Chapter 1

First Ideas

1.1 Two Partial Differential Equations

2. Verifying that the function is a solution of the heat equation is a straightforward exercise in differentiation. One way to show that $u(x,t)$ is unbounded is to observe that if $t > 0$ and $x = 2\sqrt{kt}$, then

$$u(x,t) = \frac{1}{e}t^{-3/2}$$

and this can be made as large as we like by choosing t sufficiently close to zero.

4. By the chain rule,

$$\begin{aligned} u_x &= \frac{1}{2}(f'(x-ct) + f'(x+ct)), \\ u_{xx} &= \frac{1}{2}(f''(x-ct) + f''(x+ct)), \\ u_t &= \frac{1}{2}(f'(x-ct)(-c) + f'(x+ct)(c)), \text{ and} \\ u_{tt} &= \frac{1}{2}(f''(x-ct)(-c)^2 + f''(x+ct)(c)^2). \end{aligned}$$

It is routine to verify that $u_{tt} = c^2 u_{xx}$.

7. One way to show that the transformation is one to one is to evaluate the Jacobian

$$\begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix} = b - a \neq 0.$$

Solutions Manual to Accompany Beginning Partial Differential Equations,
Third Edition. Peter V. O'Neil.

Finally, solve $\xi = a + at$, $\eta = x + bt$ for x and t to obtain the inverse transformation

$$x = \frac{1}{b-a}(b\xi - a\eta), t = \frac{1}{b-a}(\eta - \xi).$$

8. With $V(\xi, \eta) = u(x(\xi, \eta), t(\xi, \eta))$, chain rule differentiations yield:

$$u_x = V_\xi \xi_x + V_\eta \eta_x = V_\xi + V_\eta,$$
$$u_t = V_\xi \xi_t + V_\eta \eta_t = aV_\xi + bV_\eta,$$

and, by continuing these chain rule differentiations and using the product rule,

$$u_{xx} = V_{\xi\xi} + 2V_{\xi\eta} + V_{\eta\eta},$$
$$u_{tt} = a^2 V_{\xi\xi} + 2abV_{\xi\eta} + b^2 V_{\eta\eta}, \text{ and}$$
$$u_{xt} = aV_{\xi\xi} + (a+b)V_{\xi\eta} + bV_{\eta\eta}.$$

Now collect terms to obtain

$$Au_{xx} + Bu_{xt} + Cu_{tt} =$$
$$(A + aB + a^2C)V_{\xi\xi} + (2A + (a+b)B + 2abC)V_{\xi\eta} + (A + bB + b^2C)V_{\eta\eta}.$$

This, coupled with the fact that $H(x, t, u, u_x, u_t)$ transforms to some function $K(\xi, \eta, V, V_\xi, V_\eta)$, yields the conclusion.

9. From the solution of problem 8, the transformed equation is hyperbolic if $C \neq 0$ because in that case we can choose a and b to make the coefficients of $V_{\xi\xi}$ and $V_{\eta\eta}$ vanish. This is done by choosing a and b to be the distinct roots of

$$A + Ba + Ca^2 = 0 \text{ and } A + Bb + Cb^2$$

which are the same quadratic equation. For example, we could choose

$$a = \frac{-B + \sqrt{B^2 - 4AC}}{2C} \text{ and } b = \frac{-B - \sqrt{B^2 - 4AC}}{2C}.$$

If $C = 0$, use the transformation

$$\xi = t, \eta = -\frac{B}{A}x + t.$$

Now chain rule differentiations yield

$$u_x = -\frac{B}{A}V_\eta, \; u_t = V_\xi + V_\eta,$$
$$u_{xx} = \frac{B^2}{A^2}V_{\eta\eta}, \; u_{xt} = -\frac{B}{A}V_{\xi\eta} - \frac{B}{A}V_{\eta\eta}.$$

We do not need u_{tt}, because $C = 0$ in this case. Now we obtain

$$Au_{xx} + Bu_{xt} + Cu_{tt} = -\frac{B^2}{A}V_{\xi\eta},$$

yielding a hyperbolic canonical form

$$V_{\xi\eta} + K(\xi, \eta, V, V_\xi, V_\eta) = 0$$

of the given partial differential equation.

