Part I

Part Surfaces

The design, production and implementation of parts for products are common practice for most mechanical and manufacturing engineers. Any part can be understood as a solid bounded by a certain number of surfaces. Two kinds of bounding surfaces are recognized in this text: they can be either working surfaces of a part, or not working surfaces of the part. The consideration below is focused mostly on the geometry of working part surfaces.

All part surfaces are reproduced on a solid. Appropriate manufacturing methods are used for these purposes. Therefore, part surfaces are often referred to as *engineering surfaces*, in contrast to those surfaces which cannot be reproduced on a solid, and which can exist only virtually [30, 33, 34, 36, 45].

Interaction with the environment is the main purpose of all working part surfaces. Therefore, working part surfaces are also referred to as *dynamic surfaces*. Air, gases, fluids, solids and powders are good examples of the environments which part surfaces commonly interact with. Moreover, part surfaces may interact with light and other electromagnetic fields, with sound waves, etc. Favorable parameters of part surface geometry are usually outputs of a solution to complex problems in aerodynamics, hydrodynamics, contact interaction of solids with other solids, or solids with powders, etc.

In order to be able to design and produce products with favorable performance, the design and manufacture of part surfaces having favorable geometry is of critical importance. An appropriate analytical description of part surfaces is the first step to better understanding of what we need to design and how a desired part surface can be reproduced on a solid or, in other words, how a desired part surface can be manufactured.

Geometry of Surfaces: A Practical Guide for Mechanical Engineers, First Edition. Stephen P. Radzevich.

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Geometry of a Part Surface

The number of different kinds of part surfaces approaches infinity. Planes, surfaces of revolution, cylinders of general type (including, but not limited to, cylinders of revolution) and screw surfaces of constant axial pitch can all be found in the design of parts produced in industry. Examples of part surfaces are illustrated in Fig. 1.1. This figure shows part surfaces featuring simple geometry. Most surfaces of such types allow for *sliding over themselves* [33].

Part surfaces of complex geometry are widely used in practice as well. The working surface of an impeller blade is a perfect example of a part surface having complex geometry. Part surfaces of this kind are commonly referred to as *sculptured part surfaces* or *free-form part surfaces*. An example of a sculptured part surface is depicted in Fig. 1.2.

Sculptured part surfaces do not allow for *sliding over themselves*. Moreover, the parameters of local geometry of a sculptured part surface at any two infinitesimally close points within the surface patch differ from each other.

More examples of part surfaces of complex geometry can be found in various industries, in the field of design and in the production of gear cutting tools in particular [35].

1.1 On the Analytical Description of Ideal Surfaces

A smooth regular surface could be specified uniquely by two independent variables. Therefore, we give a surface P (Fig. 1.3), in most cases, by expressing its rectangular coordinates X_P, Y_P and Z_P as functions of two *Gaussian* coordinates, U_P and V_P, in a certain closed interval:

$$\mathbf{r}_P = \mathbf{r}_P(U_P, V_P) = \begin{bmatrix} X_P(U_P, V_P) \\ Y_P(U_P, V_P) \\ Z_P(U_P, V_P) \\ 1 \end{bmatrix};\ (U_{1.P} \leq U_P \leq U_{2.P}; \mathrm{V}_{1.P} \leq V_P \leq V_{2.P}) \qquad (1.1)$$

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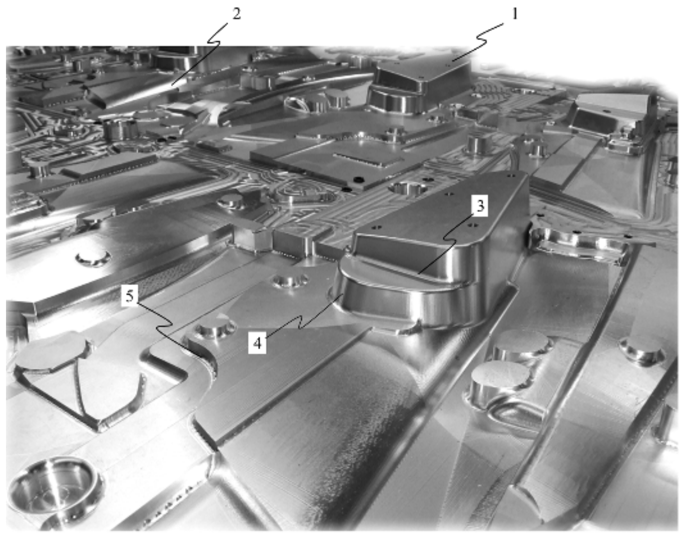

Figure 1.1 Examples of smooth regular part surfaces: a plane (1); an outer cylinder of revolution (2); an inner cylinder of revolution (3); a cone of revolution (4); a torus (5). Reproduced with permission from Industrial Model, Inc.

Here we define:

$\mathbf{r}_P$	– position vector of a point of the surface P
U_P and V_P	– curvilinear (*Gaussian*) coordinates of the point of the surface P
X_P, Y_P, Z_P	– *Cartesian* coordinates of the point of the surface P
$U_{1.P}, U_{2.P}$	– boundary values of the closed interval of the U_P-parameter
$V_{1.P}, V_{2.P}$	– boundary values of the closed interval of the V_P-parameter

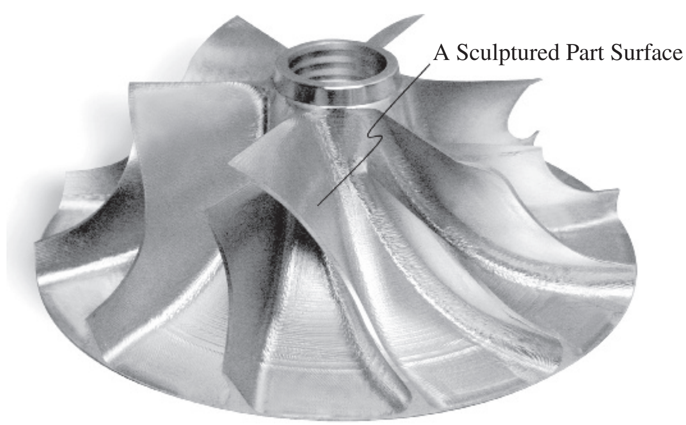

Figure 1.2 Working surface of impeller is an example of a smooth regular sculptured part surface. Reproduced from Somani Engineering.

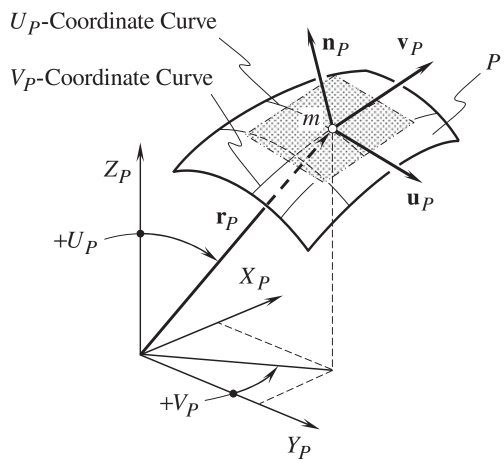

Figure 1.3 Analytical description of an ideal part surface P (adapted from [33]).

The parameters U_P and V_P must enter independently, which means that the matrix

$$\mathbf{M} = \begin{bmatrix} \dfrac{\partial X_P}{\partial U_P} & \dfrac{\partial Y_P}{\partial U_P} & \dfrac{\partial Z_P}{\partial U_P} \\ \dfrac{\partial X_P}{\partial V_P} & \dfrac{\partial Y_P}{\partial V_P} & \dfrac{\partial Z_P}{\partial V_P} \end{bmatrix} \tag{1.2}$$

has rank 2. Positions where the rank is 1 or 0 are singular points; when the rank at all points is 1, then Eq. (1.1) represents a curve.

The following notations will be convenient in the consideration below.

The first derivatives of $\mathbf{r}_P$ with respect to the *Gaussian* coordinates U_P and V_P are designated $\frac{\partial \mathbf{r}_P}{\partial U_P} = \mathbf{U}_P$ and $\frac{\partial \mathbf{r}_P}{\partial V_P} = \mathbf{V}_P$, and for the unit tangent vectors $\mathbf{u}_P = \frac{\mathbf{U}_P}{|\mathbf{U}_P|}$ and $\mathbf{v}_P = \frac{\mathbf{V}_P}{|\mathbf{V}_P|}$ correspondingly.

