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Introduction

In this chapter, a brief literature survey of analytical solutions of periodic motions in nonlinear dynamical systems will be presented. The perturbation analysis has played an important role in such an approximate analysis of periodic motions in nonlinear systems. The perturbation method, method of averaging, harmonic balance, and generalized harmonic balance will be reviewed. The application of perturbation method in time-delayed systems will be discussed briefly.

1.1 Brief History

Since the seventeenth century, there has been interest in periodic motions in dynamical systems. The Fourier series theory shows that any periodic function can be expressed by a Fourier series expansion with different harmonics. In addition to simple oscillations, there has been interest in the motions of moon, earth, and sun in the three-body problem. The earliest approximation method is the method of averaging, and the idea of averaging originates from Lagrange (1788). At the end of the nineteenth century, Poincare (1890) provided the qualitative analysis of dynamical systems to determine periodic solutions and stability, and developed the perturbation theory for periodic solutions. In addition, Poincare (1899) discovered that the motion of a nonlinear coupled oscillator is sensitive to the initial condition, and qualitatively stated that the motion in the vicinity of unstable fixed points of nonlinear oscillation systems may be stochastic under regular applied forces. In the twentieth century, one followed Poincare's ideas to develop and apply the qualitative theory to investigate the complexity of motions in dynamical systems. With Poincare's influence, Birkhoff (1913) continued Poincare's work, and proof of Poincare's geometric theorem was given. Birkhoff (1927) showed that both stable and unstable fixed points of nonlinear oscillation systems with two degrees of freedom must exist whenever their frequency ratio (or called resonance) is rational. The sub-resonances in periodic motions of such systems change the topological structures of phase trajectories, and the island chains are obtained when the dynamical systems can be renormalized with fine scales. In such qualitative and quantitative analysis, the Taylor series expansion and the perturbation analysis play an important role. However, the Taylor series expansion analysis is valid in the small finite domain under certain convergent conditions, and the perturbation analysis

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based on the small parameters, as an approximate estimate, is only acceptable for a very small domain with a short time period.

van der Pol (1920) used the averaging method to determine the periodic motions of self-excited systems in circuits, and the presence of natural entrainment frequencies in such a system was observed in van der Pol and van der Mark (1927). Cartwright and Littlewood (1945) discussed the periodic motions of the van der Pol equation and proved the existence of periodic motions. Cartwright and Littlewood (1947) discussed the periodic motions of a generalized nonlinear equation based on the similar Duffing equation. Levinson (1948) used a piecewise linear model to describe the van der Pol equation and determined the existence of periodic motions. Levinson (1949) further developed the structures of periodic solutions in such a second order differential equation through the piecewise linear model, and discovered that infinite periodic solutions exist in such a piecewise linear model. From the Levinson's results, Smale (1967) used the topological point to present the Smale horseshoe with discontinuous mappings to describe the existence of infinite periodic motions. Further, a differentiable dynamical system theory was developed. Such a theory has been extensively used to interpret the homoclinic tangle phenomenon in nonlinear dynamical systems. Smale found the infinite, many periodic motions, and a perfect minimal Cantor set near a homoclinic motion can be formed. Melnikov (1962) used the concept of Poincare (1892) to investigate the behavior of trajectories of perturbed systems near autonomous Hamiltonian systems. Melnikov (1963) further investigated the behavior of trajectories of perturbed Hamiltonian systems, and the width of the separatrix splitting was approximately estimated. The width gives the domain of the chaotic motion in the vicinity of the generic separatrix. Even if the width of the separatrix splitting was approximately estimated, the dynamics of the separatrix splitting was not developed.

Since the nonlinear phenomena was observed in engineering, Duffing (1918) used the hardening spring model to investigate the vibration of electro-magnetized vibrating beam, and after that, the Duffing oscillator has been extensively used in structural dynamics. In addition to determining the existence of periodic motions in nonlinear differential equations of the second order in mathematics, one has applied the Poincare perturbation methods for periodic motions in nonlinear dynamical systems. Fatou (1928) provided the first proof of asymptotic validity of the method of averaging through the existence of solutions of differential equations. Krylov and Bogolyubov (1935) systematically developed the method of averaging and the detailed discussion can be found in Bogoliubov and Mitropolsky (1961). The classic perturbation methods for nonlinear oscillators were presented (e.g., Stoker, 1950; Minorsky, 1962; Hayashi, 1964). Hayashi (1964) used the method of averaging and harmonic balance method to discuss the approximate periodic solutions of nonlinear systems and the corresponding stability. Nayfeh (1973) employed the multiple-scale perturbation method to develop approximate solutions of periodic motions in the Duffing oscillators. Holmes and Rand (1976) discussed the stability and bifurcation of periodic motions in the Duffing oscillator. Nayfeh and Mook (1979) applied the perturbation analysis to nonlinear structural vibrations via the Duffing oscillators, and Holmes (1979) demonstrated chaotic motions in nonlinear oscillators through the Duffing oscillator with a twin-well potential. Ueda (1980) numerically simulated chaos via period-doubling of periodic motions of Duffing oscillators.