10. In this case suppose $B^2 - 4AC = 0$. Now let

$$\xi = x,\ \eta = x - \frac{B}{2C}t.$$

Now

$$\begin{aligned} u_x &= V_\xi + V_\eta,\ u_t = -\frac{B}{2C}V_\eta, \\ u_{xx} &= V_{\xi\xi} + 2V_{\xi\eta} + V_{\eta\eta},\ u_{tt} = \frac{B^2}{4C^2}V_{\eta\eta},\ \text{and} \\ u_{xt} &= -\frac{B}{2C}V_{\xi\eta} - \frac{B}{2C}V_{\eta\eta}. \end{aligned}$$

Then

$$\begin{aligned} &Au_{xx} + Bu_{xt} + Cu_{tt} \\ &\quad = A(V_{\xi\xi} + 2V_{\xi\eta} + V_{\eta\eta}) - \frac{B^2}{2C}(V_{\xi\eta} + V_{\eta\eta}) + \frac{B^2}{4C}V_{\eta\eta} \\ &\quad = AV_{\xi\xi} + V_{\xi\eta}\left(2A - \frac{B^2}{2C}\right) + V_{\eta\eta}\left(A - \frac{B^2}{2C} + \frac{B^2}{4C}\right) \\ &\quad = AV_{\xi\xi}, \end{aligned}$$

with two terms on the next to last line vanishing because $B^2 - 4AC = 0$. This gives the canonical form

$$V_{\xi\xi} + K(\xi, \eta, V, V_\xi, V_\eta) = 0$$

for the original partial differential equation when $B^2 - 4AC = 0$.

11. Suppose now that $B^2 - 4AC < 0$. Let the roots of $Ca^2 + Ba + A = 0$ be $p \pm iq$. Let

$$\xi = x + pt,\ \eta = qt.$$

Proceeding as in the preceding two problems, we find that

$$\begin{aligned} &Au_{xx} + Bu_{xt} + Cu_{tt} \\ &\quad = (A + Bp + Cp^2)V_{\xi\xi} + (qB + 2pqC)V_{\xi\eta} + q^2V_{\eta\eta}. \end{aligned}$$

Now we need some information about p and q. Because of the way $p+iq$ was chosen,

$$C(p+iq)^2 + B(p+iq) + A = 0.$$

This gives us

$$Cp^2 - Cq^2 + Bp + A + (2Cpq + Bq)i = 0.$$

Then

$$Cp^2 - Cq^2 + Bp = 0 \text{ and } 2Cpq + Bq = 0.$$

In this case,

$$Au_{xx} + Bu_{xt} + Cu_{tt} = q^2(V_{\xi\xi} + V_{\eta\eta})$$

and we obtain the canonical form

$$V_{\xi\xi} + V_{\eta\eta} + K(\xi, \eta, V, V_\xi, V_\eta) = 0$$

for this case.

12. The diffusion equation is parabolic and the wave equation is hyperbolic.

14. $B^2 - 4AC = 33 > 0$, so the equation is hyperbolic. With

$$a = \frac{1+\sqrt{33}}{8} \text{ and } b = \frac{1-\sqrt{33}}{8}$$

the canonical form is

$$V_{\xi\eta} - \frac{16}{49\sqrt{33}}\left(\frac{-7-\sqrt{33}}{8}\xi + \frac{7-\sqrt{33}}{8}\eta\right).$$

16. With $A = 1, B = 0$, and $C = 0$, $B^2 - 4AC = -36 < 9$, so the equation is elliptic. Solve $9a^2 + 1 = 0$ to get $a = \pm i/3$. Thus use the transformation

$$\xi = x,\ \eta = \frac{1}{3}t$$

to obtain the canonical form

$$V_{\xi\xi} + V_{\eta\eta} + \xi^2 - 3\eta V = 0.$$

1.2 Fourier Series

2. $\cos(3x)$ is the Fourier series of $\cos(3x)$ on $[-\pi, \pi]$. This converges to $\cos(3x)$ for $-\pi \le x \le \pi$.

4. The Fourier series of $f(x)$ on $[-2, 2]$ is

$$\sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2}\cos(n\pi x/2),$$

converging to $1 - |x|$ for $-2 \le x \le 2$. Figure 1.1 compares a graph of $f(x)$ with the fifth partial sum of the series.

