PART I

THE BASICS OF ENUMERATIVE COMBINATORICS

1

INITIAL EnCOUNTers WITH COMBINATORIAL REASONING

1.1 INTRODUCTION

Although this text is devoted largely to enumerative combinatorics, Section 1.2 presents a brief encounter with a simple yet surprisingly versatile method to prove existence, the pigeonhole principle. Section 1.3 discusses some combinatorial construction problems associated with covering a chessboard with dominoes. In Section 1.4, we consider some number sequences that often arise in combinatorial problems such as *triangular* numbers 1,3,6,10, . . .; *square* numbers 1,4,9,16, . . .; and other *figurate numbers*, where the terminology alludes to the representation of these numbers by geometric patterns of dots. In Section 1.5, we count the number of ways a $1 \times n$ rectangle can be tiled with either unit squares of two contrasting colors or with a mixture of 1×1 squares and 1×2 dominoes. By counting the number of dots in a pattern or the number of tilings of a chessboard, we will discover several general principles of counting that are fundamental to enumerative combinatorics. In particular, we will encounter the addition and multiplication principles, which are explored in detail in Section 1.6, which concludes the chapter.

1.2 THE PIGEONHOLE PRINCIPLE

The pigeonhole principle was first applied in 1834 by Peter Dirichlet (1805–1859) to solve a problem in number theory. Soon, other mathematicians found his idea equally

Combinatorial Reasoning: An Introduction to the Art of Counting, First Edition. Duane DeTemple and William Webb.

useful and referred to it as *Dirichlet's box principle* (*Schubfachprinzip* in German). Later, in the nineteenth century, the term *pigeonhole* was used in reference to the small boxes or drawers common in desks of that century. (It may be comforting to know that envelopes, and not pigeons, are placed in the pigeonholes.)

Dirichlet's idea is simply stated as follows.

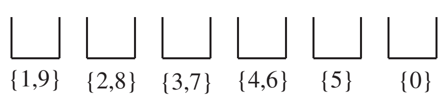
Theorem 1.1 (Pigeonhole Principle) If $n + 1$ or more objects are placed into n boxes, then at least one of the boxes contains two or more of the objects.

Proof (by contradiction). Suppose, to the contrary, that each of the n boxes contains no more than one object. Then the n boxes together contain no more than n objects, a contradiction. ■

The examples that follow show how the pigeonhole principle provides the basis for an existence proof. It is helpful to think of pigeons and pigeonholes as metaphorical terms for the objects and boxes of the theorem.

Example 1.2 In a family of seven, there must be two family members for which either the sum or difference in their ages can be given in decades, that is, as a multiple of 10.

Solution. A multiple of 10 is easy to identify by the digit 0 in the units position. If two family members have ages ending with the same digit, the difference in their ages is a multiple of 10. Also, if someone in the family has an age ending in the digit 1 and another family member's age ends with the digit 9, their sum of ages ends with the digit 0. Continuing with this type of reasoning suggests that we define the following pigeonholes:



The *pigeons* are the seven ages of the family member. When these are placed in the box labeled with the set containing the last digit of the age, the pigeonhole principle guarantees that at least one of the six boxes contains at least two people with one of the labeled ages. If these two ages happen to have the same unit's digit, then their difference is a multiple of 10. If the two ages have different last digits, these must be 1 and 9, or 2 and 8, or 3 and 7, or 4 and 6. In each case, the sum of the two ages is a multiple of 10. ■

The pigeonholes set up for Example 1.2 show why seven numbers were needed. There is no pair of numbers from the six members of the set $\{1, 2, 3, 4, 5, 10\}$ whose sum or difference is divisible by 10.

Example 1.3 There are five people in a 6 mile by 8 mile rectangular forest, each carrying a walkie-talkie with a range of 5 miles. Show that at least two of the five people can talk with one another on their walkie-talkies.

Solution. Divide the forest into four rectangular 3×4-mi plots; these are the *pigeonholes*. There are five people (the *pigeons*), so by the pigeonhole principle, at least two people are in one of the four plots. Since the maximum distance in a 3×4 rectangular plot is the 5-mi-long diagonal, these two people are within talking range. ■

The solution of a problem by means of the pigeonhole principle requires us to carry out these steps:

1. Recognize that the pigeonhole principle can be helpful.
2. Identify the pigeons and the pigeonholes.
3. Show that there are more pigeons than pigeonholes.
4. Show why the existence of two pigeons in the same pigeonhole solves the given problem.

These steps are carried out to solve the following problem.

Example 1.4 The *lattice plane* is the set of points in the Cartesian plane with integer coordinates. Given any five points of the lattice plane, show that the midpoint of some pair of points is a point in the lattice plane.

Solution. If (a, b) and (c, d) are two lattice points, their midpoint is $((a + c)/2, (b + d)/2)$, the average of the x and y coordinates. This will be a point in the lattice plane if, and only if, both $a + c$ and $b + d$ are even; that is, a and c must have the same parity, and b and d must have the same parity. This observation suggests that we use parity to define these four pigeonholes:

Box 1: x even, y even

Box 2: x even, y odd

Box 3: x odd, y even

Box 4: x odd, y odd

Since five points are placed in the four boxes, there is some box with at least two members. Both of these points have x and y coordinates with the same parity, so the midpoint of these two points has integer coordinates; that is, their midpoint is a point of the lattice plane. ■

1.2.1 Applications to Ranges and Domains of Functions

There are a number of useful variations and interpretations of the pigeonhole principle. For example, suppose that A is a set of objects and B is a set of distinct boxes. Then any placement of the objects into the boxes describes a function $f : A \to B$. If no two objects are assigned to the same box, then the function is *one-to-one*, or *injective*. The pigeonhole principle requires there to be at least as many pigeonholes as pigeons, so if $|A|$ and $|B|$ denote the respective number of elements in set A and B, we have the following theorem.

Theorem 1.5 If the function $f : A \to B$ is one-to-one, then $|A| \leq |B|$.

In the opposite direction, suppose that we have a placement of objects from set A into the boxes of set B that leaves no box empty. In other words, we have a function $f : A \to B$ that is *onto* or *surjective*, meaning that its range is all of set B. Since there must be at least as many objects as boxes, we have this theorem.

Theorem 1.6 If $f : A \to B$ is a surjective function from A onto B, then $|A| \geq |B|$.

A function that is both one-to-one and onto (i.e., is both injective and surjective), is said to be *bijective*. When the two theorems above are combined, we get the following result.

Theorem 1.7 If $f : A \to B$ is a bijective function of A onto B, then $|A| = |B|$.

Finally, we have the following result.

Theorem 1.8 If $f : A \to B$ and $|A| = |B|$, then f is one-to-one if and only if f is onto.

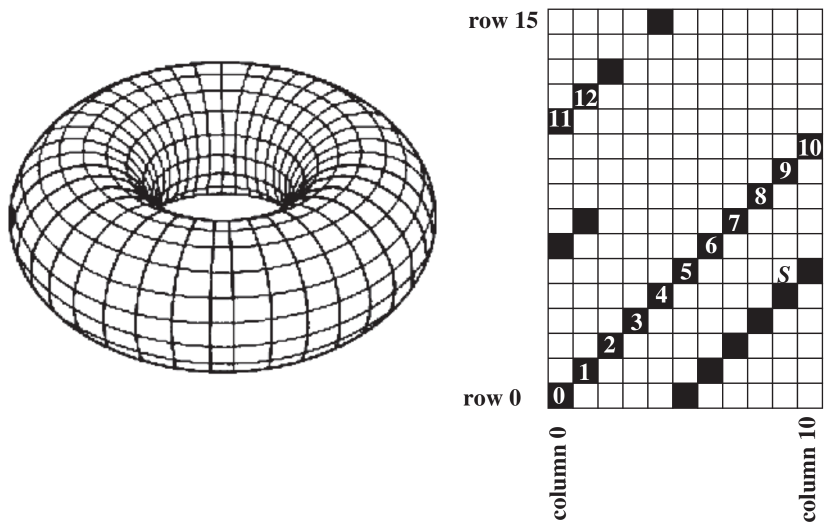
Proof. Let $|A| = |B|$ and first suppose that f is one-to-one. Then $|A| = |\text{range}(f)|$ and therefore $|\text{range}(f)| = |B|$. This shows us that $\text{range}(f) = B$, and we see that f is onto. Similarly, if f is not one-to-one, then $|A| > |\text{range}(f)|$, so $\text{range}(f)$ is a proper subset of B and f is not onto. ■

Bijections are immensely useful in combinatorial reasoning. Suppose that we have a difficult problem counting the number of elements in set A, but we can find a bijection of set A onto a set B, and B is more easily counted. Since $|A| = |B|$, our difficulties are over! This strategy is known as a *bijective proof*, and we'll see several examples of this type of combinatorial reasoning later.

1.2.2 An Application to the Chinese Remainder Theorem

Theorem 1.8 will be useful for our next application. Readers with a background or interest in number theory may find the following approach to this topic interesting. However, it is optional and can be skipped since it will not be used later.

Example 1.9 Suppose that a *torus* (i.e., the surface of a doughnut) is divided into quadrangular regions by 11 circles in one direction that are crossed by 16 circles in the orthogonal direction. If the surface is cut along one circle of each type, the surface can be unrolled and stretched to form an 11×16 rectangle partitioned into $11 \cdot 16 = 176$ squares. As shown in the diagram below, a path of squares numbered 0,1,2, . . . has been initiated that spirals around the torus, starting with square 0 in row 0 and column 0. Prove that a continuation of the spiral path covers the entire torus, so that each square is assigned a unique number 0,1,2, . . .,175.



Solution. It suffices to show that the spiral path includes all 16 of the squares located in column 0, the far left column. These are the squares numbered by the entries in the set

$$A = \{0 \cdot 11, 1 \cdot 11, 2 \cdot 11, 3 \cdot 11, \dots, 15 \cdot 11\} \tag{1.1}$$

We still need to know that no number in the left column is repeated, where the rows are numbered by the set

$$B = \{0,1,2, \dots ,15\} \tag{1.2}$$

The row of square $k \cdot 11 \in A$ is given by its remainder when divided by 16. For example, square $2 \cdot 11 = 22$ has the remainder of 6 when divided by 16, and we see from the figure that square 22 is in row 6. This suggests we consider the function $f : A \to B$ defined by mapping each element of set A to the corresponding remainder r in set B. To see why f is one-to-one, suppose that two values, say, $j \cdot 11$ and $k \cdot 11$, $j, k \in \{0,1,2, \dots .15\}$, have the same remainder r; that is, suppose that

$$j \cdot 11 = p \cdot 16 + r \text{ and } k \cdot 11 = q \cdot 16 + r \tag{1.3}$$

for the quotients p and q. When these equations are subtracted from one another, we see that

$$(j - k) \cdot 11 = (p - q) \cdot 16 \tag{1.4}$$

Since 16 divides the right side of (1.4) it must also divide the left side $(j-k)\cdot 11$. Since 16 is relatively prime to 11 (i.e., 11 and 16 have no common prime divisors), we see that 16 divides $j-k$. However, $0 \le |j-k| < 15$, and only $j-k=0$ is divisible by 16. Therefore, f is a one-to-one function that maps set A to the set B, and $|A| = |B| = 16$. We now see that each of the 16 squares in column 0 is along the spiral path, and so every square of the entire rectangle is assigned a unique number along the spiral path of squares. ■

Except for its geometric interpretation as a spiral path on a torus, Example 1.8 gives a proof of a special case of the *Chinese remainder theorem*, which first appeared in a third-century (A.D.) book *Sun Zi Suanjing* written by the Chinese mathematician Sun Tzu. It was important that $m = 11$ and $n = 16$ have no positive common divisor other than 1. More generally, we say that two integers m and n are *relatively prime* when their largest common integer divisor is 1. The following theorem can be proved similarly to the approach followed in Example 1.9.

Theorem 1.10 (Chinese Remainder Theorem) Let m and n be relatively prime positive integers, and let a and b be integers with $0 \le a < m$ and $0 \le b < n$. Then there is a unique k, $0 \le k < mn$, for which $k = mj + a$ and $k = nj' + b$ for some j and j'.

1.2.3 Generalizations of the Pigeon Principle

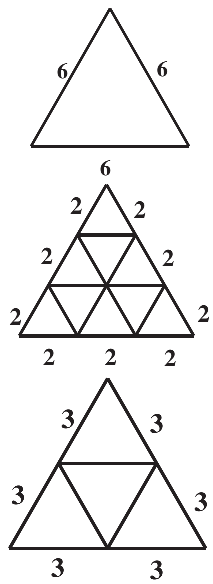
For some problems, we want to know that there are not just two but some larger number of pigeons in some box. In these cases, we can turn to a generalized version of the pigeonhole principle.

Theorem 1.11 (Generalized Pigeonhole Principles)

(a) If $nk + 1$ or more pigeons are put into n pigeonholes, then at least one pigeonhole has $k + 1$ or more pigeons.
(b) If $p_1 + p_2 + \cdots + p_n + 1$ or more pigeons are put into n pigeonholes numbered 1 through n, then, for some r, pigeonhole r has $p_r + 1$ or more pigeons.

