1

Vectors

Some physical quantities are described by scalars, e.g., density, temperature, kinetic energy. These are pure numbers, although they do have dimensions. It would make no physical sense to add a density, with dimensions of mass divided by length cubed, to kinetic energy, with dimensions of mass times length squared divided by time squared.

Vectors are mathematical objects that are associated with both a magnitude, described by a number, and a direction. An important property of vectors is that they can be used to represent physical entities such as force, momentum, and displacement. Consequently, the meaning of the vector is (in a sense we will make more precise) independent of how it is represented. For example, if someone punches you in the nose, this is a physical action that could be described by a force vector. The physical action and its result (a sore nose) are independent of the particular coordinate system we use to represent the force vector. Hence, the meaning of the vector is not tied to any particular coordinate system or description. For this reason, we will introduce vectors in *coordinate-free* form and defer description in terms of particular coordinate systems.

A vector $\mathbf{u}$ can be represented as a directed line segment, as shown in Figure 1.1. The length of the vector is its magnitude, and denoted by u or by $|\mathbf{u}|$. Multiplying a vector by a positive scalar α changes the length of the vector but not its orientation. If $\alpha > 1$, the vector $\alpha\mathbf{u}$ is longer than $\mathbf{u}$; if $\alpha < 1$, $\alpha\mathbf{u}$ is shorter than $\mathbf{u}$. If α is negative, the orientation of the vector is reversed. It is always possible to form a vector of unit magnitude by choosing $\alpha = u^{-1}$.

The addition of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written as

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \tag{1.1}$$

Although the same symbol is used as for ordinary addition, the meaning here is different. Vectors add according to the parallelogram law shown in Figure 1.2. If the "tails" of the vectors (the ends without arrows) are placed at a point, the sum is the diagonal of the parallelogram with sides formed by the vectors. Alternatively the vectors can be added by placing the "tail" of one at the "head" of the other. The sum is then the vector directed from the free "tail" to the free "head." Implicit in both of these operations is the idea that we are dealing with "free"

Fundamentals of Continuum Mechanics, First Edition. John W. Rudnicki.

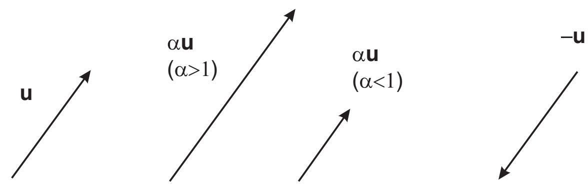

Figure 1.1 Multiplication of a vector by a scalar.

vectors. In order to add two vectors, they can be moved, keeping the length and orientation, so that the vectors can be connected head to tail. It is clear from the construction in Figure 1.2 that vector addition is commutative:

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

Note the importance of distinguishing vectors from scalars; without the bold face denoting vectors, equation (1.1) would be incorrect: the magnitude of **w** is not the sum of the magnitudes of **u** and **v**.

The parallelogram rule for vector addition follows from the nature of the physical quantities, e.g., velocity and force, that vectors represent. The rule for addition is an essential element of the definition of a vector that can distinguish them from other quantities that have both length and direction. For example, finite rotations about three orthogonal axes can be characterized by length and magnitude. Finite rotation cannot, however, be a vector because addition is not commutative. To see this, take a book with its front cover up and binding to the left. Looking down on the book, rotate it 90° counterclockwise. Now rotate the book 90° about a horizontal axis counterclockwise looking from the right. The binding should be on the bottom. Performing these two rotations in reverse order will orient the binding toward you.

Hoffmann (1975) relates the story of a tribe that thought spears were vectors because they had length and magnitude. To kill a deer to the northeast, they would throw two spears, one to the north and one to the east, depending on the resultant to strike the deer. Not surprisingly, there is no trace of this tribe, which only confirms the adage that “a little knowledge can be a dangerous thing.”

The procedure for vector subtraction follows from multiplication by a scalar and addition. To subtract **v** from **u**, first multiply **v** by -1, then add $-\mathbf{v}$ to **u**:

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

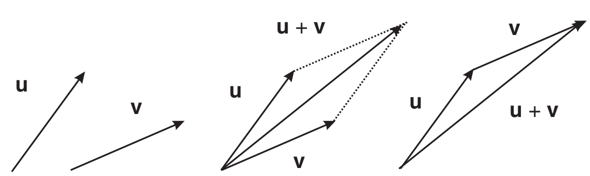

Figure 1.2 Addition of two vectors.

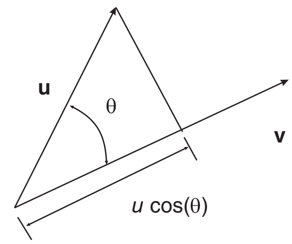

Figure 1.3 Scalar product.

