CHAPTER 1

BASIC ELECTROMAGNETIC THEORY

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Electromagnetic analysis has been an indispensable part of many engineering and scientific studies since James Clerk Maxwell published his electromagnetic theory in 1873 [1]. This is due primarily to the predictive power of Maxwell's equations as proven over the years and the pervasiveness of electromagnetic phenomena in modern technologies. Examples of these technologies are radar, remote sensing, geoelectromagnetics, bioelectromagnetics, antennas, wireless communication, optics, and high-frequency/high-speed circuits. Moreover, electromagnetic theory is valid from the static to optical regimes and from subatomic to intergalactic length scales. Therefore, electromagnetic analysis plays an important role in scientific research and engineering design.

The problem of electromagnetic analysis is actually a problem of solving a set of Maxwell's equations subject to given boundary conditions. In this chapter we review briefly some basic concepts and equations of electromagnetic theory that are used frequently in this book. Our emphasis is on the presentation of various differential equations and boundary conditions that define boundary-value problems to be solved by finite element analysis. The solution of Maxwell's equations in free space is also given in the form of an integral expression that relates the field to its source, followed by the description of Huygens's principle for calculating the exterior fields from the field on a closed surface. For a complete presentation of electromagnetic theory, the reader is referred to available textbooks [2–9]. This chapter may be skipped if the reader is familiar with the theory. Because the entire treatment of electromagnetic theory depends on vector analysis, we first review briefly the basic concepts and theorems of vector calculus.

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1.1 BRIEF REVIEW OF VECTOR ANALYSIS

Perhaps the most useful concepts in vector analysis are those of divergence, curl, and gradient. In this section we present definitions and related theorems for these operations.

Assume that f is a vector function, a quantity whose magnitude and direction vary as functions of space. The divergence of the vector function **f** is defined by the limit

$$\nabla \cdot \mathbf{f} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \left[\oiint_s \mathbf{f} \cdot \mathrm{ds} \right], \tag{1.1}$$

where s is the surface enclosing volume Δv and ds is normal to s and points outward. From the definition of divergence, we can show that

$$\iiint_V \nabla \cdot \mathrm{f}\, \mathrm{d}V = \oiint_S \mathrm{f} \cdot \mathrm{d}\mathbf{S}, \tag{1.2}$$

if the vector **f** and its first derivative are continuous in volume V as well as on its surface, S. Equation (1.2) is known as the divergence theorem or Gauss's theorem.

The curl of the vector function **f** is defined as

$$\nabla \times \mathbf{f} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \left[\oiint_s \mathrm{ds} \times \mathrm{f} \right], \tag{1.3}$$

whose magnitude in the direction of $\hat{n}$ is given by

$$\hat{n} \cdot (\nabla \times \mathbf{f}) = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\oint_c \mathbf{f} \cdot \mathrm{dl} \right], \tag{1.4}$$

where c is the contour that bounds surface Δs and $\hat{n}$ denotes the unit vector normal to Δs. The directions of $\hat{n}$ and $\hat{l}$ are related by the right-hand rule. From the definition of curl, it can be shown that

$$\iint_S (\nabla \times \mathrm{f}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathrm{f} \cdot \mathrm{dl}, \tag{1.5}$$

if the vector **f** and its first derivative are continuous on surface S as well as along contour C that bounds S. Equation (1.5) is known as Stokes's theorem.

Now let f be a scalar function of space. The gradient of the scalar function f is defined as

$$\nabla f = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \left[\oiint_s f\, \mathrm{ds} \right], \tag{1.6}$$

whose magnitude in the direction of $\hat{n}$ is given by

$$\hat{n} \cdot \nabla f = \frac{\mathrm{d}f}{\mathrm{d}n}. \tag{1.7}$$

From the definition of gradient, we can show that

$$\iiint_V \nabla f\, \mathrm{d}V = \oiint_S f\, \mathrm{d}\mathbf{S}, \tag{1.8}$$

if f and its first derivative are continuous in volume V as well as on its surface, S. Equation (1.8) is known as the gradient theorem.

Two very useful identities involving the three basic vector differential operators are

$$\nabla \times (\nabla f) = 0 \tag{1.9}$$

and

$$\nabla \cdot (\nabla \times \mathbf{f}) = 0, \tag{1.10}$$

where f and $\mathbf{f}$ are any functions that are continuous with a continuous first derivative. Both identities can be proven with the aid of the divergence and curl theorems and are easily verified in Cartesian coordinates. Another useful identity is

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla\nabla \cdot \mathbf{f} - \nabla^2 \mathbf{f}, \tag{1.11}$$

where ∇^2 is known as the Laplacian. Other useful vector identities are given in Appendix A.