Proof. Since part (a) follows from (b) by setting $p_r = k$, $1 \le r \le n$, it suffices to prove part (b). As before, it is easy to give a proof by contradiction. If each hole r has at most p_r pigeons, then all n holes together contain at most $p_1 + p_2 + \cdots + p_n$ pigeons. This contradicts the hypothesis that more than $p_1 + p_2 + \cdots + p_n$ pigeons were placed in the holes. ■

Example 1.12 An equilateral triangle has sides of length 6. Show that (a) if there are 10 points inside the triangle, then at least 2 of them are within 2 units of each other; and (b) if there are 9 points inside the triangle, there are at least 3 of them for which each is at most 3 units from each of the others.



Solution

(a) The triangle can be partitioned into nine equilateral triangles with sides of length 2. Given any 10 points within or on the large triangle, there must be 2 in the same small triangle, and these are at a distance of at most 2 units.

(b) Again using the partitioning of the large triangle into four congruent equilateral triangles of side length 3, not each of the 4 small triangles can contain just two of the nine points. Thus, some small triangle has three or more of the nine points, and each of these is at a distance of no more than 3 units from the other two points. ■

For the next example, it will be helpful to introduce some notations and terminologies that are often used in combinatorics and elsewhere in mathematics:

- The set of the first n natural numbers will be denoted by $[n] = \{1,2, \dots ,n\}$.
- A *sequence* of length n is an ordered list $(a_1, a_2, \dots , a_n)$. Equivalently, a sequence of length n is any function f: $[n] \to A$, where $f(j) = a_j$ is the jth *term* of the sequence.
- A *permutation* of $[n]$ is an ordered arrangement of the n elements of set $[n]$. Equivalently, a permutation is a bijection $\pi : [n] \to [n]$.
- A sequence $(a_1, a_2, \dots , a_n)$ of real numbers in *monotone increasing* if $a_1 < a_2 < \cdots < a_n$ and *monotone decreasing* if $a_1 > a_2 > \cdots > a_n$.

- A *subsequence* of $(a_1, a_2, \dots, a_n)$ is a sequence formed by deleting some of the terms of the given sequence but preserving the order in which the remaining terms are listed.

For example, (8,3,5,2,6,1,4,10,9,7) is a permutation of [10]. The subsequence (3,5,6,10) is monotone increasing. According to the following result, we can always find a monotone sequence of length 4 for any permutation of [10].

Example 1.13 Ten students, all of different heights, are standing shoulder to shoulder in a line. Prove that four of the students can take a step forward so that they form a line of students whose heights either decrease or increase from left to right.

Solution. Suppose that the students' heights are $h_1, h_2, \dots, h_{10}$ from left to right. Assume that there is no subsequence of four of the students (in the same left-to-right order, of course) with increasing heights. We will then show that a subsequence of four students with decreasing heights can be found. Starting with student 1 at the left, let s_1 be the largest number of students that can step forward to form a row with increasing height with student 1 at the left of the row. Similarly, let s_2 be the largest number of students that can take a step forward to form a row with increasing height with student 2 at the far left. More generally, let s_k be the largest number of students that can step forward to form a row of increasing height and with student k at the left. Since we cannot find a subsequence of four students with increasing height, we know that $1 \le s_k \le 3$, $k = 1, 2, \dots, 10$; that is, we have 10 numbers that have one of the values 1, 2, or 3. By Theorem 1.11 part (a), at least 4 of the 10 numbers are the same, say, $s_a = s_b = s_c = s_d$, $a < b < c < d$. We see that student a must be taller than student b, since otherwise student a could be added to the left of the longest increasing subsequence starting with student b, and then s_a would be larger than s_b. Thus, we have $h_a > h_b$. By the same reasoning, $h_b > h_c$ and $h_c > h_d$, so that altogether $h_a > h_b > h_c > h_d$. We see that if students *a, b, c,* and *d* take a step forward, they have decreasing heights from left to right. ■

The reasoning used in Example 1.13 can be extended to prove this theorem of Erdös and Szekeres [1].

Theorem 1.14 Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of $n + 1$ numbers that is either increasing or decreasing.

PROBLEMS

1.2.1. A bag contains seven blue, four red, and nine green marbles. How many marbles must be drawn from the bag without looking to be sure that we have drawn

(a) a pair of red marbles?

(b) a pair of marbles of the same color?

(c) a pair of marbles with different colors?

(d) three marbles of the same color?

(e) a red marble, a blue marble, and a green marble?

1.2.2. Given 10 French books, 20 Spanish books, 8 German books, 15 Russian books, and 25 Italian books, how many books must be chosen to guarantee there are

(a) twelve books of the same language?

(b) a book of each language?

1.2.3. There are 10 people at a dinner party. Show that at least two people have the same number of acquaintances at the party.

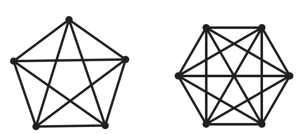
1.2.4. Given 10 distinct numbers chosen from the arithmetic sequence 1,4,7, … , $1+3k$, … , 40,43,46, prove there is at least one pair of the 10 chosen integers that has the sum 50.

1.2.5. Given any five points in the plane, with no three on the same line, show that there exists a subset of four of the points that form a convex quadrilateral.[1] [*Hint:* Consider the *convex hull* of the points; that is, consider the convex polygon with vertices at some or all of the given points that encloses all five points. This scenario can be imagined as the figure obtained by bundling the points within a taut rubber band that has been snapped around all five points. There are then three cases to consider, depending on whether the convex hull is a pentagon, a quadrilateral containing the fifth point, or a triangle containing the other two given points.]

1.2.6. Given four points on a circle, show that some three of the points lie in some closed semicircle (a closed semicircle includes its two endpoints).

1.2.7. Given five points on a sphere, show that some four of the points lie in a closed hemisphere [2].
(*Note:* A closed hemisphere includes the points on the bounding great circle.)

1.2.8. A *graph* is a set of points known as *vertices* together with a set of line segments called *edges* that connect some of the pairs of vertices. If every pair of vertices are joined by an edge, the graph is said to be *complete.* The complete graphs on five and six vertices, K_5 and K_6, are shown below:



K_5 K_6

[1]This well-known problem is called the “happy ending” problem, since two of its first investigators, Esther Klein and George Szekeres, would later be married to one another.

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(a) Show that all of the edges of K_5 can be colored blue or red so that no triangle exists in K_5 with its three edges having the same color.

(b) Show that every red and blue coloring of the edges of K_6 contains a triangle with all of its edges of the same color.

1.2.9. Suppose that 51 numbers are chosen randomly from $[100] = \{1,2, \ldots ,100\}$. Show that two of the numbers have the sum 101.

1.2.10. Assuming that there are 48 different pairs of people who know each other at a party of 20 people, show that some person has four or fewer acquaintances.

1.2.11. Choose any 51 numbers from $[100] = \{1,2, \ldots ,100\}$. Show that two of the chosen numbers are relatively prime (i.e., have no common divisor other than 1).

1.2.12. Show that any subset of eight distinct integers between 1 and 14 contains a pair of integers m and n such that m divides n.

1.2.13. Choose any 51 numbers from $[100] = \{1,2, \ldots ,100\}$. Show that there are two of the chosen numbers for which one divides the other.

1.2.14. State and prove a theorem that generalizes the results of Problems 1.2.12 and 1.2.13.

1.2.15. Consider a string of $3n$ consecutive natural numbers. Show that any subset of $n + 1$ of the numbers has two members that differ by at most 2.

1.2.16. Let $(a_1, a_2, \ldots , a_n)$ be any sequence of n natural numbers. Show that there is a subsequence of consecutive members of the sequence that is divisible by n. [*Hint:* Consider the sums $s_k = a_1 + a_2 + \cdots + a_k$.]

1.2.17. Suppose that the numbering of the squares along the spiral path shown in Example 1.9 is continued. What number k is assigned to the square S whose lower left corner is at the point (9, 5)?

1.2.18. Suppose that a torus is divided into mn quadrangular regions by m circles crossed orthogonally by n circles, as in Example 1.9. By the Chinese remainder theorem 1.10, if m and n are relatively prime, then each of the regions is reached by the unique spiral path on the torus.

(a) Using Example 1.9 as a model, draw and number the squares along the spiral path in the case that $m = 4$ and $n = 5$.

(b) How many distinct spiral paths can be found when $m = 4$ and $n = 6$?

(c) Repeat part (b) for $m = 3$ and $n = 6$.

1.2.19. Generalize the results of Problem 1.2.18.

(a) How many spiral paths exist on the torus if $m = n$?

(b) Suppose that $d \geq 2$ is the largest common divisor of m and n. How many distinct spiral paths exist on the torus?

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1.2.20. **(a)** Find a permutation of $[9] = \{1, 2, \ldots, 9\}$ for which no subsequence of length 4 is either monotone increasing or monotone decreasing (see Example 1.13).

(b) Place 10 at the right end of your sequence from part (a) and underline the four terms of an increasing subsequence.

(c) Place a 10 at the left end of your sequence from part (a) and underline the four terms of a decreasing subsequence.

1.3 TILING CHESSBOARDS WITH DOMINOES

In this section, our attention turns to combinatorial *construction*. A construction settles the question of existence in a very satisfying way, since the constructed object provides an explicit example with the required properties. Sometimes it can be shown that an object cannot possibly be constructed, so that the existence question is answered in the negative.

In each of the examples that follow, we consider how a shape formed with unit squares can be completely covered with nonoverlapping dominoes. A *domino* is a 1×2 rectangle that is viewed simply as a tile with no attention given to the dots, known as *pips,* that are imprinted on actual dominoes. For example, the 3×6 rectangle in Figure 1.1(a) can be tiled with nine dominoes in many ways, such as the tiling constructed in Figure 1.1(b). Note that a tiling must cover the entire figure and the dominoes can touch along their edges but can never overlap one another.

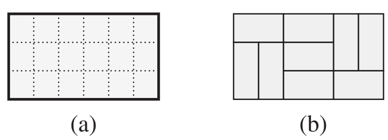


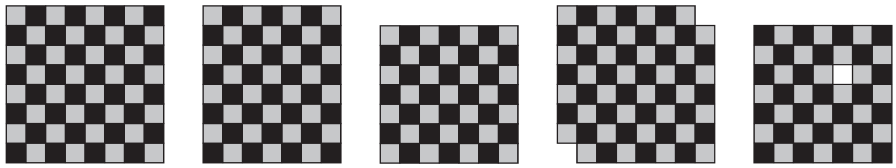
FIGURE 1.1 A 3×6 rectangle (a) and one way to tile it with dominoes (b).

In enumerative combinatorics, we would ask "in how many ways" can a given shape be tiled with dominoes, but here we will be content to consider only the existence question:

> Given a chessboard, possibly with some of its squares deleted, can we construct a tiling of the board with dominoes? If no construction is found, can it be explained why no tiling exists?

It will soon become apparent why we consider chessboards, since we will be able to take advantage of the alternating pattern of the colors of the unit squares.

Insights into the general case are often given by the examination of special cases. In the examples that follow we will consider the five chessboards shown in Figure 1.2.



(a) (b) (c) (d) (e)

FIGURE 1.2 Five examples of chessboards.

Example 1.15 Chessboard (a) is standard chessboard, with 8 unit squares in each row that are colored with an alternating pattern of black and gray colors. The bottom row can be tiled easily with 4 horizontal dominoes laid end to end. Indeed, each row can be tiled in this way, so the entire 8×8 chessboard can be tiled with horizontally aligned dominoes. Of course, there are many other ways to construct a tiling as well.

Example 1.16 If the last column of an 8×8 standard chessboard is removed, this leaves board (b) with 7 unit squares in each row. Horizontally aligned dominoes no longer can be used to tile the rows as for board (a). However, each column is 8 units high, and therefore the 8×7 chessboard can be tiled with vertically oriented dominoes.

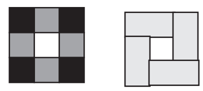
Example 1.17 Board (c) is a 7×7 board, so it has 49 unit squares, an odd number. But each domino in a tiling covers 2 unit squares. This means that any tiling by dominoes covers an even number of squares of the chessboard, so there is no possible way to tile board (c) with dominoes.

Our analysis of the first three chessboards can be generalized to rectangular chessboards of any size.

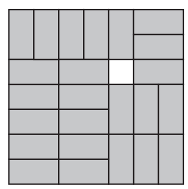
Theorem 1.18 A rectangular $m \times n$ chessboard can be tiled with dominoes if and only if at least one of its dimensions m or n is an even number.

Example 1.19 Chessboard (d) is obtained by removing two opposite corner squares from the 8×8 standard chessboard, leaving a trimmed board with 62 unit squares. It might seem, since 62 is even, that a tiling with dominoes exists. However, a closer look reveals that the 2 unit squares that were removed were both black, leaving a board with 32 gray and 30 black unit squares. But a domino, whether vertical or horizontal, simultaneously covers both a gray unit square and an adjacent black one. Thus, any trimmed chessboard cannot be tiled if it has an unequal number of gray and black unit squares. In particular, chessboard (d) cannot be tiled with dominoes.

