-2.0 (http://creativecommons.org/licenses/by/2.0)], via Wikimedia Commons; http://commons.wikimedia.org/wiki/File:Wreck_off_Coos_Bay,_Oregon.jpg)

The hypothesis of confined plasticity, that is, the plasticity that is confined to the region near the crack tip, is important for prediction of the fatigue crack growth. The first implementation of the X-FEM for fatigue fracture analysis of cracks in homogeneous, isotropic, elastic–plastic two-dimensional solids subject to mixed mode in the case of confined plasticity was presented by Elguedj, Gravouil, and Combescure (2006), where the well-known Hutchinson–Rice–Rosengren (HRR) fields were used to present the singularities in elasto-plastic fracture mechanics. In a later work, Elguedj, Gravouil, and Combescure (2007) proposed a mixed augmented Lagrangian–X-FEM method for modeling of elastic–plastic fatigue crack growth with unilateral contact on the crack faces. Prabel *et al*. (2007) presented a numerical scheme based on the X-FEM for dynamic crack propagation in elastic-plastic media using an efficient level set update on non-matching meshes. Zhang, Rong, and Li (2010) investigated the crack problem in a linear viscoelastic material using an incremental X-FEM method while ignoring the nonlinear effects caused by subcritical crack growths in the crack-tip failure zone, in which the basis functions were extracted from the viscoelastic asymptotic fields at a crack tip. Kroon (2012) presented a computational framework based on the X-FEM for high-speed crack growth in rubber-like solids for dynamic steady-state condition considering the effects of inertia, viscoelasticity, and finite strains. Pourmodheji and Mashayekhi (2012) proposed a combined X-FEM–continuum damage mechanics model to validate the ductile damage evolution experimentally measured in A533B1 steel. Broumand and Khoei (2013) presented an enriched-FEM technique for crack growth simulation in large deformation ductile fracture problems using a non-local damage-plasticity model within the X-FEM framework based on the Lemaitre damage-plasticity model. Seabra *et al*. (2013) employed an X-FEM method combined with the Lemaitre ductile damage model to simulate the crack initiation and propagation by evolution of the damage in the framework of a finite strain and a non-local integral formulation. Wang *et al*. (2013) performed a numerical simulation for rubber-modified asphalt mixtures crack growth using the X-FEM in a notched semi-circular bending test. Liu and Borja (2013) presented an explicit X-FEM framework for fault rupture dynamics accommodating bulk plasticity near the fault by comparing the performance of various plasticity models for a rock medium hosting a randomly propagating fault.

The study of fundamental phenomena in nonlinear material behavior and plasticity involves the development of computationally efficient dislocation models. The first implementation of an enriched FEM technique into a dislocation problem was performed by Ventura, Moran, and Belytschko (2005), where the regular FE solution was enriched by the closed form solution for an edge dislocation. Gracie, Ventura, and Belytschko (2007) proposed an alternative X-FEM method for modeling dislocation problems, where the tangential enrichment was used to model edge dislocations as interior discontinuities. Belytschko and Gracie (2007) presented the X-FEM for modeling dislocations in problems with multiple arbitrary material interfaces, where Peach–Koehler forces were computed using the *J*–integral method. Ventura (2008) presented an X-FEM technique based on mapping the enrichment functions onto polynomials with applications to cracks and dislocations that allows one to perform the standard Gauss quadrature in the enriched elements. Gracie, Oswald, and Belytschko (2008a) improved the X-FEM solution based on enrichments in the neighborhood of a dislocation core by applying a discontinuous jump enrichment and a singular enrichment based on closed-form, infinite-domain solutions for the dislocation core. Ventura, Gracie, and Belytschko (2009) performed a similar approach by employing a singular enrichment based on closed-form solutions for an edge dislocation around the core, and step function enrichment for the remainder of the edge dislocation. Oswald *et al*. (2009) presented further developments in the X-FEM modeling of dislocations with emphasis on problems with complex geometry, such as carbon nanotubes and thin films. Gracie and Belytschko (2009) presented a coupled atomistic–continuum method for modeling dislocations and cracks that combines the X-FEM with the bridging domain method. Oswald *et al*. (2011) presented a higher-order X-FEM method for modeling dislocations that was applicable to complex geometries, interfaces with lattice mismatch strains, and both anisotropic and spatially non-uniform material properties.

The X-FEM has been also implemented for modeling of elasto-plasticity problems in solid mechanics. Dolbow and Devan (2004) presented a geometrically nonlinear enhanced assumed strain method with discontinuous enrichment of the displacement field that exhibits locking-free response in the incompressible limit. Shamloo, Azami, and Khoei (2005) presented a computational technique based on the X-FEM in plasticity behavior of pressure-sensitive materials, where a cap plasticity model was employed in numerical simulation of powder die-pressing. Khoei, Shamloo, and Azami (2006b) employed double-surface cap plasticity together with a frictional contact model within the X-FEM framework to capture the response of initially loose metal powders to complex deformation histories encountered in the compaction process of powder. Anahid and Khoei (2008) developed an X-FEM Lagrangian formulation for modeling arbitrary interfaces in large plasticity deformations. In a later work, Khoei, Biabanaki, and Anahid (2008d) extended the X-FEM into three-dimensional Lagrangian formulation to model the arbitrary interfaces in large elasto-plastic deformations of three-dimensional solid mechanics problems. Khoei, Anahid, and Shahim (2008a) developed a computational technique based on an extended arbitrary Lagrangian–Eulerian FEM for large deformation in solid mechanic problems, in which an arbitrary Lagrangian–Eulerian technique was employed to capture the advantages of both Lagrangian and Eulerian methods and alleviate the drawbacks of the mesh distortion in Lagrangian formulation. Khoei, Biabanaki, and Anahid (2009a) presented a Lagrangian–X-FEM method for large two- and three-dimensional deformation of plasticity and contact problems. Anahid and Khoei (2010) presented an enriched arbitrary Lagrangian–Eulerian FE method for modeling of moving boundaries in large plasticity deformations. Xu, Lee, and Tan (2012) presented a two-dimensional co-rotational beam element within the X-FEM formulation for simulation of pin connections and plastic hinges by enriching both the rotation and the deflection approximations to capture the non-smoothness in both small and large deformations. Bonfils, Chevaugeon, and Moës (2012) introduced a method for treating the volumetric inequality constraint in a continuum media with a coupled X-FEM/level-set strategy to represent moving interfaces in a domain.