From the divergence theorem, one can derive some integral theorems that are used frequently in the formulation of the finite element method. If we substitute $\mathbf{f} = a\nabla b$ into (1.2), we obtain the first scalar Green's theorem

$$\iiint_V (a\nabla^2 b + \nabla a \cdot \nabla b)\, \mathrm{d}V = \oiint_S a\frac{\partial b}{\partial n}\, \mathrm{d}S, \tag{1.12}$$

where $\nabla^2 f = \nabla \cdot \nabla f$. Exchanging the positions of a and b and subtracting the resulting equation from (1.12), we obtain the second scalar Green's theorem

$$\iiint_V (a\nabla^2 b - b\nabla^2 a)\, \mathrm{d}V = \oiint_S \left(a\frac{\partial b}{\partial n} - b\frac{\partial a}{\partial n}\right) \mathrm{d}S. \tag{1.13}$$

If we substitute $\mathbf{f} = \mathbf{a} \times \nabla \times \mathbf{b}$ into (1.2), we obtain the first vector Green's theorem

$$\iiint_V [(\nabla \times \mathbf{a}) \cdot (\nabla \times \mathbf{b}) - \mathbf{a} \cdot (\nabla \times \nabla \times \mathbf{b})]\, \mathrm{d}V = \oiint_S (\mathbf{a} \times \nabla \times \mathbf{b}) \cdot \hat{n}\, \mathrm{d}S. \tag{1.14}$$

Switching the positions of $\mathbf{a}$ and $\mathbf{b}$ and subtracting the resulting equation from (1.14), we obtain the second vector Green's theorem

$$\iiint_V [\mathbf{b} \cdot (\nabla \times \nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \nabla \times \mathbf{b})]\, \mathrm{d}V$$
$$= \oiint_S (\mathbf{a} \times \nabla \times \mathbf{b} - \mathbf{b} \times \nabla \times \mathbf{a}) \cdot \hat{n}\, \mathrm{d}S. \tag{1.15}$$

These are the standard Green's theorems. They can be generalized slightly to contain another function or parameter. These generalized theorems are given in Appendix A and are actually more useful in the finite element formulation.

Exercise 1.1 Derive Gauss's theorem in (1.2) from the definition of the divergence in (1.1). Discuss the case in which $\mathbf{f}$ is discontinuous.

Exercise 1.2 Derive the alternative definition of the curl in (1.4) from the original definition given in (1.3). Derive Stokes's theorem in (1.5) from (1.4).

Exercise 1.3 Derive the alternative definition of the gradient in (1.7) from the original definition given in (1.6). Derive the gradient theorem in (1.8) from (1.6).

1.2 MAXWELL'S EQUATIONS

Maxwell's equations are a set of fundamental equations that govern all macroscopic electromagnetic phenomena. The equations can be written in both differential and integral forms, and here we present both to illustrate applications of some of the integral theorems discussed in the preceding section.

1.2.1 General Integral Form

For general time-varying fields, Maxwell's equations in integral form are given by

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Faraday's law)} \qquad (1.16)$$

$$\oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \frac{\mathrm{d}}{\mathrm{d}t} \iint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} + \iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} \qquad \text{(Maxwell–Ampère law)} \qquad (1.17)$$

$$\oiint_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \iiint_V \varrho \, \mathrm{d}V \qquad \text{(Gauss's law)} \qquad (1.18)$$

$$\oiint_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0, \qquad \text{(Gauss's law—magnetic)} \qquad (1.19)$$

where

$\mathbf{E}$ = electric field intensity (volts/meter)

$\mathbf{D}$ = electric flux density (coulombs/meter^2)

$\mathbf{H}$ = magnetic field intensity (amperes/meter)

$\mathbf{B}$ = magnetic flux density (webers/meter^2)

$\mathbf{J}$ = electric current density (amperes/meter^2)

ϱ = electric charge density (coulombs/meter^3).

In (1.16) and (1.17), S is an arbitrary open surface bounded by contour C, whereas in (1.18) and (1.19), S is a closed surface enclosing volume V.

Another fundamental equation, known as the *equation of continuity*, is

$$\oiint_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \varrho \, \mathrm{d}V. \qquad (1.20)$$

This equation, which can be derived from (1.17) and (1.18), is the mathematical form of the law of conservation of charge.

Equations (1.16)–(1.20) are valid in all circumstances regardless of the medium and the shape of the integration volume, surface, and contour. They can be considered as the fundamental equations governing the behavior of electromagnetic fields.

1.2.2 General Differential Form

Maxwell's equations in differential form can be derived from (1.16)–(1.20) by using Gauss's and Stokes's theorems. Consider a point in space where all the field quantities and their derivatives are continuous. Application of Gauss's and Stokes's theorems to (1.16)–(1.20) yields

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)} \tag{1.21}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \qquad \text{(Maxwell–Ampère law)} \tag{1.22}$$

$$\nabla \cdot \mathbf{D} = \varrho \qquad \text{(Gauss's law)} \tag{1.23}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Gauss's law—magnetic)} \tag{1.24}$$

$$\nabla \cdot \mathbf{J} = -\frac{\partial \varrho}{\partial t}. \qquad \text{(equation of continuity)} \tag{1.25}$$

Among these equations, only three are independent for the case of time-varying fields and thus are called *independent equations*. Either the first three equations, (1.21)–(1.23), or the first two equations, (1.21) and (1.22), with (1.25) can be chosen as such independent equations. The other two equations, (1.24) and (1.25) or (1.24) and (1.23), can be derived from the independent equations and thus are called *auxiliary* or *dependent equations*.

1.2.3 Electrostatic and Magnetostatic Fields

When the field quantities do not vary with time, the field is called *static*. In this case, (1.21), (1.22), and (1.25) can be written as

$$\nabla \times \mathbf{E} = 0 \tag{1.26}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{1.27}$$

$$\nabla \cdot \mathbf{J} = 0, \tag{1.28}$$

whereas (1.23) and (1.24) remain the same. It is evident that in this case there is no interaction between electric and magnetic fields; therefore, we can have separately either an *electrostatic* case described by (1.23) and (1.26) or a *magnetostatic* case described by (1.24) and (1.27), with (1.28) being a natural consequence of (1.27).