Example 1.20 Chessboard (e) is a 7×7 board has one of its black squares removed, leaving a board with 48 unit squares, 24 gray and 24 black. The reasoning that we have used to examine boards (c) and (d) is not applicable, so it may yet be possible to tile the board with dominoes. As with boards (a) and (b), we can see if a particular tiling can be constructed. Rather than try brute force on the entire board, it may be best to consider first a simpler related problem. For example, what if the center black unit square is removed from a 3×3 chessboard? The following diagram shows that the trimmed 3×3 board can tiled with four dominoes:



This example suggests that we look for a tiling that combines both horizontally and vertically aligned dominoes. Nicely enough, we quickly find a tiling of board (e):

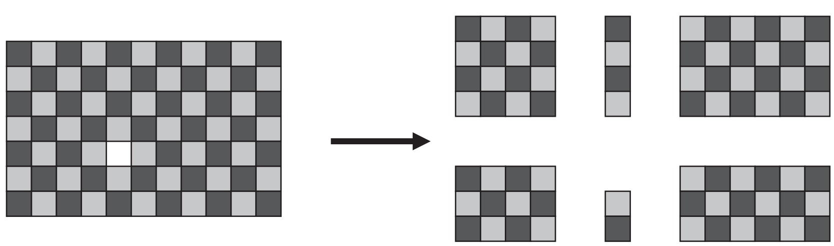


More generally, we can prove the next theorem.

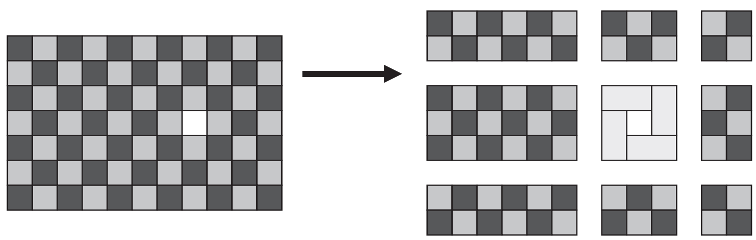
Theorem 1.21 Suppose that an $m \times n$ chessboard, m and n odd, has black corners. If any black square is removed from the board, the trimmed board can be tiled with dominoes.

Proof. Suppose that the black square that has been removed was in column i and row j. Then i and j are either both odd or both even. We consider the two cases separately.

Case 1: i and j are odd. This means the remainder of the board is a collection of rectangular boards each with at least one even dimension, so it can be tiled with dominoes by Theorem 1.18. Here is a typical example



Case 2: i and j even. This means that the 3×3 square with a deleted black center can be tiled, and that the remainder of the board is a collection of rectangular boards each with at least one even dimension, so it can be tiled with dominoes by Theorem 1.18. Here is a diagram illustrating this case:



■

It may seem curious that we have avoided a consideration of the enumerative question: How many ways can an $m \times n$ board be tiled with dominoes?

This question, although easy to ask, is not easy to answer! In 1961, the following result was derived independently by Temperley and Fisher [3] and Kasteleyn [4].

Theorem 1.22 An $m \times n$ board can be tiled by $mn/2$ dominoes in

$$\prod_{j=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{m+1}+4\cos^2\frac{k\pi}{n+1}\right)^{1/4} \tag{1.5}$$

ways.

Fortunately, the problem is much easier when m is small. Later we will introduce enumerative methods to determine the number of ways to tile the $2 \times n$ and $3 \times n$ boards with dominoes, and in Section 1.5 we will discuss the number of ways to tile $1 \times n$ boards with either colored squares or a mixture of squares and dominoes.

PROBLEMS

1.3.1. Consider an $m \times n$ chessboard, where m is even and n is odd. Prove that if two opposite corners of the board are removed, the trimmed board can be tiled with dominoes.

1.3.2. Consider an $m \times n$ chessboard, where both m and n are even. Prove that if any two unit squares of opposite color are removed, then the trimmed board can be tiled with dominoes.

1.3.3. Suppose that the lower left $j \times k$ rectangle is removed from an $m \times n$ chessboard, leaving an angle-shaped chessboard. Prove that that angular board can be tiled with dominoes if it contains an even number of squares.

1.3.4. In three dimensions, suppose that a domino consists of two unit cubes joined at their faces. What conditions on l, m, and n are necessary and sufficient for a rectangular solid of size $l \times m \times n$ to be filled with solid dominoes?

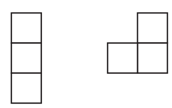
1.3.5. Consider a rectangular solid of size $l \times m \times n$, where l, m, and n are all odd positive integers. Imagine that the unit cubes forming the solid are alternately colored gray and black, with a black cube at the corner in the first column, first row, and first layer.

(a) What is the color of each of the remaining corner cubes of the solid?

(b) How can the color of the cube in column j, row k, and layer h of the solid be determined?

(c) Prove that removing any black cube leaves a trimmed solid that can be filled with solid dominoes.

1.3.6. A *tromino* is formed from three unit squares joined along common edges. There are two different trominoes, the 1×3 rectangular I tromino and the angular L tromino.

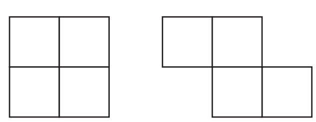


(a) Give a necessary and sufficient condition for when an $m \times n$ rectangular board can be tiled with the I tromino.

(b) Suppose that three corners are removed from a 9×9 board. Can the trimmed board be tiled with trominoes?

[*Hint:* Color the board with three colors in a pattern of alternating colors.]

1.3.7. A *tetromino* is formed with four squares joined along common edges. For example, the O and the Z tetrominoes are shown here:



(a) find the three other tetrominoes, called the I, J, and T tetrominoes.

(b) the set of five tetrominoes has a total area of 20 square units. Explain why it is not possible to tile a 4×5 square with a set of tetrominoes.

(c) show that a 4×10 rectangle can be tiled with two sets of tetrominoes.

(d) show that a 5×8 rectangle can be tiled with two sets of tetrominoes.

1.3.8. **(a)** Construct all of the tilings of a 2×4 chessboard with dominoes. [If you have skill with a computer algebra system, you may be interested in verifying that the number of tilings that you obtain is given by formula (1.5).]

(b) Repeat part (a), but for a 3×4 chessboard.

1.4 FIGURATE NUMBERS

In the fifth and sixth centuries BCE, numbers were the essence of the Pythagorean universe. Indeed, for them all objects were composed of whole numbers. A unit was not an abstraction, but was viewed as a very tiny geometric sphere that could be represented by a dot. Numbers were classified by the shape of the patterns that can be formed by the corresponding number of dots. For example, the numbers 1,3,6,10,15, … were called *triangular numbers*, because each of these numbers of dots could be arranged into a triangular pattern as shown in Figure 1.3.

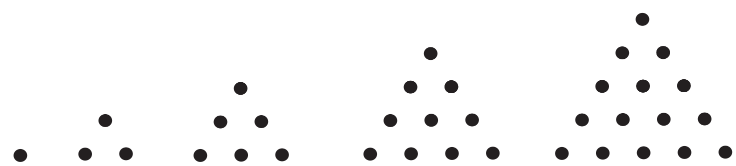


FIGURE 1.3 The first five triangular numbers 1, 3, 6, 10, and 15.

To investigate the triangular numbers in more detail, it is helpful to let t_n denote the nth triangular number. For example, $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$, and $t_5 = 15$. The triangular patterns in Figure 1.3 then make it clear, by counting the dots by rows from top to bottom, that

$$t_n = 1 + 2 + 3 + \cdots + n \tag{1.6}$$

The pattern that represents t_n consists of a triangle of t_{n-1} dots above the bottom row of n dots, so it follows that

$$t_n = t_{n-1} + n \tag{1.7}$$

This equation is a *recurrence relation* for the sequence of triangular numbers, since it is a formula for the nth triangular number that depends on the value of an earlier term in the sequence. Recurrence relations often arise in combinatorial analysis, and they will be considered in detail in Chapter 5.

Equations (1.6) and (1.7) are of interest, but neither one provides a *closed-form* expression for t_n. To obtain a simple algebraic formula for the triangular numbers, place two dot patterns side by side, with one turned upside down. This results in the pattern shown in Figure 1.4.

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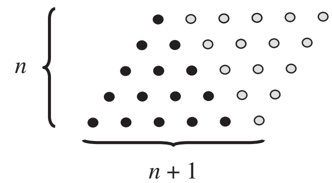


FIGURE 1.4 Dot pattern showing $t_n + t_n = n(n + 1)$.

Since each of the n rows in the combined pattern contains $n + 1$ dots, there are $n\,(n + 1)$ dots in all. Thus, we obtain the following theorem that provides a closed form expression for the nth triangular number.

Theorem 1.23 The nth triangular number $t_n = 1 + 2 + \cdots + n$ is given by

$$t_n = \tfrac{1}{2}n(n + 1) \tag{1.8}$$

For example, we see from equation (1.8) that

$$t_{100} = 1 + 2 + \cdots + 100 = \tfrac{1}{2} \cdot 100 \cdot 101 = 50 \cdot 101 = 5050$$

This sum has become well known because of its connection with a story concerning the mathematical abilities shown by Carl Friedrich Gauss (1777–1855) as a young boy. When about 10 years of age, his teacher Master Büttner asked the class to sum the numbers 1 through 100. Gauss very quickly wrote his answer on his slate, whereas his classmates continued to calculate for another hour. Master Büttner was surprised to discover that only Gauss had given the correct answer, and asked him how he obtained his result. Gauss explained that $1 + 100 = 101, 2 + 99 = 101, \ldots, 50 + 51 = 101$, so there are 50 such sums that are each 101. Therefore, the answer is $50 \cdot 101 = 5050$.

Let's now turn to the *square numbers* $s_1 = 1, s_2 = 4, s_3 = 9, s_4 = 16, s_5 = 25, \ldots, s_n = n^2$, which are represented by the dot patterns shown Figure 1.5.

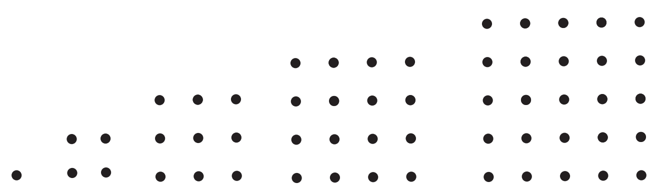


FIGURE 1.5 The first five square numbers 1, 4, 9, 16, and 25.

Once again, the dot patterns quickly reveal several properties of the square numbers. For example, the dots can be counted by diagonal rows or with angular shapes known to the Pythagoreans as *gnomens*. We also see that the square pattern that represents s_n is formed with two successive triangular patterns representing t_n and t_{n-1}.

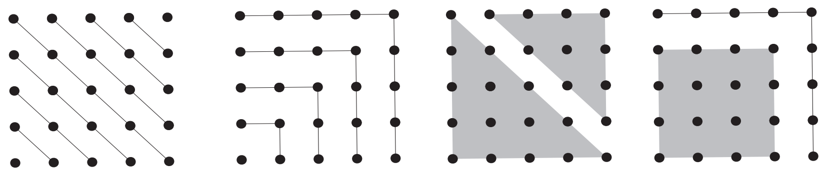


FIGURE 1.6 Discovering properties of the square numbers.

The following formulas are suggested by Figure 1.6:

$$s_n = 1 + 2 + 3 + \cdots + n + \cdots + 3 + 2 + 1 \tag{1.9}$$

$$s_n = 1 + 3 + 5 + \cdots + (2n - 1) \tag{1.10}$$

$$s_n = t_n + t_{n-1} \tag{1.11}$$

$$s_n = s_{n-1} + (2n - 1) \tag{1.12}$$

1.4.1 More General Polygonal Numbers

The Pythagoreans saw no reason to stop with the triangular and square numbers, since it was nearly as easy to arrange dots into pentagonal, hexagonal, and other polygonal patterns. For example, the pentagonal and hexagonal number patterns are shown in Figure 1.7.

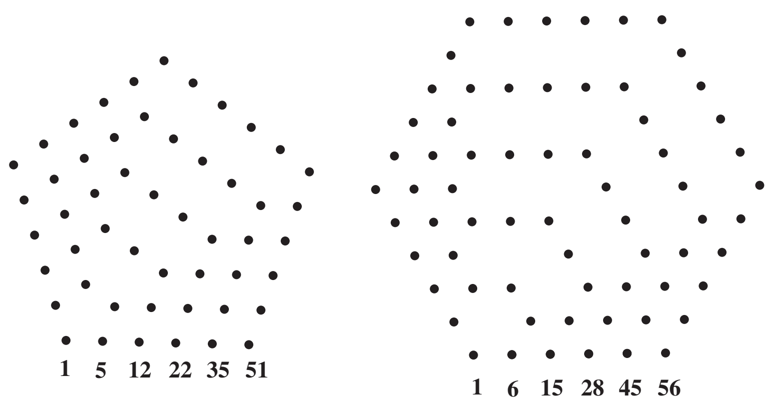


FIGURE 1.7 The pentagonal numbers $1, 5, 12, 22, 35, 51, \ldots$ and the hexagonal numbers $1, 6, 15, 28, 45, 56, \ldots$.