Based on the work of Melnikov (1963), Greenspan (1981) extended the similar ideas to the dissipative dynamical systems (also see, Greenspan and Holmes, 1983; Guckenheimer and Holmes, 1983). Further, the Melnikov method was developed for the global transversality in dissipative nonlinear systems. Once the global transversality to the separatrix exists, one thought that the Smale horseshoe presented in Smale (1967) may exist, and furthermore chaos in such a nonlinear dynamical system may occur. However, from such a prediction based on the Melnikov method, one cannot observe the global transversality in nonlinear dynamical systems. The Smale horseshoe theory may not be adequate for nonlinear dynamical systems rather than the topological structure. From the perturbation analysis, the Melnikov function was obtained for Hamiltonian systems with a small perturbation. One used such a function to analytically predict global behaviors (e.g., chaos) in the Hamiltonian systems with a small perturbation. Because of the perturbation analysis, the Melnikov method can give a reasonable analysis of the global behavior only when the perturbation is very small and close to zero. However, the perturbation is very small to zero, chaos in nonlinear dynamical systems may not occur. So the Melnikov method may not help us understand the global behaviors of nonlinear dynamical systems. Luo (1995) used the Chirikov criterion to determine Hamiltonian chaos and applied the Melnikov function to investigate the global transversality (also see, Luo and Han, 1999; Luo, 2008, 2012a). The conclusion is that the Melnikov method cannot provide an adequate prediction of chaotic motions in the dissipative system. For a better understanding of the Melnikov method, the work of Melnikov (1963) should be revisited. Melnikov (1963) presented a perturbation analysis to estimate the width of the separatrix splitting. Indeed, the width of the separatrix can be approximately estimated, but it cannot be used for predicting the existence of chaos. The Melnikov function is an approximate energy increment during a certain time period, which can be found in references (e.g., Arnold, 1964; Chirikov, 1979; Luo and Han, 2001). If the Melnikov function is zero, from a physical point of view, the system energy is conserved during a certain time period. Such a zero value of the Melnikov function does not imply that the flow has any global transversality to the separatrix. One has difficulty finding a connection from periodic motions to chaos. Thus, one continues using the perturbation analysis to determine the approximate analytical solutions of periodic motions. Coppola and Rand (1990) determined limit cycles of nonlinear oscillators through elliptic functions in the averaging method. Wang *et al*. (1992) used the harmonic balance method and the Floquet theory to investigate the nonlinear behaviors of the Duffing oscillator with a bounded potential well (also see, Kao *et al*., 1992). Luo and Han (1997) determined the stability and bifurcation conditions of periodic motions of the Duffing oscillator. However, only symmetric periodic motions of the Duffing oscillators were investigated. Luo and Han (1999) investigated the analytical prediction of chaos in nonlinear rods through the Duffing oscillator. Peng *et al*. (2008) presented the approximate symmetric solution of period-1 motions in the Duffing oscillator by the harmonic balance method with three harmonic terms. Luo (2012a) developed a generalized harmonic balance method to get the approximate analytical solutions of periodic motions and chaos in nonlinear dynamical systems. This method used the finite Fourier series to express periodic motions and the coefficients are time-varying. With averaging, a dynamical system of coefficients are obtained from which the steady-state solution are achieved and the corresponding stability and bifurcation are completed. Luo and Huang (2012a) used the generalized harmonic balance method with finite terms to obtain the analytical solution of period-1 motion of the Duffing oscillator with a twin-well potential. Luo and Huang (2012b) employed a generalized harmonic balance method to find analytical solutions of period-m motions in such a Duffing oscillator. The analytical bifurcation trees of periodic motions in the Duffing oscillator to chaos were obtained (also see, Luo and Huang, 2012c,d, 2013a,b,c, 2014). Such analytical bifurcation trees show the connection from periodic solution to chaos analytically. To better understand nonlinear behaviors in nonlinear dynamical systems, the analytical solutions for the bifurcation trees from period-1 motion to chaos in a periodically forced oscillator with quadratic nonlinearity were presented in Luo and Yu (2013a, b, c), and period-*m* motions in the periodically forced, van der Pol equation was presented in Luo and Laken (2013). The analytical solutions for the van der Pol oscillator can be used to verify the conclusions in Cartwright and Littlewood (1947) and Levinson (1949). The results for the quadratic nonlinear oscillator in Luo and Yu (2013a, b, c) analytically show the complicated period-1 motions and the corresponding bifurcation structures.