1.3.6 Shear Band Localization

The shear band localization refers to the process in which a smoothly varying deformation field suddenly gives rise to one with deformations that are highly localized in narrow bands; this phenomenon has been observed in many engineering materials including metals, concrete, soils, and rocks. Due to an excessive large aspect ratio of shear bands, numerical modeling of such problems is always challenging. Mariano and Stazi (2004) presented the interaction between a macro-crack and a population of micro-cracks using the X-FEM to a multi-field model of micro-cracked bodies by evaluating the strain localization effects around the tip of the macroscopic crack. Samaniego and Belytschko (2005) proposed the X-FEM in shear band localization, in which the transition from continuum to discontinuum was governed by the loss of hyperbolicity and the post-localization behavior of material was modeled by means of a traction–separation law obtained from a continuum J_2 flow plasticity model. Areias and Belytschko (2006b) presented a methodology based on the X-FEM to model the shear band evolution in quasi-static regime, where the FE polynomial displacement field was enriched with a fine scale function to model the high displacement gradient in a shear band. Song *et al*. (2006a) presented a computational algorithm based on the X-FEM for modeling of arbitrary dynamic crack and shear band propagation, where the discontinuity was modeled by superposed elements and phantom nodes. Réthoré, Hild, and Roux (2007c) proposed the X-FEM in conjunction with digital image correlation to capture experimentally the tangential discontinuities in shear band localization. Khoei and Karimi (2008) employed an enriched FEM technique within a higher-order continuum model based on the Cosserat continuum theory to simulate shear band localization. Sandborn and Prevost (2011) presented a numerical strategy for detecting instabilities in elasto-plastic solids by inserting a discontinuity at these instabilities, and prescribing a frictional behavior along the discontinuity. Daneshyar and Mohammadi (2013) presented a method for modeling the shear band localization with strong tangential discontinuity by means of cohesive surfaces within the X-FEM, in which a rate-independent non-associated plasticity model was incorporated along the strong discontinuity to capture the highly localized regions.

1.3.7 Fluid–Structure Interaction

Fluid–structure interaction is of great relevance in many fields of engineering as well as in the applied sciences. Hence, the evaluation of fluid–structure interaction effects and the investigation of governing physical phenomena associated with coupled systems are always challenges in problems arising in the fields of aero- and hydro-elasticity, life sciences, or bio-engineering. Legay, Chessa, and Belytschko (2006) presented an X-FEM for fluid–structure interaction with the interface and free surfaces defined by level sets, in which the fluid was treated by an Eulerian mesh and the solid by a Lagrangian mesh, and the Lagrange multiplier method and penalty method were used to couple the fluid and structure. Wang *et al*. (2008) proposed an implementation of the fluid-structure interaction using the immersed/fictitious element method for compressible fluids, in which the fictitious fluid was treated by a Lagrangian description, and for thin elements an enrichment was added in the fluid regions around the structural elements. Gerstenberger and Wall (2008a) presented a fixed grid fluid–structure interaction scheme based on the X-FEM that can be applied to the interaction of most general structures with incompressible flow, in which the extended Eulerian fluid field and the Lagrangian structural field were partitioned and iteratively coupled using a Lagrange multiplier technique for non-matching grids. In a later work, Gerstenberger and Wall (2008b) proposed two enhancements of fixed-grid methods based on a local adaptivity and a hybrid method that combines ideas from fixed-grid methods and arbitrary Lagrangian–Eulerian formulations to improve the solutions to complex fluid–structure interaction problems. Zilian and Legay (2008) introduced a weighted residual-based approach for the enriched space–time FE simulation of the interaction of fluid flow and thin flexible structures to model the coupled systems involving large structural motion and deformation of multiple-flow-immersed solid objects. Massimi, Tezaur, and Farhat (2008) proposed a discontinuous enrichment method to three-dimensional evanescent wave problems by enriching the elements with free-space solutions of evanescent wave problems to obtain the required accuracy at practical mesh resolution for fluid–fluid and fluid–solid applications.

