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INTRODUCTION

1.1 A TRANSDISCIPLINARY RESEARCH AREA

The study of extreme events has long been a very relevant field of investigation at the intersection of different fields, most notably mathematics, geosciences, engineering, and finance [1–7]. While extreme events in a given physical system obey the same laws as typical events do, extreme events are rather special from a mathematical point of view as well as in terms of their impacts. Often, procedures like mode reduction techniques, which are able to reliably reproduce the typical behavior of a system, do not perform well in representing accurately extreme events and therefore, underestimate their variety. It is extremely challenging to predict extremes in the sense of defining precursors for specific events and, on longer time scales, to assess how modulations in the external factors (e.g., climate change in the case of geophysical extremes) have an impact on their properties.

Clearly, understanding the properties of the *tail of the probability distribution* of a stochastic variable attracts a lot of interest in many sectors of science and technology because extremes sometimes relate to situations of high stress or serious hazard, so that in many fields it is crucial to be able to predict their return times in order to cushion and gauge risks, such as in the case of the construction industry, the energy sector, agriculture, territorial planning, logistics, and financial markets, just to name a few examples. Intuitively, we associate the idea of an *extreme event* to something that is either very large, or something that is very rare, or, in more practical terms, to something with a rather abnormal impact with respect to an indicator (e.g., economic

or environmental welfare) that we deem important. While overlaps definitely exist between such definitions, they are not equivalent.

An element of subjectivity is unavoidable when treating finite data—observational or synthetic—and when having a specific problem in mind: we might be interested in studying yearly or decadal temperature maxima in a given location, or the return period of river discharge larger than a prescribed value. Practical needs have indeed been crucial in stimulating the investigation of extremes, and most notably in the fields of hydrology [5] and finance [2], which provided the first examples of empirical yet extremely powerful approaches.

To take a relevant and instructive example, let us briefly consider the case of geophysical extremes, which do not only cost many human lives each year, but also cause significant economic damages [4, 8–10]; see also the discussion and historical perspective given in [11]. For instance, freak ocean waves are extremely hard to predict and can have devastating impacts on vessels and coastal areas [12–14]. Windstorms are well known to dominate the list of the costliest natural disasters, with many occurrences of individual events causing insured losses topping USD 1 billion [15, 16]. Temperature extremes, like heat waves and cold spells, have severe impacts on society and ecosystems [17–19]. Notable temperature-related extreme events are the 2010 Russian heat wave, which caused 500 wild fires around Moscow, reduced grain harvest by 30% and was the hottest summer in at least 500 years [20]; and the 2003 heat wave in Europe, which constituted the second hottest summer in this period [21]. The 2003 heat wave had significant societal consequences: for example, it caused additional deaths exceeding 70,000 [17]. On the other hand, recent European winters were very cold, with widespread cold spells hitting Europe during January 2008, December 2009, and January 2010. The increasing number of weather and climate extremes over the past few decades [22–24] has led to intense debates, not only among scientists but also policy makers and the general public, as to whether this increase is triggered by global warming.

Additionally, in some cases, we might be interested in exploring the spatial correlation of extreme events. See extensive discussion in [25, 26]. Intense precipitation events occurring at the same time within a river basin, which acts as a spatial integrator of precipitations, can cause extremely dangerous floods. Large-scale long-lasting droughts can require huge infrastructural investments to guarantee the welfare of entire populations as well as the production of agricultural goods. Extended wind storms can halt the production of wind energy in vast territories, dramatically changing the input of energy into the electric grid, with the ensuing potential risk of brown- or black-outs, or can seriously impact the air, land, and sea transportation networks. In general, weather and climate models need to resort to parametrizations for representing the effect of small-scale processes on the large-scale dynamics. Such parametrizations are usually constructed and tuned in order to capture as accurately as possible the first moments (mean, variance) of the large-scale climatic features. However, it is indeed much less clear how spatially extended extremes can be affected. Going back to a more conceptual problem, one can consider the case where we have two or more versions of the same numerical model of a fluid, which differ for the adopted spatial resolution. How can we compare the extremes of a local physical observable provided by the various versions of the model? Is there a coarse-graining procedure suited for upscaling to a common resolution the outputs

of the models, such that we find a coherent representation of the extremes? In this regard, see in [27] a related analysis of extremes of precipitation in climate models.