The unit tangent vector $\mathbf{u}_P$ (as well as the tangent vector $\mathbf{U}_P$) specifies a direction of the tangent line to the U_P-coordinate curve through the given point m on the surface P. Similarly, the unit tangent vector $\mathbf{v}_P$ (as well as the corresponding tangent vector $\mathbf{V}_P$) specifies a direction of the tangent line to the V_P-coordinate curve through that same point m on the surface P.

The significance of the unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ becomes evident from the considerations immediately following.

First, the unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ allow for an equation of the tangent plane to the surface P at m:

$$\text{Tangent plane} \Rightarrow \begin{bmatrix} [\mathbf{r}_{t.p} - \mathbf{r}_P^{(m)}] \\ \mathbf{u}_P \\ \mathbf{v}_P \\ 1 \end{bmatrix} = 0 \tag{1.3}$$

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Here we define:

$\mathbf{r}_{t.P}$ – position vector of a point of the tangent plane to the surface P at m
$\mathbf{r}_P^{(m)}$ – position vector of the point m on the surface P

Second, the unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ allow for an equation of the perpendicular, $\mathbf{N}_P$, and of the unit normal vector, $\mathbf{n}_P$, to the surface P at m:

$$\mathbf{N}_P = \mathbf{U}_P \times \mathbf{V}_P \tag{1.4}$$

$$\mathbf{n}_P = \frac{\mathbf{N}_P}{|\mathbf{N}_P|} = \frac{\mathbf{U}_P \times \mathbf{V}_P}{|\mathbf{U}_P \times \mathbf{V}_P|} = \mathbf{u}_P \times \mathbf{v}_P \tag{1.5}$$

When the order of multipliers in Eq. (1.4) [as well as in Eq. (1.5)] is chosen properly, then the unit normal vector $\mathbf{n}_P$ is pointed outward from the body side bounded by the surface P. (It should be pointed out here that the unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$, as well as the unit normal vector $\mathbf{n}_P$, are dimensionless parameters of the geometry of the surface P. This feature of the unit vectors $\mathbf{u}_P$, $\mathbf{v}_P$ and $\mathbf{n}_P$ is convenient when performing practical calculations.)

1.2 On the Difference between *Classical Differential Geometry* and *Engineering Geometry of Surfaces*

Classical differential geometry has been developed mostly for the purpose of investigation of smooth regular surfaces. Engineering geometry also deals with smooth regular surfaces. What is the difference between these two geometries?

The difference between classical differential geometry and engineering geometry of surfaces is due mostly to how surfaces are interpreted.

Only *phantom* surfaces are investigated in classical differential geometry. Surfaces of this kind do not exist physically. They can be understood as a zero-thickness film of appropriate shape. Such a film can be accessed from both sides of the surface. This causes the following indefiniteness.

As an example, consider a surface, at a certain point m, with *Gaussian* curvature $\mathscr{G}_P$ of the surface having positive value ($\mathscr{G}_P > 0$). Classical differential geometry gives no answer to the question of whether the surface P is convex or concave in the vicinity of the point m. In the first case (when the surface P is convex), the mean curvature $\mathscr{M}_P$ of the surface P at the point m is of positive value, $\mathscr{M}_P > 0$, while in the second case (when the surface P is concave), the mean curvature $\mathscr{M}_P$ of the surface P at the point m is of negative value, $\mathscr{M}_P < 0$.

A similar situation is observed when the *Gaussian* curvature $\mathscr{G}_P$ at a certain surface point is of negative value ($\mathscr{G}_P < 0$).

In classical differential geometry, the answer to the question of whether a surface is convex or concave in the vicinity of a certain point m can be given only by convention.

In turn, surfaces that are treated in engineering geometry bound a solid – a *machine part* (or *machine element*). This part can be called a *real object* (Figs 1.1 and 1.2). The real object is the bearer of the surface shape.

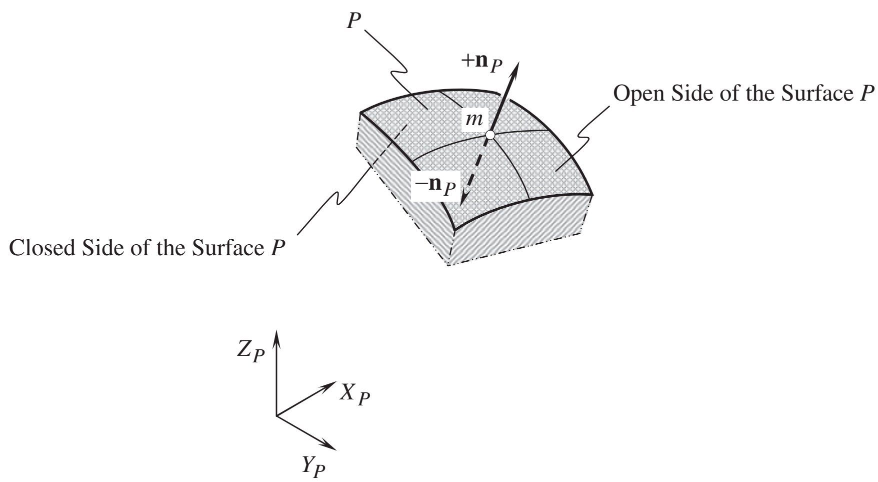

Figure 1.4 Open and closed sides of a part surface P (adapted from [33]).

Surfaces that bound real objects are accessible only from one side, as illustrated schematically in Fig. 1.4. We refer to this side of the surface as the *open side of a part surface*. The opposite side of the surface P is not accessible. Because of this, we refer to the opposite side of the surface P as the *closed side of a part surface*.

The positively directed unit normal vector $+\mathbf{n}_P$ is pointed outward from the part body, i.e. it is pointed from the body side to the void side. The negative unit normal vector $-\mathbf{n}_P$ is pointed oppositely to $+\mathbf{n}_P$.

The existence of the open and closed sides of a part surface P eliminates the problem of identifying whether a surface is convex or concave. No convention is required in this respect.

The description of a smooth regular surface in differential geometry of surfaces and in engineering geometry provides more differences between surfaces treated in these two different branches of geometry.

1.3 On the Analytical Description of Part Surfaces

Another principal difference in this respect is due to the nature of the real object. We should point out here again that a real object is the bearer of a surface shape. No real object can be machined/manufactured precisely without deviations of its actual shape from the desired shape of the real object. Smaller or larger deviations in shape of the real object from its desired shape are inevitable in nature. We won't go into detail here on the nature of the deviations. We should simply realize that such deviations always exist.

As an example, let's consider how the surface of a round cylinder is specified in differential geometry of surfaces and compare it with that in engineering geometry.

In differential geometry of surfaces, the coordinates of the current point m of the surface of a cylinder of revolution can be specified by the position vector $\mathbf{r}_m$ of the point m [Fig. 1.5(a)]. In

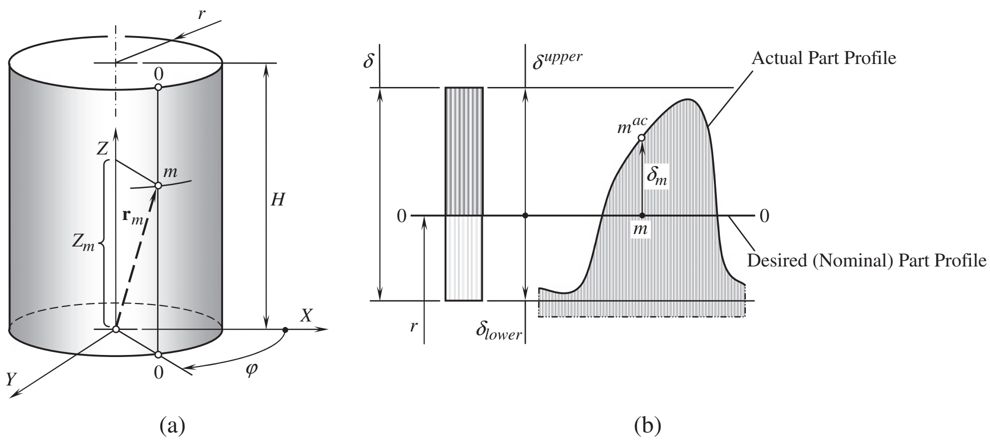

Figure 1.5 Specification of (a) an ideal and (b) a real part surface.

the case under consideration, the position vector $\mathbf{r}_m$ of a point within the surface of a cylinder of radius r, and having the Z-axis as its axis of rotation, can be expressed in matrix form as

$$\mathbf{r}_m(\varphi, Z_m) = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \\ Z_m \\ 1 \end{bmatrix} \tag{1.6}$$

Here, the surface curvilinear coordinates are denoted by φ and Z_m, accordingly. They are equivalents of the curvilinear coordinates U_P and V_P in Eq. (1.1).