For large deformations of solids in fluid–structure interaction, or fluid–solid interaction problems, the Eulerian description for the fluid and Lagrangian description for the solid are often preferable. Wang and Belytschko (2009) developed a discontinuous-Galerkin method for large deformation fluid–structure interaction problems, where the fluid–structure interface was arbitrarily aligned relative to the fluid grid. Mayer, Gerstenberger, and Wall (2009) presented a three-dimensional higher-order X-FEM method for modeling moving interfaces in fluid–structure interaction problems, which provides a method for localization of a higher order interface FE mesh in an underlying three-dimensional higher order FE mesh. Mayer *et al*. (2010) proposed a three-dimensional numerical technique to tackle the finite deformation contact of flexible solids embedded in fluid flows based on a combined X-FEM–fluid–structure-contact interaction method to compute the contact of arbitrarily moving and deforming structures embedded in an arbitrary flow field. Zilian and Netuzhylov (2010) presented a mixed-hybrid velocity-based formulation of both fluid and structure discretized by a stabilized time-discontinuous space–time FEM for analysis of thin-walled structures immersed in generalized Newtonian fluids. Wall *et al*. (2010) presented an overview on recent research activities on a fixed grid fluid-structure interaction scheme based on an X-FEM for moving interfaces on fixed Eulerian fluid grids that can be applied to the interaction of most general structures with incompressible flow. Shahmiri, Gerstenberger, and Wolfgang (2011) presented a FE embedding mesh technique based on a non-overlapping domain decomposition method to embed arbitrary fluid mesh patches into an unstructured background fluid grid. Legay (2013) employed the X-FEM for structural-acoustic problems involving immersed structures at arbitrary positions, in which the immersed structures were supposed to be thin shells and were localized in the fluid domain by a signed distance level-set.

1.3.8 Fluid Flow in Fractured Porous Media

Flow of fluids in deformable porous media has been a topic of attention in engineering science, and has been crucial for understanding and predicting the physical behavior of many problems of interest in geotechnical and petroleum engineering (Figure 1.6). The first implementation of an enriched FEM was presented by de Borst, Réthoré, and Abellan (2006) based on the PU property of FE shape functions to analyze the stress evolution and fluid flow in two-phase fluid-saturated media for a biaxial plane-strain specimen with a propagating discontinuity, for example, a crack or a shear band. Réthoré, de Borst, and Abellan (2007a) proposed this formulation for modeling dynamic shear-band propagation in a fluid-saturated medium. In a later work, Réthoré, de Borst, and Abellan (2007b) presented a two-scale approach by exploiting the PU property of the FE shape functions for fluid flow within the fractured porous media, in which the flow in the cavity of a fracture was modeled as a viscous fluid at the microscopic level. Réthoré, de Borst, and Abellan (2008) extended their model to an unsaturated porous medium by developing a two-scale model for fluid flow within a deforming unsaturated and progressively fracturing porous medium, where the flow in the cohesive crack was modeled using Darcy’s Law that takes into account changes in the permeability due to the progressive damage evolution inside the cohesive zone. Lecampion (2009) presented an X-FEM formulation for the solution of hydraulic fracture problems by introducing special tip functions encapsulating tip asymptotic functions typically encountered in hydraulic fractures. QingWen, YuWen, and TianTang (2009) proposed the X-FEM for numerical modeling of hydraulic fracturing in a gravity dam. Gracie and Craig (2010) employed the X-FEM for predicting the steady-state leakage from layered sedimentary aquifer systems perforated by abandoned wells, where the leakage of fluid between aquifers occurred through the aquitards and abandoned wells. Huang *et al*. (2011) proposed an enrichment scheme to model fractures and conduits in porous media flow problems that was able to capture effects of local heterogeneity introduced by subsurface features on the pressure solution.

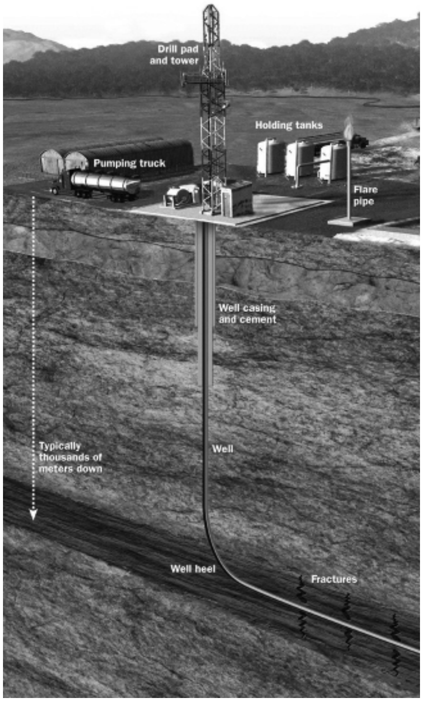

Figure 1.6 Hydraulic fracturing is the process of drilling a bore hole into a geological formation that contains *microscopic pockets* of natural gas and oil, and then fracturing the formation so that the gas and oil can escape into the bore hole and be drawn to the surface. It is a technique in which typically water is mixed with sand and chemicals, and the mixture is injected at high pressure into a wellbore to create small fractures, along which fluids such as gas, petroleum, uranium-bearing solution, and brine water may migrate to the well

