Chapter 1

First Ideas

We will begin a study of partial differential equations by deriving equations modeling diffusion processes and wave motion. These are widely applicable in the physical and life sciences, engineering, economics, and other areas. Following this, we will lay the foundations for the Fourier method, which is used to write solutions for many kinds of problems, and then solve two eigenvalue/eigenfunction problems that occur frequently when this method is used.

The chapter concludes with a proof of a theorem on the convergence of Fourier series.

1.1 Two Partial Differential Equations

1.1.1 The Heat, or Diffusion, Equation

We will derive a partial differential equation modeling heat flow in a medium. Although we will speak in terms of heat flow because it is familiar to us, the heat equation applies to general diffusion processes, which might be a flow of energy, a dispersion of insect or bacterial populations in controlled environments, changes in the concentration of a chemical dissolving in a fluid, or many other phenomena of interest. For this reason the heat equation is also called the diffusion equation.

Consider a bar of material of constant density, ρ, having uniform cross sections with area A. The lateral surface of the bar is insulated, so there is no heat loss across this surface.

Place an x-axis along the length, L, of the bar and assume that at a given time, the temperature is the same along any cross section perpendicular to this axis, although it may vary from one cross section to another. We will derive an equation for $u(x, t)$, the temperature in the cross section of the bar at x, at time t. In the context of diffusion, $u(x, t)$ is called a *density distribution function*.

Let c be the specific heat of the material of the bar. This is the amount of heat energy that must be supplied to a unit mass of the material to raise

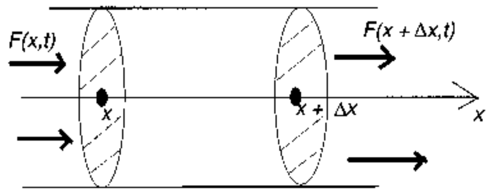

Figure 1.1: Flux in segment = rate in minus rate out.

its temperature one degree. The segment of bar between x and $x + \Delta x$ has mass $\rho A \Delta x$, and it will take approximately $\rho c A u(x,t)\Delta x$ units of heat energy to change the temperature of this segment from zero to $u(x,t)$, its temperature at time t.

The total heat energy in this segment at any time $t > 0$ is

$$E(x, \Delta x, t) = \int_x^{x+\Delta x} \rho c A u(\xi, t)\, d\xi.$$

This amount of heat energy within the segment at time t can increase in two ways: heat energy may flow into the segment across its ends (this change is the flux of the energy), and/or there may be a source or loss of heat energy within the segment. This can occur if there is, say, a chemical reaction or if the material is radioactive.

The rate of change of the temperature within the segment, with respect to time, is therefore

$$\begin{aligned} \frac{\partial E}{\partial t} &= \text{flux plus source or sink} \\ &= \int_x^{x+\Delta x} \rho c A \frac{\partial u}{\partial t}(\xi, t)\, d\xi. \end{aligned}$$

Assume for now that there is no source or loss of energy within the bar. Then

$$\text{flux } = \int_x^{x+\Delta x} \rho c A \frac{\partial u}{\partial t}(\xi, t)\, d\xi. \tag{1.1}$$

Now let $F(x,t)$ be the amount of heat energy per unit area flowing across the cross section at x at time t, in the direction of increasing x. Then the flux of the energy into the segment between x and $x + \Delta x$ at time t is the rate of flow into the segment across the section at x, minus the rate of flow out of the segment across the section at $x + \Delta x$ (Figure 1.1):

$$\text{flux } = AF(x,t) - AF(x + \Delta x, t).$$

Write this as

$$\text{flux } = -A(F(x + \Delta x, t) - F(x,t)). \tag{1.2}$$

Now recall Newton's law of cooling, which states that heat energy flows from the warmer to the cooler region, and the amount of heat energy is proportional to the temperature difference (gradient). This means that

$$F(x,t) = -K\frac{\partial u}{\partial x}(x,t).$$

The positive constant of proportionality, K, is called the *heat conductivity* of the bar. The negative sign in this equation is due to the fact that energy flows from the warmer to the cooler segment. Substitute this expression for $F(x,t)$ into equation 1.2 to obtain

$$\text{flux } = -A\left(-K\frac{\partial u}{\partial x}(x+\Delta x,t) + K\frac{\partial u}{\partial x}(x,t)\right).$$

Write this as

$$\text{flux } = \int_x^{x+\Delta x} \frac{\partial}{\partial x}\left(KA\frac{\partial u}{\partial x}(\xi,t)\right)d\xi. \tag{1.3}$$

From equations 1.1 and 1.3 for the flux, we have

$$\int_x^{x+\Delta x} \rho c A\frac{\partial u}{\partial t}(\xi,t)\,d\xi = \int_x^{x+\Delta x} \frac{\partial}{\partial x}\left(KA\frac{\partial u}{\partial x}(\xi,t)\right)d\xi.$$

Divide out the common factor A and write this equation as

$$\int_x^{x+\Delta x}\left[\rho c\frac{\partial u}{\partial t}(\xi,t) - \frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}(\xi,t)\right)\right]d\xi = 0.$$

This equation must be valid for any choices of x and $x+\Delta x$, as long as

$$0 < x < x+\Delta x < L.$$

If the integrand were nonzero at some x, then, assuming continuity of this integrand (which is reasonable on physical grounds), it would be nonzero, therefore strictly positive or strictly negative on some interval $(x, x+\Delta x)$. This would force this integral to be positive or negative, not zero, for this x and Δx, and this is a contradiction. We conclude that the integrand must be identically zero, hence

$$\rho c\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}.$$

It is convenient to denote partial derivatives using subscripts. In this notation,

$$u_t = ku_{xx}, \tag{1.4}$$

where $k = K/c\rho$ is called the *diffusivity* of the material of the bar. Equation 1.4 is the *one-dimensional heat, or diffusion, equation.* This equation, with appropriate boundary and initial conditions, models a wide range of diffusion phenomena, providing a setting for a mathematical analysis to draw conclusions about the behavior of the process under study.

If we allow for a source term $Q(x,t)$, then the heat equation is

$$u_t = ku_{xx} + Q(x,t). \tag{1.5}$$

We say that equation 1.4 is *homogeneous*. Because of the $Q(x,t)$ term, equation 1.5 is *nonhomogeneous*. Both equations are *second-order* partial differential equations because they contain at least one second derivative term, but no higher derivative. Both equations are also *linear*, which means they are linear in the unknown function and its derivatives. By contrast, the second-order partial differential equation

$$u_t = ku_{xx} + uu_x$$

is nonlinear because of the uu_x term, which allows for an interaction between the density function, u, and its rate of change with respect to x.

The linear, homogeneous heat equation $u_t = ku_{xx}$ has the important features that a finite sum of solutions and a product of a solution by a constant are again solutions. That is, if $u_1(x,y)$ and $u_2(x,y)$ are solutions, then $au_1(x,y)+bu_2(x,y)$ is also a solution for any numbers a and b. This can be verified by substituting $au_1 + bu_2$ into equation 1.4. This is not the case with the nonhomogeneous equation 1.5, as can also be seen by substitution.

Everyday experience suggests that to know the temperature in a bar of material at any time we have to have some information, such as the temperature throughout the bar at some particular time (this is an *initial condition*), together with information about the temperatures at the ends of the bar (these are *boundary conditions*). A typical initial condition has the form

$$u(x,0) = f(x) \text{ for } 0 < x < L,$$

in which $f(x)$ is a given function. *Initial* is taken as time zero as a convenience.

Boundary conditions specify conditions at end points of the space variable (or perhaps on a surface in higher dimensional models). These can take different forms. One commonly seen set of boundary conditions is

$$u(0,t) = \alpha(t), u(L,t) = \beta(t) \text{ for } 0 < x < L,$$

where $\alpha(t)$ and $\beta(t)$ are given functions. These specify conditions at the left and right ends of the material at all times.

Boundary conditions may also reflect other physical conditions at the boundary. We will see some of these when we solve specific problems in different settings.

A problem consisting of the heat equation, together with initial and boundary conditions, is called a *initial-boundary value problem for the heat equation.*

1.1.2 The Wave Equation

Imagine a string (guitar string, wire, telephone line, power line, or the like) suspended between two points. We want to describe the motion of the string

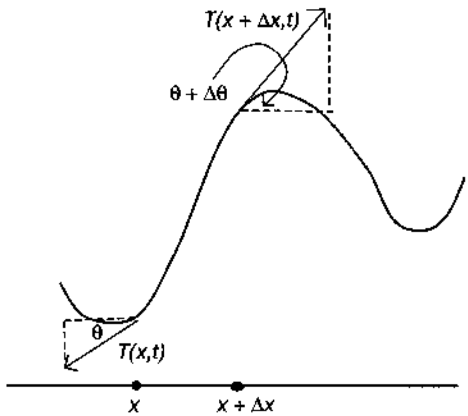

Figure 1.2: Segment of string between x and $x + \Delta x$.

if it is fixed at its ends, displaced in a specified way and released with a given velocity.

Place an x-axis along the straightened string from 0 to L, and assume that each particle of string moves only vertically in a plane. We seek a function $u(x,t)$ so that, at any time $t \geq 0$, the graph of the function $u = u(x,t)$ gives the position or shape of the string at that time. This enables us to view snapshots of the string in motion.

Begin with a simple case by neglecting damping effects, such as air resistance and the weight of the string. Let $\mathbf{T}(x,t)$ be the tension in the string at point x and time t, and assume that this acts tangentially to the string. The magnitude of this vector is $T(x,t) = \| \mathbf{T}(x,t) \|$. Also assume that the mass, ρ, per unit length is constant.