Mechanical engineers have no other option than to treat a desired (nominal) part surface P, which is given by the part blueprint, and which is specified by the tolerance for the surface P accuracy.

As manufacturing errors are inevitable, the current surface point m^{act} actually deviates from its desired location m. The position vector $\mathbf{r}_m^{act}$ of a current point m^{act} of the actual part surface deviates from $\mathbf{r}_m$ for an ideal surface point m. Without loss of generality, the surface deviations in the direction of the Z-axis are ignored. Instead, the surface deviations in the directions of the X- and Y-axes are considered.

The deviation of a point m^{act} from the corresponding surface point m that is measured perpendicular to the desired part surface P is designated as δ_m [Fig. 1.5(b)]. Formally, the position vector $\mathbf{r}_m^{act}$ of a current point m^{act} of the actual part surface can be described analytically in matrix form as

$$\mathbf{r}_m^{act}(\varphi, Z_m) = \begin{bmatrix} (r+\delta_m)\cos\varphi \\ (r+\delta_m)\sin\varphi \\ Z_m \\ 1 \end{bmatrix} \tag{1.7}$$

where the deviation δ_m is understood as a signed value. It is positive for points m^{act} located outside the surface [see Eq. (1.6)] and negative for points m^{act} located inside the surface [see Eq. (1.6)].

Unfortunately, the actual value of the deviation δ_m is never known. Thus, Eq. (1.7) cannot be used for the purpose of analytical description of real part surfaces.

In practice, the permissible deviations δ_m of surfaces in engineering geometry are limited to a certain tolerance band. An example of a tolerance band is shown schematically in Fig. 1.5(b). The positive deviation δ_m must not exceed the upper limit δ^{upper}, and the negative deviation δ_m must not be greater than the lower limit δ_{lower}. That is, in order to meet the requirements specified by the blueprint, the deviation δ_m must be within the tolerance band

$$\delta_{lower} \leq \delta_m \leq \delta^{upper} \tag{1.8}$$

The total width of the tolerance band is equal to $\delta_m = \delta^{upper} + \delta_{lower}$. In this expression for the deviation δ_m, both limits δ^{upper} and δ_{lower} are signed values. They can be either of positive value, or of negative value, as well as equal to zero.

Under such a scenario not only does the desired part surface P_{des} meet the requirements specified by the part blueprint, but any and all actual part surfaces P^{ac} located within the tolerance band $\delta_{lower} \leq \delta \leq \delta^{upper}$ meet the requirements given by the blueprint. In other words, if a surface P_δ^+ is specified by a tolerance band δ^{upper}, and a surface P_δ^- is specified by a tolerance band δ_{lower}, then an actual part surface P^{ac} is always located between the surfaces P_δ^+ and P_δ^-. And, of course, the actual part surface P^{ac} always differs from the desired part surface P_{des}. However, the deviation of the surface P^{ac} from the surface P_{des} is always the tolerance band $\delta_{lower} \leq \delta \leq \delta^{upper}$.

An intermediate summarization is as follows: we know everything about ideal surfaces, which do not exist in reality, and we know nothing about real surfaces, which exist physically (or, at least, our knowledge about real surfaces is very limited).

In addition, the entire endless surface of the cylinder of revolution is not considered in engineering geometry. Only a portion of this surface is of importance in practice. Therefore, in the axial direction, the length of the cylinder is limited to an interval $0 \leq Z_m \leq H$, where H is a pre-specified length of the cylinder of revolution.

With that said, we can now proceed with a more general consideration of the analytical representation of surfaces in engineering geometry.

1.4 Boundary Surfaces for an Actual Part Surface

Owing to the deviations, an actual part surface P^{act} deviates from its nominal (desired) surface P_{des} (Fig. 1.6). However, the deviations are within pre-specified tolerance bands. Otherwise, the real object could become useless. In practice, this particular problem is easily solved by selecting appropriate tolerance bands for the shape and dimensions of the actual surface P^{act}.

Similar to measuring deviations, the tolerances are also measured in the direction of the unit normal vector $\mathbf{n}_P$ to the desired (nominal) part surface P. Positive tolerance δ^+ is measured along the positive direction of the vector $\mathbf{n}_P$, while negative tolerance δ^- is measured along the negative direction of the vector $\mathbf{n}_P$. In a particular case, one of the tolerances, either δ^+ or δ^-, can be zero.

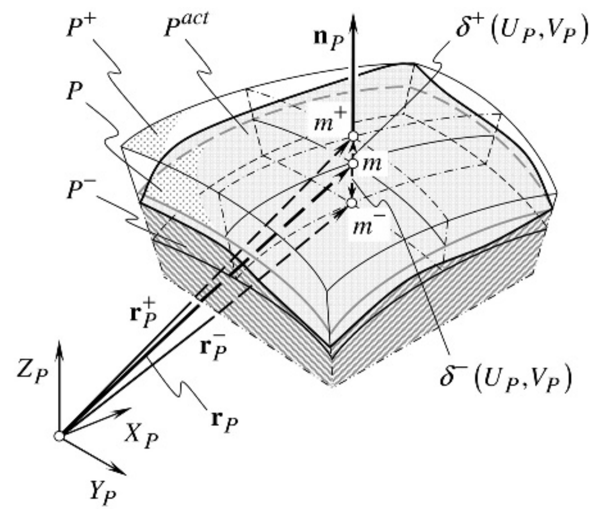

Figure 1.6 Analytical description of an actual part surface P^{act} located between the boundary surfaces P^+ and P^- (adapted from [33]).

Often, the values of the tolerance bands δ^+ and δ^- are constant within the entire patch of the surface P. However, in special cases, for example when machining a sculptured part surface on a multi-axis *NC* machine, the actual values of the tolerances δ^+ and δ^- can be set as functions of the coordinates of the current point m on the surface P. This results in the tolerances being represented in terms of U_P- and V_P-parameters of the surface P, say in the form $\delta^+ = \delta^+(U_P, V_P)$ and $\delta^- = \delta^-(U_P, V_P)$.

The endpoint of the vector $\delta^+ \cdot \mathbf{n}_P$ at a current surface point m produces the point m^+. Similarly, the endpoint of the vector $\delta^- \cdot \mathbf{n}_P$ produces the corresponding point m^-.

The surface P^+ of upper tolerance is represented by the loci of the points m^+ (i.e. by the loci of the endpoints of the vector $\delta^+ \cdot \mathbf{n}_P$). This makes it possible to have an analytical representation of the surface P^+ of upper tolerance in the form

$$\mathbf{r}_P^+(U_P, V_P) = \mathbf{r}_P + \delta^+ \cdot \mathbf{n}_P \tag{1.9}$$

Usually, the surface P^+ of upper tolerance is located above the nominal part surface P.

Similarly, the surface P^- of lower tolerance is represented by the loci of the points m^- (i.e. by the loci of the endpoints of the vector $\delta^- \cdot \mathbf{n}_P$). This also makes it possible to have an analytical representation of the surface P^- of lower tolerance in the form

$$\mathbf{r}_P^-(U_P, V_P) = \mathbf{r}_P + \delta^- \cdot \mathbf{n}_P \tag{1.10}$$

Commonly, the surface P^- of lower tolerance is located beneath the nominal part surface P.

The surfaces P^+ and P^- are the boundary surfaces. The actual part surface P^{act} is located between the surfaces P^+ and P^-, as illustrated schematically in Fig. 1.6.