Modeling of hydraulic fracture propagation has attracted considerable attention since the early 1950s because of its practical applications in a broad range of engineering areas. Hydraulic fracturing is the most commonly used stimulation technique of reservoirs, which is widely used in the petroleum industry to enhance the recovery of hydrocarbons, such as gas and oil, from underground reservoirs. Khoei and Haghighat (2011) presented an enriched FEM based on an updated Lagrangian framework combined with a generalized Newmark scheme for the time domain discretization in order to numerically simulate the saturated porous media with arbitrary material interfaces. In a later work, Khoei, Moallemi, and Haghighat (2012) presented an X-FEM technique for thermo-hydro-mechanical modeling of impermeable discontinuities in saturated porous media, where the displacement field was enriched using the Heaviside and crack tip asymptotic functions, and the pressure and temperature fields were enriched by the Heaviside and appropriate asymptotic functions. Watanabe *et al.* (2012) developed a method based on local enrichment approximations by employing the lower-dimensional interface elements to present preexisting fractures in rock material, focusing on FE analysis of coupled hydromechanical problems in discrete fracture-porous media systems. Gordeliy and Peirce (2013) presented an X-FEM method that was coupled with the elasto-hydrodynamic equations to solve elastic crack propagation due to hydraulic fracturing in an elastic medium. Mohammadnejad and Khoei (2013a) presented an X-FEM for the fully coupled analysis of deforming porous media containing weak discontinuities that interact with the flow of two immiscible, compressible wetting and non-wetting pore fluids. In a later work, Mohammadnejad and Khoei (2013b) developed a numerical model based on an extended FEM to simulate the flow of wetting and non-wetting pore fluids in progressively fracturing, partially saturated porous media, in which the mechanical and the mass transfer coupling between the crack and the porous medium surrounding the crack were taken into account. Mohammadnejad and Khoei (2013c) proposed a numerical model for the fully coupled hydro-mechanical analysis of deformable, progressively fracturing porous media using the X-FEM in conjunction with the cohesive crack model. Khoei and Vahab (2014) presented a coupled hydromechanical formulation in the framework of X-FEM method for deformable porous media subjected to crack interfaces to simulate the dynamic hydromechanical behavior of fractured porous media with opening and closing modes. Khoei *et al.* (2014b) proposed the X-FEM for mixed-mode crack growth simulation in saturated porous media using two computational algorithms to compute the interfacial forces of fluid pressure exerted on the fracture faces based on a "partitioned solution algorithm" and a "time-dependent constant pressure algorithm".

1.3.9 Fluid Flow and Fluid Mechanics Problems

Dynamic flows encompass a wide array of flows including drops, jets, plumes, thin films, and bubbles that exhibit such diverse phenomena as capillary instabilities, Homman flow, fluid bridges, particulate and bubble flows, and so on. Moreover, the interaction of different fluids is frequently observed in real problems, its numerical simulation has a long tradition and still remains an active research field. The first implementation of X-FEM in fluid flow problems was performed by Wagner *et al.* (2001) for simulation of rigid particles in Stokes flow, in which the surfaces of moving particles were not conformed to FE boundaries, and the near field form of the fluid flow about each particle was built into the FE basis using a PU enrichment. Chessa and Belytschko (2003a) presented an X-FEM method for axisymmetric two-phase flow problems with surface tension, in which the interface can move arbitrarily through the mesh and the discontinuity in the velocity gradient at the interface was modeled by a local PU. In a later work, Chessa and Belytschko (2003b) employed the X-FEM with arbitrary discontinuous gradients to two-phase immiscible flow problems, where the phase interfaces were tracked by level set functions using the same FE mesh and were updated via a stabilized conservation law. Groß and Reusken (2007) presented an extended pressure FEM for incompressible two-phase flows in which a localized force at the interface describes the effect of surface tension. Dolbow *et al.* (2008) employed the patterned-interface-reconstruction algorithm in conjunction with the X-FEM for general multi-material problems that exhibits the advantages of local, element-based interface representation, in particular the ability to enforce strict volume conservation. Fries (2009) presented an intrinsic X-FEM model for simulation of incompressible two-fluid flows, where the jumps and kinks along the interfaces in the velocity and pressure fields were modeled using an enriched PUM. Van der Bos and Gravemeier (2009) presented an enriched FEM model for simulation of the premixed combustion based on the *G*-function approach, in which a level-set or *G*–function was used to define the flame interface separating burned gas from unburned gas. Esser, Grande, and Reusken (2010) combined the X-FEM with a level set interface capturing technique for physically realistic levitated droplet problem, and applied the technique to three-dimensional simulation of a physically realistic two-phase flow problem. Abbas, Alizada, and Fries (2010) proposed an enriched FEM model for high-gradient solutions in convection-dominated problems that enables highly accurate approximations of convection-dominated problems without stabilization or mesh refinement. Sauerland and Fries (2011) investigated different enrichment schemes and time-integration schemes within the X-FEM for immiscible two-phase and free-surface flows, where the jumps and kinks in the velocity and pressure fields were captured with no restrictions on the interface topology. Cheng and Fries (2012) developed a *h*-version of the X-FEM based on a multi-level adaptive mesh refinement realized via hanging nodes on one-irregular non-conforming meshes for simulation of two-fluid incompressible flow in two and three dimensions. Liao and Zhuang (2012) presented a consistent projection-based streamline upwind/pressure stabilizing Petrov–Galerkin X-FEM to model incompressible immiscible two-phase flows, in which projections of convection and pressure gradient terms were constructed and incorporated into the stabilization formulation. Sauerland and Fries (2013) studied the issue of ill-conditioning in the X-FEM for two-phase flows, concentrating on the stable X-FEM and the application of iterative solvers.

Particulate flows arise in a wide class of research areas and industrial processes, such as fluidized suspensions, slurry transport, materials separation, rate of mixing enhancement, and so on. In fluid-rigid body problems, the field variables such as pressure and stress are discontinuous over the interface since no flow occurs inside the rigid body. Hence, modeling of the fluid flow around a stationary rigid body is crucial. Choi, Hulsen, and Meijer (2010) developed an X-FEM formulation for the direct numerical simulation of the flow of viscoelastic fluids with suspended particles, in which an arbitrary Lagrangian–Eulerian scheme was devised for moving particle problems that defines the mapping of field variables at previous time levels onto the computational mesh at the current time level. In a later work, Choi, Hulsen, and Meijer (2012) used their original formulation for simulation of the flow of a viscoelastic fluid around a stationary cylinder based on the X-FEM. Sarhangi Fard, Hulsen, and Anderson (2012a) employed the X-FEM to simulate the three-dimensional Stokes flow in complex geometries with internal moving parts and narrow gaps, in which the kinematics of the internal rigid body was enforced using a constraint Lagrangian multiplier. Sarhangi Fard *et al*. (2012b) performed a non-Newtonian viscous flow analysis in complex geometries with internal moving parts and narrow gaps based on a non-conforming mesh refinement approach and the X-FEM method. Court, Fournié, and Lozinski (2014) proposed a fictitious domain approach inspired by the X-FEM for the Stokes problem, in which the interface between the fluid and the structure was localized by a level-set function, and the Dirichlet boundary conditions were taken into account using the Lagrange multiplier.

