CHAPTER 1

THE COMPLEX NUMBERS

In this chapter, we introduce the complex numbers and their interpretation as points in a number plane, an analog to the real number line. We develop the algebraic, geometric, and topological properties of the set of complex numbers, many of which mirror those of the real numbers. These properties, especially the topological ones, are connected to sequences, and thus we conclude the chapter by studying the basic nature of sequences and series. At the conclusion of the chapter, we will possess the tools necessary to begin the study of functions of a complex variable.

1.1 Why?

Our work in this text can best be understated as follows: Let's throw $\sqrt{-1}$ into the mix and see what happens to the calculus. The result is a completely different flavor of analysis, a separate field distinguished from its real-variable sibling in some striking ways.

The use of $\sqrt{-1}$ as an intermediate step in finding solutions to real-variable problems goes back centuries. In the Renaissance, Italian mathematicians used complex numbers as a tool to find *real* roots of cubic equations. The algebraic use of complex

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numbers became much more mainstream due to the work of Leonhard Euler in the 18^{th} century and later, Carl Friedrich Gauss. Euler and Jean le Rond d'Alembert are generally credited with the first serious considerations of functions of a complex variable – the former considered such functions as an intermediate step in the calculation of certain *real* integrals, while the latter saw these functions as useful in his study of fluid mechanics.

Introducing complex numbers as a stepping stone to solve real problems is a common historical theme, and it is worth recalling how other familiar systems of numbers can be viewed to solve particular algebraic and analytic problems. The natural numbers, integers, rational numbers, and real numbers satisfy the set containments $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, but each subsequent set has characteristics not present in its predecessor. Where the natural numbers $\mathbb{N} = \{1, 2, \dots\}$ are closed under addition and multiplication, extending to the integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ provides an additive identity and inverses. The set of rational numbers $\mathbb{Q}$, consisting of all fractions of integers, has multiplicative inverses of its nonzero elements and hence is an algebraic *field* under addition and multiplication.

The move from $\mathbb{Q}$ to the real numbers $\mathbb{R}$ is more analytic than algebraic. Although $\mathbb{Q}$ is a field, it is not *complete*, meaning there are "holes" that need to be filled. For instance, consider the equation $x^2 = 2$. Since $1^2 = 1$ and $2^2 = 4$, it seems that a solution to the equation should exist and lie somewhere in between. Further analysis reveals that a solution should lie between $5/4$ and $3/2$. Successive subdivisions may be used to target where a solution should lie, but that point is not in $\mathbb{Q}$. Beyond unsolvable algebraic equations lies the number π, the ratio of a circle's circumference to its diameter, which can also be shown not to lie in $\mathbb{Q}$. The alleviation of these problems comes by allowing the set $\mathbb{R}$ of real numbers to be the *completion* of $\mathbb{Q}$. In satisfyingly imprecise terms, $\mathbb{R}$ is equal to $\mathbb{Q}$ with the "holes filled in." This is done so that the *axiom of completeness* (i.e. the least upper bound property) holds. See Appendix B for more detail. The result is that $\mathbb{R}$ is a *complete ordered field*.

The upgrade from the real numbers to the complex numbers has both algebraic and analytic motivation. The real numbers are not *algebraically* complete, meaning there are polynomial equations such as $x^2 = -1$ with no solutions. The incorporation of $\sqrt{-1}$ mentioned earlier is a direct response to this. But the work of Euler and d'Alembert shows how moving outside $\mathbb{R}$ facilitates analytic methods as well. While their work did much for bringing credibility to the use of complex numbers, it was during the 19^{th} century, in the movement to deliver rigor to mathematical analysis, that complex function theory gained its footing as a separate subject of mathematical study, due largely to the work of Augustin-Louis Cauchy, Bernhard Riemann, and Karl Weierstrass.

Function theory is the study of the calculus of complex-valued functions of a complex variable. The analysis of functions on this new domain will quickly distinguish itself from real-variable calculus. As the reader will soon see, by combining the algebra and geometry inherent in this new setting, we will be able to perform a great deal of analysis that is not available on the real domain. Such analysis will include some intuition-bending results and techniques that solve problems from calculus that are not easily accessible otherwise.

Before setting out on our study of complex analysis, we must agree on a starting point. We assume that the reader is familiar with the fundamentals of differential, integral, and multivariable calculus. The language of sets and functions is freely used; the unfamiliar reader should examine Appendix A.

Lastly, to whet our appetites for what is to come, here are a handful of exercises appearing later in the text the statements of which are understandable from calculus, but whose solutions are either made possible or much simpler by the introduction of complex numbers.