The actual part surface P^{act} cannot be represented analytically. Actually, the surface P^{act} is unknown – any surface that is located between the surfaces of upper tolerance P^+ and lower tolerance P^- meets the requirements of the part blueprint, and thus every such surface can be considered as an actual surface P^{act}. The equation of the surface P^{act} cannot be represented in the form $\mathbf{r}_P^{act} = \mathbf{r}_P + \delta^{act} \cdot \mathbf{n}_P$, because the actual value of the deviation δ^{act} at the current surface point is not known. CMM data yields only an approximation for δ^{act} as well as the corresponding approximation for P^{act}. Moreover, the parameters of the local topology of the surface P considered above cannot be calculated for the surface P^{act}. However, owing to the tolerances δ^+ and δ^- being small enough to compare the normal radii of curvature of the nominal surfaces P, it is assumed below that the surface P^{act} possesses the same geometrical properties as the surface P, and that the difference between the corresponding geometrical parameters of the surfaces P^{act} and P is negligibly small. In further consideration, this allows for a replacement of the actual surface P^{act} with the nominal surface P, which is much more convenient for performing calculations.

The consideration in this section illustrates the second principal difference between classical differential geometry and engineering geometry of surfaces.

Because of these differences, engineering geometry of surfaces often presents problems that were not envisioned in classical (pure) differential geometry of surfaces.

1.5 Natural Representation of a Desired Part Surface

The specification of a surface in terms of the first and second fundamental forms is commonly called the *natural kind of surface representation*. In general form, it can be represented by a set of two equations

$$\left.\begin{array}{r}\text{Natural form of a surface } P\\ \text{parameterization}\end{array}\right|$$

$$\Rightarrow P = P(\Phi_{1.P}, \Phi_{2.P}) \begin{cases} \Phi_{1.P} = \Phi_{1.P}(E_P, F_P, G_P) \\ \Phi_{2.P} = \Phi_{2.P}(E_P, F_P, G_P, L_P, M_P, N_P) \end{cases} \tag{1.11}$$

It was proven by *Bonnet*[1] (1867) that specification of the first and second fundamental forms determines a unique surface if the *Gaussian*[2] characteristic equation[3] and the *Codazzi*[4]–*Mainardi*[5] relationships of compatibility[6] are satisfied, and those two surfaces that have identical first and second fundamental forms must be congruent to one another [1]. (It should be mentioned here that two surfaces with identical first and second fundamental forms might also be symmetrical to one another. The interested reader is referred to special literature on differential geometry of surfaces for details about this issue.) This statement is commonly considered as the main theorem in the theory of surfaces.

[1] *Pierre Ossian* ***Bonnet*** (December 22, 1819–June 22, 1892) – a French mathematician.
[2] *Johan Carl Friedrich* ***Gauss*** (April 30, 1777–February 23, 1855) – a famous German mathematician and physical scientist.
[3] The *Gauss* equation of compatibility that follows from his famous *theorema egregium* is considered in detail in Chapter 8 [see Eq. (8.12)].
[4] *Delfino* ***Codazzi*** (March 7, 1824–July 21, 1873) – an Italian mathematician.
[5] *Gaspare* ***Mainardi*** (June 27, 1800–March 9, 1879) – an Italian mathematician.
[6] The *Codazzi–Mainardi* equations of compatibility are considered in detail in Chapter 8 [see Eqs (8.13) and (8.14)].

We should make clear what the first and second fundamental forms of a surface stand for. Both of them relate to the intrinsic geometry in the differential vicinity of a surface point.

1.5.1 First fundamental form of a desired part surface

The first fundamental form $\Phi_{1.P}$ of a smooth regular surface describes the metric properties of the surface P. Usually, it is represented as the quadratic form

$$\Phi_{1.P} \Rightarrow ds_P^2 = E_P\,dU_P^2 + 2F_P\,dU_P\,dV_P + G_P\,dV_P^2 \tag{1.12}$$

Here we define:

s_P – linear element of the surface P (s_P is equal to the length of a segment of a certain curve line on the surface P)

E_P, F_P, G_P – fundamental magnitudes of first order

Equation (1.12) is known from many advanced sources.

In engineering geometry of surfaces another form of analytical representation of the first fundamental form $\Phi_{1.P}$ is proven to be useful:

$$\Phi_{1.P} \Rightarrow ds_P^2 = [dU_P \quad dV_P \quad 0 \quad 0] \cdot \begin{bmatrix} E_P & F_P & 0 & 0 \\ F_P & G_P & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} dU_P \\ dV_P \\ 0 \\ 0 \end{bmatrix} \tag{1.13}$$

This kind of analytical representation of the first fundamental form $\Phi_{1.P}$ was proposed by Radzevich [32]. The practical advantage of Eq. (1.13) is that it can easily be incorporated into computer programs in which multiple coordinate system transformations are used. The last is vital for many CAD/CAM applications.

The fundamental magnitudes of the first order E_P, F_P and G_P can be calculated from the following equations:

$$E_P = \mathbf{U}_P \cdot \mathbf{U}_P \tag{1.14}$$

$$F_P = \mathbf{U}_P \cdot \mathbf{V}_P \tag{1.15}$$

$$G_P = \mathbf{V}_P \cdot \mathbf{V}_P \tag{1.16}$$

The fundamental magnitudes E_P, F_P and G_P of the first order are functions of U_P- and V_P-parameters of the surface P. In general, these relationships can be represented in the form

$$E_P = E_P(U_P, V_P) \tag{1.17}$$

$$F_P = F_P(U_P, V_P) \tag{1.18}$$

$$G_P = G_P(U_P, V_P) \tag{1.19}$$

The fundamental magnitudes E_P and G_P are always positive ($E_P > 0$, $G_P > 0$), while the fundamental magnitude F_P can be equal to zero ($F_P \geq 0$). This results in the first fundamental form always being non-negative ($\Phi_{1.P} \geq 0$).

The first fundamental form $\Phi_{1.P}$ yields computation of the following major parameters of geometry of the surface P:

(a) length of a curve–line segment on the surface P;
(b) square of the surface P portion bounded by a closed curve on the surface;
(c) angle between any two directions on the surface P.

Owing to the first fundamental form representing the length of a curve-line segment, it is always non-negative, i.e. the inequality $\Phi_{1.P} \geq 0$ is always observed.

The discriminant H_P of the first fundamental form $\Phi_{1.P}$ can be calculated from the equation

$$H_P = \sqrt{E_P G_P - F_P^2} \tag{1.20}$$

It is assumed here and below that the discriminant H_P is always non-negative, i.e. $H_P = +\sqrt{E_P G_P - F_P^2}$.

Having the fundamental magnitudes of the first order E_P, F_P and G_P calculated makes possible easy calculation of the following parameters of geometry of a part surface P.

The length s of a curve segment $U_P = U_P(t)$, $V_P = V_P(t)$, $t_0 \leq t \leq t_1$ is given by the equation

$$s = \int_{t_0}^{t} \sqrt{E_P \left(\frac{dU_P}{dt}\right)^2 + 2F_P \frac{dU_P}{dt}\frac{dV_P}{dt} + G_P \left(\frac{dV_P}{dt}\right)^2} dt \tag{1.21}$$

The value of the angle θ between two specified directions through a certain point m on the surface P can be calculated from one of the following equations:

$$\cos\theta = \frac{F_P}{\sqrt{E_P G_P}} \tag{1.22}$$

$$\sin\theta = \frac{H_P}{\sqrt{E_P G_P}} \tag{1.23}$$

$$\tan\theta = \frac{H_P}{F_P} \tag{1.24}$$

For the calculation of square S_P of a surface patch Σ, which is bounded by a closed line on the surface P, the following equation is commonly used:

$$S_P = \iint_{\Sigma} \sqrt{E_P G_P - F_P^2} dU_P dV_P \tag{1.25}$$

The fundamental form $\Phi_{1.P}$ remains the same while the surface is bending. This is another important feature of the first fundamental form $\Phi_{1.P}$. The feature can be employed to design 3D CAM for finishing of a turbine blade with an abrasive strip as a cutting tool.

1.5.2 Second fundamental form of a desired part surface

The second fundamental form $\Phi_{2.P}$ describes the curvature of a smooth regular surface P.