Let's denote the nth *r-gonal number* by $p_n^{(r)}$. For example, Figure 1.7 shows us that $p_3^{(5)} = 12$ is the third pentagonal number, and $p_5^{(6)} = 45$ is the fifth hexagonal number.

To obtain a closed-form expression for $p_n^{(r)}$, we can subdivide the dot pattern into a row of n dots together with $r - 2$ triangular arrays. There are $r - 2$ triangular patterns of dots that each contain t_{n-1} dots. The hexagonal case of this subdivision is shown in Figure 1.8.

FIGURE 1.8 A hexagonal dot pattern shows that $p_5^{(6)} = 5 + (6-2)t_{5-1}$.

In general, the following theorem expresses the nth r-gonal number in terms of triangular numbers.

Theorem 1.24 The nth r-gonal number is given by

$$p_n^{(r)} = n + (r-2)t_{n-1}, n \geq 1 \tag{1.13}$$

where t_{n-1} is the $(n-1)$st triangular number.

In view of (1.8), some algebraic rearrangement gives us the equivalent closed-form expressions in the next theorem.

Theorem 1.25 The nth r-gonal number is given by the closed-form expressions

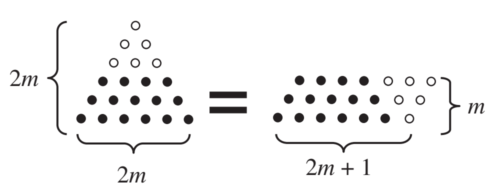
$$\begin{aligned} p_n^{(r)} &= \frac{n}{2}[n(r-2)-(r-4)] \\ &= \frac{n}{2}[(n-1)r-2(n-2)] \end{aligned} \tag{1.14}$$

These two formulas suggest that we define $p_0^{(r)} = 0$.

There are several useful variations of dot patterns beyond the polygonal numbers. In particular, the *centered polygonal numbers* are introduced in the following problem set. Dot patterns will also be used extensively in Chapter 6 when we investigate partition numbers.

PROBLEMS

1.4.1. The following diagram illustrates that $t_{2m} = m(2m+1)$:



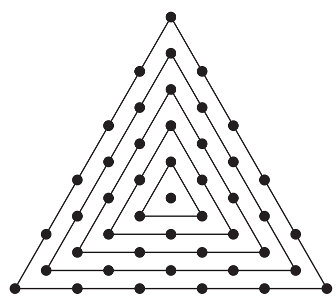
Create a similar diagram that illustrates the formula $t_{2m+1} = (2m+1)(m+1)$.

1.4.2. **(a)** Use dot patterns to show that the square of an even integer is a multiple of 4. [In number-theoretic terms, the square of any even integer is said to be *congruent to* 0 *modulo* 4, which is written as $(2n)^2 \equiv 0 (\text{mod } 4)$.]

(b) Verify your result of part (a) with algebra.

1.4.3. Use both algebra and dot patterns to show that the square of an odd integer is congruent to 1 modulo 8; that is, show that $s_{2n+1} = 8u_n + 1$ for some integer u_n. Be sure to identify the integer u_n by its well-known name.

1.4.4. The *centered triangular numbers* are obtained by starting with a single dot and then surrounding it by triangles with 2, 3, 4, 5, ... dots per side. The following diagram shows that the first five centered triangular numbers are 1, 4, 10, 19, 31, and 46:



(a) use the diagram to show that the nth centered triangular number is given by $1 + 3t_n$.

(b) derive a closed form expression for the nth centered triangular number.

1.4.5. The *centered square numbers* are obtained much like the centered triangle numbers of Problem 1.4.4, except that squares with an increasing number of dots per side surround a center dot.

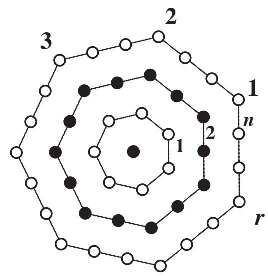
(a) Create a diagram that shows the sequence of centered square numbers begining with 1, 5, 13, 25, and 41.

(b) Color the dots in the diagram from part (a) to show that the nth centered square number is given by $(n + 1)^2 + n^2$.

(c) Shade your diagram from part (a) to shows that every centered square number is congruent to 1 modulo 4.

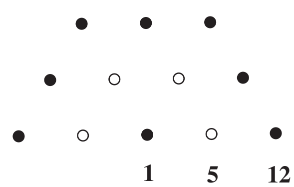
(d) Verify part (c) with algebra.

1.4.6. The *centered polygonal numbers* are obtained much like the centered triangle and centered square numbers of Problems 1.4.4 and 1.4.5, except that polygons with an increasing number of dots per side surround a center dot:



Explain why the nth centered r-gonal number can be mathematically represented by $c_n^{(r)} = rt_n + 1$.

1.4.7. The first three *trapezoidal numbers* are 1, 5, and 12, as shown by the dot pattern here.



(a) Continue the trapezoidal pattern to find the next three trapezoidal numbers.

(b) Draw some lines on your diagram from part (a) to explain why the trapezoidal numbers are simply an alternative pattern for the pentagonal numbers $p_1 = 1, p_2 = 5, p_3 = 12, \ldots$.

(c) Use the trapezoidal diagram to show why each pentagonal number is the sum of a triangular number and a square number. Give an explicit formula for p_n in terms of the triangular and square numbers.

(d) The trapezoidal diagram shows that each pentagonal number is the difference of two triangular numbers. Determine the two triangular numbers corresponding to p_n and express this result in a formula.

(e) Construct a diagram showing that each pentagonal number is one-third of a triangular number. Give an explicit formula of this property.

1.4.8. A domino is a 1×2 rectangular tile, divided by a centerline into two unit squares. In a double-6 set, each square is imprinted with a pattern ranging from 0 to 6 dots that determines the *suit* of the domino. For example, the 2–5 domino has 2 pips on one square and 5 pips on the other, so it belongs to both the two suit and the five suit. A *double* domino, such as the 6–6 domino, belongs to just one suit.

(a) How many dominoes are in a double-6 set? Note that there is just one domino for each pair of suits, since a $p - q$ domino is the same as the $q - p$ domino.

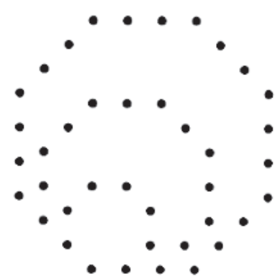
(b) How many total pips are on the complete double-6 set?

1.4.9. Dominoes, as described in Problem 1.4.8, also come in double-9, double-12, double-15, and even double-18 sets. Consider, more generally, a double-n set, so each half-domino is imprinted with 0 to n pips.

(a) Derive a formula for the number of dominoes in a double-n set. Use the formula to determine the number of dominoes in a double-n set for $n =$ 6, 9, 12, 15, and 18.

(b) Derive a formula for the total number of pips in a double-n set. Use the formula to determine the total number of pips in a double-n set for $n =$ 6, 9, 12, 15, and 18.

1.4.10. Trace the octagonal dot pattern and provide shading (see Figure 1.8) to show that $p_4^{(8)} = 4 + (8-2)t_{4-1} = 4 + 6 \cdot 6 = 40$:



1.5 COUNTING TILINGS OF RECTANGLES

In this section, we will investigate the number of ways that a $1 \times n$ rectangular board can be tiled with either squares of two colors or a mixture of 1×1 squares and 1×2 dominoes. We will discover several useful principles of combinatorial reasoning.

1.5.1 Tiling a Rectangle with Squares of Two Colors

For small values of n it is not unreasonable to draw all of the tilings with squares of two colors and then directly count the number of ways this can be done. Our hope is to discover and utilize patterns to obtain general answers. Figure 1.9 shows all of the tilings of a $1 \times n$ rectangular board with gray and white square tiles in the cases $n =$ 1, 2, and 3.

If we let $T^{(n)}$ denote the number of ways to tile a board of length n with tiles of two colors, we see that $T^{(1)} = 2, T^{(2)} = 4, T^{(3)} = 8$. This suggests that the number of

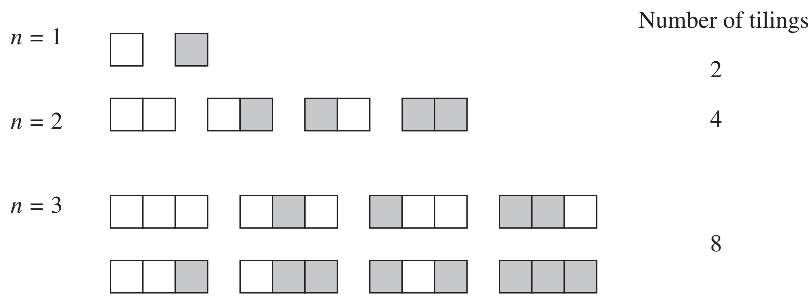


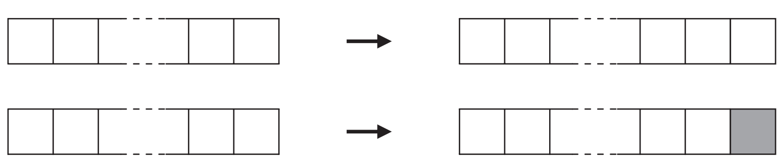
FIGURE 1.9 Tilings by squares of two colors.

tilings doubles each time the length of the board increases by one; thus it appears that we have the recurrence relation

$$T^{(n)} = 2T^{(n-1)} \tag{1.15}$$

Note that this equation will hold for all integers $n \geq 1$ if we define $T^{(0)} = 1$.

To understand why equation (1.15) holds, we see that there are two ways to extend a tiling of a board of length $n - 1$ to become a tiling of a board of length n, namely, these, where we add either a white or a gray square to the right end:



The two methods never form the same tiling, since one extension gives a tiling that ends with a white square and the other extension ends with a gray square. Moreover, *all* tilings of a board of length n are obtained in this way, since given any tiling of length n, we can delete its rightmost square to create a tiling of a board of length $n - 1$. In summary, *all* of the tilings of length n can be separated into two disjoint classes depending on the color of the rightmost square—either white or gray—and the two classes together contain all of the $T^{(n)}$ tilings of a board of length n. Thus, we have derived the recursion relation $T^{(n)} = T^{(n-1)} + T^{(n-1)} = 2T^{(n-1)}$ that we had guessed earlier.

In later chapters, we will consider a variety of methods to solve recurrence relations, but the doubling relation can be solved easily by iterating the recurrence until we arrive at an expression containing the known value $T^{(0)} = 1$:

$$T^{(n)} = 2T^{(n-1)} = 2\left(2T^{(n-2)}\right) = 2\left(2\left(2T^{(n-3)}\right)\right) = \cdots = 2^n T^{(0)} = 2^n. \tag{1.16}$$

We have proved the following theorem.

Theorem 1.26 There are

$$T^{(n)} = 2^n \tag{1.17}$$

ways to tile a rectangular board of length n with white and gray squares.

This answer is unexpectedly simple, which suggests there may be a more direct derivation of the result. Let's consider the board of length 4, as shown in Figure 1.10.

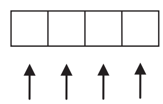


FIGURE 1.10 There are 4 two-way choices to make: cover each of the four cells of the board with a white or gray tile.

The arrows in the figure highlight where we can decide what color square to use, white or gray. Since each of the four choices can go either of two ways, there are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ ways to tile a board of length 4. The same reasoning can be applied to count the number of tilings of a board of any length n; we have n choices that can each be made in two ways, so the number of tilings of a $1 \times n$ board with squares of two colors is

$$2 \times 2 \times 2 \times \cdots \times 2 = 2^n$$

The reasoning we have used is called the *multiplication principle* of combinatorics, and will be explored in detail in the next section of this chapter. This principle can be used to solve some related tiling problems, as demonstrated with these two questions:

Question 1. In how many ways can a $1 \times n$ rectangular board be tiled so that each cell of the board is covered with a red, green, or blue 1×1 square tile?

Answer 1. There is a three-way choice to be made n times, so the board can be tiled in $3 \times 3 \times \cdots \times 3 = 3^n$ ways.

Question 2. In how many ways can a $1 \times n$ rectangular board be tiled so that the first cell of the board is covered with a red square tile, the second cell is covered with either a green or blue tile, and more generally the jth cell is covered by a square tile with j choices of its color?

Answer 2. There are j choices for the color of the tile that covers the jth cell, $1 \leq j \leq n$, so the board can be tiled in $1 \times 2 \times 3 \times \cdots \times n$ ways.