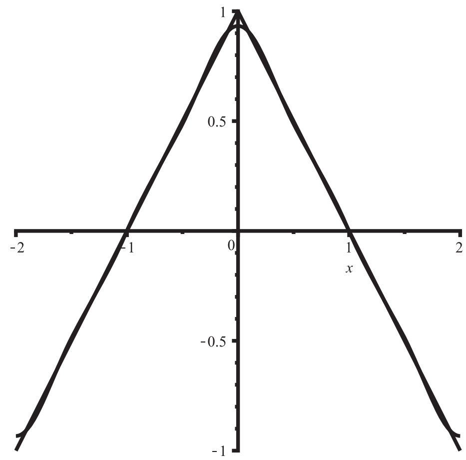

Figure 1.1: $f(x)$ and the 5th partial sum of the Fourier series in Problem 4.

6. The Fourier series is

$$\frac{2}{\pi} + \frac{4}{3\pi}\cos(x) - \sin(x)$$
$$+ \sum_{n=2}^{\infty} \frac{4(-1)^{n+1}}{\pi(4n^2-1)}\cos(nx).$$

Figure 1.2 compares a graph of the function with the fifth partial sum of the series.

8. The Fourier series converges to

$$\begin{cases} \cos(x) & \text{for } -2 < x < 1/2, \\ \sin(x) & \text{for } 1/2 < x < 2, \\ (\cos(2)+\sin(2))/2 & \text{for } x = \pm 2. \end{cases}$$

10. The series converges to

$$\begin{cases} 1 & \text{for } -2 < x < 0, \\ -1 & \text{for } 0 < x, 1/2, \\ x^2 & \text{for } 1/2 < x < 2, \\ 0 & \text{at } x = 0, \\ -3/8 & \text{at } x = 1/2, \\ 5/2 & \text{at } x = \pm 2. \end{cases}$$

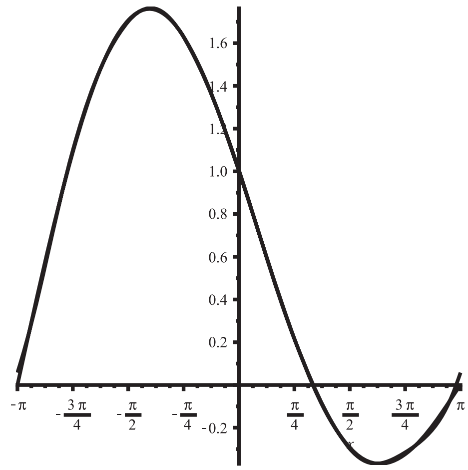

Figure 1.2: $f(x)$ and the 5th partial sum of the Fourier series in Problem 6.

12. The series converges to

$$\begin{cases} 1-x & \text{for } -3 < x < -1/2, \\ 2+x & \text{for } -1/2 < x < 1, \\ 4-x^2 & \text{for } 1 < x < 2, \\ 1-x-x^2 & \text{for } 2 < x < 3, \\ 3/2 & \text{at } x = -1/2, \\ 3 & \text{at } x = 1, \\ -5/2 & \text{at } x = 2, \\ -7/2 & \text{at } x = \pm 3. \end{cases}$$

14. Multiply by $f(x)$ to obtain

$$\begin{aligned}(f(x))^2 =& \frac{1}{2}a_0 f(x) \\ &+ \sum_{n=1}^{\infty} \left(a_n f(x)\cos(n\pi x/L) + b_n f(x)\sin(n\pi x/L)\right).\end{aligned}$$

Integrate term by term:

$$\int_{-L}^{L} (f(x))^2\,dx = \frac{1}{2}a_0 \int_{-L}^{L} f(x)\,dx$$
$$+ \sum_{n=1}^{\infty} \left(a_n \int_{-L}^{L} f(x)\cos(n\pi x/L)\,dx + b_n \int_{-L}^{L} f(x)\sin(n\pi x/L)\,dx \right).$$

Then

$$\int_{-L}^{L} (f(x))^2\,dx = \frac{1}{2}a_0(La_0) + \sum_{n=1}^{\infty} L(a_n^2 + b_n^2).$$

Upon division by L, this yields Parseval's equation.

16. The cosine series is

$$\sum_{n=1}^{\infty} \frac{4\sin(n\pi/2)}{n\pi}\cos(n\pi x/2),$$

converging to 1 for $0 \le x < 1$, to -1 for $1 < x \le 2$, and to 0 at $x = 1$. Figure 1.3 compares the function to the 100th partial sum of this cosine expansion.