Consider a point K on a smooth regular part surface P (Fig. 1.7). The location of the point K is specified by the coordinates U_P and V_P. A line through the point K is located entirely within the surface P. A nearby point m is located within the line through the point K. The location of the point m is specified by the coordinates $U_P + dU_P$ and $V_P + dV_P$ as it is infinitesimally close to the point K. The closest distance of approach of the point m to the tangent plane through the point K is expressed by the second fundamental form $\Phi_{2.P}$. Torsion of the curve Km is ignored. Therefore, the distance a is assumed equal to zero ($a = 0$).

Usually, it is represented as the quadratic form (Fig. 1.7)

$$\Phi_{2.P} \Rightarrow -d\mathbf{r}_P \cdot d\mathbf{n}_P = L_P\, dU_P^2 + 2M_P\, dU_P\, dV_P + N_P\, dV_P^2 \tag{1.26}$$

Equation (1.26) is known from many advanced sources.

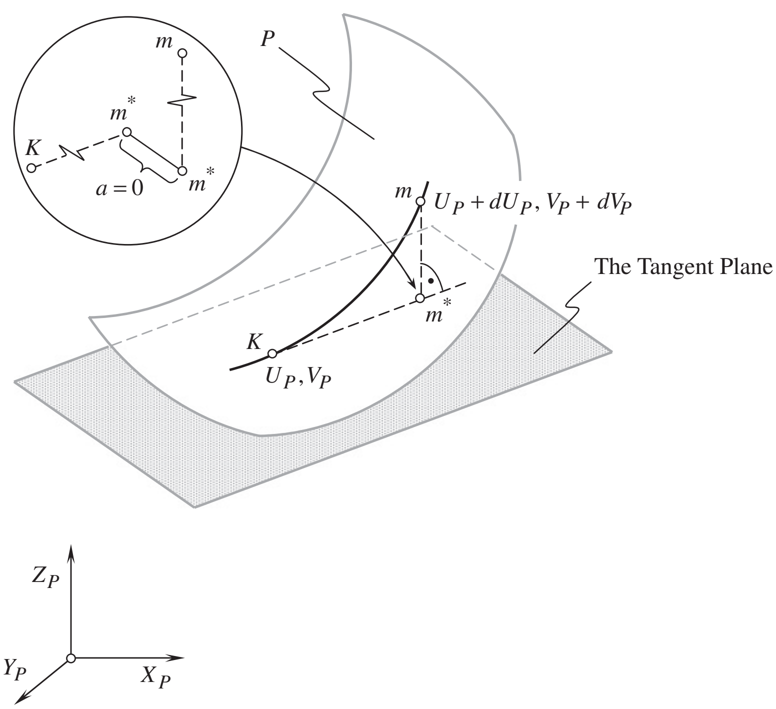

Figure 1.7 Definition of second fundamental form $\Phi_{2.P}$ of a smooth regular part surface P.

In engineering geometry of surfaces another form of analytical representation of the second fundamental form $\Phi_{2.P}$ is proven to be useful:

$$\Phi_{2.P} \Rightarrow [dU_P \quad dV_P \quad 0 \quad 0] \cdot \begin{bmatrix} L_P & M_P & 0 & 0 \\ M_P & N_P & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} dU_P \\ dV_P \\ 0 \\ 0 \end{bmatrix} \tag{1.27}$$

This kind of analytical representation of the second fundamental form $\Phi_{2.P}$ was proposed by Radzevich [32]. Similar to Eq. (1.13), the practical advantage of Eq. (1.27) is that it can easily be incorporated into computer programs in which multiple coordinate system transformations are used. The last is vital for many CAD/CAM applications.

In Eq. (1.27), the parameters L_P, M_P and N_P designate fundamental magnitudes of the second order.

The fundamental magnitudes of the second order can be computed from the following equations:

$$L_P = \frac{\frac{\partial \mathbf{U}_P}{\partial U_P} \times \mathbf{U}_P \cdot \mathbf{V}_P}{\sqrt{E_P G_P - F_P^2}} \tag{1.28}$$

$$M_P = \frac{\frac{\partial \mathbf{U}_P}{\partial V_P} \times \mathbf{U}_P \cdot \mathbf{V}_P}{\sqrt{E_P G_P - F_P^2}} = \frac{\frac{\partial \mathbf{V}_P}{\partial U_P} \times \mathbf{U}_P \cdot \mathbf{V}_P}{\sqrt{E_P G_P - F_P^2}} \tag{1.29}$$

$$N_P = \frac{\frac{\partial \mathbf{V}_P}{\partial V_P} \times \mathbf{U}_P \cdot \mathbf{V}_P}{\sqrt{E_P G_P - F_P^2}} \tag{1.30}$$

The fundamental magnitudes L_P, M_P and N_P of the second order are also functions of U_P- and V_P-parameters of the surface P. These relationships in general can be represented in the form

$$L_P = L_P(U_P, V_P) \tag{1.31}$$

$$M_P = M_P(U_P, V_P) \tag{1.32}$$

$$N_P = N_P(U_P, V_P) \tag{1.33}$$

The discriminant T_P of the second fundamental form $\Phi_{2.P}$ can be computed from the equation

$$T_P = \sqrt{L_P N_P - M_P^2} \tag{1.34}$$

Implementation of the first, $\Phi_{1.P}$, and of the second, $\Phi_{2.P}$, fundamental forms of a smooth regular part surface P makes possible a significant simplification when performing calculation of parameters of the surface geometry.

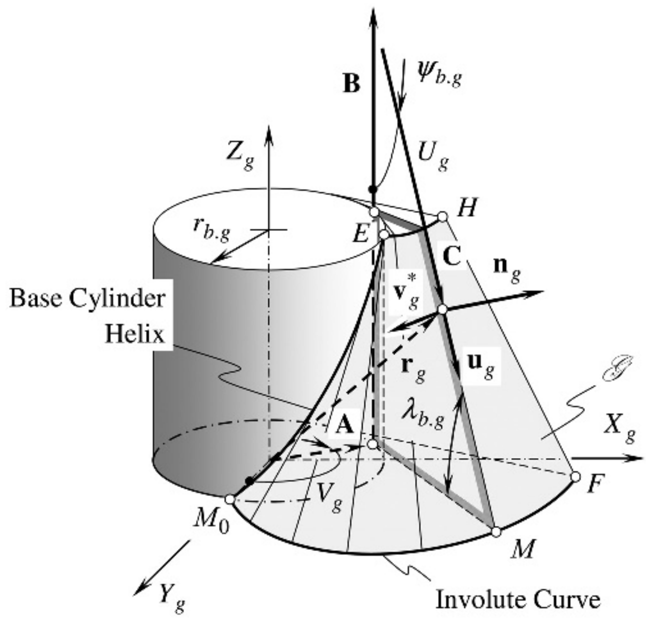

Figure 1.8 Screw involute surface of a helical gear tooth (adapted from [33]).

1.5.3 Illustrative example

Let's consider an example of how an analytical representation of a surface in a *Cartesian* coordinate system can be converted into the natural parameterization of that same surface [33, 34, 36].

A screw involute surface $\mathscr{G}$ of a gear tooth is described analytically in a *Cartesian* coordinate system $X_g Y_g Z_g$ (Fig. 1.8).