In recent years, time-delayed systems are of great interest since such systems extensively exist in engineering (e.g., Tlusty, 2000; Hu and Wang, 2002). The infinite dimensional state space causes a significant difficulty in understanding such a time-delayed problem. One tried to work on numerical methods to get the corresponding complicated behaviors. On the other hand, one is interested in the stability and bifurcation of equilibriums of the time-delayed systems (e.g., Stepan, 1989; Sun, 2009; Insperger and Stepan, 2011). In addition, one is also interested in periodic solutions in time-delayed dynamical systems. Perturbation methods have been used in recent years for such periodic motions in delayed dynamical systems. For instance, the approximate solutions of the time-delayed nonlinear oscillator were investigated by the method of multiple scales (e.g., Hu, Dowell, and Virgin 1998; Wang and Hu, 2006). The harmonic balance method was also used to determine approximate periodic solutions for delayed nonlinear oscillators (e.g., MacDonald, 1995; Liu and Kalmar-Nagy, 2010; Leung and Guo, 2012). However, such approximate solutions of periodic motions in the time-delayed oscillators are based on one or two harmonic terms, which are not accurate enough. In addition, the corresponding stability and bifurcation analysis of such approximate solutions may not be adequate. In this book, an alternative way of finding the accurate analytical solutions of periodic flows in time-delayed dynamical systems will be presented. This method is without any small-parameter requirement. In addition, this approach can also be applicable to the coefficient varying with time.

1.2 Book Layout

In this book, a new analytical method will be presented for analytical solutions of periodic motions in nonlinear dynamical systems with/ without time delay. The basic theory of nonlinear systems will be briefly introduced. The analytic method based on the generalized harmonic balance will be comprehensively discussed, and this method will be applied to nonlinear dynamical systems to find the periodic motions analytically and to determine the analytical bifurcation trees of periodic motions to chaos. The main body in this book is summarized as follows:

- In Chapter 2, the basic theory of nonlinear dynamical systems will be introduced. Local theory, global theory, and bifurcation theory of nonlinear dynamical systems will be briefly discussed. The stability switching and bifurcation on specific eigenvectors of the linearized system of a nonlinear system at equilibrium will be discussed. The higher-order singularity and stability for nonlinear systems on the specific eigenvectors will be developed.
- In Chapter 3, from Luo (2012a), the analytical dynamics of periodic flows and chaos in nonlinear dynamical systems will be presented. The analytical solutions of periodic flows

and chaos in autonomous systems will be discussed first, and the analytical dynamics of periodically forced nonlinear dynamical systems will be presented. The analytical solutions of periodic motions in free and periodically forced vibration systems will be presented. In a similar fashion, the analytical solutions of periodic flows for time-delayed nonlinear systems will be presented with/without periodic excitations, and time-delayed nonlinear vibration systems will be also discussed for engineering application. The analytical solutions of periodic flows and chaos are independent of the small parameters, which are different from the traditional perturbation methods. The methodology presented herein will end the history of chaos being numerically simulated only.

- In Chapter 4, from the idea of Luo (2012a, 2013), period-*m* flows to quasi-periodic flows in nonlinear dynamical systems will be presented. The analytical solutions of quasi-periodic flows in autonomous systems will be discussed, and the analytical solutions of quasi-periodic flows in periodically forced nonlinear dynamical systems will be presented. The analytical solutions of quasi-periodic motions in free and periodically forced vibration systems will be presented. The analytical solutions of quasi-periodic flows for time-delayed nonlinear systems will be presented with/without periodic excitations, and time-delayed nonlinear vibration systems will be discussed as well.
- In Chapter 5, analytical solutions for period-*m* motions in a periodically forced, quadratic nonlinear oscillator will be presented through the Fourier series solutions with finite harmonic terms, and the stability and bifurcation analyses of the corresponding period-1 motions will be carried out. There are many period-1 motions in such a nonlinear oscillator, and the parameter map for excitation amplitude and frequency will be developed for different period-1 motions. For each period-1 motion branch, analytical bifurcation trees of period-1 motions to chaos will be presented. For a better understanding of complex period-*m* motions in such a quadratic nonlinear oscillator, trajectories, and amplitude spectrums will be illustrated numerically.
- In Chapter 6, analytical solutions for period-*m* motions in a time-delayed, nonlinear oscillator will be presented through the Fourier series, and the stability and bifurcation analyses of the corresponding periodic motions will be presented through the eigenvalue analysis. Analytical bifurcation trees of periodic motions to chaos will be presented through the frequency-amplitude curves. Trajectories and amplitude spectrums of periodic motions in such a time-delayed nonlinear system will be illustrated numerically for a better understanding of time-delayed nonlinear dynamical systems.