1.2.4 Time-Harmonic Fields

When field quantities in Maxwell's equations are harmonically oscillating functions with a single frequency, the field is referred to as *time-harmonic*. By using the complex phasor notation [3], (1.21), (1.22), and (1.25) can be written in a simplified form as

$$\nabla \times \mathbf{E} = -\mathrm{j}\omega \mathbf{B} \tag{1.29}$$

$$\nabla \times \mathbf{H} = \mathrm{j}\omega \mathbf{D} + \mathbf{J} \tag{1.30}$$

$$\nabla \cdot \mathbf{J} = -\mathrm{j}\omega \varrho, \tag{1.31}$$

where the time convention $e^{j\omega t}$ is used and suppressed and ω is angular frequency. It is evident that in this case, the electric and magnetic fields must exist simultaneously, and they interact with each other; it is also evident that the static case is the limit of the time-harmonic case as the frequency ω approaches zero.

The use of time-harmonic fields is not as restrictive as it first appears. By using Fourier analysis, any time-varying field can be expressed in terms of time-harmonic components via the Fourier transforms

$$\mathbf{E}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}(\omega)\, \mathrm{e}^{\mathrm{j}\omega t}\, \mathrm{d}\omega \tag{1.32}$$

and

$$\mathbf{E}(\omega) = \int_{-\infty}^{\infty} \mathbf{E}(t)\, \mathrm{e}^{-\mathrm{j}\omega t}\, \mathrm{d}t. \tag{1.33}$$

Therefore, if the time-harmonic expression of a field is known for any ω, its counterpart in the time domain can be obtained by evaluating (1.32).

Exercise 1.4 Show that the Fourier transforms defined in (1.32) and (1.33) do not violate causality. In other words, show that $\mathbf{E}(t_1)$ as evaluated from (1.32) using $\mathbf{E}(\omega)$ given by (1.33) does not depend on $\mathbf{E}(t_2)$ for $t_2 > t_1$.

1.2.5 Constitutive Relations

The three independent equations among the five Maxwell's equations described earlier are in an *indefinite* form because the number of equations is less than the number of unknowns. Maxwell's equations become *definite* when *constitutive relations* between the field quantities are specified. The constitutive relations describe the macroscopic properties of the medium being considered. For a simple medium, they are

$$\mathbf{D} = \epsilon\mathbf{E} \tag{1.34}$$

$$\mathbf{B} = \mu\mathbf{H} \tag{1.35}$$

$$\mathbf{J} = \sigma\mathbf{E}, \tag{1.36}$$

where the constitutive parameters ϵ, μ, and σ denote, respectively, the permittivity (farads/meter), permeability (henrys/meter), and conductivity (siemens/meter) of the medium. These parameters are tensors for anisotropic media and scalars for isotropic media. For inhomogeneous media, they are functions of position, whereas for homogeneous media, they are all constants.

1.3 SCALAR AND VECTOR POTENTIALS

To solve Maxwell's equations, one may first convert the first-order differential equations involving two field quantities into second-order differential equations involving only one field quantity. This is demonstrated here by considering the electrostatic and magnetostatic cases.

1.3.1 Scalar Potential for the Electrostatic Field

As we mentioned earlier, the electrostatic field is governed by (1.23) and (1.26). The latter can be satisfied by representing the electric field $\mathbf{E}$ as

$$\mathbf{E} = -\nabla\phi, \tag{1.37}$$

where ϕ is called the electric *scalar potential*. Substituting (1.37) into (1.23) with the aid of (1.34), one obtains

$$-\nabla \cdot (\epsilon \nabla \phi) = \varrho, \tag{1.38}$$

which is the second-order differential equation governing ϕ. Equation (1.38) is the well-known *Poisson equation*.

1.3.2 Vector Potential for the Magnetostatic Field

The magnetostatic field is governed by (1.24) and (1.27). Equation (1.24) can be satisfied by representing the magnetic flux density $\mathbf{B}$ as

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{1.39}$$

where $\mathbf{A}$ is called the magnetic *vector potential*. Substituting (1.39) into (1.27) with the aid of (1.35) yields the second-order differential equation

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J}. \tag{1.40}$$

This equation, however, does not determine $\mathbf{A}$ uniquely because, if $\mathbf{A}$ is a solution to (1.40), any function that can be expressed as $\mathbf{A}' = \mathbf{A} + \nabla f$ is also a solution, regardless of the form of f. Thus to determine $\mathbf{A}$ uniquely, one must impose a condition on its divergence. Such a condition is called a *gauge condition*, and a natural choice for this condition is

$$\nabla \cdot \mathbf{A} = 0. \tag{1.41}$$

It is important to understand that $\mathbf{B}$ is always unique even if $\mathbf{A}$ is not. Therefore, if our objective is to calculate $\mathbf{B}$, it is not necessary to impose the gauge condition.

These discussions are pertinent to the static case. In the time-harmonic case, the electric and magnetic fields can also be represented by introducing a scalar and a vector potential, in a manner similar to the formulation in this section [2]. However, this will not be discussed here because in this book we work directly with the electric and magnetic fields for time-harmonic problems.

Exercise 1.5 Formulate time-harmonic fields in terms of a vector and a scalar potential and discuss possible choices for a gauge condition.