1.3.10 Phase Transition and Solidification

Numerical modeling of melting/solidification or dissolution/precipitation is significant in the evaluation of operating conditions and designs of equipment configuration in these technologies. However, modeling the moving solid–liquid interface in such phase transition is a challenging problem. Phase transition represent diffuse material interfaces across which several fields may exhibit sharp gradients, and even discontinuities, as the interface thickness vanishes and becomes "sharp". Hence, the evaluation and enforcement of interfacial conditions is an important consideration for generating accurate approximations to solutions of boundary value problems involving the evolution of sharp interfaces. Merle and Dolbow (2002) proposed the X-FEM for modeling thermal problems with moving heat sources and phase boundaries to capture the highly localized, transient solution in the vicinity of a heat source or material interface. Chessa, Smolinski, and Belytschko (2002) presented an enriched FEM for the numerical solution of phase change problems, where the phase interface was evolved by the use of artificial heat capacity technique that was described by a level set function. Ji, Chopp, and Dolbow (2002) presented a hybrid numerical method based on the X-FEM for modeling the evolution of sharp phase interfaces on fixed grids with reference to solidification problems, where the temperature field was evolved according to the classical heat conduction in two sub-domains separated by a moving freezing front. Ji and Dolbow (2004) considered a problem stemming from phase transitions in stimulus-responsive hydrogels, wherein a sharp interface separates swelled and collapsed phases, and presented that as the reciprocal interfacial mobility vanishes, it plays the role of a penalty parameter enforcing a pure Dirichlet constraint, eventually triggering oscillations in the interfacial velocity. Zabaras, Ganapathysubramanian, and Tan (2006) studied dendritic solidification of pure materials from an under-cooled melt by using a coupled X-FEM–LSM for modeling the thermal problem and a volume-averaged stabilized formulation for modeling fluid flow.

In the solidification process, complications are described as isothermal phase change problems that are characterized mainly by the material melting temperature and its latent heat, in which an inherent difficulty with these problems is the discontinuity in the temperature gradient at the solidification front. Uchibori and Ohshima (2012) presented a numerical analysis based on a moving boundary X-FEM technique for modeling melting/solidification problems to simulate the discontinuous temperature gradient in the element crossed by the solid–liquid interface. Zhou and Qi (2010) proposed the X-FEM within the ABAQUS^{TM} software to simulate the discontinuous interface in the liquid–solid forming process, where the discontinuous interface in the liquid–solid forming processes was handled using the LSM.

Duddu *et al*. (2011) presented a sharp-interface numerical formulation based on the X-FEM–LSM using an Eulerian description for modeling diffusional evolution of precipitates produced by phase transformations in elastic media. Skrzypczak (2012) presented a mathematical model for modeling sharp interfaces in solidification process of pure metals, where the interface motion was described by the level set function. Ghoneim, Hunedy, and Ojo (2013) proposed a numerical simulation based on an interface-enriched extended FE–LSM to study the solute-induced melting of additive powder particles during transient liquid phase bonding. Cosimo, Fachinotti, and Cardona (2013) presented an enriched FE formulation for solving isothermal phase change problems, where the discontinuity in the temperature gradient at the solidification front was modeled by enriching the FE space through a function whose definition includes a gradient discontinuity.

1.3.11 Thermal and Thermo-Mechanical Problems

Most of the critical engineering components are generally exposed to both thermal and mechanical loading during their service life, such as combustion chambers of internal combustion engines, nuclear reactor components, spacecraft, blade casing in thermal power plants, and so on. There are a number of thermal and thermo-mechanical problems in the engineering and materials science communities that are characterized by the presence of a highly localized moving heat source. Michlik and Berndt (2006) proposed a thermo-mechanical X-FEM analysis for modeling thermal barrier coatings in order to provide useful versatility in prediction of effective thermal conductivity and the in-plane Young's modulus of multi-layered thermal barrier coatings. Duflot (2008) employed the X-FEM for the analysis of steady-state thermo-elastic problems in cracked structures, where both thermal and mechanical fields are enriched in order to represent the discontinuous temperature, heat flux, displacement, and traction across the isothermal crack surface. Fagerstorm and Larsson (2008) presented a thermo-mechanically coupled interface fracture formulation based on discontinuous representation for temperature and displacements fields pertinent to the FPZ into a cohesive zone. Zamani, Gracie, and Eslami (2010) proposed the X-FEM to predict the SIFs for thermo-elastic crack problems using higher-order terms of the thermo-elastic asymptotic crack-tip fields for the enrichment of temperature and displacement fields. In a later work, Zamani and Eslami (2010) extended their model in simulation of problems with stationary cracks under dynamic thermo-mechanical loading to model the effect of mechanical and thermal shocks on a body with stationary cracks. Lee, Yang, and Maute (2011) presented an X-FEM method for the analysis of heat conduction at submicron scales of geometrically complex nano-structured materials. Yvonnet *et al*. (2011) proposed a computational procedure based on a coupled X-FEM–LSM for modeling the Kapitza thermal resistance at an arbitrarily shaped interface. Fan *et al*. (2012) employed the X-FEM to investigate the effect of thermally grown oxide on multiple surface cracking behavior in an air plasma sprayed thermal barrier coating system. Hosseini, Bayesteh, and Mohammadi (2013) proposed a computational method based on the X-FEM for fracture analysis of isotropic and orthotropic functionally graded materials under mechanical and steady-state thermal loadings. Yu and Gong (2013) employed the X-FEM for modeling the temperature field in heterogeneous materials, where the temperature field was enriched by incorporating the level-set enrichment function for the element containing material interfaces. Macri and Littlefield (2013) presented a multi-scale technique based on an enriched partition of unity approach that incorporates the thermal effects occurring on the micro-structure into the global model for simulation of heterogeneous materials undergoing substantial thermal stresses.