Apply Newton's second law of motion to the segment of string between x and $x+\Delta x$. This states that the net force on the segment due to the tension is equal to the acceleration of the center of mass of the segment times the mass of the segment. This is a vector equation, meaning that we can match the horizontal components and the vertical components of both sides. Looking at the vertical components in Figure 1.2 gives us approximately

$$T(x + \Delta x, t)\sin(\theta + \Delta\theta) - T(x,t)\sin(\theta) = \rho(\Delta x)u_{tt}(\overline{x}, t),$$

in which $\overline{x}$ is the center of mass of this segment of string. Then

$$\frac{T(x + \Delta x, t)\sin(\theta + \Delta\theta) - T(x,t)\sin(\theta)}{\Delta x} = \rho u_{tt}(\overline{x}, t).$$

The vertical component, $v(x,t)$, of the tension is

$$v(x,t) = T(x,t)\sin(\theta).$$

Then

$$\frac{v(x+\Delta x,t)-v(x,t)}{\Delta x}=\rho u_{tt}(\overline{x},t).$$

Let $\Delta x \to 0$. Then $\overline{x} \to x$, and this equation yields

$$v_x = \rho u_{tt}.$$

The horizontal component of the tension is

$$h(x,t) = T(x,t)\cos(\theta).$$

But

$$v(x,t) = h(x,t)\tan(\theta) = h(x,t)u_x,$$

so

$$v_x = (hu_x)_x = \rho u_{tt}.$$

By assumption, the horizontal component of the tension on the entire segment of string is zero:

$$h(x+\Delta x,t) - h(x,t) = 0.$$

Therefore, $h(x,t)$ is independent of x, and

$$(hu_x)_x = hu_{xx}.$$

Then

$$hu_{xx} = \rho u_{tt}.$$

Or, in its more traditional form,

$$u_{tt} = c^2 u_{xx}, \tag{1.6}$$

where $c^2 = h/\rho$. Equation 1.6 is the *one-dimensional wave equation* (one space dimension).

If a *forcing term* is included to allow other forces acting on the string, then the wave equation may take the form

$$u_{tt} = c^2 u_{xx} + P(x,t).$$

As with the heat equation, we attempt to solve the wave equation subject to initial and boundary conditions specifying the position of the string at time $t = 0$, and the forces that set the string in motion.

The boundary conditions if the ends of the string are fixed are

$$u(0,t) = u(L,t) = 0 \text{ for } t > 0.$$

We will also see variations on these boundary conditions. For example, if the ends are in motion, with their positions at time t given as functions of t, then

$$u(0,t) = \alpha(t), u(L,t) = \beta(t) \text{ for } t > 0,$$

for some given functions $\alpha(t)$ and $\beta(t)$.

Initial conditions take the form

$$u(x,0) = \varphi(x) \text{ and } u_t(x,0) = \psi(x) \text{ for } 0 < x < L,$$

specifying the initial position and velocity of the string. Equation 1.6, together with boundary and initial conditions, is called an *initial-boundary value problem for the wave equation*.

As we develop methods of solving these and other partial differential equations, under a variety of initial and boundary conditions, we will also explore properties of solutions and questions such as the sensitivity of solutions to small perturbations of initial and boundary conditions.

Problems for Section 1.1

1. Show that
$$u(x,t) = \cos(\alpha\pi x)e^{-\alpha^2\pi^2 t}$$
is a solution of the heat equation with $k = 1$, on any interval $[0, L]$.

2. Show that
$$u(x,t) = t^{-3/2}e^{-x^2/4kt}$$
is a solution of $u_t = ku_{xx}$ for $x > 0, t > 0$. Show also that this solution is unbounded.

3. Show that
$$u(x,t) = a\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$
satisfies wave equation 1.6, with a any constant, c and L positive constants, and n any positive integer.

4. Let f be a differentiable function of a single variable, defined on the entire real line. Show that
$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct))$$
is a solution of the wave equation $u_{tt} = c^2 u_{xx}$ for all x and t, and that $u(x,0) = f(x)$.

5. Let $\psi(x)$ be continuous on the real line. Let
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.$$
Show that $u(x,t)$ satisfies the wave equation and that
$$u_t(x,0) = \psi(x) \text{ for } 0 < x < L.$$

6. Let φ and ψ be continuous on $[0, L]$. Let

$$u(x,t) = \frac{1}{2}(\varphi(x-ct) + \varphi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds.$$

   Show that $u(x,t)$ satisfies the wave equation and also the initial conditions $u(x,0) = \varphi(x)$ and $u_t(x,0) = \psi(x)$.

Problems 7–12 deal with a classification of second-order partial differential equations that are linear with constant coefficients in the second derivative terms. Such an equation has the form

$$Au_{xx} + Bu_{xt} + Cu_{tt} + H(x,t,u,u_x,u_t) = 0. \tag{1.7}$$

A, B, and C are constants; A and B are not both zero; and $H(x,t,u,u_x,u_t)$ is any function of x, t, u, u_x, and u_t. Thus the equation may not be linear in the first derivative terms or terms involving u. It is always possible to transform equation 1.7 to one of three standard, or canonical, forms. These problems explore how to do this.

7. Start with a change of variables

$$\xi = x + at, \eta = x + bt.$$

   Show that this transformation from the x, t-plane to a ξ, η-plane is invertible if $a \neq b$, and that

$$x = \frac{1}{b-a}(b\xi - a\eta), t = \frac{1}{b-a}(\eta - \xi).$$

8. Let $u(x(\xi,\eta), t(\xi,\eta)) = V(\xi,\eta)$, obtained by substituting for x and y in terms of ξ and η in equation 1.7. Show that the resulting partial differential equation for V is

$$\begin{aligned}&(A + aB + a^2C)V_{\xi\xi} + (2A + (a+b)B + 2abC)V_{\xi\eta}\\ &+ (A + bB + b^2C)V_{\eta\eta} + K(\xi,\eta,V,V_\xi,V_\eta) = 0.\end{aligned} \tag{1.8}$$

   Hint: Use the chain rule to compute u_{xx}, u_{xt}, and u_{tt} in terms of partial derivatives of $V(\xi,\eta)$.

9. Suppose $B^2 - 4AC > 0$. Try to choose a and b to make the coefficients of $V_{\xi\xi}$ and $V_{\eta\eta}$ vanish. This requires that we solve for a and b so that

$$Ca^2 + Ba + A = 0 \text{ and } Cb^2 + Bb + A = 0.$$

   Notice that a and b both satisfy the same quadratic equation, having coefficients A, B, and C. Show that, if $C \neq 0$, then equation 1.7 transforms to

$$V_{\xi\eta} + K(\xi,\eta,V,V_\xi,V_\eta) = 0.$$

by choosing

$$a = \frac{-B + \sqrt{B^2 - 4AC}}{2C} \text{ and } b = \frac{-B - \sqrt{B^2 - 4AC}}{2C}.$$

In this case we say that equation 1.7 is *hyperbolic*. The transformed equation is the *canonical form of the hyperbolic equation*.

If $C = 0$, show that we can choose

$$\xi = t, \eta = -\frac{B}{A}x + t$$

to obtain the hyperbolic canonical form.

10. Show that, if $B^2 - 4AC = 0$, then by choosing $a = 0$ and $b = -B/2C$, equation 1.7 transforms to

$$V_{\xi\xi} + K(\xi, \eta, V, V_\xi, V_\eta) = 0.$$

In this case equation 1.7 is called *parabolic* and the transformed equation is called the *canonical form of the parabolic equation*.

11. Finally, suppose $B^2 - 4AC < 0$. Now the roots of $Ca^2 + Ba + A = 0$ are complex, say $p \pm iq$. Define the transformation

$$\xi = x + pt, \eta = qt.$$

and show that this transforms equation 1.7 to

$$V_{\xi\xi} + V_{\eta\eta} + K(\xi, \eta, V, V_\xi, V_\eta) = 0.$$

In this case, equation 1.7 is said to be *elliptic* and the transformed equation is the *canonical form of the elliptic equation*.

12. Classify the diffusion equation and the wave equation as being elliptic, parabolic, or hyperbolic.

In each of problems 13–17, classify the partial differential equation and determine its canonical form.

13. $4u_{xx} - 2u_{xt} + u_{tt} + 2u_x - xu = 0.$

14. $2u_{xx} + u_{xt} - 4u_{tt} + x + t = 0.$

15. $u_{xx} - 3u_{xt} - xu = 0.$

16. $u_{xx} + 9u_{tt} + x^2 - tu = 0.$

17. $u_{xx} - 2u_{xt} + 3u_{tt} + 12u^2 = 0.$

1.2 Fourier Series

In attempting to solve problems involving the heat equation, French mathematician Joseph Fourier (1768–1830) announced that he could write solutions by expanding the initial temperature function in an infinite series of sines and/or cosines of different frequencies. Because nearly any function (for example, a differentiable function) could be an initial temperature function, this led to the astounding assertion that almost any function one could think of had such a trigonometric series representation. This was too much for the rest of the scientific community to accept.

Nevertheless, Fourier's method did appear to solve significant problems. Intensive research, carried out in the eighteenth and nineteenth centuries, justified Fourier's claims, and Fourier series now have many applications. In this section we outline the fundamental idea of a Fourier series, enabling us to use these series to solve initial-boundary value problems.

1.2.1 The Fourier Series of a Function

Given $f(x)$ defined on $[-L, L]$, we want to choose numbers $a_0, a_1, \cdots$ and $b_1, b_2, \cdots$ such that

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{1.9}$$

on this interval. This is not always possible, but we will explore the idea to see when it might work.

Fourier was not the first to imagine such a thing. The great Swiss mathematician Leonhard Euler (1707–1783) devised a way of calculating the $a_n's$ and $b_n's$ in the series 1.9. While lacking in rigor, Euler's approach is interesting and actually leads to the correct choice of the coefficients. We follow Euler's reasoning here, with a proof given in section 1.4.