The sine series is

$$\sum_{n=1}^{\infty} \frac{1}{n\pi}(-4\cos(n\pi/2) + 2(1 + (-1)^n))\sin(n\pi x/2),$$

converging to 0 at the end points and at 1, and to the function for $0 < x < 1$ and $1 < x < 2$. Figure 1.4 is the 100th partial sum of this sine series.

18. The cosine expansion is

$$1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(-1 + (-1)^n)\cos(n\pi x).$$

This converges to $f(x)$ on $[0, 1]$. Figure 1.5 compares the function with the 10th partial sum of this cosine series.

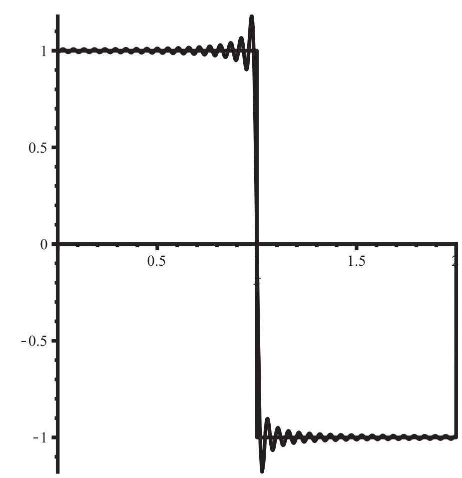

Figure 1.3: $f(x)$ and the 100th partial sum of the cosine series in Problem 16.

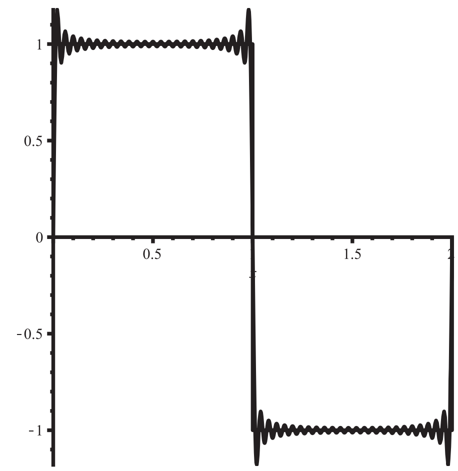

Figure 1.4: $f(x)$ and the 100th partial sum of the sine expansion in Problem 16.

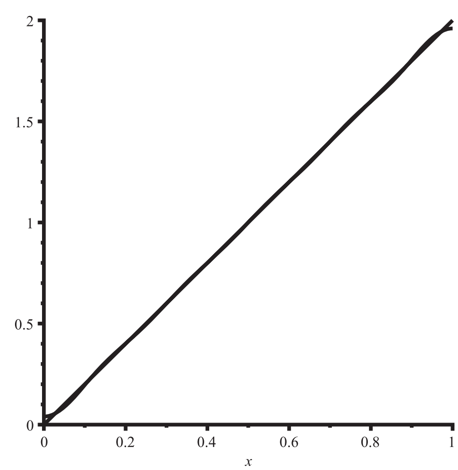

Figure 1.5: $f(x)$ and the 10th partial sum of the cosine series in Problem 18.

The sine expansion is

$$\sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1}\sin(n\pi x),$$

converging to 0 at $x = 0$ and $x = 1$, and to $2x$ for $0 < x < 1$. Figure 1.6 compares the function with the 50th partial sum of this sine expansion.

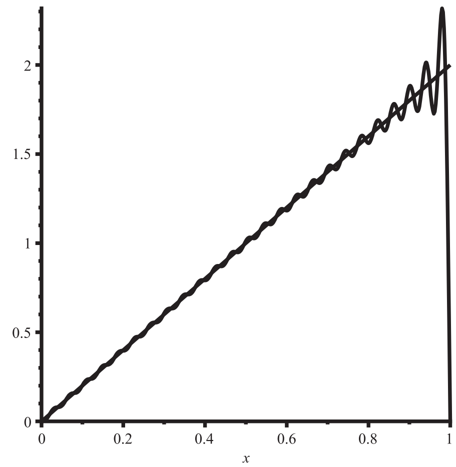

Figure 1.6: $f(x)$ and the 50th partial sum of the sine expansion in Problem 18.