When we talk about the impacts of geophysical extremes, a complex portfolio of aspects need to be considered, so the study of extremes leads naturally to comprehensive transdisciplinary areas of research. The impacts of geohazards depend strongly not only on the magnitude of the extreme event, but also on the vulnerability of the affected communities. Some areas, for example, coasts, are especially at risk of high-impact geophysical hazards, such as extreme floods caused by tsunamis, storm surges, and freak waves. Delta regions of rivers face additional risks due to flooding resulting from intense and extensive precipitation events happening upstream the river basin, maybe at distances of thousands of kilometers. Sometimes, storm surges and excess precipitation act in synergy and create exceptional coastal flooding. Mountain areas are in turn, extremely sensitive to flash floods, landslides, and extreme solid and liquid precipitation events.

When observing the impacts of extreme events on the societal fabric, it can be noticed that a primary role is played by the level of resilience and preparedness of the affected local communities. Such levels can vary enormously, depending on many factors including the availability of technology; social structure; level of education; quality of public services; presence of social and political tensions, including conflicts; gender relations; and many others [28–30]. Geophysical extremes can wipe out or substantially damage the livelihood of entire communities, leading in some cases to societal breakdown and mass migration, as, for example, in the case of intense and persistent droughts. Prolonged and extreme climate fluctuations are nowadays deemed to be responsible for causing or accelerating the decline of civilizations—for example, the rapid collapse of the Mayan empire in the XI century, apparently fostered by an extreme multidecadal drought event [31]. Cold spells can also have severe consequences. An important but not so well-known example is the dramatic impacts of the recurrent ultra cold winter *Dzud* events in the Mongolian plains, which lead to the death of livestock due to starvation, and have been responsible for causing in the past the recurrent waves of migration of nomadic Mongolian populations and their clash with China, Central Asia, and Europe [32, 33]. The meteorological conditions and drivers of *Dzud* events are basically uninvestigated.

Nowadays, public and private decision makers are under great uncertainty and need support from science and technology to optimally address how to deal with forecasts of extreme events in order to address questions such as: How to evacuate a coastal region forecasted to be flooded as a result of a storm surge; and how to plan for successive severe winter conditions affecting Europe's transportation networks? How to minimize the risk of drought-induced collapse in the availability of staple food in Africa? How to adapt to climate change? Along these lines, today, a crucial part of advising local and national governments is not only the prediction of natural hazards, but also the communication of risk to a variety of public and private stakeholders, as, for example, in the health, energy, food, transport, and logistics sectors [23].

Other sectors of public and private interest where extremes play an important role are finance and (re-)insurance. Understanding and predicting events like the crash of the New York Stock Exchange of October 1987 and the Asian crisis have become extremely important for investors and institutions. The ability to assess the very high quantiles of a probability distribution, and delve into low-probability events is of great

interest, because it translates into the ability to deal efficiently with extreme financial risks, as in the case of currency crises, stock market crashes, and large bond defaults, and, in the case of insurance, of low probability/high risk events [2].

The standard way to implement risk-management strategies has been, until recently, based on the value-at-risk (VaR) approach [34]. The VaR approach typically aims at estimating the worst anticipated loss over a given period with a given probability assuming regular market conditions. The basic idea is to extract information from the typical events to predict the properties of the tails of the probability distribution. The VaR method has been recently criticized because of various limitations. In many applications, simple normal statistical models are used, resulting in an underestimation of the risk associated with the high quantiles in the common (in the financial sector) case where fat-tailed distributions are present. More sophisticated statistical models partially address this problem, but, since they are based on fitting distributions like the Student- t or mixtures of normals, the properties of the nontypical events are poorly constrained. Nonparametric methods, instead, cannot be used beyond the range of observed values, and therefore, it is virtually impossible to have any predictive power for assessing the probability of occurrence of out-of-sample events [35].

Intuitively, it seems impossible to be able to predict the probability of occurrence of events larger than what has already been observed, and, in general, of events that are extremely untypical. The key idea is to focus on the tail of the distribution, by constructing a statistical model aimed at describing only the extreme data, so that the fitting procedure is tailored to what one is interested in [36, 37]. In other terms, this requires separating clearly typical events from nontypical—extreme—events, disregarding entirely the former, and attributing to the latter *special* features. One needs to note that, by the very nature of the procedure, introducing spurious events in the group of selected extremes (the so-called *soft extremes*) may lead to substantial biases in the procedure of statistical fitting.