Euler's approach was based on some easily derived trigonometric integrals. If n and k are positive integers, then

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx = 0,$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{k\pi x}{L}\right) dx = 0 \text{ if } n \neq k,$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx = 0 \text{ if } n \neq k.$$

These are called *orthogonality relations* for reasons that will be clarified when we treat eigenfunction expansions in Chapter 7.

In addition,

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx = L \text{ for } n = 1, 2, \cdots .$$

Now assume equation 1.9, and suppose that we can interchange the summation and an integration. In this case,

$$\begin{aligned}
&\int_{-L}^{L} f(x)\,dx \\
&= \int_{-L}^{L} \frac{1}{2}a_0\,dx + \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx + b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) dx\right] \\
&= La_0,
\end{aligned}$$

because the integrals of $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ over $[-L, L]$ are all zero. The integrated equation therefore reduces to

$$\int_{-L}^{L} f(x)\,dx = La_0,$$

from which we conclude that

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx. \tag{1.10}$$

This is a formula for a_0. Next we want to obtain formulas for a_k with $k = 1, 2, \cdots$. Let k be any positive integer. Multiply equation 1.9 by $\cos(k\pi x/L)$ and integrate to obtain

$$\begin{aligned}
&\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right) dx \\
&= \int_{-L}^{L} \frac{1}{2}a_0 \cos\left(\frac{k\pi x}{L}\right) dx \\
&+ \sum_{n=1}^{\infty}\left[a_n \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right) dx + b_n \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right) dx\right].
\end{aligned}$$

Now,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx = 0.$$

Further, by the orthogonality relations, all of the integrals in the summation are zero except for the integral

$$\int_{-L}^{L} \cos\left(\frac{k\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right) dx,$$

which occurs when $n = k$. This integral equals L. We therefore have

$$\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx = a_k L,$$

from which

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right) \text{ for } k = 1, 2, 3, \cdots. \tag{1.11}$$

Notice that this reproduces the formula for a_0 when $k = 0$.

Similarly, if we multiply equation 1.9 by $\sin(k\pi x/L)$ and integrate term by term, all terms vanish except the $n = k$ term in the integrals of the sine terms, and we obtain

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx. \tag{1.12}$$

Equations 1.10–1.12 are the *Fourier coefficients* of $f(x)$ on $[-L, L]$. When these Fourier coefficients are used, the series on the right side of equation 1.9 is called the *Fourier series* of $f(x)$ on $[-L, L]$.

Now we must be careful not to overreach. Although we have a plausible rationale for the selection of the Fourier coefficients of a function, we have no reason to believe that this Fourier series actually converges to the function at all (or any!) points of the interval. The following two examples are revealing in this regard.

Example 1.1 Let

$$f(x) = \begin{cases} 0 & \text{for } -3 \le x < 0, \\ 2 + x & \text{for } 0 \le x \le 3. \end{cases}$$

We will write the Fourier series of $f(x)$ on $[-3, 3]$. Compute the coefficients:

$$\begin{aligned} a_0 &= \frac{1}{3}\int_{-3}^{3} f(x)\,dx \\ &= \frac{1}{3}\int_{0}^{3} (2 + x)\,dx = \frac{7}{2}, \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{1}{3}\int_{0}^{3} (2 + x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{3((-1)^n - 1)}{n^2\pi^2}, \end{aligned}$$

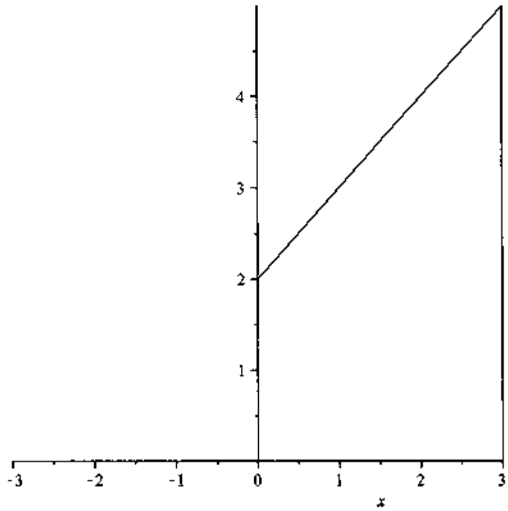

Figure 1.3: Graph of $f(x)$ from example 1.1.

and

$$\begin{aligned} b_n &= \frac{1}{3}\int_{-3}^{3} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{1}{3}\int_{0}^{3} (2+x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2-5(-1)^n}{n\pi}. \end{aligned}$$

The Fourier series of $f(x)$ on $[-3,3]$ is

$$\frac{7}{4}+\sum\left[\frac{3((-1)^n-1)}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right)+\frac{2-5(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right].$$

Figure 1.3 is a graph of the function, and Figures 1.4 and 1.5 compare the function with the 10th and 50th partial sums, respectively, of its Fourier series. These graphs suggest that the series converges to $f(x)$ for $-3 < x < 0$ and for $0 < x < 3$. However, at $x = 0$, the series does not appear to converge to $f(0)$, which is 2. And at both 3 and -3, the Fourier series is the same:

$$\frac{7}{4}+\sum_{n=1}^{\infty}\left(\frac{3((-1)^n-1)}{n^2\pi^2}(-1)^n\right).$$

This series cannot converge to both $f(-3) = 0$ and to $f(3) = 5$.

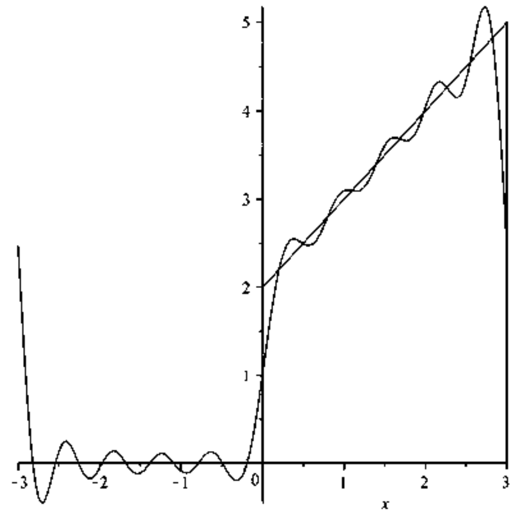

Figure 1.4: Comparison of $f(x)$ with the 10th partial sum in example 1.1.

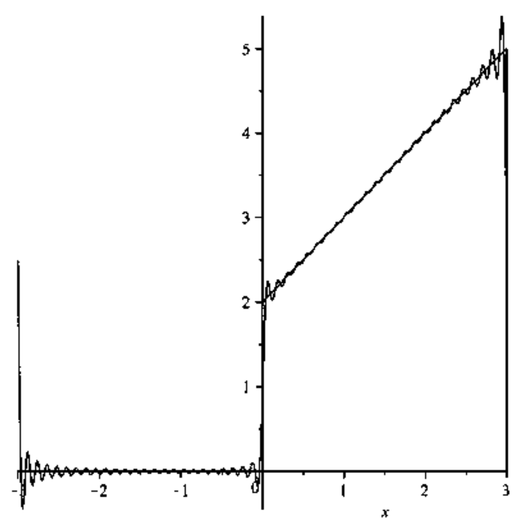

Figure 1.5: Comparison of $f(x)$ with the 50th partial sum in example 1.1.

Example 1.2 Let

$$g(x) = \begin{cases} -1 & \text{for } -2 \le x < 1, \\ 3 & \text{for } 1 \le x < 3/2, \\ -5 & \text{for } 3/2 \le x \le 2. \end{cases}$$

The Fourier coefficients of $g(x)$ on $[-2, 2]$ are

$$\begin{aligned} a_0 &= -2, \\ a_n &= \frac{1}{n\pi}(-4\sin(n\pi/2) + 8\sin(3n\pi/4)), \\ b_n &= \frac{4}{n\pi}((-1)^n + \cos(n\pi/2) - 2\cos(3n\pi/4)). \end{aligned}$$

The Fourier series is

$$-1 + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}(-4\sin(n\pi/2) + 8\sin(3n\pi/4))\cos\left(\frac{n\pi x}{2}\right)\right.$$
$$\left. + \frac{4}{n\pi}((-1)^n + \cos(n\pi/2) - 2\cos(3n\pi/4))\sin\left(\frac{n\pi x}{2}\right)\right].$$

Figure 1.6 is a graph of this function, and Figures 1.7 and 1.8 are graphs of the tenth and fiftieth partial sums of the Fourier series, respectively. It does appear that the series converges to $g(x)$ for $-2 < x < 1$, $1 < x < 3/2$, and $3/2 < x < 2$. However, it is not clear what the series converges to at $x = -2, 1, 3/2$ or 2. And, as we saw in example 1.1, this Fourier series is the same at both end points of the interval, even though $g(-2) \neq g(2)$.

Because of examples like these, we need something to tell us the sum of a Fourier at points on the interval. One criterion for convergence is in terms of the familiar notions of continuity and differentiability. We say that $f(x)$ is *piecewise continuous* on $[a, b]$ if the following three conditions are satisfied:

1. $f(x)$ is continuous at all but possibly finitely many points of $[a, b]$.

2. If there is a point c with $a < c < b$ at which $f(x)$ is discontinuous, then $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ are both finite. That is, $f(x)$ has finite one-sided limits at every point of discontinuity interior to the interval (if there are any such points).

3. $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ are both finite. This means that, at the end points of the interval, the function has finite limits as x approaches the end point from within the interval.

These conditions mean that any discontinuities the function has on the interval are *jump discontinuities*, so-called because the graph has a gap or jump at such a point. The function of example 1.1 has a jump discontinuity at $x = 0$ (Figure 1.3), while the function of example 1.2 has jump discontinuities at $x = 1$ and $x = 3/2$ (Figure 1.6). Both of these functions are piecewise continuous on their interval of definition.