Forthcoming Exercises

1. Derive triple angle identities for $\sin 3\theta$ and $\cos 3\theta$. [Section 2.4, Exercise 6]
2. Find a continuous one-to-one planar transformation that maps the region lying inside the circles $(x-1)^2 + y^2 = 4$ and $(x+1)^2 + y^2 = 4$ onto the upper half-plane $y \geq 0$. [Section 2.6, Exercise 13]
3. Find the radius of convergence of the Taylor series expansion of the function

$$f(x) = \frac{\sin x}{1 + x^4}$$

about $a = 2$. [Section 3.2, Exercise 3]

4. Verify the summation identity, where $c \in \mathbb{R}$ is a constant.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + c^2} = \begin{cases} \dfrac{\pi^2}{6} & \text{if } c = 0, \\[2ex] \dfrac{\pi}{2c} \coth \pi c - \dfrac{1}{2c^2} & \text{if } c \neq 0 \end{cases}$$

[Section 5.3, Exercise 11]

5. Evaluate the following integrals, where $n \in \mathbb{N}$ and $a, b > 0$.

(a) $\displaystyle\int_0^{2\pi} \cos^n t \, dt,$ [Section 2.8, Exercise 4]

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{x(x^2 + b^2)} \, dx,$ [Section 5.4, Exercise 7]

(c) $\displaystyle\int_0^{\infty} \frac{\sqrt[n]{x}}{x^2 + a^2} \, dx,$ [Section 5.4, Exercise 10]

6. Find a real-valued function u of two variables that satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

inside the unit circle and continuously extends to equal 1 for points on the circle with $y > 0$ and 0 for points on the circle with $y < 0$. [Section 6.2, Exercise 3]

1.2 The Algebra of Complex Numbers

As alluded to in Section 1.1, we desire to expand from the set of real numbers in a way that provides solutions to polynomial equations such as $x^2 = -1$. One may be tempted to simply define a number that solves this equation. The drawback to doing so is that the negative of this number would also be a solution, and this could cause some ambiguity in the definition. We therefore choose a different method.

1.2.1 Definition. A *complex number* is an ordered pair of real numbers. The set of complex numbers is denoted by $\mathbb{C}$.

By definition, any $z \in \mathbb{C}$ has the form $z = (x, y)$ for numbers $x, y \in \mathbb{R}$. What distinguishes complex numbers from their counterparts, the two-dimensional vectors in $\mathbb{R}^2$, is their algebra – specifically, their multiplication.

1.2.2 Definition. If (a, b) and (c, d) are complex numbers, then we define the algebraic operations of *addition* and *multiplication* by

$$\begin{aligned} (a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc). \end{aligned}$$

Clearly, $\mathbb{C}$ is closed under both of these operations. (Adding or multiplying two complex numbers results in another complex number.)

Notice that if $a, b \in \mathbb{R}$, then $(a, 0) + (b, 0) = (a + b, 0)$ and $(a, 0)(b, 0) = (ab, 0)$. Therefore $a \mapsto (a, 0)$ is a natural algebraic embedding of $\mathbb{R}$ into $\mathbb{C}$. Accordingly, it is natural to write a for the complex number $(a, 0)$, and in this way, we consider $\mathbb{R} \subseteq \mathbb{C}$.

For any complex number $z = (x, y)$,

$$z = (x, 0) + (0, 1)(y, 0) = x + (0, 1)y.$$

In other words, each complex number can be written uniquely in terms of its two real components and the complex number $(0, 1)$. This special complex number gets its own symbol.

1.2.3 Definition. The *imaginary unit* is the complex number $i = (0, 1)$. A complex number z expressed as

$$z = x + iy \tag{1.2.1}$$

is said to be in *rectangular form.*

Because every complex number can be written uniquely as above, we (usually) refrain from using the ordered pair notation in favor of using the rectangular form. Notice that i is a solution to the equation $z^2 = -1$.

It is left as an exercise to verify that 0 is the additive identity and 1 is the multiplicative identity, every member of $\mathbb{C}$ has an additive inverse, both operations are associative and commutative, multiplication distributes over addition, and if $z \neq 0$ is written as in (1.2.1), then it has the multiplicative inverse

$$z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right) \tag{1.2.2}$$

in $\mathbb{C}$. This shows that $\mathbb{C}$ is a *field.*

1.2.4 Definition. For a complex number z written as in (1.2.1), we call the real numbers x and y the *real part* and *imaginary part* of z, respectively, and use the symbols $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. If $\operatorname{Re} z = 0$, then z is called *imaginary* (or *purely imaginary*). The *conjugate* of z is the complex number $\overline{z} = x - iy$. The *modulus* (or *absolute value*) of z is the nonnegative real number $|z| = \sqrt{x^2 + y^2}$.