1.4 WAVE EQUATIONS

As just mentioned, we deal with the time-harmonic case directly in terms of the electric and magnetic fields. For this, it is necessary to derive from Maxwell's equations, which involve *both* electric and magnetic fields, the governing differential equations involving only *either* field.

1.4.1 Vector Wave Equations

The differential equation for $\mathbf{E}$ can be obtained by eliminating $\mathbf{H}$ from (1.29) and (1.30) with the aid of the constitutive relations (1.34)–(1.36). Doing this, one obtains

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) - \omega^2 \epsilon \mathbf{E} = -j\omega \mathbf{J}. \tag{1.42}$$

Similarly, one can eliminate $\mathbf{E}$ to find the equation for $\mathbf{H}$ as

$$\nabla \times \left(\frac{1}{\epsilon}\nabla \times \mathbf{H}\right) - \omega^2 \mu \mathbf{H} = \nabla \times \left(\frac{1}{\epsilon}\mathbf{J}\right). \tag{1.43}$$

These equations are called *inhomogeneous vector wave equations.* It is evident that the solution of (1.42) also satisfies (1.23), and the solution of (1.43) also satisfies (1.24).

1.4.2 Scalar Wave Equations

In electromagnetic analysis, whenever possible we simplify problems by using a two-dimensional model to approximate a three-dimensional problem. Assume that the fields and the associated medium have no variation with respect to one Cartesian coordinate, say the z-coordinate. It can then be shown that the z-components of (1.42) and (1.43) become

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{\mu_r}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mu_r}\frac{\partial}{\partial y}\right) + k_0^2 \epsilon_r\right] E_z = \mathrm{j} k_0 Z_0 J_z \tag{1.44}$$

and

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{\epsilon_r}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\epsilon_r}\frac{\partial}{\partial y}\right) + k_0^2 \mu_r\right] H_z = -\frac{\partial}{\partial x}\left(\frac{1}{\epsilon_r}J_y\right) + \frac{\partial}{\partial y}\left(\frac{1}{\epsilon_r}J_x\right), \tag{1.45}$$

respectively, where ϵ_r $(= \epsilon/\epsilon_0)$ and μ_r $(= \mu/\mu_0)$ denote the relative permittivity and relative permeability, respectively, which are assumed here to be complex scalar functions of position; k_0 $(= \omega\sqrt{\epsilon_0 \mu_0})$ is the wavenumber in free space; Z_0 $(= \sqrt{\mu_0/\epsilon_0})$ is the intrinsic impedance of free space; and ϵ_0 $(= 8.854 \times 10^{-12}$ farad/meter) and μ_0 $(= 4\pi \times 10^{-7}$ henry/meter) are the permittivity and permeability of free space. Equations of the type of (1.44) and (1.45) are called *inhomogeneous scalar wave equations* or *Helmholtz equations.*

1.5 BOUNDARY CONDITIONS

While there are many functions that satisfy the governing differential equations in a domain of interest, only one of them is the real solution to a specific problem. To determine this solution, one must know the *boundary conditions* associated with the domain. In other words, a complete description of an electromagnetic problem should include information about both differential equations and boundary conditions. In this section we present some boundary conditions that apply to many practical problems. These can be derived from Maxwell's equations in integral form (1.16)–(1.19).

1.5.1 At the Interface between Two Media

At a source-free interface between two media, say medium 1 and medium 2, the fields must satisfy four conditions, given by

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{1.46}$$

$$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0 \tag{1.47}$$

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \tag{1.48}$$

$$\hat{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0, \tag{1.49}$$

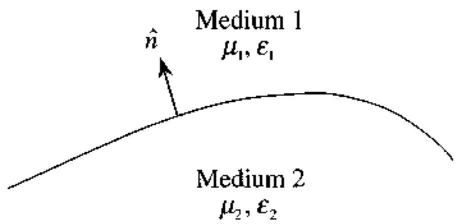

Figure 1.1 Interface between two media.

where $\hat{n}$ is the unit vector normal to the interface pointing from medium 2 into medium 1 (Fig. 1.1). These four equations are also known as *field continuity conditions*. Among these four conditions, only two are independent: one from (1.46) and (1.49), and the other from (1.47) and (1.48).

Note that in (1.47) and (1.48) it is assumed that neither surface currents nor surface charges exist at the interface. If there exists a surface electric current density $\mathbf{J}_s$ and a surface charge density ϱ_s, these two equations must be modified as

$$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \varrho_s \tag{1.50}$$

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s. \tag{1.51}$$

Exercise 1.6 Derive the boundary conditions (1.50) and (1.51) from Maxwell's equations in integral form (1.18) and (1.17).

1.5.2 At a Perfectly Conducting Surface

The boundary conditions can be simplified when one of the media, say medium 2, becomes a perfect conductor. Since a perfect conductor cannot sustain internal fields, (1.46) becomes

$$\hat{n} \times \mathbf{E} = 0 \tag{1.52}$$

and (1.49) reduces to

$$\hat{n} \cdot \mathbf{B} = 0, \tag{1.53}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the fields exterior to the conductor and $\hat{n}$ is the normal pointing away from the conductor. Note that in this case the boundary can always support a surface current ($\mathbf{J}_s = \hat{n} \times \mathbf{H}$) and a surface charge ($\varrho_s = \hat{n} \cdot \mathbf{D}$).