The product of the first n positive integers occurs frequently in combinatorics, so it is given a special name and notation:

$$n! = n \times (n-1) \times \cdots \times 3 \times 2 \times 1 \text{ is } n \textit{ factorial}$$

1.5.2 Tiling a $1 \times n$ Rectangle with Exactly r Gray Squares

As before, we will count the number of ways to tile a board with white and gray squares, but now we will restrict the number of squares of each color. Let $C(n, r)$ denote the number of ways to tile a $1 \times n$ board using r gray squares and $n - r$ white squares. Some simple drawings should provide the entries in the following table. We have set $C(0,0) = 1$ to provide a more complete table:

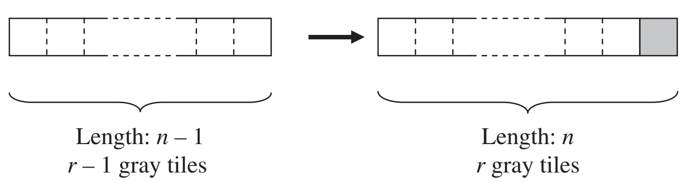
$C(n, r)$	Number of Gray Squares in Tiling r				
Length of board n	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$
$n = 0$	1				
$n = 1$	1	1			
$n = 2$	1	2	1		
$n = 3$	1	3	3	1	
$n = 4$	1	4	6	4	1

It follows from Theorem 1.26 that the sum of the entries in row n is 2^n, and so we have the identity

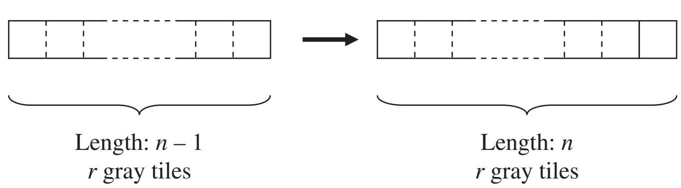
$$\sum_{r=0}^{n} C(n, r) = 2^n \tag{1.18}$$

We might also note from the table that the sum of two adjacent values in a row is the entry in the next row just below the right-hand summand. This could have been anticipated, since there are two distinct ways to create a tiling of length n that has r gray squares:

1. Given any of the $C(n - 1, r - 1)$ tilings of length $n - 1$ with $r - 1$ gray tiles, add a gray tile at the right to create a tiling of length n with r gray squares:



2. Given any of the $C(n - 1, r)$ tilings a tiling of length $n - 1$ and r gray tiles, add a white tile at the right to create a tiling of length n still with r gray tiles:



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The two methods never form the same tiling, since we create tilings that end in squares of different colors. Moreover, all of the tilings of length n with r gray squares are formed in this way, since any tiling necessarily ends with either a gray or a white tile. Therefore, the set of all tilings of length n with r gray squares splits into two disjoint subsets, one with $C(n-1, r-1)$ elements and the other with $C(n-1, r)$ elements. We have proved the following recursive identity:

$$C(n, r) = C(n-1, r-1) + C(n-1, r) \tag{1.19}$$

This identity makes it easy to extend the table of values for as many rows as we wish, as shown here:

					r				
	1								
	1	1							
	1	2	1						
	1	3	3	1					
n	1	4	6	4	1				
	1	5	10	10	5	1			
	1	6	15	20	15	6	1		
	1	7	21	35	35	21	7	1	
	1	8	28	56	70	56	28	8	1

This tabulation is the famous *Pascal triangle*,[2] for which the entry in row n and column r is the *binomial coefficient* denoted by $C(n,r)$ or by $\binom{n}{r}$. For example, $\binom{8}{3} = 56$ is the entry in row 8 and column 3 of the tabulation above. Both of the indices n and r begin with 0.

The C symbol refers to its meaning as a *combination*; that is, $C(n,r)$ is the number of ways in which a subset of r objects can be chosen from a given set of n distinct objects, where the order in which objects are selected is unimportant. It is helpful to use the letter C as a reminder that we are counting the ways to *choose* a subset of a certain size, and the symbols $C(n,r)$ and $\binom{n}{r}$ are usefully read as "n choose r." Combinations and the binomial coefficients play a major role in combinatorial analysis, and their study will be taken up in detail in later chapters. In particular, later we will obtain a formula for the binomial coefficients, namely

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1.20}$$

[2]Blaise Pascal (1623–1662) was a French mathematician, physicist, inventor, essayist, philosopher, and theologian. As a teenager, he invented a calculating machine called the *Pascaline*. He wrote fundamental works on projective geometry and probability, and later in his a life a treatise on the triangular array that we now call *Pascal's triangle*.

where

$$0! = 1, \quad k! = k(k-1)(k-2)\cdots 3\cdot 2\cdot 1 \tag{1.21}$$

are the *factorials* introduced earlier.

For our tiling problem, $C(n,r)$ is the number of ways to choose the r cells to be covered with gray tiles on a $1 \times n$ board, leaving the remaining $n - r$ squares of the board to be covered with white tiles. For example, to tile a 1×7 board, there are $\binom{7}{1} = 7$ (i.e., 7 choose 1) ways to choose one cell covered by a gray tile. To form a tiling with 2 gray tiles, there are $\binom{7}{2}$ (i.e., 7 choose 2) ways to choose the 2 cells covered by gray tiles. Since $\binom{7}{2} = 21$, as seen from Pascal's triangle or from the factorial expression

$$\binom{7}{2} = \frac{7!}{2!5!} = \frac{7\cdot 6}{2\cdot 1} = 21$$

we see that a 1×7 board can be tiled in 21 ways using 2 gray and 5 white tiles. Similarly, there are 35 ways to tile a board of length 7 with 3 gray tiles and 4 white tiles.

The reasoning above can be applied generally to boards of any length, giving us the following theorem.

Theorem 1.27 The number of tilings of a $1 \times n$ board with r gray tiles and $n - r$ white tiles is given by the binomial coefficient (n choose r):

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1.22}$$

Counting tilings has provided us with two identities, (1.18) and (1.19) for the binomial coefficients that will be encountered again later. Combinatorialists generally favor writing "n choose r" with the $\binom{n}{r}$ style of notation, so the identities take the form

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n \tag{1.23}$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad \text{(Pascal's identity)} \tag{1.24}$$

Both of these identities were derived with combinatorial reasoning; that is, they were obtained using the principles of counting rather than, say, algebraic calculation. It would be more difficult to prove identity (1.23) algebraically using formulas (1.20) and (1.21), although Pascal's identity can be verified in a few lines of calculation. To the combinatorialist, a combinatorial proof is much the preferred approach, with algebra used sparingly. Combinatorial proofs will be used frequently throughout this text.

1.5.3 Tiling a 1 × *n* Rectangle with Squares and Dominoes

Figure 1.11 shows the ways that a 1 × 1, 1 × 2, 1 × 3, and 1 × 4 board can be tiled with 1 × 1 squares and 1 × 2 dominoes.

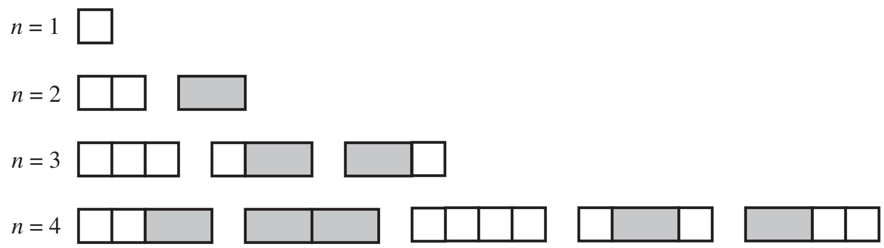
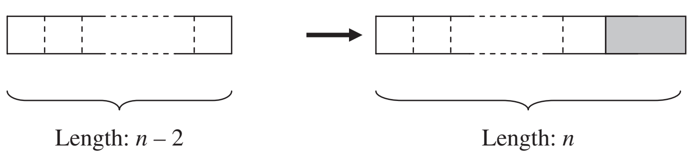


FIGURE 1.11 Tilings by squares and dominoes.

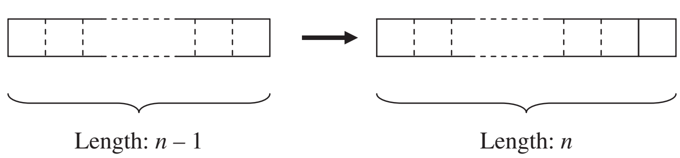
To see a pattern, it helps to check that there are 8 ways to tile a board of length 5. If we let f_n denote the number of tilings of a $1 \times n$ board, we see that $f_1 = 1, f_2 = 2$, $f_3 = 3, f_4 = 5$, and $f_5 = 8$.

Although the evidence is minimal, so far it appears that the sum of two successive terms of the sequence gives the next term in the sequence. To verify that this property holds in general, we can proceed by analogy with the approach that was successful for the tiling of a board with colored squares. That is, we want to see how the tilings of boards of length $n - 2$ and $n - 1$ can be modified to form all of the tilings of a board of length *n*. As before, it seems helpful to add an additional tile at the right of the board in one of these two distinct ways:

1. Given a tiling of length $n - 2$, add a domino at the right to create a tiling of length *n*:



2. Given a tiling of length $n - 1$, add a square at the right to create a tiling of length *n*:



The two methods never form the same tiling, since we create a tiling ending in either a domino or a square. Moreover, any tiling must end with either a square or domino, so *all* of the tilings of length n are created in *exactly one* of these two ways. Thus, we have the following recurrence relation:

$$f_n = f_{n-2} + f_{n-1} \tag{1.25}$$

Using the beginning values, $f_1 = 1, f_2 = 2$, we can use the recurrence formula (1.25) to extend the sequence as far as we wish. For example, the first 10 terms of the sequence are given in the following table:

n	1	2	3	4	5	6	7	8	9	10
f_n	1	2	3	5	8	13	21	34	55	89

This is the famous sequence of *Fibonacci numbers*,[3] which are most often defined by the following recurrence relation and initial conditions:

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2 \tag{1.26}$$

We see that $f_1 = F_2 = 1, f_2 = F_3 = 2$, and more generally that $f_n = F_{n+1}$. The numbers in the sequence f_n are often called the *combinatorial Fibonacci numbers,* since they frequently appear in counting situations.

The following theorem relates the number of tilings by squares and dominoes to the Fibonacci numbers.

Theorem 1.28 The number of tilings f_n of a $1 \times n$ board with squares and dominoes is given by the Fibonacci number F_{n+1}.

Often combinatorial problems come in different disguises, but they are the same problems beneath the disguises. For example, suppose that we wish to know the number of sequences of length n where each term is a 0 or a 1. Such a sequence—known as a *binary sequence*—is just a way to describe how a $1 \times n$ board can be tiled with gray and white tiles, where each 0 corresponds to a white tile and each 1 corresponds to a gray one. Thus, by Theorem 1.26, there are 2^n binary sequences of length n. Similarly, to determine how many binary sequences of length n have r ones and $n - r$ zeros, we would use the same method as used to find the number of tilings of a $1 \times n$ board with r gray tiles and $n - r$ white tiles. In Problem 1.5.11, you will discover how to answer this question: *How many binary sequences of length n have no two consecutive ones?*

[3]Leonardo Fibonacci (ca. 1170–1250), also known as Leonardo of Pisa, is most famous for his book *Liber Abaci,* which introduced Indo-Arabic mathematics and notations to European scholars. The number sequence that bears his name was first introduced in a combinatorial problem related to breeding rabbits under some idealized conditions.

PROBLEMS

1.5.1. **(a)** Extend Figure 1.9 to depict the set of 16 tilings of a board of length 4, where each tile is either gray or white.

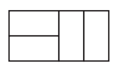
(b) Explain why it is easy to use the 8 tilings of boards of length 3 to draw all of the tiled boards of length 4.

1.5.2. Explain how to obtain the tilings of a 1×4 board with 2 gray squares by modifying the tilings of the 1×3 boards that have either 1 or 2 gray squares.

1.5.3. Use formulas (1.20) and (1.21) to prove Pascal's identity (1.24).

1.5.4. Show that there are 8 ways to tile a 1×5 rectangular board with squares and dominoes.

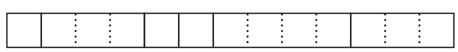
1.5.5. **(a)** Find all of the ways that a 2×4 rectangular board can be tiled with 1×2 dominoes. Here is one way to tile the board:



Draw all of the ways to tile a 2×4 board with dominoes.

(b) How many ways can a $2 \times n$ board be tiled with dominoes?

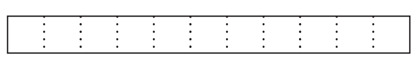
1.5.6. Suppose that a *train* of length n is a tiling of a $1 \times n$ rectangle by $1 \times r$ rectangles called *cars*, where the length of a car is any positive integer r. For example, here is a train of length 13 formed with 6 cars of lengths 1, 3, 1, 1, 4, and 3, with total length $1 + 3 + 1 + 1 + 4 + 3 = 13$:



(a) Let $t^{(n)}$ denote the number of trains of length n. Determine $t^{(1)}, t^{(2)}, t^{(3)}, t^{(4)}$ by constructing a figure showing all of the trains.