The goal of this introduction is to motivate the reader to delve into the mathematics of extremes by presenting some of the most interesting challenges in various areas of scientific research where extremes play a major role. In this sense, we stick to the history of the field, where mathematical findings and relevant applications have gone hand in hand since several decades. In the following sections, we introduce the main themes of this book, try to clarify the main ideas that we will develop, and underline the most problematic aspects related to the development of a rigorous theory of extremes for dynamical systems as well as to its possible use in the study of specific mathematical and more applied problems.

1.2 SOME MATHEMATICAL IDEAS

One can safely say that in the case of extremes, as in many other sectors of knowledge, the stimuli leading to the mathematical theory have come from the need to systematize and understand current technological applications at a deeper level. A more complete and rigorous mathematical framework is also instrumental in defining more powerful and more robust algorithms and approaches to studying time series and improving our ability to describe, study, and predict extremes, and, eventually,

cushioning their impacts. This book aims at providing a mathematical point of view on extremes that are able to relate their features to the dynamics of the underlying system. Obviously, in order to develop a mathematical theory of extremes, it is necessary to carefully define the *rules of the game*, that is, lay out criteria clearly separating the extremes from regular events. Moreover, it is crucial to construct a conceptual framework that can be easily adapted to many different applications while, at the same time, is able to deliver as many results as possible that are universal within some suitable limits.

The quest for some level of universality, apart from being of obvious mathematical interest, is very relevant when using exact mathematical results for interpreting data coming from observations or from numerical simulations. In fact, the presence of universal mathematical properties gives more robustness to the procedures of statistical inference. It is clear that the tantalizing goal of constructive credible estimates for very high quantiles and for the probability of occurrence of yet unobserved events requires one to provide arguments to define the properties for the tails of distribution under very general conditions, and, possibly, of an explicit universal functional form describing them.

The classical construction of the mathematical theory of extremes and the definition of extreme value laws (EVLs) result from the generalization of the intuitive points of view—extreme as large and extreme as rare—introduced before. Following [38], one considers a random variable (r.v.) X and investigates the conditions under which one can construct the properties of the r.v. M_n given by the maximum of a sequence of n independent and identically distributed (i.i.d.) r.v. $X_j, j = 1, \dots, n$, in the limit $n \rightarrow \infty$. This is an extremely powerful and fruitful approach to the problem, which we will discuss later in detail. One can find that, under rather general conditions and using a suitable procedure of rescaling, it is possible to construct such a limit law for M_n . In practice, one finds a general three-parameter statistical model for fitting the empirical probability distribution of the *block maxima* (BM), which are constructed from a time series of length $s = n \cdot k$ by chopping into k (long) blocks of length n , and taking the maximum of each block. We refer to such a model as Generalized Extreme Value (GEV) distribution.

The GEV distribution provides a generalization of the Frechét, Gumbel, and Weibull distributions, which have long been used for studying extreme events in many fields of science and technology. Nowadays, *GEV*-fitting is one of the most common methods for dealing with extremes in many applications. Giving a meaning to the statistics of, for example, the annual maxima of surface temperature in a given location requires taking implicitly or explicitly the BM point of view [39]. The sign of the *shape parameter* determines the qualitative properties of the extremes, If the shape parameter is positive (Frechét distribution), the extremes are unlimited and; if the shape parameter is negative (Weibull distribution), the extremes are upper limited, with the Gumbel distribution (vanishing shape parameter) being the limiting case lying in-between. The *location parameter*, instead, describes how typically large extremes are, while the *scale parameter* indicates the variability of the extremes.

A crucial aspect is that, under the same mathematical conditions allowing for the definition of the limit laws for the variables M_n , it is possible to look at the problem from a complementary point of view. One finds a one-to-one correspondence

between the statistical properties of the BM in the limit of very large n , and those of the *above-threshold* events. These can be described by the conditional probability of X , given that X itself exceeds a certain threshold value T , for very large T . One considers the case of T approaching infinity if the distribution of X is not bounded from above, or, in the opposite case, of T approaching the maximum of the support of the probability distribution of X . One can prove that *these* maxima are distributed according to the two-parameter generalized Pareto distribution (GPD) [1, 40, 41]. This point of view leads to looking at extremes using the *peaks over threshold* (POT) method. When one looks at the risk of occurrence of negative anomalies of input of wind energy into the electric grid larger than a given alert level T , the POT point of view is taken [42]. When performing POT statistical inference, one derives the values of the two parameters of the corresponding GPD, namely, the *shape* and *scale* parameters, with a similar meaning as for the GEV case. See [43] for an elegant and comprehensive discussion of advanced uses of the POT methods for the relevant case of rainfall data.