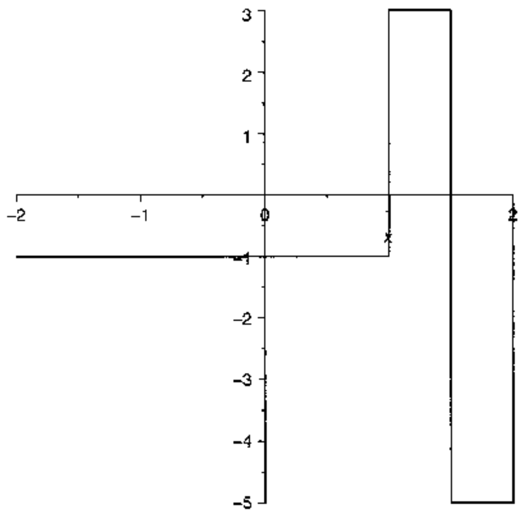

Figure 1.6: Graph of $g(x)$ of example 1.2.

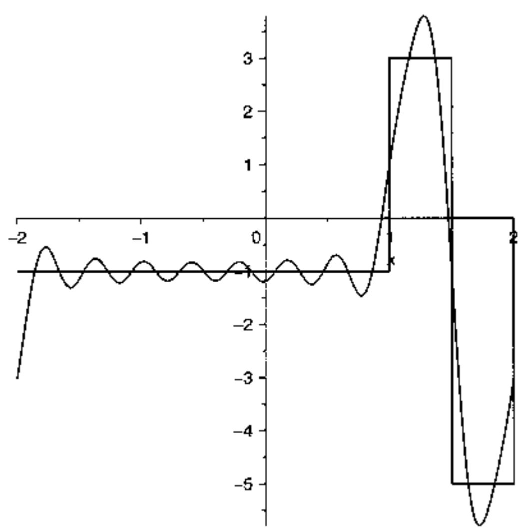

Figure 1.7: Comparison of $g(x)$ with the 10th partial sum in example 1.2.

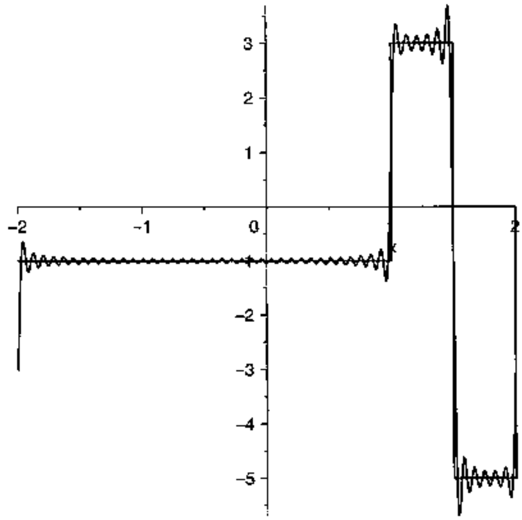

Figure 1.8: Comparison of $g(x)$ with the 50th partial sum in example 1.2.

$f(x)$ is *piecewise smooth* on $[a, b]$ if $f(x)$ and its derivative $f'(x)$ are both piecewise continuous on the interval.

Piecewise smooth means that the graph has a continuous tangent at all but finitely many points, and any discontinuities of the function exhibit themselves in finite jumps or gaps in the graph. The functions of examples 1.1 and 1.2 are piecewise smooth.

Finally, we will use the standard notation

$$f(x-) = \lim_{h \to 0+} f(x-h) \text{ and } f(x+) = \lim_{h \to 0+} f(x+h).$$

$f(x-)$ is the *left limit* of the function at x, and $f(x+)$ is the *right limit* at x. The plus and minus signs in the notation refer only to left and right limits, and x itself may be positive, negative or zero.

In example 1.1,

$$f(0-) = 0 \text{ and } f(0+) = 2,$$

while for $-3 < x < 0$ and $0 < x < 3$, $f(x-) = f(x+) = f(x)$. Further, $f(-3+) = 0$ and $f(3-) = 5$.

At any point at which the function is continuous, the left and right limits equal the function value at the point.

In example 1.2,

$$g(1-) = \lim_{h \to 0+} g(1-h) = -1$$

while

$$g(1+) = \lim_{h \to 0+} g(1+h) = 3.$$

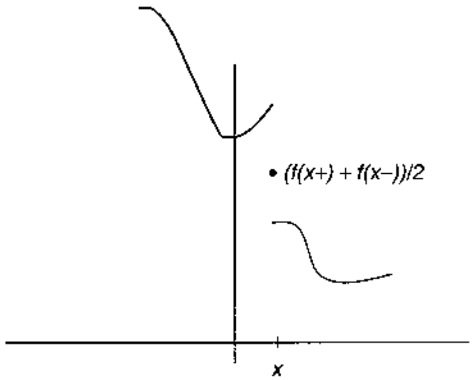

Figure 1.9: Convergence of a Fourier series at a jump discontinuity.

For $-2 < x < 1, 1 < x < 3/2$ and $3/2 < x < 2$, $g(x-) = g(x+) = g(x)$ because $g(x)$ is continuous on these intervals. And, $g(-2+) = -1$ and $g(2-) = -5$.

With these ideas and notation, we can state the following.

Theorem 1.1 (Convergence of Fourier Series) Let $f(x)$ be piecewise smooth on $[-L, L]$. If $-L < x < L$, then the Fourier series of $f(x)$ on this interval converges to

$$\frac{1}{2}(f(x-) + f(x+)).$$

Further, at both $x = L$ and $x = -L$, the Fourier series converges to

$$\frac{1}{2}(f(-L+) + f(L-)).$$

Figure 1.9 displays this behavior. If the function has a jump discontinuity at x, then the graph has a gap at x and the Fourier series converges to the average of the left and right limits of the function at x. This is the point midway between the ends of the graph at the gap. At any x where the function is continuous, the series converges to $f(x)$, because at such a point, $f(x-) = f(x+) = f(x)$.

At both end points L and $-L$, the Fourier series converges to the average of the left limit of the function at L, and the right limit at $-L$.

In example 1.1, the Fourier series of $f(x)$ converges to

$$\begin{cases} 0 & \text{for } -3 < x < 0, \\ 1 & \text{at } x = 0, \\ 2 + x & \text{for } 0 < x < 3 \\ 5/2 & \text{at } x = -3 \text{ and at } x = 3. \end{cases}$$

This conclusion at the end points 3 and -3 follows from the facts that $f(-3+) = 0$ and $f(3-) = 5$, and the average of these limits is $5/2$.

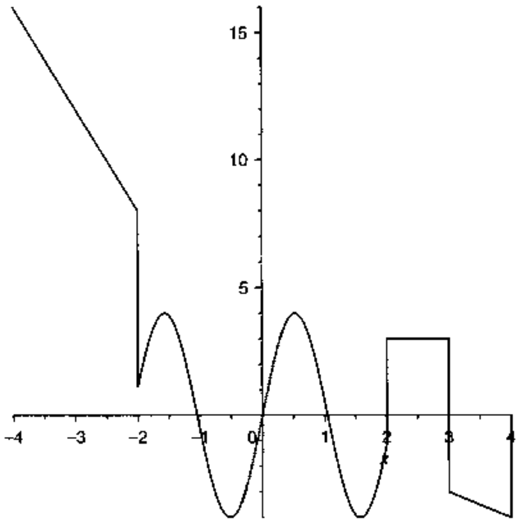

Figure 1.10: The function $w(x)$ of example 1.3.

In example 1.2, the Fourier series of $g(x)$ converges to

$$\begin{cases} -1 & \text{for } -2 < x < 1, \\ 1 & \text{at } x = 1, \\ 3 & \text{for } 1 < x < 3/2, \\ -1 & \text{at } x = 3/2, \\ -5 & \text{for } 3/2 < x < 2 \\ -3 & \text{at } x = -2 \text{ and at } x = 2. \end{cases}$$

The conclusion at the end points follows from the facts that $g(-2+) = -1$ and $g(2-) = -5$, and the average of -1 and -5 is -3.

Example 1.3 Let

$$w(x) = \begin{cases} -4x & \text{for } -4 \le x < -2, \\ 4\sin(3x) & \text{for } -2 \le x < 2, \\ 3 & \text{for } 2 \le x < 3, \\ -x & \text{for } 3 \le x \le 4. \end{cases}$$

Figure 1.10 is a graph of this function. The Fourier series of $w(x)$ on $-[4, 4]$

converges to

$$\begin{cases} -4x & \text{for } -4 < x < -2, \\ \frac{1}{2}(8 - 4\sin(6)) & \text{at } x = -2, \\ 4\sin(3x) & \text{for } -2 < x < 2, \\ \frac{1}{2}(3 + 4\sin(6)) & \text{at } x = 2, \\ -x & \text{for } 3 < x < 4 \\ 6 & \text{at } x = -4 \text{ and at } x = 4. \end{cases}$$

The conclusion at the end points follows from the calculation

$$w(-4+) = -4(-4) = 16 \text{ and } w(4-) = -4.$$

We do not have to compute the Fourier coefficients of $w(x)$ to draw these conclusions.

1.2.2 Fourier Sine and Cosine Series

In solving partial differential equations on an interval $[0, L]$, we will often need to expand a function in a series of just sines, or just cosines, on this half-interval.

The key to such a sine or cosine expansion is to recall some facts about even and odd functions. A function $f(x)$ defined on $[-L, L]$ is called an *even function* if

$$f(-x) = f(x) \text{ for } 0 < x \leq L.$$

Figure 1.11 shows a typical graph of an even function. The part of the graph to the left of the vertical axis is a reflection across this axis of the part to the right. (Fold the paper along the vertical axis and trace the part of the graph for $x > 0$). Examples of even functions are $x^2, x^6, \cos(x)$, and e^{x^2}.