(b) Guess and then prove a formula for $t^{(n)}$.

1.5.7. The following train (see Problem 1.5.6 for the definition of a train) has just one car of length 13:



(a) in how many ways can a train of length 13 be formed with 2 cars?

(b) why are there $\binom{12}{4}$ trains of length 13 that can be formed with 5 cars?

(c) generalize your answer to part (b) to give a binomial coefficient that expresses the number of trains of length n with r cars.

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1.5.8. There are 4 ways to express 3 as an ordered sum of positive integers, namely, $3, 1 + 2, 2 + 1$, and $1 + 1 + 1$.

(a) In how many ways can 4 be expressed as an ordered sum?

(b) In how many ways can a positive integer n be expressed as an ordered sum of positive integers? (*Suggestion:* See Problem 1.5.6 and Problem 1.5.7.)

1.5.9. There are 4 ways to express 5 as a sum of 2 ordered summands, namely $4 + 1, 3 + 2, 2 + 3$, and $1 + 4$.

(a) In how many ways can 5 be expressed as a sum of 3 ordered summands? (See Problem 1.5.8.)

(b) In how many ways can a positive integer n be expressed as a sum of k summands?

1.5.10. Suppose that a *g-tiling* is a tiling of a rectangular board by squares, dominoes, or trominoes, where a tromino is a 1×3 tile. Let $G^{(n)}$ denote the number of g-tilings of a board of size $1 \times n$.

(a) Determine $G^{(1)}, G^{(2)}, G^{(3)}, G^{(4)}$.

(b) Explain how to associate the g-tilings of length 4 with the tilings of length 1, 2, and 3, and obtain an expression for $G^{(4)}$ in terms of $G^{(1)}, G^{(2)}, G^{(3)}$.

(c) Derive a recurrence relation for $G^{(n)}$, where $n \geq 4$.

1.5.11. How many binary sequences of length n have no two consecutive ones? (A binary sequence in an ordered list of ones and zeros, such as 100101001.) For example, there are five binary sequences of length 3 with no two consecutive ones, namely, 000, 100, 010, 001, and 101.

1.6 ADDITION AND MULTIPLICATION PRINCIPLES

In enumerative combinatorics, the goal is to determine the number of members of a set S, as determined by the properties that determine membership in the set. For example, S may be the set of (1) dots that can be arranged into a geometric figure of a certain shape, (2) tilings of a $1 \times n$ rectangle by squares of two colors, or (3) tilings of a chessboard with dominoes. The number of elements of set S is denoted by $|S|$.

1.6.1 Addition Principles

In several of the problems addressed in the last section, it was helpful to separate the members of a set S into two subsets, say, sets A and B. For example, we separated the tilings of a rectangle into two subsets; one subset contained those tilings ending with a white tile and the second subset contained the tilings ending with a gray tile. To account for every member of S, it is important that *every* member of S either belong to either A or B. In other words, we required $S = A \cup B$. But we also want to count every element just once, so no element of S can belong to both A and B; that is, we require A and B to be disjoint sets, so that $A \cap B = \varnothing$. When these two conditions are met, we say that the subsets A and B are a *partition* of set S. The number of elements in S is simply the sum of the number of elements in the subsets A and B. This important idea is known as the *addition principle* of combinatorics.

Theorem 1.29 (Addition Principle) Let $S = A \cup B$, where $A \cap B = \varnothing$. Then

$$|S| = |A \cup B| = |A| + |B|. \tag{1.27}$$

Example 1.30 The traditional soccer ball, first made in Denmark around 1950, is a spherical polyhedron known as a *truncated icosahedron*. The surface of the ball has 20 white hexagonal panels and 12 black pentagonal panels (see diagram). What is the total number of panels of the ball?

Solution. Each panel of the ball is either hexagonal or pentagonal, so the set of hexagonal panels and the set of pentagonal panels partition the set of all of the panels of the ball. By the addition principle, there are $20 + 12 = 32$ panels in all. ■

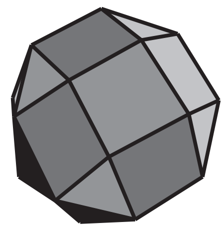
For some problems, the addition principle is inverted. As a quick example, if there are 15 members of the math club and 9 members are women, then there are $15 - 9 = 6$ male members. More generally, we have this theorem, where $S - A$ denotes the difference of sets defined by $S - A = \{x \in S \mid x \notin A\}$.

Theorem 1.31 (Subtraction Principle) Let $A \subseteq S$ and $B = S - A$. Then

$$|B| = |S - A| = |S| - |A| \tag{1.28}$$

Proof. Since A and $B = S - A$ are a partition of the set S, we have $|S| = |A| + |B| = |A| + |S - A|$. Solving this equation for $|B|$ gives equation (1.28). ■

Example 1.32 The figure shown is called the *gyroelongated square bicupola*, a polyhedron with 34 faces that are either squares or equilateral triangles. If there are 10 square faces, how many faces are triangles?



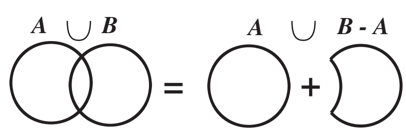
Solution. Let F, S, and T denote the sets of all faces, square faces, and triangular faces, respectively. Thus $T \subset F$ and $T = F - S$. According to the subtraction principle, there are $|T| = |F - S| = |F| - |S| = 34 - 10 = 24$ triangular faces. ■

In the next example, the addition principle is extended to a union of two sets that are not necessarily disjoint. If $S = A \cup B$, where $A \cap B \neq \emptyset$, then $|S|$ is no longer given by the sum $|A| + |B|$ since each element $x \in A \cap B$ is counted once as a member of A and a second time as a member of B. To compensate, we can subtract $|A \cap B|$ from the sum $|A| + |B|$ so that each member of S is counted exactly once. As a result, we have this theorem, which we also prove more formally by applying the subtraction principle.

Theorem 1.33 (Addition Principle for Unions of Two Sets) Let $S = A \cup B$. Then

$$|S| = |A \cup B| = |A| + |B| - |A \cap B| \tag{1.29}$$

Proof (via the Subtraction Principle). The Venn diagram below makes it clear that $S = A \cup B = A \cup (B - A)$, so S is partitioned by the disjoint sets A and $B - A$:



Therefore $|S| = |A| + |B - A|$ by the addition principle (1.27). But $A \cap B \subseteq B$, so by the subtraction principle $|B - A| = |B| - |A \cap B|$. Combining equations, we get $|S| = |A \cup B| = |A| + |B - A| = |A| + |B| - |A \cap B|$. ■

Example 1.34 Let H be the set of hearts, and F be the set of face cards from an ordinary 52-card deck. How many cards in the deck are either a heart or a face card?

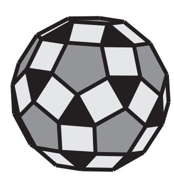
Solution. We need to determine $|S|$, where $S = H \cup F$. There are $|H| = 13$ hearts, and $|F| = 12$ face cards with 3 face cards in each of the 4 suits. In particular, there are 3 hearts that are also face cards, so $|H \cap F| = 3$. We can now apply formula (1.29) to get $|H \cup F| = |H| + |F| - |H \cap F| = 13 + 12 - 3 = 22$. ■

In Chapter 4, we will consider the principle of inclusion–exclusion (PIE), which extends Theorem 1.33 to a formula that counts the number of elements in a set given as a union $S = A_1 \cup A_2 \cup \cdots \cup A_n$ of arbitrarily many sets. For now, however, suppose that each element of S belongs to exactly one of the subsets. In this case we say that $A_1, A_2, \ldots, A_n$ are *pairwise disjoint*, so that $A_j \cap A_k = \emptyset$, $j \neq k$. When $S = A_1 \cup A_2 \cup \cdots \cup A_n$ is a union of pairwise disjoint sets, we say that the sets $A_1, A_2, \ldots, A_n$ are a *partition* of set S. We can again find the number of members of S by summing the number of members of each set of the partition, giving us this theorem.

Theorem 1.35 (Extended Addition Principle) Let $S = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_j \cap A_k = \emptyset$, $j \neq k$. Then

$$|S| = |A_1| + |A_2| + \cdots + |A_n| \tag{1.30}$$

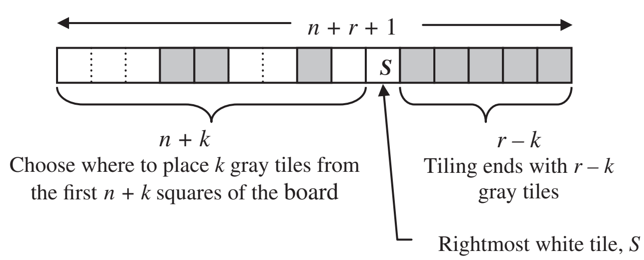
Example 1.36 The *rhombicosidodecahedron* is a polyhedron with 20 triangular faces, 30 square faces, and 12 pentagonal faces. Therefore the total number of faces is $20 + 40 + 12 = 62$ faces:



Example 1.37 In Section 1.5, we saw that $C(n,r)$ is the number of the ways to tile a rectangular $1 \times n$ board with n tiles, r of which are gray and the rest white. Use the tiling model to show that

$$C(n + r + 1, r) = C(n, 0) + C(n + 1, 1) + \cdots + C(n + r, r) \tag{1.31}$$

Solution. The left side of (1.31) counts the number of ways to tile a $1 \times (n + r + 1)$ board with r gray tiles, and $n + 1$ white tiles. We must show the sum on the right side of (1.31) gives the same count. Suppose that the rightmost white square, call it S, is in position $n + k + 1$, where $k = 0, 1, \ldots, r$. To the right of S is a string of $r - k$ consecutive gray tiles, and the k other gray tiles must be located among the $n + r$ tiles to the left of S:



There are $C(n + k, k)$ ways to tile the first $n + k$ squares of the board using k gray tiles, so this is the number of tilings for which the S has position $n + k + 1$. By noting the position of the rightmost white square, we have partitioned the tiling into disjoint sets. By the extended addition principle, we see that the number of tilings is given by $\sum_{k=0}^{r} C(n+k, k)$, which is just the sum on the right of (1.31). ■

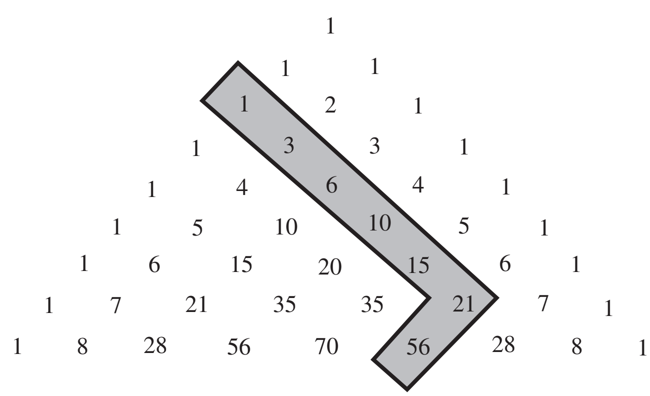
Identity (1.31) is most often written using the notation $C(n, r) = \binom{n}{r}$, acquiring the form shown in the following theorem.

Theorem 1.38 (Hockey Stick Identity 1)

$$\binom{n+r+1}{r} = \binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} \tag{1.32}$$

The name *hockey stick* becomes evident if a loop is drawn in Pascal's triangle that tightly encloses the binomial coefficients that appear in identity (1.32). Here is the hockey stick identity that corresponds to the case $n = 2$, $r = 5$:

$$56 = \binom{8}{5} = \binom{2}{0} + \binom{3}{1} + \binom{4}{2} + \binom{5}{3} + \binom{6}{4} + \binom{7}{5} = 1 + 3 + 6 + 10 + 15 + 21$$



A second hockey stick identity can be proved by partitioning the set of tilings according to the position of the rightmost gray tile of any tiling of a board of length $n + 1$ with $r + 1$ gray squares (see Problem 1.6.8).

1.6.2 Multiplication Principle

In how many ways can you flip a coin and roll a six-sided die? The answer is certainly not 8, since there are six outcomes for the die when the coin lands heads and 6 additional outcomes for the die when the coin lands tails. Altogether there are

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$12 = 2 \cdot 6$ ways to flip the coin and roll the die. We see that the number of outcomes is given by the *product* of choices, not the sum.

If $C = \{\text{H,T}\}$ is the set of outcomes of a coin flip, and $D = \{1,2,3,4,5,6\}$ the set of outcomes of a roll of the die, then the set of outcomes of the coin-and-die experiment is described by the set of ordered pairs in the set

$$\begin{aligned} S = C \times D = \{&(\text{H}, 1), (\text{H}, 2), (\text{H}, 3), (\text{H}, 4), (\text{H}, 5), (\text{H}, 6) \\ &(\text{T}, 1), (\text{T}, 2), (\text{T}, 3), (\text{T}, 4), (\text{T}, 5), (\text{T}, 6)\} \end{aligned} \tag{1.33}$$

The set shown in (1.33) is the Cartesian product of the sets C and D. More generally, we have the following definition.