It is remarkable that while for a given series of i.i.d. r.v. X_j , the probability distributions of POT events and of the BM are different, also in the limit, because the procedure of selection of extremes is fundamentally different, the two points of view are in some sense equivalent. In other terms, we have universal properties that do not depend on the procedure of selection of the extremes. More specifically, if one learns the properties of extremes defined as maxima taken over large blocks, the properties of extremes as events above a (high) threshold can be automatically deduced, and the other way around as well. In fact, the shape parameters of the GEV and GPD are the same. Additionally, one can write explicit relationships by linking the GEV's scale and location parameters on one side with the GPD scale parameter and threshold T (which acts as a hidden parameter) so that a, one-to-one correspondence between the two distributions can be found [44].

There has long been a very lively debate on whether the BM or POT method is better suited for treating finite time series coming from social, engineering, or natural systems. Most importantly, the existence of the corresponding well-defined and universal parametric probability distributions implies that if we are able to obtain a robust fit for the extremes of an observable, given a set of observations, we will be able to predict the probability of occurrence (or the return time) of events larger than anything observed, with an accuracy that depends on the quality of the fit. This clarifies why the existence of universality fosters predictive power.

1.3 SOME DIFFICULTIES AND CHALLENGES IN STUDYING EXTREMES

1.3.1 Finiteness of Data

It is important to note that, despite the powerful mathematics behind the EVLs, not all problems related to extreme events can be tackled following the approaches mentioned above. Difficulties emerge, for example, when considering finite data samples and attempting to derive via statistical inference the models of the underlying EVLs.

While the relationship between *very large* and *very rare* events is far from obvious, it seems hard to conceive—or provide any useful definition of—an extreme as a fairly

common event, so that, by construction, the statistical inference of extremes always suffers from a relative (or serious) lack of data for fitting purposes [3]. The problem is more easily seen taking the BM point of view and the classical case of a time series X_j , $1 \leq j \leq s = nk \gg 1$. Assuming that each entry of the time series is the realization of i.i.d. stochastic variables, we need to divide the nk entries into k blocks, each of size n , and perform our statistical inference over the M_j , $1 \leq j \leq k$. Since we are targeting extremes, we clearly have to consider the case $n \gg 1$. Moreover, since we need to perform statistical inference using the GEV model, we definitely need $k \gg 1$ in order to have sufficient robustness. In particular, fitting an p -parameter probability distribution requires $O(10^p)$ independent data [45]; hence, considering the GEV method, we need $k = O(10^3)$ candidates as true extremes. Assuming that yearly maxima are reasonable candidates for extremes of temperature at a given location, what said implies that we need time series covering $O(10^3)$ years to perform a reasonable GEV-based analysis of extremes. It is immediately apparent that available observational data—which cover at most three centuries for some meteorological stations (and neglecting all problems of homogenization)—are not appropriate, while one immediately sees the potential of using ultralong numerical simulations with climate models.

If data are abundant, one can think of many possible options for dividing a time series of length nk into k blocks, each of length n . One can proceed as follows: larger and larger values of n are considered, until for $n > n_{crit}$ the estimates of the GEV parameters (and in particular of the shape parameter) are stable (and the goodness of fit is high). This allows us to say that we have reached—for all practical purposes—the asymptotic limit of the theory. Further increasing the value of n makes our fitting procedure worse, because we reduce the value of k , thus increasing the uncertainty in the parameters' estimate [46]. The basic problem with the BM method is that many excellent candidates for extremes are removed from the analysis because only the maximum of each block is retained in the subsequent statistical inference procedure. Interestingly, in many applications the POT approach is preferred to the BM approach for fitting time series because it provides more efficient use of data and has better properties of convergence when finite datasets are considered [3]. A comprehensive treatment of optimal threshold selection procedures for the POT method is presented in [43].