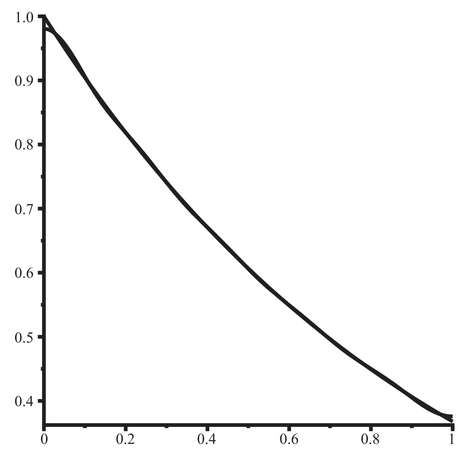

Figure 1.7: $f(x)$ and the 10th partial sum of the cosine series in Problem 20.

20. The cosine expansion is

$$1-\frac{1}{e}+\sum_{n=1}^{\infty}\frac{2}{1+n^2\pi^2}(1-e^{-1}(-1)^n)\cos(n\pi x),$$

converging to e^{-x} for $0 \leq x \leq 1$. Figure 1.7 shows the function and the 10th partial sum of this series.

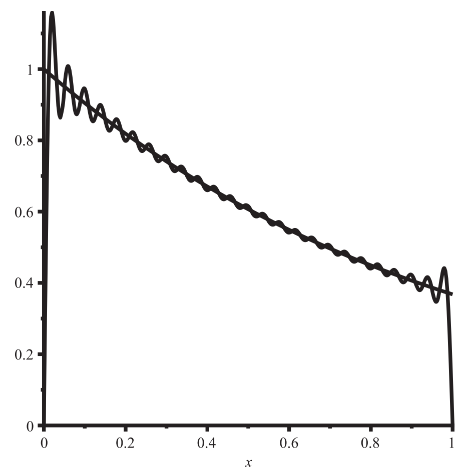

Figure 1.8: $f(x)$ and the 50th partial sum of the sine expansion in Problem 20.

The sine expansion is

$$\sum_{n=1}^{\infty} \frac{2n\pi}{1+n^2\pi^2}(1-e^{-1}(-1)^n)\sin(n\pi x).$$

This series converges to 0 at $x = 0$ and at $x = 1$, and to e^{-x} for $0 < x < 1$. Figure 1.8 shows the 50th partial sum.

22. The cosine expansion is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}(2\cos(n\pi/2) - (1+(-1)^n))\cos(n\pi x/2),$$

converging to $f(x)$ on $[0, 2]$. Figure 1.9 shows graphs of the function and the 10th partial sum of this cosine series.

The sine series is

$$\sum_{n=1}^{\infty} \frac{16\sin(n\pi x/2)}{n^2\pi^2}\sin(n\pi x/2),$$

converging to $f(x)$ on $[0, 2]$. The function and the 10th partial sum of this sine series are shown in Figure 1.10.

23. Expand $f(x) = \sin(x)$ in a cosine series on $[0, \pi]$:

$$\sin(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{-2(1+(-1)^n)}{\pi(n^2-1)}\cos(nx).$$

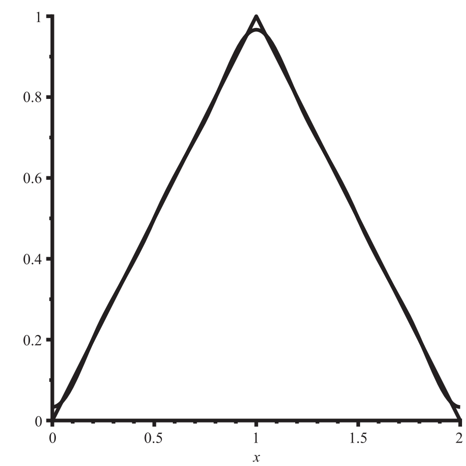

Figure 1.9: $f(x)$ and the 10th partial sum of the cosine series in Problem 22.

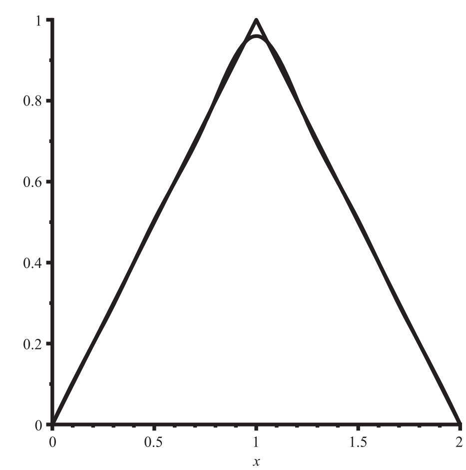

Figure 1.10: $f(x)$ and the 50th partial sum of the sine expansion in Problem 22.