There are two ways to multiply vectors: the scalar or dot product and the vector or cross product. The scalar product is given by

$$\mathbf{u} \cdot \mathbf{v} = uv\cos(\theta) \tag{1.2}$$

where θ is the angle between **u** and **v**. As indicated by the name, the result of this operation is a scalar. As shown in Figure 1.3, the scalar product is the magnitude of **v** multiplied by the projection of **u** onto **v**, or vice versa. The definition (1.2) combined with rules for vector addition and multiplication of a vector by a scalar yield the relation

$$(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2)\cdot\mathbf{v} = \alpha\mathbf{u}_1\cdot\mathbf{v}+\beta\mathbf{u}_2\cdot\mathbf{v}$$

where α and β are scalars and $\mathbf{u}_1$ and $\mathbf{u}_2$ are vectors.

If $\theta = \pi$ in (1.2) the two vectors are opposite in sense, i.e., their arrows point in opposite directions. If $\theta = \pi/2$ or $-\pi/2$, the scalar product is zero and the two vectors are *orthogonal*. Although the scalar product is zero neither **u** nor **v** is zero. If, however,

$$\mathbf{u} \cdot \mathbf{v} = 0 \tag{1.3}$$

for any vector **v** then $\mathbf{u} = 0$.

The other way to multiply vectors is the vector or cross product. The result is a vector

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} \tag{1.4}$$

The magnitude is $w = uv\sin(\theta)$, where θ is again the angle between **u** and **v**. As shown in Figure 1.4, the magnitude of the cross product is equal to the area of the parallelogram

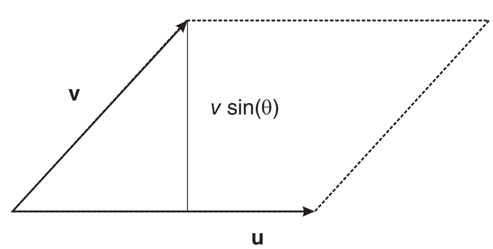

Figure 1.4 Magnitude of the vector or cross product.

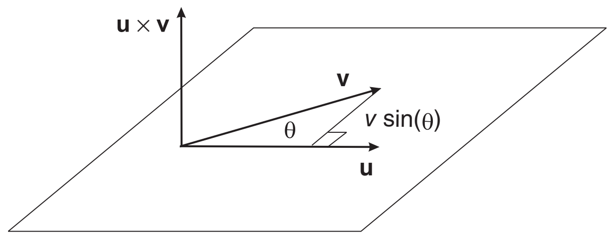

Figure 1.5 Direction of vector or cross product.

formed by **u** and **v**. As depicted in Figure 1.5, the direction of **w** is perpendicular to the plane formed by **u** and **v** and the sense is given by the *right hand rule*: If the fingers of the right hand are in the direction of **u** and then curled in the direction of **v**, the thumb of the right hand is in the direction of **w**. The three vectors **u**, **v**, and **w** are said to form a right-handed system.

The triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is equal to the volume of the parallelepiped formed by **u**, **v**, and **w** if they are right-handed and the negative of the volume if they are not (Figure 1.6). The parentheses in this expression may be omitted because it makes no sense if the dot product is taken first: the result is a scalar and the cross product is an operation between two vectors.

Now consider the triple vector product $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$. The vector $\mathbf{v} \times \mathbf{w}$ must be perpendicular to the plane containing **v** and **w**. Hence, the vector product of $\mathbf{v} \times \mathbf{w}$ with another vector **u** must result in a vector that is in the plane of **v** and **w**. Consequently, the result of this operation may be represented as

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \alpha \mathbf{v} + \beta \mathbf{w} \tag{1.5}$$

where α and β are scalars.

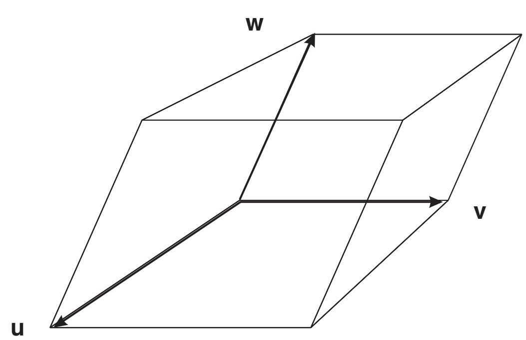

Figure 1.6 Triple scalar product.

1.1 Examples

1.1.1

Show that if the triple scalar product vanishes

$$\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = 0 \tag{1.6}$$

the three vectors are coplanar.

The *vector* product $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$. If the triple scalar product vanishes, then $\mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$ and hence is in the plane of $\mathbf{u}$ and $\mathbf{v}$. Consequently, $\mathbf{w}$ can be expressed as a linear combination of the other two, e.g., $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$ where α and β are scalars (as long as $\mathbf{u}$ and $\mathbf{v}$ are not collinear).

1.1.2

Show that if $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$ the triple scalar product of the three vectors vanishes.