If data are relatively scarce, one is bound to relax the criteria for defining extremes (thus considering soft extremes (e.g., taking n not too large, or choosing a moderate threshold T). As discussed in [46, 47], softening the criteria for choosing extremes leads to biases in the estimates of the EVL distribution parameters, the basic reason being that we quickly corrupt the statistics with many bogus entries. Therefore, in some cases one needs to take a more heuristic point of view and construct empirical measures of extremes, defined, for example, as events above a given percentile—say 95th—of the underlying probability distribution. This is, in fact, the standard point of view taken in most climate-related investigations [23]. Unfortunately, as soon as these—pragmatic and useful—points of view are taken, we lose universality and, consequently, predictive power. This demonstrates that it is important to not only understand what the limiting EVLs of given stochastic processes are, but also to evaluate how fast the convergence of finite-data estimates is [46]. The reader is encouraged to look at [48] for an in-depth analysis of extremes in simple atmospheric models,

with a detailed discussion of how fast the statistics of BM converges to the GEV distribution family depending on the choice of observable.

1.3.2 Correlation and Clustering

Apart from the elements of subjectivity and data requirements in defining how large or how rare an event has to be in order to be called a *true extreme*, many additional aspects need to be dealt with when looking at many situations of practical interest. In particular, it is extremely important to investigate the *recurrence* properties of the extremes. This entails assessing whether they come as statistically independent events, or whether they form clusters, that is, groups of extreme events concentrated in time. The two scenarios relate to two different underlying dynamical processes, where the occurrence of extremes is related or not to the presence of persistent patterns of dynamics, and to memory effects, and, in terms of risk, imply entirely different strategies of adaptation, mitigation, and response.

The classical EVT is basically unaltered if the stochastic variables X_j 's, instead of being independent, are weakly correlated, meaning that the correlation between the variables X_j and X_k decreases *rapidly enough* with $|j - k|$. In the presence of short range (i.e., small $|j - k|$) strong correlations between X_j and X_k , the GEV- and GPD-based approaches are not equivalent, the basic reason being that the POT method is bound to select clusters of extremes, which are instead automatically neglected in the BM procedure [3]. As a result, one can prove that when estimating the shape parameter with the POT and BM method using the same time series, one expects to obtain different estimates, with the POT method giving biased results. At the practical level, this may result in errors in the estimate of the return times of extreme events. The *extremal index* (EI), which is the inverse of the average cluster size [49], is the most important indicator of how important clustering is in a given time series, and various statistical methods have been devised to optimally estimate its value [50]. In order to use the POT approach, we need to post-process the data by performing *declustering*. Commonly used declustering procedures are based on the idea of discarding all the elements of a cluster except the largest one. After this treatment of data, the POT method typically delivers the same estimates of the shape parameter as the BM method [3].

Taking a concrete example where these issues play a key role, one may want to accommodate situations where the occurrence of an extreme is not exclusively related to the occurrence of a large event, but to the persistence of the considered observable above a certain threshold for an extended period of time, so that clustering of individual events is crucial. This is exactly how heat stress indices are defined by the World Health Organization (WHO) in relation to the persistence of high temperature, because the human body is well suited to resist short spells of high temperatures, but has instead great problems in dealing with extended period of physical stress due to heat. See also the definition of heat wave in [51]. Similarly, food security is strongly affected by prolonged heat and drought stress in some key regions in the world and contingency plans, based on risk reduction and insurance-based methods, are continuously updated to take into account the time scale of the persistence of extreme conditions [9, 52].

Some explicit results have been presented in the physical literature regarding extremes of time series resulting from stochastic variables X_j featuring long-term correlations. In particular, one observes that the recurrence times of occurrence of the above threshold events, instead of being Poisson distributed (which, roughly speaking, implies that occurrence of one extreme does not influence the occurrence of another extreme), follow a stretched exponential statistics, with ensuing implications on the possibility of predicting extremes [53, 54]. A detailed discussion of the performance and limitations of the POT and BM methods in the context of time series featuring substantial long-term correlations is given in [7].

1.3.3 Time Modulations and Noise

Often, the parameters or the boundary conditions of a system, change with time: what is the best way to analyze extremes in a context like this? The usual setting of EVT is based upon assuming stationarity in the stochastic variables. When performing statistical inference from data, is it more efficient to remove trends in the time series of the considered observables and then study the extremes of the obtained detrended time series? Or should we analyze the original time series, and use time as a covariate? How do we remove periodic components in the time-series of a process (e.g., energy consumption) influenced by, for example, the daily and seasonal cycle?