If $f(x)$ is even on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

This is because the area under the graph to the right of the vertical axis equals the area under the graph to the left.

We call $f(x)$ an *odd function* if

$$f(-x) = -f(-x) \text{ for } 0 < x \leq L.$$

Figure 1.12 shows the graph of a typical odd function. The graph to the left of the vertical axis is the reflection through the origin of the graph to the right of the vertical axis. That is, fold the graph for $x > 0$ over the vertical axis, then fold again over the horizontal axis. Examples of odd functions are $x^3, \sin(x)$, and $x^7 - 4x$.

If $f(x)$ is odd on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\,dx = 0$$

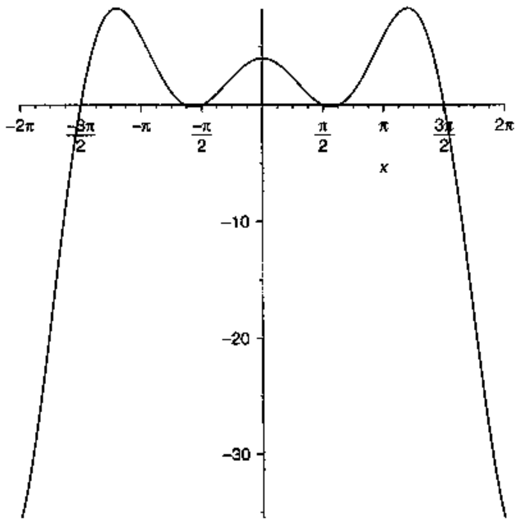

Figure 1.11: A typical even function, symmetric about the vertical axis.

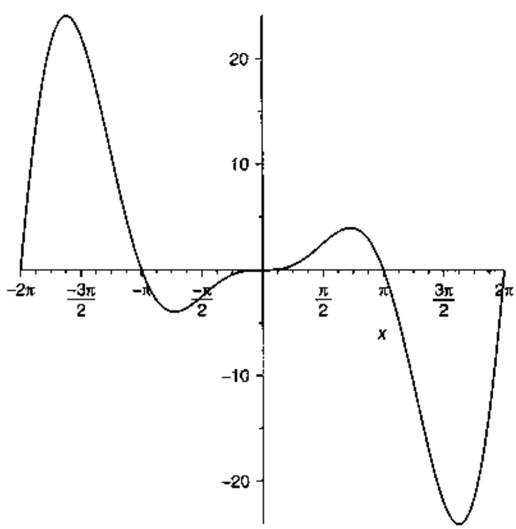

Figure 1.12: A typical odd function, symmetric through the origin.

because the area under the graph to the right of the vertical axis is the negative of the area to the left.

Notice that:

a product of odd functions is even (for example, $x^3x^5 = x^8$),

a product of even functions is even ($x^4x^2 = x^6$), and

a product of an even and an odd function is odd ($x^6x^3 = x^9$).

Now go back to the general setting of a function $g(x)$ defined on $[0, L]$. Extend $g(x)$ to an even function $G_e(x)$ on $[-L, L]$ by defining

$$G_e(x) = \begin{cases} g(x) & \text{for } 0 \leq x \leq L, \\ g(-x) & \text{for } -L \leq x < 0. \end{cases}$$

This defines $G_e(x)$ on $[-L, L]$ by leaving $g(x)$ alone for $0 \leq x \leq L$, and reflecting the graph of $g(x)$ on $[0, L]$ across the vertical axis to define it for $-L \leq x < 0$.

Expand $G_e(x)$ in a Fourier series on $[-L, L]$. Because $G_e(x)\cos(n\pi x/L)$ is an even function and $G_e(x) = g(x)$ for $0 \leq x \leq L$, the cosine coefficients in this expansion are

$$\begin{aligned} A_n &= \frac{1}{L}\int_{-L}^{L} G_e(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_{0}^{L} G_e(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_{0}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx. \end{aligned}$$

Further, $G_e(x)\sin(n\pi x/L)$ is odd on $[-L, L]$, so the coefficients of the sine terms vanish:

$$B_n = \frac{1}{L}\int_{-L}^{L} G_e(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = 0.$$

The Fourier series of $G_e(x)$ on $[-L, L]$ is, therefore,

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right),$$

containing only cosine terms. Because $G_e(x) = g(x)$ on $[0, L]$, this is a cosine expansion of $g(x)$ on $[0, L]$.

In summary, the *Fourier cosine series* of $g(x)$ on $[0, L]$ is

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right), \tag{1.13}$$

in which

$$A_n = \frac{2}{L}\int_{0}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \text{ for } n = 0, 1, 2, 3, \cdots. \tag{1.14}$$

Using $G_e(x)$ on $[-L, L]$ and the convergence theorem, we can determine the sum of this cosine expansion on $[0, L]$. Assuming that $g(x)$ is piecewise smooth, use the fact that $G_e(x) = g(x)$ for $0 \leq x \leq L$ to conclude that the cosine series converges to

$$\frac{1}{2}(g(x-) + g(x+)) \text{ if } 0 < x < L.$$

For $x = 0$, compute the limits:

$$\begin{aligned} G_e(0+) &= \lim_{h \to 0+} G_e(0 + h) = \lim_{h \to 0+} G_e(h) \\ &= \lim_{h \to 0+} g(x) = g(0+) \end{aligned}$$

and

$$\begin{aligned} G_e(0-) &= \lim_{h \to 0+} G_e(0 - h) = \lim_{h \to 0+} G_e(-h) \\ &= \lim_{h \to 0+} g(h) = g(0+). \end{aligned}$$

Therefore, at $x = 0$, the cosine expansion of $g(x)$ converges to

$$\frac{1}{2}(G(0-) + G(0+)) = \frac{1}{2}(g(0+) + g(0+)) = g(0+).$$

A similar argument shows that, at $x = L$, the cosine series for $g(x)$ on $[0, L]$ converges to $g(L-)$.

The even extension of $g(x)$ to $G_e(x)$ was a device used to obtain this half-interval expansion of $g(x)$ from the already known Fourier series of $G_e(x)$ on $[L, L]$. In computing the coefficients A_n in a cosine expansion, we need only $g(x)$ and do not have to explicitly construct $G_e(x)$. Just write the series 1.13, with coefficients from equation 1.14.

Example 1.4 (A Fourier Cosine Expansion) Let $g(x) = e^x$ for $0 \leq x \leq 2$. From equation 1.14, the cosine coefficients of $g(x)$ on this interval are

$$A_0 = \frac{2}{2} \int_0^2 e^x \, dx = e^2 - 1$$

and, for $n = 1, 2, \cdots$,

$$A_n = \frac{2}{2} \int_0^2 e^x \cos\left(\frac{n\pi x}{L}\right) dx = \frac{4}{4 + n^2\pi^2}((-1)^n e^2 - 1).$$

Further, using the convergence theorem, this series will converge to e^x for $0 \leq x \leq 2$:

$$e^x = \frac{1}{2}(e^2 - 1) + \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2}((-1)^n e^2 - 1) \cos\left(\frac{n\pi x}{2}\right).$$

Figure 1.13 shows a graph of $g(x) = e^x$ compared with the 10th partial sum of this cosine series on $[0, 2]$. This cosine expansion appears to converge very quickly to $g(x)$.

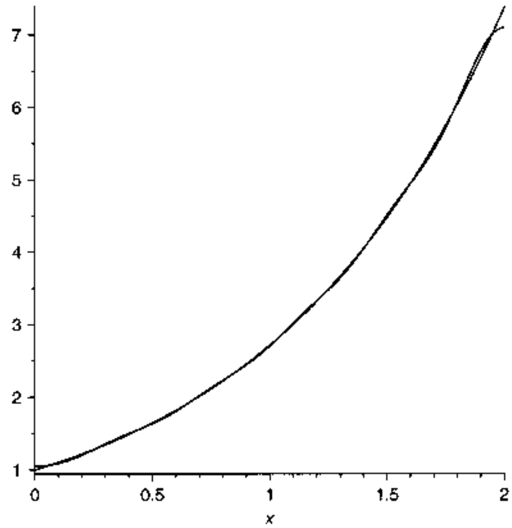

Figure 1.13: $g(x)$ compared with the 10th partial sum of the cosine expansion in example 1.4.

We can develop a Fourier sine expansion of $g(x)$ on $[0, L]$ by a similar tactic. Now, however, because the sine terms in the series are odd functions, extend $g(x)$ to an odd function $G_o(x)$ on $[-L, L]$ by setting

$$G_o(x) = \begin{cases} g(x) & \text{for } 0 \leq x \leq L, \\ -g(-x) & \text{for } -L \leq x < 0. \end{cases}$$

$G_o(x)$ is an odd function and $G_o(x) = g(x)$ on $[0, L]$. Further, the Fourier series for $G_o(x)$ on $[-L, L]$ will contain only sine terms, because the coefficients of the cosine terms involve integrals of $G_o(x)\cos(n\pi x/L)$, and these integrals are zero because this is an odd function on $[-L, L]$.

In summary, the *Fourier sine expansion* of $g(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right), \tag{1.15}$$

where

$$B_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx \text{ for } n = 1, 2, \cdots . \tag{1.16}$$

As with cosine series, we do not actually have to write out $G_o(x)$ to compute these coefficients.