Definition 1.39 The *Cartesian product* of sets A and B, denoted by $S = A \times B$, is the set of all ordered pairs whose first coordinate is a member of set A and whose second coordinate is from set B:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

The number of ordered pairs is the product of the number of choices $|A|$ for the first entry and the number of choices $|B|$ for the second entry of the ordered pair. Therefore, we have the following theorem.

Theorem 1.40 (Multiplication Principle for a Cartesian Product) For any two sets A and B, the number of ordered pairs in the Cartesian product $S = A \times B$ is given by

$$|S| = |A \times B| = |A||B| \tag{1.34}$$

It is very important to note that the number of choices available for the second entry in each ordered pair is independent of the choice made for the first entry of the ordered pair. For example, we were *not* asked to roll a 12-sided die if the coin landed tails and otherwise roll a six-sided die if the coin landed heads.

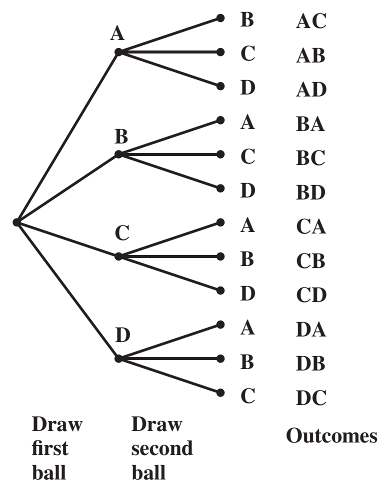
To form the Cartesian product, it was helpful to create a sequence of two choices: first, choose the first coordinate and then choose the second coordinate. In other words, we may formulate a two-stage experiment that results in all of the possibilities. The total number of possibilities is then the product of the number of choices available at each stage, *where the number of choices available for the second stage is independent of which choice was made in the first stage.*

The following example illustrates this type of reasoning, where a diagram known as a *possibility tree* is a useful way to visualize the succession of stages.

Example 1.41 A bag contains 4 balls, labeled A, B, C, and D. In how many ways can two balls be drawn from the bag, where the first drawn ball is not replaced into the bag?

Solution. Four possible balls can be drawn first: ball A, B, C, or ball D. If ball A is drawn first, there are 3 ways—B or C or D—to draw the next ball. Indeed, there are always 3 ways to draw the second ball independently of which ball was drawn first.

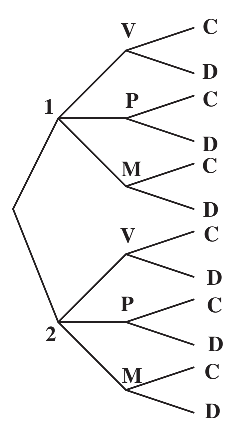
The $12 = 4 \cdot 3$ ways to draw the two balls without replacement is illustrated with this possibility tree:



If the first ball were replaced before the second ball is drawn, there would always be four choices for the second ball and $16 = 4 \cdot 4$ ways to draw the two balls with replacement. Therefore, $S = \{A, B, C, D\}$ is the set of choices for both the first and second ball drawn, and the number of ways to draw the balls with replacement is $|S \times S| = |S||S| = 4^2 = 16$. ■

The following example is solved with a three-stage process.

Example 1.42 At Henry's Ice Cream Shop, ice cream can be ordered in two sizes—one or two scoops, in three flavors—vanilla, peach, or mint—and in two containers—a dish or a cone. How many ways can ice cream be purchased at Henry's? See the following diagram:



Solution. Let $R = \{1, 2\}$ be the set of available sizes; $S = \{\text{V, P, M}\}$, the set of flavor choices; and $T = \{\text{D, C}\}$, the set of containers. An order is placed by choosing the size, flavor, and container, and can be visualized by this possibility tree that shows there are 12 ways to place an order.

Alternatively, each order is described by a ordered triple in the Cartesian product $R \times S \times T$, and $|R \times S \times T| = |R||S||T| = 2 \cdot 3 \cdot 2 = 12$. For example, (2,P,D) corresponds to two scoops of peach ice cream in a dish. ■

The reasoning we have used for two- and three-stage processes extends to multistage processes. However, it is important to observe that the number of choices at each successive stage cannot depend on which choices have been made in any of the earlier stages. The particular set of choices at any stage is allowed to vary according to previous choices, but the number of choices at each stage must be independent of the choices made earlier.

Theorem 1.43 (Multiplication Principle for a Multistage Process) Suppose that a multistage experiment consists of a sequence of n steps, for which there are

1. p_1 ways to perform step 1
2. p_2 ways to perform step 2 (independently of how step 1 was performed)
3. p_3 ways to perform step 3 (independently of how steps 1 and 2 were performed)
   ...

n. p_n ways to perform step n (independently of how the preceding steps were performed)

Then there are $p_1 p_2 \cdots p_n$ ways to carry out all n steps of the multistage experiment.

The following examples illustrate how to apply the multiplication principle.

Example 1.44 In Washington State, a car license is an arrangement of three letters followed by four digits.

(a) What is the number of possible licenses?

(b) What is the number of possible licenses if no digit or letter can be repeated?

Solution. Any license plate can be viewed as a seven-step process in which the leftmost symbol is chosen, then the second symbol, and so on, finishing with the choice made for the seventh symbol at the far right.

(a) There are 26 letters and 10 digits, so the number of possible license plates is

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175{,}760{,}000$$

by the multiplication principle.

(b) The second letter cannot repeat the first, so there are 25 choices available. Likewise, the third letter cannot repeat either of the two letters that have been selected already, so there are 24 remaining choices. Similar considerations apply to the selection of the digits.

By the multiplication principle, there are $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 78{,}624{,}000$ license plates with no repeated symbol. ■

Example 1.45 Eight people—Alice, Bob, Carly, Dave, . . . , Hal—arrive at the tennis courts. In how many ways can they

(a) form pairs to play four singles matches?
(b) divide up to play two games of doubles?

Solution. Imagine that you are the match organizer. Formulate a sequence of steps that creates all of the possible matches for either singles or doubles.

(a) Let's choose the pairs of opponents in succession, beginning with the choice of an opponent for Alice. There are seven possible opponents for Alice, any one of Bob, Carly, . . . , or Hal. After Alice and her opponent leave to start their match, six people remain. Choose any one of them to pick his or her opponent in 5 ways. This leaves a group of four, and you can ask one of them to choose an opponent in 3 ways. This leaves just two people, with one way for them to become opponents. Altogether, the four-stage process shows that there are $7 \cdot 5 \cdot 3 \cdot 1 = 105$ ways to divide eight people into four pairs.
(b) The four steps used in part (a) to form the singles matches can just as well be used to form four pairs of doubles partners. However, as a fifth step, we can ask Alice and her partner to choose one of the three other teams as their opposing team. The two remaining teams will then form the other doubles match. We now see that there are $7 \cdot 5 \cdot 3 \cdot 3 \cdot 1 = 315$ ways to form the two doubles matches. ■

The reasoning in Example 1.45(a) applies to any even number of players, say, $2n$, to show that the number of ways to arrange singles matches among $2n$ players is

$$(2n-1)!! = (2n-1)(2n-3)\cdots(3)(1) \tag{1.35}$$

where the decreasing product of positive odd positive integers is sometimes called a *double factorial.* The double factorial of an even integer is given by

$$(2n)!! = (2n)(2n-2)\cdots(4)(2) \tag{1.36}$$

so that the double factorial $m!!$ is defined for all positive integers. Note that $(n!)! \neq n!!$.

In the next example, we consider a sequence with terms that are each either a 0 or a 1. Since only two symbols appear, we recall that this is called a *binary sequence*. For example, 101011 is a binary sequence of length 6.

Example 1.46 What is the number of binary sequences

(a) of length 6?
(b) of length n?

Solution. There is a two-way choice—either 0 or 1—for each term of the sequence, and each choice is independent of the choices made for the previous terms. This gives us the answers

(a) $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^6$
(b) $2 \cdot 2 \cdot \cdots \cdot 2 = 2^n$ ■

Binary sequences arise frequently as a way to convey or code information. For example, they may code a binary (base two) numeral such as

$$101011_2 = 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 = 32 + 0 + 8 + 0 + 2 + 1 = 43_{10}$$

The binary sequences of length 6 correspond to the $2^6 = 64$ integers 0,1, … , 63.

In the next example, binary sequences provide a way to count the number of subsets of a finite set.

Example 1.47 What is the number of subsets of the six-element set $S = \{a, b, c, d, e, f\}$, including the empty subset $\emptyset$ and S itself?

Solution. Since $|S| = 6$, any subset of S can be associated with a binary sequence of length 6 by assigning a 1 if the element is included in the subset and a 0 if not. For example, the binary sequence 101011 corresponds to the subset $\{a,c,e,f\}$. Similarly, 000000 corresponds to the empty subset and 111111 to S itself. This association describes a bijection (i.e., a one-to-one matching) of binary sequences of length 6 and the subsets of a six-element set, so, from Example 1.46, we see there are 2^6 subsets of the six element set S. ■

The reasoning in Example 1.47 applies equally well to any finite set with n members, giving us the following theorem.

Theorem 1.48 A set S with $|S| = n$ members has 2^n subsets.

Just as the addition principle can be inverted to become the subtraction principle, the multiplication principle can be inverted to become the *division principle*.

Theorem 1.49 (Division Principle) Let set S have the partition $S = A_1 \cup A_2 \cup \cdots \cup A_k$, where the pairwise disjoint sets $A_1, A_2, \ldots, A_k$ each have the same number of members, say, $r = |A_1| = |A_2| = \cdots = |A_k|$. Then there are $k = |S|/r$ subsets in the partition, or equivalently, there are $r = |S|/k$ members of each set of the partitioning family.

Proof. Applying the extended addition principle in Theorem 1.35, we obtain

$$|S| = |A_1| + |A_1| + \cdots + |A_k| = rk$$ ∎

Example 1.50 A child has five plastic letters and has written PANDA. How many words[4] can be written with different arrangements of the same set of plastic letters? For example, AANPD and DNAPA are two more words.

Solution. Suppose, temporarily, that all of the letters have different colors. For example, one A is red and the other A is blue. To form a word, we have a five-step process as we choose letters from left to right; that is, we can form a set S of words with $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ members. Suppose that A_1 is the subset of words with the red A to the left of the blue A, and A_2 is the subset with the reversed order. By switching the two As, we see that $|A_1| = |A_2|$. Thus, by the division principle, there are $5!/2 = 120/2 = 60$ different words that can be formed if the two As have the same color. ∎

1.6.3 Combining the Addition and Multiplication Principles

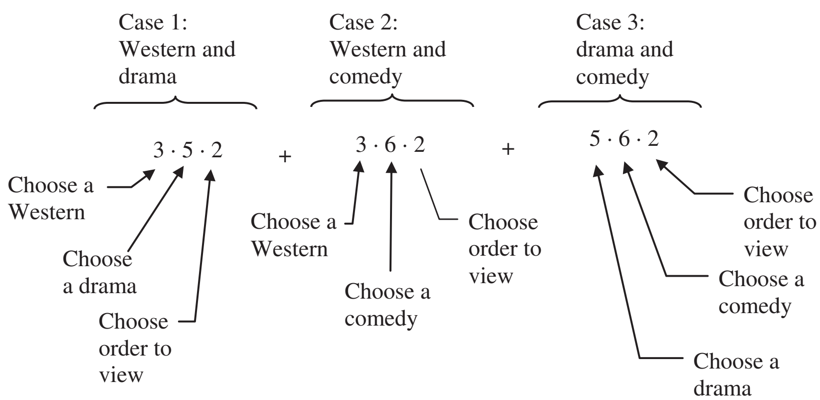
Many combinatorial problems are solved using both the addition and multiplication principles.

Example 1.51 Mischa is spending a long weekend at a beach house, and wants to watch a video each evening on Friday and Saturday. The beach house has a video library containing three Westerns, five dramas, and six comedies from which to choose. How many ways can Mischa choose to watch two movies of different genres?

Solution. There are three cases, depending on which genres are chosen. Once the two genres are chosen, she can perform a three-stage process in choosing a video of each genre and choosing the order in which they will be watched. Here is a useful

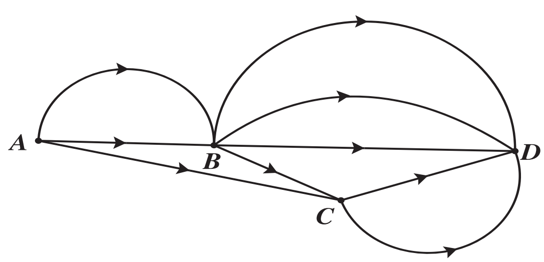
[4]In the present context, we use the term *word* either to refer to a specific English word (e.g., panda) or to a permutation of letters forming a nonword (e.g., dnapa).

way to show both the three cases and the choices made in each case:



It is now clear that Mischa has $3 \cdot 5 \cdot 2 + 3 \cdot 6 \cdot 2 + 5 \cdot 6 \cdot 2 = 30 + 36 + 60 = 126$ ways to watch a video each night. ■

Example 1.52 The diagram below shows a system of roads from Abbottsville (A) to Dusty (D), some of which pass through Baker (B) or Cavendish (C). How many routes, always following the arrows, extend from Abbotsville to Dusty?