1.3.12 Plates and Shells

Plates and shells are widely used in thin-walled structures such as aircraft fuselages subjected to bending and pressure loadings (Figure 1.7). Through-the-thickness cracks may develop when these structures are subjected to cyclic loading, and the determination of mixed-mode SIFs is critical to the modeling of

Figure 1.7 The Impact Dynamics Research Facility is used by NASA to conduct crash testing of full-scale aircraft under controlled conditions. The aircraft are swung by cables from an A-frame structure that is approximately 400 ft long and 230 ft high. (Source: NASA; http://commons.wikimedia.org/wiki/File:Impact_Landing_Dynamics_Facility_Crash_Test_-_GPN-2000-001907.jpg)

fatigue crack propagation. Despite the practical importance, relatively little research has focused on developing robust numerical methods to determine fracture parameters and simulate crack growth in thin plates. The first implementation of an X-FEM for modeling cracks and crack growth in plates was performed by Dolbow, Moës, and Belytschko (2000b) in the context of the Mindlin–Reissner plate theory. Liang *et al*. (2003) developed an X-FEM model to evolve patterns of multiple cracks in a brittle thin film bonded to an elastic substrate, in which the film was susceptible to subcritical cracking, obeying a kinetic law that relates the velocity of each crack to its energy release rate. Huang *et al*. (2003a) employed the X-FEM to compute the steady-state energy release rate of channeling cracks in thin films, where the driving force for channeling cracks was obtained as a function of elastic mismatch, crack spacing, and the thickness ratio between the substrate and the film. Bachene, Tiberkak, and Rechak (2009a) and Bachene *et al*. (2009b) proposed the X-FEM to model vibrations of cracked plates based on Mindlin plate theory, where the effects of shear deformation and rotatory inertia were included in the development of the model. Lasry *et al*. (2010) presented an X-FEM for simulation of thin cracked plates under bending loads based on the Kirchhoff–Love theory that is well suited to very thin plates commonly used, for instance, in aircraft structures. Fan *et al*. (2011) investigated multiple cracking behavior in a thin elastic film bonded to a thick elastic substrate using an X-FEM method, in which the SIFs were obtained using a periodic FEM for the cracked film/substrate system. Natarajan *et al*. (2011) applied the X-FEM to model the linear free flexural vibrations of cracked functionally graded material plates based on the first-order shear deformation theory, and performed a parametric study on the influence of gradient index, crack location, crack length, crack orientation, and thickness on the natural frequencies of FGM plate. Lasry, Renard, and Salaün (2012) proposed the X-FEM for modeling bending plates with through-the-thickness cracks based on the Kirchhoff–Love plate model, where the reduced Hsieh–Clough–Tocher triangles and reduced Fraejis de Veubeke–Sanders quadrilaterals were used for the numerical discretization. Xu, Lee, and Tan (2013a, b) presented the X-FEM for modeling of a plate element with high gradient zones in both rotation and deflection displacement fields in the vicinity of a yield line in a plate structure with elasto-plastic material.

A large number of industrial and engineering structures, such as aircraft fuselages, storage tanks, ship hulls, and pipes, are made of plates and shells. Existence of defects and cracks in such structures may lead to substantial decrease in load capacity, fatigue crack propagation, leak before breakage, and even structural collapse. The out-of-plane effects in large thin walled structures frequently require the use of a robust dedicated method in order to properly capture the details of the three-dimensional crack tip conditions. Areias and Belytschko (2005b) developed a numerical procedure for the analysis of arbitrary crack propagation in sandwich shells, in which a new enrichment of the rotation field was proposed that satisfies the director inextensibility condition. Wyart *et al*. (2007, 2009) employed the sub-structured FE/X-FEM approach to compute the SIFs in large aircraft thin walled structures containing cracks. Bayesteh and Mohammadi (2011) studied the effect of crack tip enrichment functions in the X-FEM analysis of shells by evaluating fracture mechanics parameters such as the SIF, crack-tip opening displacement, and crack-tip opening angle. Larsson, Mediavilla, and Fagerström (2011) presented a formulation based on the cohesive zone concept applied to a kinematically consistent shell model enhanced with an X-FEM based discontinuous kinematical representation. Liu, Zhang, and Zheng (2012) investigated the plastic collapse and crack behavior of steel pressure vessels and piping using the X-FEM to provide a fundamental support for safety evaluation and life prediction of pressurized structures.