1.5.3 At an Imperfectly Conducting Surface

When medium 2 is an imperfect conductor, it can be shown that the electric and magnetic fields at the surface of the conductor are approximately related by

$$\mathbf{E} - (\hat{n} \cdot \mathbf{E})\hat{n} = \eta Z_0 \hat{n} \times \mathbf{H} \tag{1.54}$$

or alternatively,

$$\hat{n} \times \mathbf{E} = \eta Z_0 \left[(\hat{n} \cdot \mathbf{H})\hat{n} - \mathbf{H}\right], \tag{1.55}$$

where $\eta = \sqrt{\mu_{r2}/\epsilon_{r2}}$ is the normalized intrinsic impedance of medium 2 [10, 11]. Equation (1.54) or (1.55) is called an *impedance boundary condition*. It can be written in a more standard form as

$$\frac{1}{\mu_{r1}}\hat{n} \times (\nabla \times \mathbf{E}) - \frac{jk_0}{\eta}\hat{n} \times (\hat{n} \times \mathbf{E}) = 0 \tag{1.56}$$

or

$$\frac{1}{\epsilon_{r1}}\hat{n} \times (\nabla \times \mathbf{H}) - jk_0\eta\, \hat{n} \times (\hat{n} \times \mathbf{H}) = 0, \tag{1.57}$$

which is known as a mixed boundary condition or a boundary condition of the third kind.

In the two-dimensional case, the impedance boundary condition can be written as

$$\frac{1}{\mu_{r1}}\frac{\partial E_z}{\partial n} = \frac{jk_0}{\eta}E_z \tag{1.58}$$

for the case of $\mathbf{E} = \hat{z}E_z$ (usually referred to as the E_z-polarization case), and

$$\frac{1}{\epsilon_{r1}}\frac{\partial H_z}{\partial n} = jk_0\eta H_z \tag{1.59}$$

for the case of $\mathbf{H} = \hat{z}H_z$ (often referred to as the H_z-polarization case).

1.5.4 Across a Resistive and Conductive Sheet

An electrically resistive sheet is a thin sheet of electric current with density proportional to the tangential electric field at its surface [10]. Based on (1.46) and (1.51), the boundary conditions across such a sheet are

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \tag{1.60}$$

and

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \tag{1.61}$$

with

$$\hat{n} \times (\hat{n} \times \mathbf{E}) = -R\mathbf{J}_s, \tag{1.62}$$

where R is the resistivity in ohms per square. Substituting (1.62) into (1.61), we obtain

$$\hat{n} \times (\hat{n} \times \mathbf{E}) = -R\,\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2). \tag{1.63}$$

Resistive sheets are useful in numerical simulations because they can be used to approximate thin dielectric layers with

$$R = \frac{Z_0}{jk_0(\epsilon_r - 1)\tau}, \tag{1.64}$$

where ϵ_r is the relative permittivity and τ is the thickness of the dielectric layer.

The electromagnetic dual of a resistive sheet is a magnetically conductive sheet supporting only a magnetic current. The boundary conditions across its surface are

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \tag{1.65}$$

and

$$\hat{n} \times (\hat{n} \times \mathbf{H}) = -G\,\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2), \tag{1.66}$$

where G denotes the conductivity in siemens per square. A conductive sheet is useful because it can be used to approximate a thin magnetic layer with

$$G = \frac{Y_0}{\mathrm{j}k_0(\mu_\mathrm{r} - 1)\tau}, \tag{1.67}$$

where $Y_0 = 1/Z_0$ and μ_r is the relative permeability of the layer.

In general, a resistive and a conductive sheet can be combined to model a thin material layer whose permittivity and permeability both differ from those of the surrounding medium [10]. Such a model can simplify numerical analysis for engineering applications.

1.6 RADIATION CONDITIONS

When the outer boundary of a domain recedes to infinity, the domain is called *unbounded* or *open*. A condition must be specified at this outer boundary to obtain a unique solution for the problem. Such a condition is referred to as a *radiation condition.*

Assuming that all sources and objects are immersed in free space and located within a finite distance from the origin of a coordinate system, the electric and magnetic fields are required to satisfy

$$\lim_{r \to \infty} r \left[\nabla \times \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \mathrm{j}k_0 \hat{r} \times \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \right] = 0, \tag{1.68}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Equation (1.68) is usually referred to as the *Sommerfeld radiation condition* for general three-dimensional fields. For two-dimensional fields, the Sommerfeld radiation condition becomes

$$\lim_{\rho \to \infty} \sqrt{\rho} \left[\frac{\partial}{\partial \rho} \begin{pmatrix} E_z \\ H_z \end{pmatrix} + \mathrm{j}k_0 \begin{pmatrix} E_z \\ H_z \end{pmatrix} \right] = 0, \tag{1.69}$$

where $\rho = \sqrt{x^2 + y^2}$.

Equations (1.68) and (1.69) are exactly valid at infinity. In numerical analysis, it is often desirable to reduce the size of a computational domain by using a finite boundary to truncate an infinite domain. When applied at such a finite boundary, (1.68) and (1.69) can be regarded as first-order approximations with limited accuracy. Better accuracy can be achieved by developing higher-order approximations (Chapter 9).