The equation of the screw involute surface $\mathscr{G}$ is represented in matrix form as

$$\mathbf{r}_g(U_g, V_g) = \begin{bmatrix} r_{b.g} \cos V_g + U_g \cos \psi_{b.g} \sin V_g \\ r_{b.g} \sin V_g - U_g \sin \psi_{b.g} \sin V_g \\ r_{b.g} \tan \psi_{b.g} - U_g \sin \psi_{b.g} \\ 1 \end{bmatrix} \tag{1.35}$$

where we define:

$r_{b.g}$ – radius of the base cylinder of the screw involute surface $\mathscr{G}$ of the gear tooth
$\psi_{b.g}$ – base helix angle of the screw involute surface $\mathscr{G}$ of the gear tooth

This equation allows for calculation of the two tangent vectors $\mathbf{U}_g(U_g, V_g)$ and $\mathbf{V}_g(U_g, V_g)$ that are correspondingly equal:

$$\mathbf{U}_g = \begin{bmatrix} \cos\psi_{b.g}\sin V_g \\ -\cos\psi_{b.g}\cos V_g \\ -\sin\psi_{b.g} \\ 1 \end{bmatrix} \tag{1.36}$$

$$\mathbf{V}_g = \begin{bmatrix} -r_{b.g}\sin V_g + U_g\cos\psi_{b.g}\cos V_g \\ r_{b.g}\cos V_g + U_g\cos\psi_{b.g}\sin V_g \\ r_{b.g}\tan\psi_{b.g} \\ 1 \end{bmatrix} \tag{1.37}$$

Substituting the calculated vectors $\mathbf{U}_g$ and $\mathbf{V}_g$ into Eqs (1.14) through (1.16), one can come up with formulae for the calculation of fundamental magnitudes of the first order:

$$E_g = 1 \tag{1.38}$$

$$F_g = -\frac{r_{b.g}}{\cos\psi_{b.g}} \tag{1.39}$$

$$G_g = \frac{U_g^2\cos^4\psi_{b.g} + r_{b.g}^2}{\cos^2\psi_{b.g}} \tag{1.40}$$

These equations can be substituted directly into Eq. (1.12) for the first fundamental form:

$$\Phi_{1.g} \Rightarrow dU_g^2 - 2\frac{r_{b.g}}{\cos\psi_{b.g}}dU_g dV_g + \frac{U_g^2\cos^4\psi_{b.g} + r_{b.g}^2}{\cos^2\psi_{b.g}}dV_g^2 \tag{1.41}$$

The calculated values of the fundamental magnitudes E_g, F_g and G_g can also be substituted into Eq. (1.13) for the quadratic form $\Phi_{1.g}$. In this way, the matrix representation of the first fundamental form $\Phi_{1.g}$ can be obtained. The interested reader may wish to complete this formulae transformation on his/her own.

The discriminant H_g of the first fundamental form $\Phi_{1.g}$ of the surface $\mathscr{G}$ can be calculated from the formula

$$H_g = U_g\cos\psi_{b.g} \tag{1.42}$$

In order to derive an equation for the second fundamental form $\Phi_{2.g}$ of the gear tooth surface $\mathscr{G}$, the second derivatives of $\mathbf{r}_g(U_g, V_g)$ with respect to U_g- and V_g-parameters are required. The equations for the vectors $\mathbf{U}_g$ and $\mathbf{V}_g$ derived above make possible calculation of the required derivatives:

$$\frac{\partial \mathbf{U}_g}{\partial U_P} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{1.43}$$

$$\frac{\partial \mathbf{U}_g}{\partial V_g} \equiv \frac{\partial \mathbf{V}_g}{\partial U_g} = \begin{bmatrix} \cos\psi_{b.g}\cos V_g \\ \cos\psi_{b.g}\sin V_g \\ 0 \\ 1 \end{bmatrix} \tag{1.44}$$

$$\frac{\partial \mathbf{V}_g}{\partial V_g} = \begin{bmatrix} -r_{b.g}\cos V_g - U_g\cos\psi_{b.g}\sin V_g \\ -r_{b.g}\sin V_g + U_g\cos\psi_{b.g}\cos V_g \\ 0 \\ 1 \end{bmatrix} \tag{1.45}$$

Further, substitute these derivatives [see Eqs (1.43) through (1.45)] and Eq. (1.20) into Eqs (1.28) through (1.30). After the necessary formulae transformations are complete, Eqs (1.28) through (1.30) cast into a set of formulae for the calculation of fundamental magnitudes of the second order of the screw involute surface $\mathscr{G}$:

$$L_g = 0 \tag{1.46}$$

$$M_g = 0 \tag{1.47}$$

$$N_g = -U_g \sin\psi_{b.g}\cos\psi_{b.g} \tag{1.48}$$

After substituting Eqs (1.46) through (1.48) into Eqs (1.28) through (1.30), one can obtain an equation for the calculation of the second fundamental form of the screw involute surface $\mathscr{G}$ of a gear tooth:

$$\Phi_{2.g} \Rightarrow -d\mathbf{r}_g \cdot d\,\mathbf{N}_g = -U_g \sin\psi_{b.g}\cos\psi_{b.g} dV_g^2 \tag{1.49}$$

Similar to the derivation of Eq. (1.41), the calculated values of the fundamental magnitudes L_g, M_g and N_g can be substituted into Eq. (1.27) for the quadratic form $\Phi_{2.g}$. In this way, the matrix representation of the first fundamental form $\Phi_{2.g}$ can be calculated as well. The interested reader may wish to complete this formulae transformation on his/her own.

The discriminant T_g of the second fundamental form $\Phi_{2.g}$ of the screw involute surface $\mathscr{G}$ is equal to

$$T_g = \sqrt{L_g M_g - N_g^2} = 0 \tag{1.50}$$

The derived set of six equations for the calculation of the fundamental magnitudes

$E_g = 1$	$L_g = 0$
$F_g = -\dfrac{r_{b.g}}{\cos\psi_{b.g}}$	$M_g = 0$
$G_g = \dfrac{U_g^2 \cos^4\psi_{b.g} + r_{b.g}^2}{\cos^2\psi_{b.g}}$	$N_g = -\,U_g \sin\psi_{b.g} \cos\psi_{b.g}$

represents a natural kind of parameterization of the part surface P. All major elements of geometry of the gear tooth surface can be calculated based on the fundamental magnitudes of the first $\Phi_{1.g}$ and of the second $\Phi_{2.g}$ order. Location and orientation of the surface $\mathscr{G}$ are the two parameters that remain indefinite as yet.

Once a surface is represented in natural form, i.e. is expressed in terms of six fundamental magnitudes of the first and second order, then further calculation of the parameters of the surface P gets much easier. In order to demonstrate a significant simplification of the calculation of the parameters of the geometry of the surface P, numerous useful equations are presented below within the body of the text as an example.

1.6 Elements of Local Geometry of a Desired Part Surface

Part surfaces of various complexities are used in present practice. Some part surfaces feature simple geometry, such as cylinders of revolution, cones of revolution, planes, some kinds of surfaces of revolution, some kinds of screw surfaces. Other part surfaces, for example sculptured part surfaces, feature complex geometry. It often happens that the analytical description of the local geometry of sculptured part surfaces works perfectly when evaluating their performance capability. Bearing this in mind, the main elements of a surface local geometry are outlined briefly below.

1.6.1 Unit tangent vectors

At any point m of a smooth regular surface P, unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ can be constructed.

In case a part surface P is given by an equation in matrix representation [see Eq. (1.1)], tangent vectors $\mathbf{U}_P$ and $\mathbf{V}_P$ to the surface P at an arbitrary point m can be expressed in terms of the first derivatives of the position vector of a point $\mathbf{r}_P$ with respect to the curvilinear coordinates U_P and V_P accordingly:

$$\mathbf{U}_P = \frac{\partial\,\mathbf{r}_P}{\partial\,U_P} \tag{1.51}$$

$$\mathbf{V}_P = \frac{\partial\,\mathbf{r}_P}{\partial\,V_P} \tag{1.52}$$

Having the tangent vectors $\mathbf{U}_P$ and $\mathbf{V}_P$ calculated makes it possible to calculate the unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ respectively:

$$\mathbf{u}_P = \frac{\mathbf{U}_P}{|\mathbf{U}_P|} \tag{1.53}$$

$$\mathbf{v}_P = \frac{\mathbf{V}_P}{|\mathbf{V}_P|} \tag{1.54}$$

The unit tangent vector $\mathbf{u}_P$ (as well as the tangent vector $\mathbf{U}_P$) specifies a direction of the tangent line to the U_P-coordinate curve through the given point m on the surface P. Similarly, the unit tangent vector $\mathbf{v}_P$ (as well as the corresponding tangent vector $\mathbf{V}_P$) specifies a direction of the tangent line to the V_P-coordinate curve through that same point m on the surface P.

The significance of the unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ becomes evident from the considerations immediately below.