1.3.13 Contact Problems

Numerical modeling of engineering contact problems is one of the most difficult and demanding tasks in computational mechanics. Frictional contact can be observed in many problems; such as: crack propagation, metal forming operation, drilling pile, and so on (Figure 1.8). Hence, much attention must be given to the numerical research aspects of this complex problem. The first implementation of an extended finite element framework in a contact problem was performed by Dolbow, Moës, and Belytschko (2001) for modeling of crack growth with frictional contact on the crack faces. Khoei and Nikbakht (2006, 2007) developed an enriched FEM for modeling of frictional contact problems based on the penalty method in order to simulate the frictional behavior of contact between two bodies. Khoei, Shamloo, and Azami (2006b) employed a cap plasticity model in conjunction with a frictional contact model within the X-FEM framework to simulate the compaction of powder die-pressing. Vitali and Benson (2006) presented an X-FEM formulation for contact in multi-material arbitrary Lagrangian–Eulerian calculations. Liu and Borja (2008) presented an incremental quasi-static contact algorithm for path-dependent frictional crack propagation in the framework of the X-FEM, where the contact constraint was embedded within a localized element by penalty method. Giner *et al*. (2008b) performed a numerical analysis of complete sliding contact and its associated singularity based on the partition of unity FEM, in which the enriched functions were derived from the analytical expression of the asymptotic displacement field in the vicinity of the contact corner. Khoei, Biabanaki, and Anahid (2009a) presented a Lagrangian–X-FEM method for three-dimensional modeling of large-plasticity deformations and contact problems. Nistor *et al*. (2009) proposed an approach to couple the X-FEM with the Lagrangian large sliding frictionless contact algorithm, in which a hybrid X-FEM contact element was introduced in the framework of a contact search algorithm allowing for an update of contacting surfaces pairing.

The contact problem generally suffers from a numerical instability similar to that encountered in incompressible elasticity, in which the normal contact pressure exhibits spurious oscillation. This oscillation does not go away with mesh refinement, and in some cases it even gets worse as the mesh is refined. There are several stabilized approaches introduced in the literature to address the problem of contact pressure oscillation. Béchet, Moës, and Wohlmuth (2009a) introduced a stable Lagrange multiplier space to impose stiff interface conditions within the context of the X-FEM. Becker, Burman, and Hansbo (2009) presented a Nitsche X-FEM for incompressible elasticity with discontinuous elasticity modulus that satisfies the inf–sup condition for stabilized methods related to the non-mixed constant-strain method. Zilian and Fries (2009) proposed an approach for the imposition of constraints along moving or fixed immersed interfaces in the context of the X-FEM, in which the use of Lagrange multipliers or penalty

Figure 1.8 A crash test is a form of destructive testing usually performed in order to ensure safe design standards in crash worthiness and crash compatibility for various modes of transportation or related systems and components. There are various types of crash test, such as frontal-impact test (in the photograph), offset crash test, side-impact crash test, and roll-over crash test. (Source: StaraBlazkova at the Czech language Wikipedia [GFDL (www.gnu.org/copyleft/fdl.html) or CC-BY-SA-3.0 (http://creativecommons.org/licenses/by-sa/3.0/)], from Wikimedia Commons; http://commons.wikimedia.org/wiki/File:Muzeum_of_trans_Gla_crash_test.jpg)

methods was circumvented by a localized mixed hybrid formulation of the model equations. Vitali and Benson (2009) presented an approach for classical kinetic friction laws based on a multi-material arbitrary Lagrangian–Eulerian formulation, where the nodal tangent relative velocities between two materials containing more than one material in their support were updated through nodal accelerations to account for kinetic friction effects. Liu and Borja (2009) presented an X-FEM approach for the simulation of slow-rate frictional faulting in geologic media incorporating bulk plasticity and variable friction, in which the bulk plasticity was included in the formulation to trigger fault rupture and its extension as plastic yielding was localized at the fault tip. In a later work, Liu and Borja (2010a) proposed an X-FEM framework for frictional crack problems to accommodate finite deformation and bulk inelasticity, in which conditions for contact and frictional sliding were imposed in the current configuration, and integration on all types of nonlinearity, including material, geometric, and contact search, were performed implicitly. Liu and Borja (2010b) also employed a stabilized FE formulation based on the polynomial pressure projection technique that was used successfully for the Stokes equation and for coupled solid-deformation-fluid diffusion using low-order mixed finite elements.

The enforcement of the contact condition is still a challenging aspect of the contact problem, whether it be in the context of classical nonlinear contact mechanics in which element sides are aligned to the contact faces, or in the framework of the assumed enhanced strain, or X-FEM in which contact faces are allowed to pass through and cut the interior of finite elements. Khoei and Mousavi (2010) presented a node-to-segment contact algorithm based on the X-FEM to model the large deformation/large sliding contact problem using the penalty approach. Pierrès, Baietto, and Gravouil (2010) developed an X-FEM for three-dimensional modeling of the cracked structure and the crack interface as two independent global and local problems characterized by different length scales and different behaviors, in which the interface was linked to the global problem in a weak sense in order to avoid instabilities in the contact solution. Mueller-Hoeppe, Wriggers, and Loehnert (2012) proposed an X-FEM formulation for crack face contact in terms of a penalty formulation for normal contact, in which the contact surface discretization was based on tetrahedrization according to the level set field. Laursen *et al*. (2012) proposed a mortar contact formulation for deformable-deformable contact that presents promising results for development of a numerical method allowing imposition of contact constraints with the X-FEM method. Khoei, Hirmand, and Vahab (2014a) presented an augmented Lagrangian formulation for modeling frictional discontinuities in the framework of X-FEM method, in which the nonlinear contact behavior was modeled based on an active set strategy to fulfill the Kuhn–Tucker inequalities within the iterations of Newton–Raphson procedure. Khoei and Vahab (2014) presented an approach to model the contact behavior in hydromechanical deformable porous media subjected to crack interfaces in the framework of an X-FEM.