Some of these aspects are dealt with in [3, 55]: it is proposed to treat time as covariate and construct in this way a time-dependent description of extremes. See also [56], where this method is compared with what is obtained by dividing the time series to be studied in smaller blocks, and then performing standard EVT parameter inference in each block separately assuming stationarity, as proposed in [57]. Recently, [58] have proposed new statistical methods for constructing efficient time-dependent EVT models from nonstationary time series, while [59] have underlined the limitations of this approach when forcing terms have different time scales.

This issue is of extreme relevance and urgency, for example, with regard to the very active field of investigation of the meteo-climatic extremes in the context of the changes in the properties of the climate system due to anthropogenic influence. In most cases, for the reasons outlined above, the investigation of changes in extremes is performed by looking at heuristic indicators, such as changes in the probability of occurrence of empirically defined above-thresholds events [60] or in the value of very low and high quantiles [61]. Though it is becoming more common in geophysical literature to make explicit—for example, [62]—or implicit—for example, [63]—use of EVT. See also [4, 5, 64] for comprehensive reviews.

Another aspect to be mentioned is the role of noise or finite precision in observations. When taking into account real measuring devices, we also face the problem of errors—uncertainties and biases—in recording data. Therefore, observational noise needs to be addressed in order to connect mathematical results to inferences from data [65]. On a related note, [66–68] concluded that there is a substantial impact of finite precision (typically, 1 mm) on the rain gauge readings on the fitting procedures followed for reconstructing the statistical properties of rainfall data.

1.4 EXTREMES, OBSERVABLES, AND DYNAMICS

The open issues and practical problems mentioned above clarify that it is necessary to develop a comprehensive mathematical view of extremes in order to achieve flexibility and predictive power in many real-life problems.

Traditionally, the theory of extreme events has been developed in the context of probability theory and stochastic processes, as suggested by the definitions provided above. Nonetheless, many of the examples we have hinted at suggest that one needs to move in the direction of extending and adapting the classical results of extreme value theory for investigating the extremes of observables of suitably defined dynamical systems. The reader is encouraged to look at [69] for a comprehensive presentation of the theory of dynamical systems, and into [70, 71] for a point of view specifically tailored for linking dynamical systems and (nonequilibrium) statistical mechanics.

Roughly speaking, the conceptual bridge relies on considering a given (continuous or discrete time) dynamical system as the generator of infinite stochastic variables, each defined by the time series of a given *observable*, that is, a sufficiently regular function mapping the trajectory of the point representing the evolution of the system in the phase space into the real numbers, and then studying the extremes of such observables. Such a point of view, first proposed in [72], has the crucial advantage of giving the possibility of relating the properties of the extremes to the underlying dynamics of the system generating them, thus providing a natural link between a probability theory and a dynamical systems theory and connecting, in practical terms, forward numerical simulations of—possibly complex—systems with the statistical inference of the properties of extremes from a time series.

Moreover, it provides the perfect setting for studying the properties of extremes generated by numerical models, which are finite-precision (and thus noisy) implementations of systems ranging from simple, low-dimensional maps up to discretized (in time and space) representations of the evolution of continuum systems, such as fluids. It is clear that by considering dynamical systems with different properties of decay of correlations, one mirrors precisely the conditions of weak versus strong correlations of stochastic variables described above. This substantiates the idea of clusters of extreme events, and can define cases where the one-to-one equivalence between BM and POT approaches is broken, therefore requiring additional mathematical subtlety [49, 73, 74].

A key ingredient of a theory of extremes that incorporates dynamics and recurrences is the choice of the observable. This provides a new dimension of the problem, and requires the scientist to define what a good, meaningful, useful, and well-behaved observable is. Moreover, given a numerical model, one can practically explore many of the aspects related to temporal or spatial coarse graining just by constructing in rather simple ways the observable of interest. This issue naturally provides a more statistical mechanical, physical setting to the problem of extremes, paving the way for fascinating applications where extremes can be used as indicators of the structural properties of a system, defining new, powerful methods to study its tipping points [75].

We shall see that, by looking at extremes, one can learn about the qualitative properties of the dynamical system generating them, for example, by learning

whether it features regular or chaotic motions [76], and, under suitable circumstances, understand basic information on the geometry of the attractor and on the Lyapunov exponents determining the growth or decay or errors in given directions. Therefore, extremes act as a probe of a dynamical system, and, when suitable observables are considered, they define a natural microscope to look at the details of the invariant measure, to the point of providing alternative ways to compute the Hausdorff dimension of the attractor.