Since $1 + (-1)^n = 0$ if n is odd, we need only to retain the even positive integers in the sum. Replace n with $2n$ to write

$$\sin(x) = \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2 - 1)} \cos(2nx).$$

Now choose $x = \pi/2$.

1.3 Two Eigenvalue Problems

2. Eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2,\ X_n(x) = \cos\left(\frac{(2n-1)\pi x}{2L}\right).$$

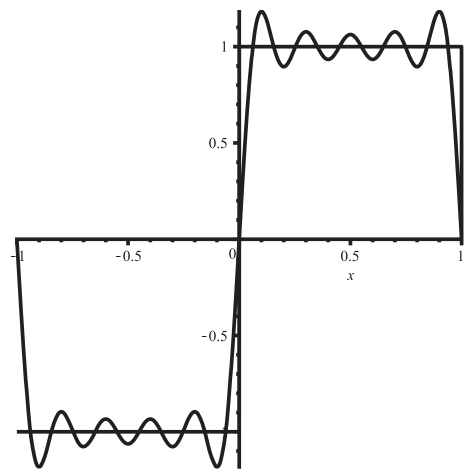

Figure 1.11: $f(x)$ and the 10th partial sum.

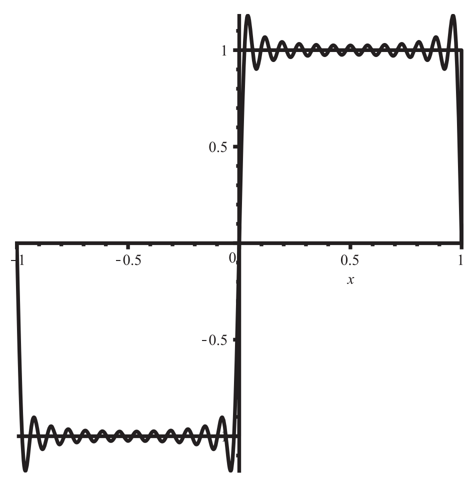

Figure 1.12: $f(x)$ and the 25th partial sum.

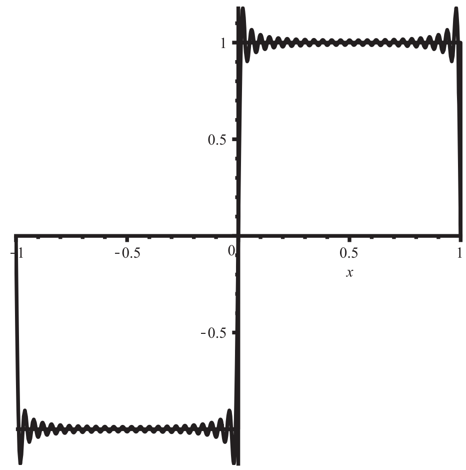

Figure 1.13: $f(x)$ and the 50th partial sum.

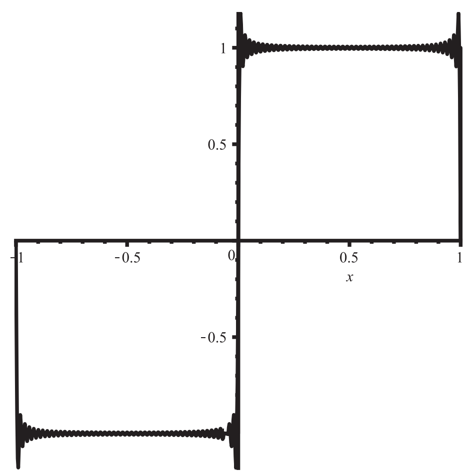

Figure 1.14: $f(x)$ and the 100th partial sum.

4. Eigenvalues and eigenfunctions are

$$\lambda_n = \alpha_n^2,\ X_n(x) = \sin(\alpha_n x),$$

where α_n is the nth positive root (in increasing order) of the equation $\tan(\alpha L) = -2\alpha$.

1.4 A Proof of the Convergence Theorem

The Fourier series of $f(x)$ on $[-1, 1]$ is

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - (-1)^n) \sin(n\pi x).$$

Figures 1.11–1.14 show the function and the nth partial sum for $n = 10, 25, 50, 100$, respectively.