Assuming that $g(x)$ is piecewise smooth on $[0, L]$, this sine series will converge to

$$\frac{1}{2}(g(x-) + g(x+)) \text{ for } 0 < x < L.$$

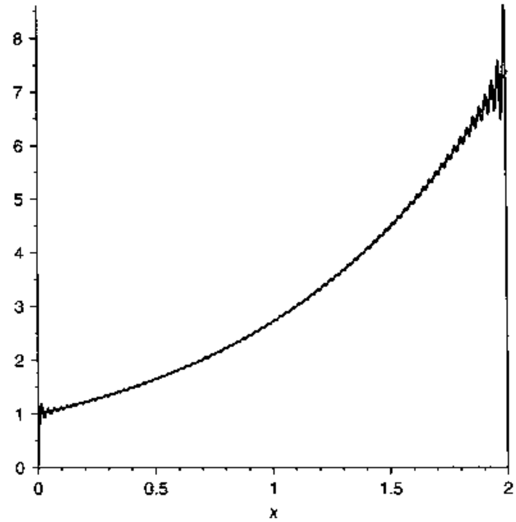

Figure 1.14: e^x compared with the 150th partial sum of the sine expansion in example 1.5.

Without any computation, a sine series converges to 0 at $x = 0$ and at $x = L$, because $\sin(0) = 0$ and $\sin(n\pi) = 0$ for every integer n.

Example 1.5 (A Fourier Sine Expansion) We will write a Fourier sine series for e^x on $[0, 2]$. The coefficients are

$$B_n = \int_0^2 e^x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2n\pi}{4 + n^2\pi^2}\left(1 - (-1)^n e^2\right).$$

This sine series converges to e^x for $0 < x < 2$ (but not at 0 or 2), so we can write

$$e^x = \sum_{n=1}^{\infty} \frac{2n\pi}{4 + n^2\pi^2}\left(1 - (-1)^n e^2\right) \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < 2.$$

Figure 1.14 shows graphs of $g(x)$ and the 150th partial sum of this sine expansion. Contrast this with the much more rapid convergence of the cosine expansion of this function in example 1.4.

Problems for Section 1.2

In each of problems 1–6, write the Fourier series of the function, and determine the sum of this series on the interval. Compare graphs of some partial sums of the series with a graph of the function.

1. $f(x) = -x$ for $-1 \le x \le 1$.

2. $f(x) = \cos(3x)$ for $-\pi \le x \le \pi$.

3. $f(x) = \sin(2x)$ for $-2 \le x \le 2$.

4. $f(x) = 1 - |x|$ for $-2 \le x \le 2$.

5. $f(x) = \begin{cases} -4 & \text{for } -\pi \le x < 0, \\ 4 & \text{for } 0 \le x \le \pi. \end{cases}$

6. $f(x) = \cos(x/2) - \sin(x)$ for $-\pi \le x \le \pi$

In each of problems 7–12, determine the sum of the Fourier series of the function on the interval. In doing this, it is not necessary to compute the Fourier coefficients.

7. $f(x) = \begin{cases} 2x & \text{for } -3 \le x \le 0, \\ 0 & \text{for } -2 < x < 1, \\ x^2 & \text{for } 1 \le x \le 3. \end{cases}$

8. $f(x) = \begin{cases} \cos(x) & \text{for } -2 \le x \le 1/2, \\ \sin(x) & \text{for } 1/2 \le x \le 2. \end{cases}$

9. $f(x) = \begin{cases} -x & \text{for } -4 \le x < 2, \\ 2 & \text{for } 0 \le x \le 4. \end{cases}$

10. $f(x) = \begin{cases} 1 & \text{for } -2 \le x \le 0, \\ -1 & \text{for } 0 < x < 1/2, \\ x^2 & \text{for } 1/2 \le x \le 2. \end{cases}$

11. $f(x) = \begin{cases} \cos(\pi x) & \text{for } -2 \le x < 0, \\ x & \text{for } 0 \le x \le 2. \end{cases}$

12. $f(x) = \begin{cases} 1 - x & \text{for } -3 <\le x \le -1/2, \\ 2 + x & \text{for } -1/2 < x \le 1, \\ 4 - x^2 & \text{for } 1 < x \le 2, \\ 1 - x - x^2 & \text{for } 2 < x \le 3. \end{cases}$

13. Sum both of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Hint: Expand $f(x) = \frac{1}{2}x^2$ in a Fourier series on $[-\pi, \pi]$. Now make choices of x to obtain these series.

14. Suppose

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

for $-L \leq x \leq L$. Multiply this equation by $f(x)$ and assume that the resulting expression can be integrated term by term to derive *Parseval's equation:*

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L}(f(x))^2\,dx.$$

In each of problems 15–22, find the Fourier cosine series and the Fourier sine series for $f(x)$ on the interval. Determine what each series converges to on this interval. Compare graphs of some partial sums of this series with a graph of the function.

15. $f(x) = 4$ for $0 \leq x \leq 3$.

16. $f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ -1 & \text{for } 1 < x \leq 2. \end{cases}$

17. $f(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \pi/2, \\ \sin(x) & \text{for } \pi/2 < x \leq \pi. \end{cases}$

18. $f(x) = 2x$ for $0 \leq x \leq 1$.

19. $f(x) = x^2$ for $0 \leq x \leq 2$.

20. $f(x) = e^{-x}$ for $0 \leq x \leq 1$.

21. $f(x) = \sin(3x)$ for $0 \leq x \leq \pi$.

22. $f(x) = \begin{cases} x & \text{for } 0 \leq x \leq 1, \\ 2 - x & \text{for } 1 \leq x \leq 2. \end{cases}$

23. Sum the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

Hint: Expand $\sin(x)$ in a cosine series on $[0, \pi]$ and evaluate this series at an appropriately chosen point.

1.3 Two Eigenvalue Problems

In solving initial-boundary value problems, we will often encounter the problem:

$$X'' + \lambda X = 0; X(0) = X(L) = 0. \tag{1.17}$$

We want to find numerical values of the constant λ such that there are nontrivial (not identically zero) solutions $X(x)$ of problem 1.17. Such values of λ are called *eigenvalues* of this problem, and corresponding nontrivial solutions $X(x)$ are *eigenfunctions*.

To find the eigenvalues and eigenfunctions of problem 1.17, consider separately the cases that λ is zero, negative, or positive.

If $\lambda = 0$, then $X'' = 0$, so $X(x) = cx + d$ for some numbers c and d. Now

$$X(0) = d = 0$$

and

$$X(L) = cL = 0 \text{ implies that } c = 0.$$

This means that $X(x) = 0$ for all x. The only solution for $X(x)$ in this case is the trivial solution, so 0 is not an eigenvalue of this problem.

If $\lambda < 0$, then we may write $\lambda = -\alpha^2$, where $\alpha > 0$. Now

$$X'' - \alpha^2 X = 0,$$

with general solution

$$X(x) = ce^{\alpha x} + de^{-\alpha x}.$$

Then

$$X(0) = c + d = 0 \text{ implies that } c = -d,$$

so

$$X(x) = c(e^{\alpha x} - e^{-\alpha x}) = 2c\sinh(\alpha x).$$

And,

$$X(L) = 2c\sinh(\alpha L) = 0.$$

Since $\alpha L > 0$, $\sinh(\alpha L) > 0$, so $c = 0$ and $X(x)$ is the trivial function. This problem has no negative eigenvalue.

Finally, if $\lambda > 0$, write $\lambda = \alpha^2$, with $\alpha > 0$. Then

$$X'' + \alpha^2 X = 0,$$

with solutions of the form

$$X(x) = a\cos(\alpha x) + b\sin(\alpha x).$$

Now

$$X(0) = a = 0$$

so $X(x) = b\sin(\alpha x)$. Then

$$X(L) = b\sin(\alpha L) = 0.$$

If $b = 0$ we again have only the trivial solution and have run out of cases. Thus attempt to find solutions with $b \neq 0$. This will require that we choose α so that

$$\sin(\alpha L) = 0.$$

We can manage this if αL is an integer multiple of π, leading us to choose

$$\alpha = \frac{n\pi}{L} \text{ for } n = 1, 2, \cdots .$$

Denote

$$\lambda_n = \frac{n^2\pi^2}{L^2} \text{ for } n = 1, 2, 3, \cdots .$$

These are the eigenvalues of the problem 1.17, indexed by n. Corresponding to each eigenvalue λ_n, we have an eigenfunction

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

or any nonzero constant multiple of this function.

We will also frequently encounter the problem:

$$X'' + \lambda X = 0; X'(0) = X'(L) = 0. \tag{1.18}$$

As with problem 1.17, consider cases on λ.

If $\lambda = 0$, then $X(x) = cx + d$ for some constants c and d. Now $X'(0) = c = 0$, and $X(x) = d$ is a solution. Unlike problem 1.17, 0 is an eigenvalue of this problem, with nonzero constant eigenfunctions.

If $\lambda < 0$, write $\lambda = -\alpha^2$, with $\alpha > 0$, to obtain the general solution

$$X(x) = ce^{\alpha x} + de^{-\alpha x}$$

of the differential equation. Now

$$X'(0) = \alpha c - \alpha d = 0$$

so $c = d$ and

$$X(x) = c(e^{\alpha x} + e^{-\alpha x}) = 2c\cosh(\alpha x).$$

Then

$$X'(L) = 2c\alpha\sinh(\alpha L) = 0.$$

But $\alpha L > 0$, and $\sinh(\alpha L) > 0$, so $c = 0$ and this case has only the trivial solution. Problem 1.18 has no negative eigenvalue.