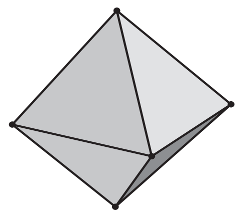


Solution. There are three cases: ABD, ACD, and $ABCD$, with two routes from A to B, one direct route from A to C, one direct route from B to C, three direct routes from B to D, and two direct routes from C to D. Therefore, there are $2 \cdot 3 + 1 \cdot 2 + 2 \cdot 1 \cdot 2 = 12$ ways to go from Abbottsville to Dusty. ■

Example 1.52 solves a problem in *path counting*, a topic of considerable interest and usefulness in combinatorial reasoning. Path counting will be revisited later, when we choose a variety of different patterns for the edges that a path must follow.

PROBLEMS

1.6.1. The 2006 World Cup used a soccer ball called the +*teamgeist* (*team spirit*, with the + sign to allow the German word to be copyrighted). Mathematically, the ball is a spherical analog of the truncated octahedron, obtained by starting with the octahedron of eight triangles as shown, then replacing (truncating) each corner with a square, and finally rounding the faces to become spherical. What counting principle can be used to determine the number of panels on the +teamgeist soccer ball?



1.6.2. The 2010 World Cup held in South Africa used the *jabulani* soccer ball, where the name means *celebrate* in the Zulu (Bantu) language. There are exactly eight panels on the ball, including four rounded triangular panels. What counting principle can be used to determine the number of nontriangular panels of a jabulani ball?

1.6.3. There are two types of seams of the traditional soccer ball shown in Example 1.30: those that separate two hexagonal panels and those that separate a pentagonal panel from a hexagonal panel. Find the total number of seams on the ball, and the number that separate one hexagonal panel from another.

1.6.4. How many ways can you put on a pair of socks and a pair of shoes?

1.6.5. Maria likes to order double-scoop ice cream cones, with chocolate or strawberry on the bottom, and chocolate, vanilla, or mint on the top. Describe the ways in which Maria can order her ice cream cones with a Cartesian product, and count the number of types of cones that she likes.

1.6.6. Suppose that 13 players arrive at the tennis courts. In how many ways can they

(a) split up to form 6 games of singles, with one player sitting out?

(b) split up to for 3 games of doubles, with one player sitting out?

1.6.7. Generalize the results of Example 1.45 and Problem 1.6.6; that is, provide formulas for the number of ways to split up m people into singles and doubles matches. For doubles, create as many matches as possible and then set up a singles match if enough people remain not already playing doubles.

1.6.8. Consider the tilings of a $1 \times (n+1)$ board with $r+1$ gray tiles.

(a) Explain why $\binom{r+k}{r}$ tilings have the rightmost gray tile in cell $r+k+1$, where $k = 0, 1, 2, \ldots, n-r$.

(b) Use part (a) to prove the following (hockey stick identity 2):

$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}$$

1.6.9. Construct Pascal's triangle showing rows 0 through 6.

(a) Draw a loop around the terms in the hockey stick identity (1.32) for the case $n = 3$ and $r = 2$, showing that a hockey stick shape in formed.

(b) Repeat part (a) but for the hockey stick identity of Problem 1.6.8 in the case that $n = 5$ and $r = 3$.

(c) How do the terms in the handle of the hockey stick relate to the term in the blade of the hockey stick?

1.6.10. The Math Club needs to choose a president and treasurer, where the two officers cannot be from the same year in school. The club has four seniors, seven juniors, and six sophomores. In how many ways can the two officers be chosen?

1.6.11. There are three roads from Sylvan to Tacoma, four roads from Tacoma to Umpqua, and two roads from Sylvan directly to Umpqua. How many routes, with no backtracking, can be taken from Sylvan to Umpqua?

1.7 SUMMARY AND ADDITIONAL PROBLEMS

- *The Pigeonhole Principle.* The simplest statement of the principle is as follows: If more than n pigeons are placed into n pigeonholes, at least one hole is occupied by two or more pigeons. More generally, if more than $p_1 + p_2 + \cdots + p_n$ pigeons are placed into holes 1,2, … , n, then some hole k contains more than p_k pigeons. The pigeonhole principle is an important strategy to prove the existence of some mathematical object, but little if any information about how to construct the object is given.
- *Chessboard Tilings with Dominoes.* The approach taken was constructive. For example, since each domino covers two unit squares, an $m \times n$ board must have an even number of unit squares, thus requiring that one or both of m and n be even. Indeed, this condition is also sufficient since it is easy to construct the tiling in which every domino is aligned with the side of even dimension. For a board with odd dimensions, with alternate squares colored black and gray, trimming any square with the color of the corner squares leaves a board that can be tiled with dominoes. The proof was again constructive; the trimmed board can always be partitioned into rectangles each with an even dimension.
- *Figurate Numbers.* This section investigated enumerative problems revolving around the question "How many dots are in this sequence of patterns?" Several important principles of counting were found to be helpful:

The nth term of the sequence may be related in a simple way to earlier terms in the sequence; that is, the sequence was described by a recurrence relation.

The complex dot pattern could be partitioned into simpler dot patterns with known formulas; therefore, a formula could be obtained for the nth term of the complex pattern.

- *Counting Tilings of Rectangles.* The central problem was to count the number of ways that a $1 \times n$ board can be tiled with tiles of a specified type. In particular, if r of the unit squares were to be covered with a gray tile, and the remaining $n - r$ unit square covered with a white tile, it was critical to know how many ways r units squares can be chosen from the set of n unit squares. This number was called the *combination* $C(n,r)$ that is read as "n choose r" and often written in the alternative notation $\binom{n}{r}$. Using the tiling model, it was shown that there are 2^n ways to tile a $1 \times n$ board with any number $r = 0, 1, 2, \ldots, n$ gray tiles; that is, $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. Also, any tiling of a $1 \times n$ board with r gray tiles end with either a white or a gray tile, so the $1 \times (n-1)$ boards to the left of the rightmost have either r or $r - 1$ gray tiles. Indeed, all such $1 \times (n-1)$ boards are accounted for, and it follows that

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

a relation known as *Pascal's identity*. Finally, it was shown that the number ways to tile an $1 \times n$ board with a sequence of squares and dominoes is given by the *combinatorial Fibonacci number* f_n, where $f_n = F_{n+1}$ and $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, \ldots$ are the standard Fibonacci numbers.
- *Addition and Multiplication Principles.* To count the number of elements in a set S, it is helpful to partition S into pairwise disjoint subsets $A_1, A_2, \ldots, A_k$ for which $S = A_1 \cup A_2 \cup \cdots \cup A_k$, $A_i \cap A_j = \emptyset$, $i \neq j$. Then $|S| = |A_1| + |A_2| + \cdots + |A_k|$, a result known as the *addition principle*. If each element of S is obtained as the result of a n-stage process in which there are p_j ways to carry out stage j independently of the way any previous stage was carried out, then the number of outcomes in S is given by $p_1 p_2 \cdots p_n$, a result known as the *multiplication principle.* Often, both the addition and the multiplication principle are used together to solve a combinatorial problem.

PROBLEMS

1.7.1. **(a)** Let A be any set of 51 numbers chosen from [100]. Show that two members of A differ by 50.

(b) State and prove a generalization of the result of part (a).

1.7.2. Let m be an odd number. Show that m is a divisor of at least one of the terms in the sequence $(1,3,7,15,\ldots,2^m - 1)$ whose jth term is $a_j = 2^j - 1$.

1.7.3. Show that any set of 10 natural numbers, each between 1 and 100, contains two disjoint subsets with the same sum of its members.

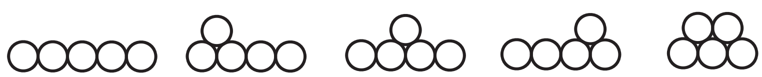
1.7.4. Given $nk + 1$ or more pigeons placed into n pigeonholes, show that the average number of pigeons per hole is larger than k.

1.7.5. A traditional dartboard divides the circular board into 20 sectors that are numbered clockwise from the top with the sequence

20 - 1 - 18 - 4 - 13 - 6 - 10 - 15 - 2 - 17 - 3 - 19 - 7 - 16 - 8 - 11 - 14 - 9 - 12 – 5

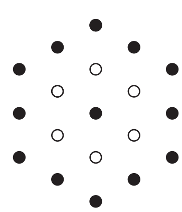
There is considerable variation in the sum of three successive numbers, from $23 = 1 + 18 + 4$ to $42 = 19 + 7 + 16$. Can the numbers 1 through 20 be rearranged so that the sum of each group of three successive numbers is smaller than 32?

1.7.6. Imagine large dots represent stacked logs as seen from their ends. It is required that the stack be either one or two rows high, that the logs in the lower row have no gap between adjacent logs, and that each of the logs in the upper row must touch two logs in the lower row. The diagram that follows shows all five ways to stack five logs:



Investigate the sequence of numbers that gives the number of ways to stack n logs into piles that meet the criteria described above. Justify your answer.

1.7.7. The centered hexagon numbers (or hex numbers) H_n are obtained by starting with a single dot and then surrounding it by hexagons with 6,12,18,… dots on its sides. The diagram shows that $H_0 = 1$, $H_1 = 7$, and $H_2 = 19$:

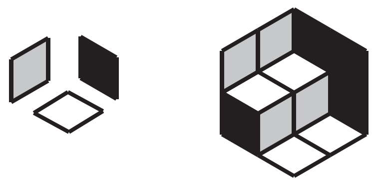


(a) extend the diagram with two more surrounding hexagons to determine H_3 and H_4.

(b) derive a formula that gives H_n in terms of the triangular numbers.

(c) obtain an expression for H_n as a function of n.

(d) suppose that *tridominoes* are formed with a pair of equilateral triangles joined along a common edge, and are colored gray, white, or black according to their orientation. The figure here shows the three types of tridominoes and one way they can be used to tile the hex pattern for the H_2 array of dots.



Show that every hex pattern of H_n dots can be tiled with tridominoes, and give the number of tridominoes of each color that are used in the tiling. (*Suggestion:* The hex numbers might also be called the *corner* numbers!)

1.7.8. Recall that $\binom{8}{3}$ is the number of ways to tile a 1×8 board with 3 gray and 5 white squares.

(a) Using the tiling model, explain why $\binom{8}{3} = \binom{8}{5}$.

(b) Generalize your answer to part (a) to explain why

$$\binom{n}{r} = \binom{n}{n-r}$$

1.7.9. Use the result of Problem 1.7.8 to show that hockey stick identity 1 (1.32) can be rewritten to become

$$\binom{n+1}{r+1} = \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}$$

(hockey stick identity 2).

1.7.10. A $2 \times n$ rectangular board is to be tiled with three gray squares and $2n - 3$ white squares.

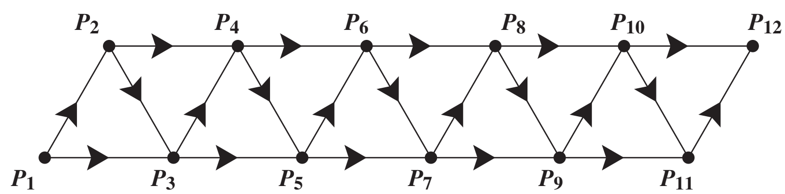
(a) How many tilings have all three gray squares in the same row?

(b) How many tilings have no row that is all white?

(c) Use your answers to the parts above to give an identity involving the binomial coefficient $\binom{2n}{3}$.

1.7.11. **(a)** How many paths extend from point P_1 to each of the points $P_2, P_3, \ldots, P_{12}$ in the following directed graph? Each step along any path must be in the direction indicated by the arrow. For example, there are two paths from

P_1 to P_3:



(b) Imagine extending this graph to an arbitrary number of points $P_1, P_2, \ldots, P_n$. What famous number sequence gives the number of paths to the points $P_1, P_2, \ldots, P_n$? Provide a justification for your answer.

1.7.12. How many ways can some coins be removed from a coin purse containing 3 pennies, 7 nickels, 4 dimes, and 2 quarters? The coins of the same denomination are considered to be identical, and the order of removal is not important.

1.7.13. There are six positive divisors of $12 = 2^2 \times 3^1$, namely, $1 = 2^0 \times 3^0$, $2 = 2^1 \times 3^0$, $4 = 2^2 \times 3^0$, $3 = 2^0 \times 3^1$, $6 = 2^1 \times 3^1$, and $12 = 2^2 \times 3^1$. What is the number of positive divisors of these integers?

(a) 660 **(b)** $2^5 \times 3^7 \times 11^2 \times 23^4$ **(c)** 10^{100}

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