1.7 FIELDS IN AN INFINITE HOMOGENEOUS MEDIUM

In a general inhomogeneous medium, (1.42) and (1.43) do not have a closed-form solution, and a numerical method is often required to seek their approximate solution. If, however, the problem at hand involves an infinite homogeneous medium, where ϵ and μ are constant, closed-form solutions may be obtained. Take free space as example, where $\epsilon = \epsilon_0$ and $\mu = \mu_0$. In such a medium, (1.42) and (1.43) become

$$\nabla^2\mathbf{E} + k_0^2\mathbf{E} = \mathrm{j}\omega\mu\mathbf{J} + \frac{1}{\epsilon_0}\nabla\varrho \tag{1.70}$$

and

$$\nabla^2\mathbf{H} + k_0^2\mathbf{H} = -\nabla \times \mathbf{J}, \tag{1.71}$$

with the use of (1.23) and (1.24). It is important to recognize that (1.70) is equivalent to (1.42) only when it is solved in conjunction with (1.23); similarly, (1.71) is equivalent to (1.43) only when it is solved together with (1.24). Using the principle of linear superposition, one can find the solution to (1.70) and (1.71) as

$$\mathbf{E}(\mathbf{r}) = \iiint_V \left[-\mathrm{j}\omega\mu_0\mathbf{J}(\mathbf{r}') - \frac{1}{\epsilon_0}\nabla'\varrho(\mathbf{r}')\right] G_0(\mathbf{r},\mathbf{r}')\,\mathrm{d}V' \tag{1.72}$$

and

$$\mathbf{H}(\mathbf{r}) = \iiint_V \left[\nabla' \times \mathbf{J}(\mathbf{r}')\right] G_0(\mathbf{r},\mathbf{r}')\,\mathrm{d}V', \tag{1.73}$$

where $G_0(\mathbf{r},\mathbf{r}')$ denotes the point-source response, known as the *scalar Green's function* (see Appendix C), given by

$$G_0(\mathbf{r},\mathbf{r}') = \frac{e^{-\mathrm{j}k_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{1.74}$$

Equations (1.72) and (1.73) can also be written as

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \iiint_V \left[-\mathrm{j}\omega\mu_0\mathbf{J}(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}') + \frac{1}{\epsilon_0}\varrho(\mathbf{r}')\nabla'G_0(\mathbf{r},\mathbf{r}')\right]\mathrm{d}V' \\ &= \left(-\mathrm{j}\omega\mu_0 + \frac{1}{\mathrm{j}\omega\epsilon_0}\nabla\nabla\cdot\right)\iiint_V \mathbf{J}(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}')\,\mathrm{d}V' \end{aligned} \tag{1.75}$$

and

$$\mathbf{H}(\mathbf{r}) = \iiint_V \mathbf{J}(\mathbf{r}') \times \nabla'G_0(\mathbf{r},\mathbf{r}')\,\mathrm{d}V' = \nabla \times \iiint_V \mathbf{J}(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}')\,\mathrm{d}V', \tag{1.76}$$

with the use of vector identities and integral theorems.

Therefore, if a source is known in an infinite homogeneous medium, the fields can readily be calculated by evaluating appropriate integrals over the source. These results are also applicable to surface or line sources provided that the volume integrals are replaced by the corresponding surface or line integrals. They can also be reduced for two-dimensional problems by using the relation

$$\int_{-\infty}^{\infty} G_0(\mathbf{r},\mathbf{r}')\,\mathrm{d}z' = G_0(\boldsymbol{\rho},\boldsymbol{\rho}'), \tag{1.77}$$

where

$$G_0(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{1}{4\mathrm{j}}H_0^{(2)}(k_0|\boldsymbol{\rho}-\boldsymbol{\rho}'|), \tag{1.78}$$

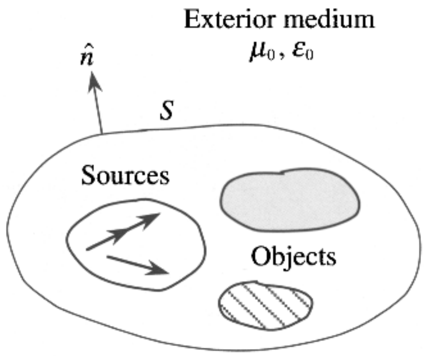

Figure 1.2 Huygens's principle.

in which $\boldsymbol{\rho} = x\hat{x} + y\hat{y}$ and $H_0^{(2)}(k_0|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ is the zeroth-order Hankel function of the second kind.

Exercise 1.7 In a homogeneous and source-free region, (1.43) is reduced to $\nabla \times (\nabla \times \mathbf{H}) - k^2\mathbf{H} = 0$ and (1.71) is reduced to $\nabla^2\mathbf{H} + k^2\mathbf{H} = 0$. Show that $\mathbf{H} = \hat{x}H_0 \mathrm{e}^{-\mathrm{j}kx}$ does not satisfy the former equation, but satisfies the latter. Show that this field does not satisfy (1.24) either. This exercise demonstrates that the solutions to (1.70) and (1.71) do not necessarily satisfy Maxwell's equations.

Exercise 1.8 Show that (1.75) and (1.76) are equivalent to (1.72) and (1.73) by using vector identities and integral theorems.

1.8 HUYGENS'S PRINCIPLE

The preceding section provided expressions to calculate the fields radiated by a given current source. However, in the finite element method we usually solve for the fields surrounding an object instead of the currents induced on the object. Therefore, it would be useful to have expressions to calculate the fields everywhere based on knowledge of the fields surrounding an object. Huygens's principle provides such relations.