If $\lambda > 0$, set $\lambda = \alpha^2$, with $\alpha > 0$. Now $X'' + \alpha^2 X = 0$, with the general solution

$$X = c\cos(\alpha x) + d\sin(\alpha x).$$

Now $X'(0) = d\alpha = 0$, so

$$X(x) = c\cos(\alpha x).$$

Then

$$X'(L) = -\alpha c \sin(\alpha L) = 0.$$

To obtain a nontrivial solution, we need $c \neq 0$. This forces us to choose α so that $\sin(\alpha L) = 0$. As in problem 1.17, αL must be a positive integer multiple of π, say $\alpha L = n\pi$. Then α must be chosen as

$$\alpha = \frac{n\pi}{L} \text{ for } n = 1, 2, 3, \cdots .$$

For each positive integer n,

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

is an eigenvalue of problem 1.18, with corresponding eigenfunction

$$X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \text{ for } n = 1, 2, \cdots .$$

To summarize, the eigenvalues and eigenfunctions of problem 1.18 are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, X_n(x) = \cos\left(\frac{n\pi x}{L}\right) \text{ for } n = 0, 1, 2, \cdots .$$

We are now prepared to solve some important problems involving partial differential equations.

Problems for Section 1.3

Find the eigenvalues and eigenfunctions of each of the following problems.

1. $X'' + \lambda X = 0; X(0) = X'(L) = 0.$
2. $X'' + \lambda X = 0; X'(0) = X(L) = 0.$
3. $X'' + \lambda X = 0; X(0) = X(L), X'(0) = X'(L).$
4. $X'' + \lambda X = 0; X(0) = 0, X(L) + 2X'(L) = 0.$

1.4 A Proof of the Convergence Theorem

We will prove the Fourier convergence theorem, providing an understanding of why the series converges to the left and right limit of the function at interior points of the interval.

1.4.1 The Role of Periodicity

Recall that $f(x)$ is *periodic* if $f(x)$ is defined for all real numbers x and, for some positive number p,

$$f(x) = f(x+p)$$

for all x. We call p a *period* of f. If p is a period, so are $2p$, $3p$, and so on. When we speak of *the* period, or *fundamental period* of a function, we mean its smallest period. For example, $\sin(x)$ has period $2n\pi$ for any positive integer n, because

$$\sin(x + 2n\pi) = \sin(x) \text{ for all } x.$$

However, 2π is the smallest period of $\sin(x)$, so we say that $\sin(x)$ has fundamental period 2π.

Of course, "most" functions (for example, polynomials and exponential functions) are not periodic.

Now consider an issue we avoided in informally developing the idea of a Fourier series. For Fourier expansions on an interval $[-L, L]$, or perhaps $[0, L]$, we considered function values $f(x)$ only for x in this interval. Indeed, this is the way we look at things when we use Fourier series to solve initial-boundary value problems involving partial differential equations. If we are studying heat conduction in a bar of metal, we place the bar along an axis and do not think about values of x outside this segment.

However, even though we think of the function as living on an interval in such applications, the Fourier series on $[-L, L]$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right],$$

which is not only defined on the entire real line but is periodic of period $2L$! How can $f(x)$, which is defined only on an interval, be equal to a periodic function (its Fourier series)?

The answer is that the Fourier series actually represents not the function, but its periodic extension to the entire real line.

We can see this with a simple example. Let $f(x) = x$ for $-1 \leq x < 1$. The familiar graph is shown in Figure 1.15. Figure 1.16 shows its periodic extension $\tilde{f}$ to the entire line. This extension is done by sliding the graph forward from $[-1, 1)$ onto the intervals $[1, 3), [3, 5), [5, 7), \cdots$ and backward onto the intervals $[-3, -1), [-5, -3), [-7, -5), \cdots$, as in the diagram. Then $f(x) = \tilde{f}(x)$ for $-1 \leq x < 1$, but $\tilde{f}(x)$ continues on to be defined for all x. Further, $\tilde{f}$ is periodic of period 2, just as the Fourier expansion of $f(x)$ on $[-1, 1]$ is.

In making such an extension, a subtlety appears. Because $\tilde{f}$ is periodic, we cannot assign functions values to $\tilde{f}$ arbitrarily at different points. Given any a, we necessarily have $\tilde{f}(a) = \tilde{f}(a + np)$ for any integer n.

In this example, where $\tilde{f}$ has period $2, \tilde{f}(x) = \tilde{f}(x + 2n)$ for every integer n. In particular,

$$\tilde{f}(-1) = \tilde{f}(-1 + 2) = \tilde{f}(1).$$

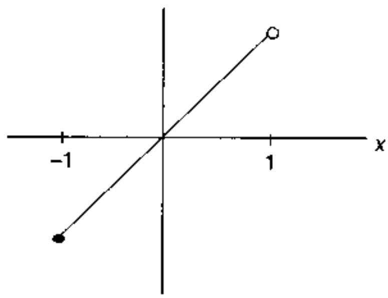

Figure 1.15: Graph of $f(x) = 1$ for $-1 \leq x < 1$.

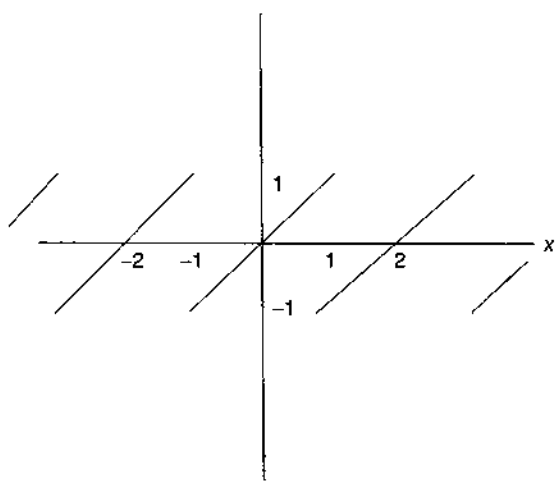

Figure 1.16: Periodic extension $\tilde{f}(x)$ of the graph of Figure 1.15.

It was in anticipation of this that we defined $f(x)$ for $-1 \leq x < 1$ instead of on $-1 \leq x \leq 1$. Once $\tilde{f}(-1)$ is specified, then $\tilde{f}(1)$ is predetermined.

Now notice that, with $\tilde{f}(x)$ defined for all x, there are no endpoints to distinguish in stating a convergence theorem. At *every* real number x, the Fourier series of $\tilde{f}(x)$ on $[-1, 1]$ converges to $(\tilde{f}(x-) + \tilde{f}(x+))/2$.

In particular, at $x = -1$, $\tilde{f}(-1+) = -1$ and $\tilde{f}(-1-) = 1$, so at -1 the series converges to

$$\frac{1}{2}(\tilde{f}(-1+) + \tilde{f}(1-)) = 0.$$

And at $x = 1$, $\tilde{f}(1-) = 1$ and $\tilde{f}(1+) = -1$, so the series also converges to

$$\frac{1}{2}(\tilde{f}(-1+) + \tilde{f}(1-)) = 0.$$

Armed with this point of view, we now develop the machinery needed to prove the convergence theorem, beginning with Dirichlet's formula. We will let $L = \pi$ and work on $[-\pi, \pi)$. This simplifies the frequently encountered expression $n\pi x/L$ to just nx.

1.4.2 Dirichlet's Formula

Convergence of any series depends on the convergence of its sequence of partial sums. Let $f(x)$ be periodic of period 2π. The Nth partial sum of the Fourier expansion of $f(x)$ on $[-\pi, \pi]$ is

$$S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)),$$

in which the $a_n's$ and $b_n's$ are the Fourier coefficients of the function on the interval. We rewrite $S_N(x)$ in a way that will help us determine its limit as $N \to \infty$.

Insert the Fourier coefficients of $f(x)$ into $S_N(x)$ to write:

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)\,d\xi \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{N}\left[\int_{-\pi}^{\pi} f(\xi)\cos(n\xi)\,d\xi\cos(nx) + \int_{-\pi}^{\pi} f(\xi)\sin(n\xi)\,d\xi\sin(nx)\right] \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\left[\frac{1}{2} + \sum_{n=1}^{N}(\cos(n\xi)\cos(nx) + \sin(n\xi)\sin(nx))\right]d\xi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\xi)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos(n(\xi - x))\right]d\xi. \qquad (1.19) \end{aligned}$$

It is possible to derive a simple expression for the sum in square brackets. Let $y = \xi - x$ and let

$$\sigma = \frac{1}{2} + \sum_{n=1}^{N} \cos(ny).$$

Multiply σ by $2\sin(y/2)$ to obtain

$$\begin{aligned}
2\sigma \sin(y/2) &= \sin(y/2) + 2\sum_{n=1}^{N} \cos(ny)\sin(y/2) \\
&= \sin(y/2) + \sum_{n=1}^{N} [\sin((n+1/2)y) - \sin((n-1/2)y)] \\
&= \sin(y/2) + [\sin(3y/2) - \sin(y/2)] + [\sin(5y/2) - \sin(3y/2)] + \cdots \\
&+ [\sin((N-1/2)y) - \sin((N-3/2)y)] \\
&+ [\sin((N+1/2)y) - \sin((N-1/2)y)].
\end{aligned}$$

This is a telescoping sum, with all terms except one canceling on the right. We obtain

$$2\sigma \sin(y/2) = \sin((N+1/2)y).$$

Then

$$\sigma = \frac{\sin((N+1/2)(\xi - x))}{2\sin((\xi - x)/2)},$$

provided that $\sin((\xi - x)/2) \neq 0$. Inserting this result into equation 1.19, we have

$$S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\xi) \frac{\sin((N+1/2)(\xi - x))}{2\sin((\xi - x)/2)}\, d\xi. \tag{1.20}$$

Put $t = \xi - x$ into equation 1.20 to obtain

$$S_N(x) = \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} f(x+t) \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt.$$

Because $f(x)$ is periodic of period 2π, this integrand also has period 2π and we can carry out the integration over any interval of length 2π. In particular, we can write

$$S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \frac{\sin((N+1/2)t)}{2\sin(t/2)}\, dt. \tag{1.21}$$

This is *Dirichlet's formula*. The function

$$\frac{\sin((N+1/2)t)}{2\sin(t/2)}$$

is called the *Dirichlet kernel*. It has the following property, which we will use shortly.