An especially important role is played by observables whose extremes correspond to close returns of the orbit near a reference point [49, 74, 77, 78]. Interestingly, perturbing systems with noise allows the establishment of EVLs for such observables also in the case of deterministic quasi-periodic motion and removes clusters of extreme events when strong correlations are otherwise present [79].

Recurrence-based techniques have also been shown to be directly usable for studying the properties of extremes in climatic time series [80]. Nonetheless, in many practical applications, one is interested in studying a different sort of observables, the so-called *physical observables* [44, 81], which *a priori* have nothing to do with the recurrence of an orbit near a given reference point, but rather describe macroscopic or anyway physically relevant properties of the system. As a simple example, one may consider the extremes of the energy [82] or of the temperature [48] in a model of geophysical fluid. The extremes of physical observables permit the study of rather sophisticated aspects of the geometry of the attractor of the underlying system, providing a formidable tool for analyzing the properties of the unstable and stable components of the tangent space.

One can formulate the problem of studying, at least heuristically, extremes for non-stationary systems by taking into consideration some recent results of nonequilibrium statistical mechanics and dynamical systems theory. This can involve the construction of a response theory for Axiom A dynamical systems to predict the change in the expectation value of suitably defined observables resulting from weak perturbations that are also time dependent, for example, such as small changes in a parameter [83, 84]. In order to apply these results when assessing the time-dependent properties of extremes—see Section 1.3.3—one needs to construct observables that can represent the extreme events of interest, and then apply the response theory to compute their change as a result of the perturbation. Finally, the computed response can be reformulated in terms of time-dependent correction to the value of the EVL parameters [44]. An interesting aspect is that, since extremes are in this case described by the universal parametric family of EVLs, one could draw the rather counter-intuitive conclusions that, in some sense, the response of extremes to perturbations could be a better-posed problem than studying the response of the statistics of the *bulk* of the events [56, 82]. In practical terms, this gives a framework for the rigorous questions mentioned before in this introduction, such as determining how extremes change when time-dependent perturbations are added to the system [56], as in the case of changes in climatic extremes in an overall changing climate [85–87]. Apart from its relevance in terms of applications, the mathematical interest in this regard is considerable.

A different yet related dynamical point of view on extremes of nonstationary systems is based upon considering the mathematical construction of the so-called *pullback* attractor [88–90], sometimes referred to as *snapshot* attractor [91], which is

basically the time-dependent generalization of the usual attractor appearing in many dissipative chaotic system [70, 71], and enjoys a lot of the useful properties of the latter, except time invariance. The time-dependent measure supported on the pullback attractor at time t is constructed by evolving from an infinitely distant past a Lebesgue measure with compact support. This procedure, in practical numerical applications, corresponds to starting to run in the distant past a large number of simulations with different initial conditions, and observing the resulting ensemble at time t [89]. The time-dependent properties of extremes can then be assessed studying the properties of such an ensemble [59, 92–94].

This setting suggests the possibility of achieving predictability of extremes in a statistical sense, that is, developing tools for understanding how their properties change as a result of changing parameters of the system that generates them, which is clearly a major scientific challenge in, for example, climate science [4, 23], where *big data* are being advocated [10]. Our ability to predict the occurrence of individual extreme events efficiently is still modest, see some examples in [4]. A crucial aspect is that it is not easy to anticipate (we are not talking about *ex-post* analysis) the dynamical causes leading to an extreme event. As clarified in [95, 96], (finite-time) Lyapunov exponents and related dynamical objects have an important role in assessing the potential predictability of a chaotic system *for typical conditions* [97], in terms of allowing for effectively extremes with a certain lead time. Some authors have proposed ingenious methods for detecting precursors [98, 99], but still a comprehensive understanding of this problem is missing.

1.5 THIS BOOK

The scope for the lines of investigation described above is immense, and what we are proposing in this book is a limited perspective resulting from the collective effort of a group of authors coming together and joining forces from a rather diverse spectrum of scientific expertise, ranging from probability theory, to dynamical systems; from statistical mechanics, to geophysical fluid dynamics; from theoretical physics, to time series analysis. Without hoping or aiming to be either comprehensive or conclusive, this book wishes to provide a new perspective on extreme events, a transdisciplinary field of research of interest for mathematicians, natural scientists, statisticians, engineers, and social scientists.