Substituting $\mathbf{w}$ into (1.6) yields zero because the scalar products of $\mathbf{u} \times \mathbf{v}$ with $\mathbf{v}$ and with $\mathbf{u}$ are zero.

Exercises

1.1 Explain (in words and/or diagrams) why

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

and that

$$\mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{v} = 0$$

where $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

1.2 Explain (in words and/or diagrams) why

$$\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \times \mathbf{w} \cdot \mathbf{u} = \mathbf{w} \times \mathbf{u} \cdot \mathbf{v}$$

but that a minus sign is introduced if the order of any two vectors is reversed.

1.3 Explain why $\mathbf{u} \times (\mathbf{v} \times \mathbf{u})$ is orthogonal to $\mathbf{u}$ and show that α and β in (1.5) are then related by

$$\alpha v \cos(\theta) + \beta u = 0$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

1.4 Prove that if (1.3) is satisfied for *any* vector $\mathbf{v}$ then $\mathbf{u} = \mathbf{0}$.

1.5 Show that

(a) $$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = u^2 - v^2$$

(b) $$(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v}) = -2\mathbf{u}\times\mathbf{v}$$

1.6 Consider the plane triangle shown in Figure 1.7 with sides of lengths a, b, and c and angles α, β, and γ opposite sides a, b, and c, respectively. Use *coordinate-free vector methods* to prove (do not use geometry or, if you know it, index notation)

(a) law of cosines:

$$a^2 + b^2 - 2ab\cos(\gamma) = c^2$$

(b) law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

[Hint: Use scalar and vector products.]

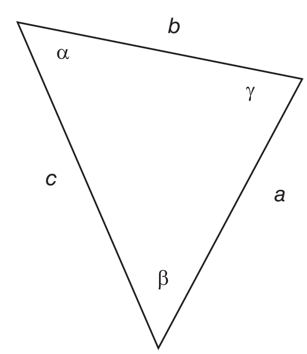

Figure 1.7 Diagram for Problem 1.6.

1.7 Let **a**, **b**, and **c** be non-coplanar vectors that form three edges of a tetrahedron (see Figure 1.8). Let $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ be the outward unit normals to the faces formed by each pair of vectors and let S_1, S_2, and S_3 be the corresponding areas. Show that the product of the unit vector normal to the fourth face and the area of the face is given by

$$\mathbf{n}S = -(\mathbf{n}_1 S_1 + \mathbf{n}_2 S_2 + \mathbf{n}_3 S_3)$$

1.8 Determine α and β in (1.5) (in terms of **u**, **v**, **w**, and scalar and cross products).

1.9 A line in direction **l** is defined by the vector relation

$$\mathbf{u} = \mathbf{a} + \mathbf{l}s$$

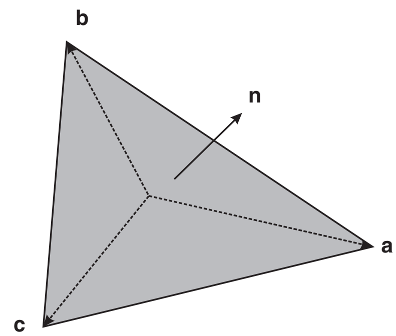

Figure 1.8 Diagram for Problem 1.7.

where $\mathbf{l}$ is a unit vector and s is a scalar parameter $-\infty < s < \infty$. Show that this will intersect a second line $\mathbf{v} = \mathbf{b} + \mathbf{m}s$, where $\mathbf{m}$ is a unit vector, if

$$\mathbf{a} \cdot (\mathbf{l} \times \mathbf{m}) = \mathbf{b} \cdot (\mathbf{l} \times \mathbf{m})$$

and determine their point of intersection, i.e. values of s for each line at the intersection.

1.10 Find the equation of the line that passes through two given points A and B located relative to a point O by two vectors $\mathbf{u}$ and $\mathbf{v}$ (Figure 1.9).

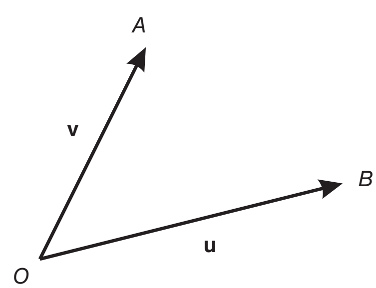

Figure 1.9 Diagram for Problem 1.10.

1.11 If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are not coplanar, then it is possible to find scalars α, β, and γ such that any arbitrary vector $\mathbf{z}$ can be expressed as $\mathbf{z} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$. Determine α, β, and γ (in terms of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$). What happens if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar?

1.12 Find an expression for a unit vector that lies in the intersection of the plane of $\mathbf{u}$ and $\mathbf{v}$ with the plane of $\mathbf{x}$ and $\mathbf{y}$.

Reference

Hoffmann B 1975 *About Vectors*. Dover.