1.6.2 Tangent plane

The calculated unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ allow for an equation of the tangent plane to the surface P at m:

$$\begin{bmatrix} [\mathbf{r}_{t.p} - \mathbf{r}_P^{(m)}] \\ \mathbf{u}_P \\ \mathbf{v}_P \\ 1 \end{bmatrix} = 0 \tag{1.55}$$

Here we define:

$\mathbf{r}_{t.P}$ – position vector of a point of the tangent plane to the surface P at m
$\mathbf{r}_P^{(m)}$ – position vector of the point m on the surface P

1.6.3 Unit normal vector

At any point of a smooth regular surface P, the unit normal vector $\mathbf{n}_P$ can be constructed. The calculated unit tangent vectors $\mathbf{u}_P$ and $\mathbf{v}_P$ allow for an equation of the unit normal vector $\mathbf{n}_P$ to the surface P at m:

$$\mathbf{n}_P = \frac{\mathbf{N}_P}{|\mathbf{N}_P|} = \frac{\mathbf{U}_P \times \mathbf{V}_P}{|\mathbf{U}_P \times \mathbf{V}_P|} = \mathbf{u}_P \times \mathbf{v}_P \tag{1.56}$$

When the order of the multipliers in Eq. (1.56) is chosen properly, the unit normal vector $\mathbf{n}_P$ points outward from the body side bounded by the surface P.

1.6.4 Unit vectors of principal directions on a part surface

At any point m on a smooth regular surface P there exist two directions, in which the normal curvature of the surface reaches extreme values. These directions are commonly called the *principal directions* on a surface P.

Commonly, the vectors of the principal directions on a surface P are designated $\mathbf{T}_{1.P}$ and $\mathbf{T}_{2.P}$. The vectors $\mathbf{T}_{1.P}$ and $\mathbf{T}_{2.P}$ are located within a tangent plane through the point m. They are perpendicular to one another ($\mathbf{T}_{1.P} \perp \mathbf{T}_{2.P}$).

The normal curvature of the surface P in the direction specified by the tangent vector $\mathbf{T}_{1.P}$ is of maximum value, while the normal curvature of that same surface in the direction specified by the tangent vector $\mathbf{T}_{2.P}$ is of minimum value.

For the calculation of vectors $\mathbf{T}_{1.P}$ and $\mathbf{T}_{2.P}$ of principal directions through a given point m on the surface P, the fundamental magnitudes of the first order E_P, F_P, G_P and of the second order L_P, M_P, N_P are used.

The vectors $\mathbf{T}_{1.P}$ and $\mathbf{T}_{2.P}$ of principal directions can be calculated as roots of the equation

$$\begin{vmatrix} E_P\, dU_P + F_P\, dV_P & F_P\, dU_P + G_P\, dV_P \\ L_P\, dU_P + M_P\, dV_P & M_P\, dU_P + N_P\, dV_P \end{vmatrix} = 0 \tag{1.57}$$

The first principal plane section $C_{1.P}$ is orthogonal to P at m, and passes through the vector $\mathbf{T}_{1.P}$ of the first principal direction. The second principal plane section $C_{2.P}$ is orthogonal to P at m, and passes through the vector $\mathbf{T}_{2.P}$ of the second principal direction.

In engineering geometry of surfaces it is often preferred not to use the tangent vectors $\mathbf{T}_{1.P}$ and $\mathbf{T}_{2.P}$ of the principal directions, but to treat the unit tangent vectors $\mathbf{t}_{1.P}$ and $\mathbf{t}_{2.P}$ of the principal directions instead. The unit tangent vectors $\mathbf{t}_{1.P}$ and $\mathbf{t}_{2.P}$ are calculated from the equations

$$\mathbf{t}_{1.P} = \frac{\mathbf{T}_{1.P}}{|\mathbf{T}_{1.P}|} \tag{1.58}$$

$$\mathbf{t}_{2.P} = \frac{\mathbf{T}_{2.P}}{|\mathbf{T}_{2.P}|} \tag{1.59}$$

respectively.

1.6.5 Principal curvatures of a part surface

Two normal curvatures of a surface P measured in the principal plane sections $C_{1.P}$ and $C_{2.P}$ are commonly referred to as *principal curvatures* of the surface. Principal curvatures of a smooth regular surface P are denoted by $k_{1.P}$ and $k_{2.P}$ accordingly. We should stress here that the inequality

$$k_{1.P} > k_{2.P} \tag{1.60}$$

is always observed at any and all regular points on a part surface P.

In degenerate cases, e.g. at all points on a sphere, as well as at all points on a plane, all normal curvatures at a surface point are equal to one another. In these degenerate cases principal directions on a surface cannot be identified.

At a specified point m on a smooth regular part surface P, the principal curvatures $k_{1.P}$ and $k_{2.P}$ of the surface are calculated as roots of the square equation

$$\begin{vmatrix} L_P - E_P k_P & M_P - F_P k_P \\ M_P - F_P k_P & N_P - G_P k_P \end{vmatrix} = 0 \tag{1.61}$$

In exploded form, Eq. (1.61) can be rewritten as

$$(E_P G_P - F_P^2)k_P^2 - (E_P N_P - 2F_P M_P + G_P L_P)k_P + (L_P N_P - M_P^2) = 0 \tag{1.62}$$

The principal radii of curvature $R_{1.P}$ and $R_{2.P}$ are reciprocal to the corresponding principal curvatures $k_{1.P}$ and $k_{2.P}$ of the surface P at that same point m. Thus, the principal radii of curvature $R_{1.P}$ and $R_{2.P}$ can be expressed in terms of the corresponding principal curvatures $k_{1.P}$ and $k_{2.P}$ accordingly:

$$R_{1.P} = \frac{1}{k_{1.P}} \tag{1.63}$$

$$R_{2.P} = \frac{1}{k_{2.P}} \tag{1.64}$$

Use of Eqs (1.63) and (1.64) makes it possible to compose an equation for the calculation of principal radii of curvature $R_{1.P}$ and $R_{2.P}$ similar to Eq. (1.61) that is used for the calculation of the principal curvatures $k_{1.P}$ and $k_{2.P}$ of the surface P at a point m. In exploded form, such an equation can be rewritten as

$$R_P^2 - \frac{E_P N_P - 2F_P M_P + G_P L_P}{T_P} R_P + \frac{H_P}{T_P} = 0 \tag{1.65}$$

Here, H_P is the discriminant of the first order [see Eq. (1.20)] and T_P is the discriminant of the second order [see Eq. (1.34)] of the surface P at a point m. (*Reminder:* algebraic values of the radii of principal curvature $R_{1.P}$ and $R_{2.P}$ are related to each other by $R_{2.P} > R_{1.P}$.)

The normal curvature k_P of a surface P at an arbitrary direction through a point m can be calculated from the equation

$$k_P = \frac{\Phi_{2.P}}{\Phi_{1.P}} \tag{1.66}$$

In case an angle θ between the normal plane section C_P through the point m and the first principal plane $C_{1.P}$ is known, then the *Euler*[7] equation for the calculation of k_P

$$k_P = k_{1.P} \cos^2\theta + k_{2.P} \sin^2\theta \tag{1.67}$$

[7] *Leonhard* ***Euler*** (April 15, 1707–September 18, 1783) – a famous Swiss mathematician and physicist.

can conveniently be used (here, θ is the angle that the normal plane section C_P makes with the first principal plane section $C_{1.P}$; in other words, $\theta = \angle(\mathbf{t}_P, \mathbf{t}_{1.P})$ with $\mathbf{t}_P$ designating the unit tangent vector within the normal plane section C_P).

Equation (1.67) can also be rewritten in the form

$$k_P = H_P + \frac{k_{1.P} - k_{2.P}}{2} \cos 2\theta \tag{1.68}$$

One more equation is of practical importance:

$$\tau_P = (k_{2.P} - k_{1.P}) \sin\theta \cos\theta \tag{1.69}$$

This equation is commonly called the *Sophie Germain equation* (or *Bertrand*[8] *equation* in another interpretation). In this equation, the torsion τ_P of a surface point m is expressed in terms of the principal curvatures $k_{1.P}$ and $k_{2.P}$, and of the angle θ.

The curvature of a surface in a plane section at an angle υ in relation to the corresponding normal plane section can be calculated from the *Meusnier*[9] formula

$$k_{P.\upsilon} = \frac{k_P}{\cos\upsilon} \tag{1.70}$$

This equation can also be expressed in terms of corresponding radii of curvature:

$$R_{P.\upsilon} = R_P \cos\upsilon \tag{1.71}$$

1.6.6 Other parameters of curvature of a part surface

In addition to the normal curvature k_P, and to the principal curvatures $k_{1.P}$ and $k_{2.P}$ at a point m of a smooth regular part surface P, several other parameters of curvature of a part surface are used in practice.