1.3.14 Topology Optimization

Continuum structural optimization methods are of great interest in engineering applications for their benefits in helping engineers achieve better designs and save time. Given the significance of the shape representation in structural shape and topology optimization problems, the LSM can be efficiently implemented in structural optimization problems. Belytschko, Xiao, and Parimi (2003c) presented an approach for topology optimization based on an implicit function description of the surface of the design, in which the implicit function was constrained by upper and lower bounds, so that only a band of nodal variables needs to be considered in each step of the optimization. Sy and Renard (2010) proposed a method based on a fictitious domain approach for structural optimization with a coupling between shape and topological gradients, in which the fictitious domain approach was inspired by the X-FEM. Wei, Wang, and Xing (2010) employed the X-FEM in conjunction with the LSM to solve structural shape and topology optimization problems, and showed that the X-FEM leads to more accurate results compared to the FEM solution without increasing the mesh density and DOF. Edke and Chang (2011) performed a design process that supports the shape optimization of structural components under a two-dimensional mixed mode fracture for maximum service life, in which the process incorporates the X-FEM–LSM for crack growth modeling without remeshing. Kreissl and Maute (2012) presented an optimal design of fluidic devices subjected to incompressible flow at low Reynolds numbers using a LSM to describe the fluid–solid interface geometry, in which the fluid flow was modeled by the incompressible Navier–Stokes equations within the X-FEM. Li, Wang, and Wei (2012) proposed the X-FEM for LSM structural optimization with a partition integral method, in which the X-FEM integral scheme without quadrature subcells and a higher order element X-FEM scheme were employed.

1.3.15 Piezoelectric and Magneto-Electroelastic Problems

Materials with a strong piezoelectric effect are generally used in various applications, as sensors, actuators, or transducers. These applications range from sub-millimeter length scales in microelectromechanical systems up to large scales in the design of smart electromechanical structures, such as wings in the aerospace industry. As for regular materials subjected to high mechanical stresses, the knowledge of fracture behavior for these smart materials is often crucial for design of parts under high electrical and mechanical loading. Béchet, Scherzer, and Kuna (2009b) proposed an application of the X-FEM to the analysis of fracture in piezoelectric materials by introducing new enrichment functions suitable for cracks in piezoelectric structures. Verhoosel, Remmers, and Gutiérrez (2010) employed a PU-based cohesive zone FEM to mimic crack nucleation and propagation in a piezoelectric continuum, in which a multi-scale framework was proposed to appropriately represent the influence of the microstructure on the response of a miniaturized component. Rochus *et al*. (2011) investigated various X-FEM techniques to solve the electrostatic problem when the electrostatic domain was bounded by a conducting material in order to accurately evaluate the electrostatic forces acting on the devices. Rojas-Díaz *et al*. (2011) studied the static fracture analysis of two-dimensional linear magneto-electroelastic solids within the X-FEM framework considering the media possessing fully coupled piezoelectric, piezomagnetic, and magneto-electric effects. Bhargava and Sharma (2011) proposed the X-FEM in a two-dimensional finite piezoelectric media weakened by a crack, where the four-fold standard enrichment functions were taken in conjugation with the interaction integral to evaluate the intensity factors. Bhargava and Sharma (2012) introduced a generalized set of crack tip enrichment functions by redefining the existing basis functions, and presented the efficiency of these enrichment functions by validation of the Griffith crack in an infinite domain with the energy norm and the convergence of SIFs. Nguyen-Vinh *et al*. (2012) presented an X-FEM formulation for dynamic fracture of piezoelectric materials that was applied to mode I and mixed mode fractures for quasi-steady cracks. Kästner *et al*. (2013) presented an application of bilinear and biquadratic X-FEM formulations for modeling weak discontinuities in magnetic and coupled magneto-mechanical boundary value problems, where the level set representation of curved interfaces was used to resolve the location of curved interfaces and the discontinuous physical behavior.

1.3.16 Multi-Scale Modeling

Simulations with atomistic resolution of dislocation cores and crack fronts are critical to a more fundamental understanding of the physics of plasticity and failure. However, even the treatment of submicron cracks and dislocation loops by atomistic methods is generally not feasible because of the large number of atoms required. While concurrent models can deal with defects of moderate size, in the order of hundreds of Angstroms, this does not suffice for many dislocation and crack problems of interest. A key need is for methods that can apply atomistic models where needed and apply continuum models to the remainder of the domain with the capability to model the discontinuities associated with cracks and dislocations. Guidault *et al.* (2008) proposed a multi-scale technique for crack propagation using two strategies; a micro–macro approach using a domain decomposition method to account for the efficient treatment of the global and local effects due to the crack, and a local enrichment method on the basis of an X-FEM technique to describe the geometry of the crack independently of the mesh. Gracie and Belytschko (2009) presented a coupled atomistic–continuum method for modeling dislocations and cracks that combines the X-FEM with the bridging domain method, where the multi-scale strategy was used to model the crack surfaces and slip planes in the continuum, and the bridging domain method was employed to link the atomistic model with the continuum. Aubertin, Réthoré, and de Borst (2010) presented a multi-scale method that couples a molecular dynamics approach for describing fracture at the crack-tip with an X-FEM for discretizing the remainder of the domain to simulate dynamic fracture in an efficient manner on basis of elementary physical principles. Kästner, Haasemann, and Ulbricht (2011) proposed a multi-scale simulation of fiber-reinforced polymers, where the heterogeneous material structure in a representative volume element was modeled by the X-FEM. Loehnert, Prange, and Wriggers (2012) presented a discretization error controlled adaptive multi-scale technique for an accurate simulation of micro-structural effects within a macroscopic component using the corrected X-FEM, in which the incorporation of micro-structural features such as micro-cracks was achieved by means of the multi-scale projection method. Macri and Littlefield (2013) presented a multi-scale technique for modeling heterogeneous materials undergoing substantial thermal stresses based on an enriched PU approach that incorporates the thermal effects occurring on the microstructure into the global model.