One can safely say that the main difference between this book and many other excellent monographs in the literature, ranging from rather abstract mathematical formulations of the theory of extremes [1, 100], to sophisticated presentation of algorithms for defining, detecting, and performing statistical inference of extremes [3, 50], to specific applications [2], is the focus on *dynamics*. In other words, we do not take extremes as results of a *black box*—a stochastic process whose origin we might not necessarily be interested in *per se*—but rather explore the links between the (typically chaotic) system under investigation and the generating process leading to the extreme and the extreme event itself. The freedom of looking at different extremes is guaranteed by the possibility of choosing different observables, which may be tailored for looking at local, global, or recurrence properties of the system and for focusing on specific regions of its attractor. This perspective leads to

considering extremes as a revealing source of information on the microscopic or macroscopic properties of the system, and, hence, is naturally suited for improving our understanding of its statistical mechanics. As opposed to many contributions in the scientific literature, our emphasis will not be on presenting ideas for optimizing statistical inference procedures, even if we will present examples and ideas in this direction. Though, we hope to contribute to providing useful guidance for statistical inference procedures by clarifying what they *should* find for given systems.

The main motivation for the approach we propose here comes mostly, but not exclusively, from a more general interest by the authors in exploring the fruitful and emerging *nexus* between mathematics and geophysical fluid dynamics, which has recently received global accolade with the Mathematics for Planet Earth international initiative (<http://www.mpe2013.org>), and in particular of the program *Mathematics for the Fluid Earth* (<http://www.newton.ac.uk/event/MFE>) [101] held at the Newton Institute for Mathematical Science in Cambridge (UK); see also the recent review [102]. Moreover, the authors hope to contribute to stimulating the development of new, effective, and robust methods for studying extremes in a meteo-climatic context, thus contributing to the global effort for adapting to climate change and climate-related risk. This book tries to provide stimulations, hints, and new results having in mind a readership of (applied) mathematicians, statisticians, theoretical physicists, and experts in probability, stochastic processes, statistical mechanics, time series analysis, and (geophysical) fluid dynamics.

The structure of the book can be described as follows:

- In Chapter 2 we present an overview of general laws and concepts used for describing rare events and introduce some terminology.
- In Chapter 3 the basics of classical extreme value theory are introduced, concentrating on results that are useful for developing a dynamical systems perspective. In this part of the book, there is no reference to dynamics, whereas everything is formulated exclusively in terms of stochastic processes.
- Chapters 4 and 5 constitute the core of the book. In the former, we introduce dynamical systems as generators of random processes and present a description of a variety of methods and approaches to derive EVLs for the so-called *distance observables*. In the latter, we construct the correspondence between EVLs and hitting/return time statistics in uniformly hyperbolic, nonuniformly hyperbolic, and quasi-periodic systems.
- In Chapter 6 we focus on specific dynamical systems of special interest, and study in detail the role of the decay of correlations in establishing EVLs and relate it to the chaotic nature of the dynamics, and investigate the rate of convergence of the statistics of extremes to the asymptotic EVLs. We also introduce and discuss the properties of the extremes of the so-called *physical observables*.
- In Chapter 7 we tackle the important problem of the impact of noise on the statistics of extremes in dynamical systems, treating the case of random perturbations to the dynamics and of observational noise.
- In Chapter 8 we take the point of view of statistical mechanics and, using Axiom A systems as the mathematical framework, we discuss extremes in the context of high-dimensional dynamical systems, introducing the so-called *physical*

observables, relating the properties of the extremes to the dynamical properties of the underlying system, and proposing a framework for a response theory of extremes.

- In Chapter 9 we move our focus to studying the procedures for statistical inference and present instructive applications of the theory in numerical simulations, investigating the role of finite size effects in the inference. We also present examples of how EVT can be used to derive relevant information on the geometrical and dynamical properties of the underlying system, and investigate how the presence of noise impacts the statistics of extremes.
- Chapter 10 focuses on physically oriented applications of EVT, showing how it can be used for detecting tipping points in multistable systems and, additionally, for providing a rigorous characterization of the properties of temperature fields in the present and changing climate conditions.
- Chapter 11 contains the concluding remarks of the book and provides indications for future research activities in the field.
- Appendix A includes a few MATLAB[®] numerical codes used for producing some of the numerical results contained in the book, which are distributed for the benefit of the readers.

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This book is dedicated to friendship and to our friends.