Consider a surface S enclosing the source of radiation and any other objects so that the infinite space exterior to the surface is homogeneous (Fig. 1.2). If the fields on the surface are known, the fields outside the surface are given by

$$\mathbf{E}(\mathbf{r}) = \oiint_S \{-\mathrm{j}\omega\mu_0 [\hat{n}' \times \mathbf{H}(\mathbf{r}')] G_0(\mathbf{r}, \mathbf{r}') + [\hat{n}' \cdot \mathbf{E}(\mathbf{r}')] \nabla' G_0(\mathbf{r}, \mathbf{r}') + [\hat{n}' \times \mathbf{E}(\mathbf{r}')] \times \nabla' G_0(\mathbf{r}, \mathbf{r}')\} \, \mathrm{d}S' \tag{1.79}$$

and

$$\mathbf{H}(\mathbf{r}) = \oiint_S \{\mathrm{j}\omega\epsilon_0 [\hat{n}' \times \mathbf{E}(\mathbf{r}')] G_0(\mathbf{r}, \mathbf{r}') + [\hat{n}' \cdot \mathbf{H}(\mathbf{r}')] \nabla' G_0(\mathbf{r}, \mathbf{r}') + [\hat{n}' \times \mathbf{H}(\mathbf{r}')] \times \nabla' G_0(\mathbf{r}, \mathbf{r}')\} \, \mathrm{d}S', \tag{1.80}$$

where $\hat{n}'$ denotes the unit vector normal to S at $\mathbf{r}'$ and pointing toward the exterior region. These results can be derived rigorously using Green's theorems (Chapter 10).

If we let $\hat{n}' \times \mathbf{H}(\mathbf{r}') = \mathbf{J}_s(\mathbf{r}')$, $\mathbf{E}(\mathbf{r}') \times \hat{n}' = \mathbf{M}_s(\mathbf{r}')$, $\hat{n}' \cdot \epsilon_0 \mathbf{E}(\mathbf{r}') = \varrho_{e,s}(\mathbf{r}')$, and $\hat{n}' \cdot \mu_0 \mathbf{H}(\mathbf{r}') = \varrho_{m,s}(\mathbf{r}')$, then (1.79) and (1.80) can be written as

$$\mathbf{E}(\mathbf{r}) = \oiint_S \Big[-\mathrm{j}\omega\mu_0 \mathbf{J}_s(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}') + \frac{1}{\epsilon_0}\varrho_{e,s}(\mathbf{r}')\nabla' G_0(\mathbf{r},\mathbf{r}') - \mathbf{M}_s(\mathbf{r}') \times \nabla' G_0(\mathbf{r},\mathbf{r}')\Big] \mathrm{d}S' \tag{1.81}$$

and

$$\mathbf{H}(\mathbf{r}) = \oiint_S \Big[\mathbf{J}_s(\mathbf{r}') \times \nabla' G_0(\mathbf{r},\mathbf{r}') - \mathrm{j}\omega\epsilon_0 \mathbf{M}_s(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}') + \frac{1}{\mu_0}\varrho_{m,s}(\mathbf{r}')\nabla' G_0(\mathbf{r},\mathbf{r}')\Big] \mathrm{d}S'. \tag{1.82}$$

A comparison of these two equations with (1.75) and (1.76) indicates that $\mathbf{J}_s$ and $\mathbf{M}_s$ are equivalent to surface electric and magnetic currents, and $\varrho_{e,s}$ and $\varrho_{m,s}$ are equivalent to surface electric and magnetic charges. Because of this interpretation, (1.81) and (1.82) are also known as the *surface equivalence principle.*

Again, all of these results can be reduced to two dimensions using the relation in (1.77). In particular, for both E_z- and H_z-polarization cases, we have

$$\phi(\boldsymbol{\rho}) = \oint_\Gamma \left[\phi(\boldsymbol{\rho}') \frac{\partial G_0(\boldsymbol{\rho},\boldsymbol{\rho}')}{\partial n'} - G_0(\boldsymbol{\rho},\boldsymbol{\rho}') \frac{\partial \phi(\boldsymbol{\rho}')}{\partial n'} \right] \mathrm{d}\Gamma', \tag{1.83}$$

where $\phi = E_z$ for E_z-polarization, $\phi = H_z$ for H_z-polarization, and Γ is a contour that encloses the source of the field as well as all possible objects.

Exercise 1.9 Derive (1.83) from (1.79) and (1.80) by using (1.77). Rewrite the right-hand side of (1.83) in terms of equivalent surface currents for both E_z- and H_z-polarizations.

1.9 RADAR CROSS SECTIONS

One of the emphases of this book is the treatment of open-region scattering problems. An important parameter that characterizes scattering by an object is called *radar cross section* [12]. This parameter is defined for plane-wave incidence.

In the three-dimensional case, the radar cross section is defined by

$$\begin{aligned}\sigma(\theta,\varphi;\theta^{\mathrm{inc}},\varphi^{\mathrm{inc}}) &= \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}^{\mathrm{sc}}(r,\theta,\varphi)|^2}{|\mathbf{E}^{\mathrm{inc}}(\theta^{\mathrm{inc}},\varphi^{\mathrm{inc}})|^2} \\ &= \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{H}^{\mathrm{sc}}(r,\theta,\varphi)|^2}{|\mathbf{H}^{\mathrm{inc}}(\theta^{\mathrm{inc}},\varphi^{\mathrm{inc}})|^2},\end{aligned} \tag{1.84}$$

where $(\mathbf{E}^{\mathrm{sc}}, \mathbf{H}^{\mathrm{sc}})$ denote the scattered field observed in the direction (θ, φ) and $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ denote the incident field from the direction $(\theta^{\mathrm{inc}}, \varphi^{\mathrm{inc}})$. When the incident and observation directions are the same, σ is called the *monostatic* or *backscatter* radar cross section; otherwise, it is referred to as the *bistatic* radar cross section.