Mean curvature of a surface

The mean curvature at a surface point is defined as half the sum of the principal curvatures at that same surface point m. Some researchers prefer to define the mean curvature not as half the sum, but as the sum of principal curvatures at a surface point m. Under such a scenario the mean curvature $\mathscr{M}_P$ is specified as $\mathscr{M}_P = k_{1.P} + k_{2.P}$. An equation for $\mathscr{M}_P$ that is equivalent to Eq. (1.73) can be rewritten in the form $\mathscr{M}_P = \frac{E_P N_P - 2F_P M_P + G_P L_P}{(E_P G_P - F_P^2)}$.

By definition, the mean curvature $\mathscr{M}_P$ is equal to

$$\mathscr{M}_P = \frac{k_{1.P} + k_{2.P}}{2} \tag{1.72}$$

[8] *Joseph Louis François* ***Bertrand*** (March 11, 1822–April 5, 1900) – a French mathematician.

[9] *de La Place Jean Baptiste Marie* ***Meusnier*** (June 19, 1754–June 13, 1793) – a French mathematician.

The mean curvature can also be expressed in terms of fundamental magnitudes of the first and second order:

$$\mathscr{M}_P = \frac{E_P N_P - 2\,F_P M_P + G_P L_P}{2\,(E_P G_P - F_P^2)} \tag{1.73}$$

Gaussian curvature of a surface

The *Gaussian* curvature (or, in other words, full curvature) at a surface point is defined as a product of principal curvatures at that same surface point m. By definition, the *Gaussian* curvature $\mathscr{G}_P$ is equal to

$$\mathscr{G}_P = k_{1.P} \cdot k_{2.P} \tag{1.74}$$

The *Gaussian* curvature can also be expressed in terms of fundamental magnitudes of the first and second order:

$$\mathscr{G}_P = \frac{L_P N_P - M_P^2}{E_P G_P - F_P^2} \tag{1.75}$$

Equation (1.73) for mean curvature $\mathscr{M}_P$ together with Eq. (1.75) for *Gaussian* curvature $\mathscr{G}_P$ makes it possible to compose a quadratic equation

$$k_P^2 - 2\,\mathscr{M}_P k_P + \mathscr{G}_P = 0 \tag{1.76}$$

for the calculation of principal curvatures $k_{1.P}$ and $k_{2.P}$.

On solution of Eq. (1.76) with respect to $k_{1.P}$ and $k_{2.P}$, the principal curvatures $k_{1.P}$ and $k_{2.P}$ can be expressed in terms of the mean curvature $\mathscr{M}_P$ and of the *Gaussian* curvature $\mathscr{G}_P$:

$$k_{1.P} = \mathscr{M}_P + \sqrt{M_P^2 - \mathscr{G}_P} \tag{1.77}$$

$$k_{2.P} = \mathscr{M}_P - \sqrt{M_P^2 - \mathscr{G}_P} \tag{1.78}$$

Absolute curvature of a surface

In some applications it could be reasonable to specify the local geometry of a surface by means of absolute curvature. By definition, the absolute curvature $\tilde{A}_P$ at a point m of a smooth regular part surface P is equal to

$$\tilde{A}_P = |k_{1.P}| + |k_{2.P}| \tag{1.79}$$

The absolute curvature $\tilde{A}_P$ at a point m of a smooth regular part surface P can be expressed in terms of fundamental magnitudes of the first and second order, as well as in terms of the mean curvature $\mathscr{M}_P$ and *Gaussian* curvature $\mathscr{G}_P$ at a surface point m [see Eqs (1.77) and (1.78)].

Shape operator of a surface

The *shape operator* is a generalized measure of concavity and convexity of a surface point m. *Weingarten*[10] is credited with the concept of the shape operator of a surface, which is also often referred to as the *shape index* or *Weingarten map*.

The differential structure of a surface is captured by the local *Hessian* matrix, which may be approximated in terms of surface normals by

$$\mathscr{H} = \begin{bmatrix} -\left(\dfrac{\partial \mathbf{n}_P}{\partial x}\right)_x & -\left(\dfrac{\partial \mathbf{n}_P}{\partial x}\right)_y \\ -\left(\dfrac{\partial \mathbf{n}_P}{\partial y}\right)_x & -\left(\dfrac{\partial \mathbf{n}_P}{\partial y}\right)_y \end{bmatrix} \tag{1.80}$$

where subscripts "x" and "y" denote the x and y components of the parameterized vector velocity.

The principal curvatures of the part surface are the eigenvalues of the *Hessian* matrix, found by solving the equation

$$|\mathscr{H} - k\,\mathbf{I}| = 0 \tag{1.81}$$

for k, where $\mathbf{I}$ is the identity matrix.

By definition, the shape operator $\mathscr{O}_P$ is the differential of the *Gauss* map of the surface. The shape operator $\mathscr{O}_P$ is a generalized measure of concavity and convexity.

The determinant of the shape operator at a point is the *Gaussian* curvature, but it also contains other information, since the mean curvature is half the trace of the shape operator. The eigenvectors and eigenvalues of the shape operator at each surface point determine the directions in which the surface bends at each point.

Koenderink and *van Doorn* developed a single-value, angular measure to describe the local surface topology in terms of the principal curvatures.

The shape operator is given in terms of the components of the first and second fundamental forms by *Weingarten equations*

$$\mathscr{O}_P = \frac{\begin{vmatrix} G_P L_P - F_P M_P & G_P M_P - F_P N_P \\ E_P M_P - F_P L_P & E_P N_P - F_P M_P \end{vmatrix}}{E_P G_P - F_P^2} \tag{1.82}$$

The shape operator can also be expressed in terms of principal curvatures at a surface point m:

$$\mathscr{O}_P = -\frac{2}{\pi} \arctan \frac{k_{1.P} + k_{2.P}}{k_{1.P} - k_{2.P}} \tag{1.83}$$

[10]*Julius* ***Weingarten*** (March 2, 1836–June 16, 1910) – a German mathematician.

and may be expressed in terms of the surface normal:

$$\mathscr{S}_P = -\frac{2}{\pi} \arctan \frac{\left(\frac{\partial \mathbf{n}_P}{\partial x}\right)_x + \left(\frac{\partial \mathbf{n}_P}{\partial x}\right)_y}{\sqrt{\left[\left(\frac{\partial \mathbf{n}_P}{\partial x}\right)_x - \left(\frac{\partial \mathbf{n}_P}{\partial x}\right)_y\right]^2 + 4\left(\frac{\partial \mathbf{n}_P}{\partial x}\right)_y \left(\frac{\partial \mathbf{n}_P}{\partial x}\right)_x}} \tag{1.84}$$

The shape operator varies from -1 to $+1$. It describes the local shape at a surface point independent of the scale of the surface. A shape operator value of $+1$ corresponds to a concave local portion of the surface P for which the principal directions are unidentified, and thus the normal radii of curvature in all directions are identical to each other. A shape operator of 0 corresponds to a saddle-like local portion of the surface P with principal curvatures of equal magnitude but opposite sign.

Curvedness of a surface

The *surface curvedness* is another measure that is derived from the surface principal curvatures. By definition, the surface curvedness $\mathscr{R}_P$ is equal to

$$\mathscr{R}_P = \sqrt{\frac{k_{1.P}^2 + k_{2.P}^2}{2}} \tag{1.85}$$

The curvedness describes the scale of the surface P, independent of its shape.

These quantities $\mathscr{S}_P$ and $\mathscr{R}_P$ are the primary differential properties of a smooth regular part surface. Note that they are properties of the surface itself and do not depend upon its parameterization, except for a possible change of sign.

In order to get a profound understanding of differential geometry of surfaces, the interested reader may wish to go to advanced monographs in the field. Systematic discussion of the topic is available from many sources. The author would like to direct the reader's attention to the monographs by *doCarmo* [5], *Struik* [52] and others.

The elements of a surface local geometry considered briefly above make it possible to introduce a definition of the term *sculptured part surface* P.

Definition 1.1 *The sculptured part surface P is a smooth regular surface, whose major parameters of local geometry differ from each other in the differential vicinity of any two points infinitely close to each other.*

The given definition of the term *sculptured part surface* P is of critical importance for further discussion. It is instructive to point out here that a sculptured part surface P does not allow for *sliding over itself*.