Lemma 1.1

$$\frac{1}{\pi}\int_{-\pi}^{0}\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt = \frac{1}{\pi}\int_{0}^{\pi}\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt = \frac{1}{2}.$$

Proof Let $f(x) = 1$. The Fourier coefficients of $f(x)$ on $[-\pi, \pi]$ are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} d\xi = 2$$

and, for $n = 1, 2, \cdots$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(n\xi)\,d\xi = 0 \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(n\xi)\,d\xi = 0.$$

Therefore, for this function, $S_N(x) = 1$ and Dirichlet's formula 1.21 becomes:

$$\frac{1}{\pi}\int_{-\pi}^{0}\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt + \frac{1}{\pi}\int_{0}^{\pi}\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt = 1. \tag{1.22}$$

But if we let $t = -w$ in the left integral in equation 1.22, we obtain

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{0}\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt &= \frac{1}{\pi}\int_{\pi}^{0}\frac{\sin((N+1/2)w)}{2\sin(w/2)}(-1)\,dw \\ &= \frac{1}{\pi}\int_{0}^{\pi}\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dw.\end{aligned}$$

This means that the integrals on the left in equation 1.22 are equal. Because their sum is 1, each integral equals 1/2, as was to be shown.

1.4.3 The Riemann-Lebesgue Lemma

We will prove a result needed to compute the limit of $S_N(x)$ as $N \to \infty$.

Lemma 1.2 (Riemann-Lebesgue) Let g be piecewise continuous on $[a, b]$. Then

$$\lim_{\omega\to\infty}\int_a^b g(t)\sin(\omega t)\,dt = 0.$$

Proof Suppose first that g is continuous on $[a, b]$. Let

$$I = \int_a^b g(t)\sin(\omega t)\,dt.$$

Let $t = \xi + \pi/\omega$, with ω chosen large enough that $b - \pi/\omega \geq a$. Then

$$I = \int_{a-\pi/\omega}^{b-\pi/\omega} g(\xi + \pi/\omega)\sin(\omega\xi + \pi)\,d\xi = -\int_{a-\pi/\omega}^{b-\pi/\omega} g(\xi + \pi/\omega)\sin(\omega\xi)\,d\xi.$$

To maintain t as the variable of integration, replace ξ with t in the last integral:

$$I = -\int_{a-\pi/\omega}^{b-\pi/\omega} g(t+\pi/\omega)\sin(\omega t)\,dt.$$

Add this expression for I to the definition of I to write

$$\begin{aligned} 2I &= \int_a^b g(t)\sin(\omega t)\,dt - \int_{a-\pi/\omega}^{b-\pi/\omega} g(t+\pi/\omega)\sin(\omega t)\,dt \\ &= \int_a^{b-\pi/\omega} [g(t) - g(t+\pi/\omega)]\sin(\omega t)\,dt \\ &+ \int_{b-\pi/\omega}^{a-\pi/\omega} g(t)\sin(\omega t)\,dt - \int_{a-\pi/\omega}^{a} g(t+\pi/\omega)\sin(\omega t)\,dt. \end{aligned} \tag{1.23}$$

Because g is continuous on a closed interval $[a, b]$, there is some number M such that $|g(t)| \leq M$ on $[a, b]$. Then

$$\left|\int_{b-\pi/\omega}^{b} g(t)\sin(\omega t)\,dt\right| \leq M\frac{\pi}{\omega}$$

and

$$\left|\int_{a-\pi/\omega}^{a} g(t+\pi/\omega)\sin(\omega t)\,dt\right| \leq M\frac{\pi}{\omega}.$$

For the remaining integral in equation 1.23, use the fact that g is uniformly continuous on $[a, b]$. Let $\epsilon > 0$. Then there is some $\delta > 0$ such that

$$|g(x) - g(y)| < \frac{1}{3}\epsilon \text{ if } |x - y| < \delta.$$

Then

$$|g(t) - g(t+\pi/\omega)| < \frac{1}{3}\epsilon \text{ if } \frac{\pi}{\omega} < \delta.$$

Therefore, if $\omega > \pi/\delta$, and ω is also large enough that $b - \pi/\omega \geq a$ and $M\pi/\omega < \epsilon/3$, we have from equation 1.23 and the bounds just obtained that

$$|2I| < M\frac{\pi}{\omega} + M\frac{\pi}{\omega} + \frac{\epsilon}{3} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

But then

$$|I| < \frac{\epsilon}{2} < \epsilon \text{ if } \omega > \pi/\delta.$$

This proves that

$$\lim_{\omega\to\infty} I = 0.$$

Now we must confront the case that g has a finite number of jump discontinuities in $[a, b]$, say at $t_1 < t_2, < \cdots < t_k$. Write I as a sum of integrals, over $[a, t_1], [t_1, t_2], \cdots, [t_k, b]$. By redefining $g(t)$ at the end points of each of these intervals, if necessary, we can write I as a finite sum of integrals, each having the same form as I, but each having a continuous integrand. (This redefinition of $g(t)$ at finitely many points of $[a, b]$ does not affect the value of the integrals). By what we have just shown, each of these integrals has limit 0 as $\omega \to \infty$, completing the proof of the lemma.

1.4.4 Proof of the Convergence Theorem

We will prove that, if f is piecewise smooth on $[-\pi, \pi]$ and periodic of period 2π, then the Fourier series of $f(x)$ on this interval converges at x to

$$\frac{1}{2}\left(f(x-) + f(x+)\right).$$

The argument is essentially the one used by Dirichlet. Use Dirichlet's formula and Lemma 1.1 to write

$$\begin{aligned}
S_N(x) &= \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt \\
&+ \frac{1}{\pi}\int_{0}^{\pi} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{0} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt - \frac{1}{2}f(x-) + \frac{1}{2}f(x-) \\
&+ \frac{1}{\pi}\int_{0}^{\pi} f(x+t)\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt - \frac{1}{2}f(x+) + \frac{1}{2}f(x+) \\
&= \frac{1}{\pi}\int_{-\pi}^{0} [f(x+t) - f(x-)]\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt + \frac{1}{2}f(x-) \\
&+ \frac{1}{\pi}\int_{0}^{\pi} [f(x+t) - f(x+)]\frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt + \frac{1}{2}f(x+).
\end{aligned} \tag{1.24}$$

To complete the proof, it is enough to show that each of the last two integrals in equation 1.24 has limit zero as $N \to \infty$. In this event, we will have shown

$$S_N(x) \to \frac{1}{2}\left(f(x-) + f(x+)\right).$$

To prove that the last integral has limit zero, let

$$g(t) = \frac{f(x+t) - f(x+)}{2\sin(t/2)} \text{ for } 0 < t \leq \pi.$$

Using the fact that $f'(x)$ is piecewise continuous on $[-\pi, \pi]$, observe that

$$\begin{aligned}
\lim_{t\to 0+} g(t) &= \lim_{t\to 0} \frac{f(x+t) - f(x+)}{2\sin(t/2)} \\
&= \lim_{t\to 0+} \frac{f(x+t) - f(x+)}{t}\,\frac{t/2}{\sin(t/2)} \\
&= \lim_{t\to 0+} \frac{f(x+t) - f(x+)}{t}\,\lim_{t\to 0+}\frac{t/2}{\sin(t/2)} \\
&= f'(x+)\cdot 1 = f'(x+).
\end{aligned}$$

Now define

$$g(0) = f'(0+).$$

Because f is piecewise smooth on $[-\pi, \pi]$ and periodic of period 2π, g is piecewise smooth on $[0, \pi]$. By the Riemann-Lebesgue lemma, with $\omega = N + 1/2$,

$$\begin{aligned}
&\lim_{\omega\to\infty} \int_0^\pi g(t)\sin(\omega t)\,dt \\
&= \lim_{N\to\infty} \int_0^\pi |f(x+t) - f(x+)| \frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt = 0.
\end{aligned}$$

This proves that the last integral in equation 1.24 has limit zero as $N \to \infty$. By a similar argument, we can also conclude that

$$\lim_{N\to\infty} \int_{-\pi}^0 |f(x+t) - f(x-)| \frac{\sin((N+1/2)t)}{2\sin(t/2)}\,dt = 0.$$

This proves that

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\left(f(x+) - f(x-)\right),$$

as we wanted to show.

Problem for Section 1.4

At points of discontinuity, Fourier series exhibit the Gibbs phenomenon, named for the Yale mathematician Josiah Willard Gibbs (1839–1903), who was the first to offer an explanation. To see the Gibbs phenomenon in a specific case, let

$$f(x) = \begin{cases} -1 & \text{for } -1 \le x \le 0, \\ 1 & \text{for } 0 < x \le 1. \end{cases}$$

This function has a jump discontinuity at $x = 0$, and the Fourier series of $f(x)$ on $[-1, 1]$ converges to 0 there.

Graph $f(x)$ and partial sums $S_N(x)$ of this Fourier series, for $N = 10$, 20, 40, 60, and 100. You will see the partial sums approach closer to the graph of the function for $-1 < x < 0$ and $0 < x < 1$. However, at 0, the partial sums appear to exhibit oscillations of a height that do not decrease as x approaches 0 (even though the length of interval to the left or right of 0 on which these oscillations occur becomes shorter). This is the *Gibbs phenomenon*, and it is seen in the convergence of Fourier series at jump discontinuities.

In the late 18th century, it was thought that, if a series of functions converges to a function, then the graphs of the partial sums would vary from the limit function by less and less as $N \to \infty$. The Gibbs phenomenon showed that this is not true in the way it was then understood. The length of interval on which the oscillations occur does shorten and tends to zero as $n \to \infty$, but the oscillations remain at about the same height.