The radar cross section is often normalized to either λ^2 or m^2. The unit for σ/λ^2 is dBsw and for σ/m^2 is dBsm when expressed in the logarithmic scale. To incorporate information about polarization, the radar cross section can also be defined by

$$\sigma_{pq}(\theta,\varphi;\theta^{\mathrm{inc}},\varphi^{\mathrm{inc}}) = \lim_{r\to\infty} 4\pi r^2 \frac{|E_p^{\mathrm{sc}}(r,\theta,\varphi)|^2}{|E_q^{\mathrm{inc}}(\theta^{\mathrm{inc}},\varphi^{\mathrm{inc}})|^2}, \tag{1.85}$$

where p and q represent either θ or φ.

To evaluate the radar cross section, we have to compute the scattered field in the far-field zone with $r \to \infty$ using expressions such as (1.81) and (1.82). These expressions can be simplified greatly for $r \to \infty$. For example, under this condition, (1.81) and (1.82) become

$$\mathbf{E}(\mathbf{r}) \approx \frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{4\pi r} \oint\!\!\oint_S \{\mathrm{j}\omega\mu_0 \hat{r} \times [\hat{r} \times \mathbf{J}_s(\mathbf{r}')] + \mathrm{j}k_0 \hat{r} \times \mathbf{M}_s(\mathbf{r}')\}\, \mathrm{e}^{\mathrm{j}k_0 \hat{r}\cdot\mathbf{r}'} \mathrm{d}S' \tag{1.86}$$

and

$$\mathbf{H}(\mathbf{r}) \approx \frac{\mathrm{e}^{-\mathrm{j}k_0 r}}{4\pi r} \oint\!\!\oint_S \{\mathrm{j}\omega\epsilon_0 \hat{r} \times [\hat{r} \times \mathbf{M}_s(\mathbf{r}')] - \mathrm{j}k_0 \hat{r} \times \mathbf{J}_s(\mathbf{r}')\}\, \mathrm{e}^{\mathrm{j}k_0 \hat{r}\cdot\mathbf{r}'} \mathrm{d}S'. \tag{1.87}$$

In the two-dimensional case, the radar cross section is defined by

$$\begin{aligned}\sigma(\varphi,\varphi^{\mathrm{inc}}) &= \lim_{\rho\to\infty} 2\pi\rho \frac{|\mathbf{E}^{\mathrm{sc}}(\rho,\varphi)|^2}{|\mathbf{E}^{\mathrm{inc}}(\varphi^{\mathrm{inc}})|^2} \\ &= \lim_{\rho\to\infty} 2\pi\rho \frac{|\mathbf{H}^{\mathrm{sc}}(\rho,\varphi)|^2}{|\mathbf{H}^{\mathrm{inc}}(\varphi^{\mathrm{inc}})|^2},\end{aligned} \tag{1.88}$$

which is also referred to as *scattering width* or *echo width* in the monostatic case. It is often normalized to λ, with a unit of dBw, or m, with a unit of dBm, when expressed in the logarithmic scale. The scattered field in the far-field zone can be evaluated using expressions such as (1.83). Under the condition $\rho \to \infty$, equation (1.83) can be simplified as

$$\phi(\boldsymbol{\rho}) \approx \sqrt{\frac{\mathrm{j}k_0}{8\pi\rho}}\mathrm{e}^{-\mathrm{j}k_0\rho} \oint_\Gamma \left[\hat{\rho}\cdot\hat{n}'\phi(\boldsymbol{\rho}') - \frac{1}{\mathrm{j}k_0}\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'}\right] \mathrm{e}^{\mathrm{j}k_0\hat{\rho}\cdot\boldsymbol{\rho}'}\,\mathrm{d}\Gamma', \tag{1.89}$$

by using the asymptotic expression for the Hankel function.

Exercise 1.10 Derive (1.86) and (1.87) from (1.81) and (1.82) when $r \to \infty$. Derive (1.89) from (1.83) when $\rho \to \infty$.

1.10 SUMMARY

In this chapter we first reviewed the basic concepts and integral theorems in vector analysis. We then presented Maxwell's equations in integral form as the fundamental equations, from which Maxwell's equations in differential form and boundary conditions were derived. The electrostatic problem was formulated in terms of a scalar potential, and the magnetostatic problem was formulated in terms of a vector potential. However, the time-harmonic problem was formulated directly in terms of the electric or magnetic field, although it was also possible to use scalar and vector potentials for formulation. The corresponding partial differential equations were derived in the form of the Poisson, Helmholtz, and vector wave equations. This was followed by the presentation of a variety of boundary conditions and radiation conditions. These will all be used in the finite element analysis of electromagnetic problems.

This chapter also reviewed the integral expressions that relate the field to its source in free space and Huygens's principle for calculating exterior fields from the field on a closed surface. These will be used frequently in the chapters to follow. Finally, we presented the definition of radar cross section and its calculation because it is used often to present the results of the finite element analysis of scattering problems.

Although this chapter has not reviewed some other important concepts, such as the duality principle, uniqueness theorem, reciprocity theorem, and various equivalence principles, a good comprehension of these concepts can help greatly in computational electromagnetics research. The reader is encouraged to consult References 2–9 for this purpose.

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