It is a direct calculation to verify the relationship

$$|z|^2 = z\overline{z} \tag{1.2.3}$$

for all $z \in \mathbb{C}$. Other useful identities involving moduli and conjugates of complex numbers are left to the exercises.

1.2.5 Example. The identity (1.2.3) is useful for finding the rectangular form of a complex number. For instance, consider the quotient

$$z = \frac{1+2i}{2+i}.$$

To find the expressions from Definition 1.2.4, we multiply by the conjugate of the denominator over itself,

$$z = \frac{1+2i}{2+i}\,\frac{2-i}{2-i} = \frac{4+3i}{5},$$

to get a positive denominator. We see that $\operatorname{Re} z = 4/5$, $\operatorname{Im} z = 3/5$, $|z| = 1$, and $\overline{z} = (4-3i)/5$.

Summary and Notes for Section 1.2.

The set of complex numbers $\mathbb{C}$ consists of ordered pairs of real numbers. The real numbers are the those complex numbers of the form $(x, 0)$ for $x \in \mathbb{R}$, and $i = (0, 1)$ is the imaginary unit. Algebraic operations are defined to make $\mathbb{C}$ a field and so that $i^2 = -1$. We write the complex number $z = (x, y)$ in the rectangular form $z = x + iy$, and the conjugate of z is $\overline{z} = x - iy$.

In their attempts to find real solutions to cubic equations, Italian mathematicians found it necessary to manipulate complex numbers. Perhaps the first to consider them was Gerolamo Cardano in the 16th century, who named them “fictitious numbers.” Rafael Bombelli introduced the algebra of complex numbers shortly thereafter. At that time, the square roots of negative numbers were just manipulated as a means to an end. The ordered pair definition can be traced to William Rowan Hamilton almost three centuries later.

Exercises for Section 1.2.

1. For the following complex numbers z, calculate $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, and $\overline{z}$.

   (a) $z = 3 + 2i$

   (b) $z = \dfrac{1+i}{i}$

(c) $z = \dfrac{2-i}{1+i} + i$

(d) $z = (4+2i)\overline{(3+i)}$

2. Verify the following algebraic properties of $\mathbb{C}$.

   (a) The complex numbers 0 and 1 are the additive and multiplicative identities of $\mathbb{C}$, respectively.

   (b) Each $z \in \mathbb{C}$ has an additive inverse.

   (c) Addition and multiplication of complex numbers is associative. In other words, $z + (w+v) = (z+w)+v$ and $z(wv) = (zw)v$ for all $z, w, v \in \mathbb{C}$.

   (d) Addition and multiplication of complex numbers is commutative. That is, $z + w = w + z$ and $zw = wz$ for all $z, w \in \mathbb{C}$.

   (e) Multiplication of complex numbers distributes over addition. That is, $a(z+w) = az + aw$ for all $a, z, w \in \mathbb{C}$.

   (f) If $z \in \mathbb{C}$ is nonzero, then its multiplicative inverse is as given in (1.2.2).

3. $\triangleright$ Verify the following identities involving the conjugate.

   (a) For each $z \in \mathbb{C}$, $\overline{\overline{z}} = z$.

   (b) For each $z \in \mathbb{C}$,
$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \qquad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

   (c) For all $z, w \in \mathbb{C}$,
$$\overline{z+w} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{z}\,\overline{w}.$$

4. $\triangleright$ Verify the following identities involving the modulus. For each, let $z, w \in \mathbb{C}$.

   (a) $|zw| = |z||w|$

   (b) $|z/w| = |z|/|w|$ if $w \neq 0$

   (c) $|\overline{z}| = |z|$

   (d) $-|z| \leq \operatorname{Re} z \leq |z|$

5. $\triangleright$ Prove that for all $z, w \in \mathbb{C}$,
$$|z+w|^2 = |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2.$$

6. Let $p \colon \mathbb{C} \to \mathbb{C}$ be a polynomial. That is, $p(z) = \sum_{k=0}^{n} a_k z^k$ for $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. A *root* (or *zero*) of p is a number $r \in \mathbb{C}$ such that $p(r) = 0$. Show that if $a_0, \ldots, a_n \in \mathbb{R}$, then $\overline{r}$ is a root of p whenever r is a root of p.

7. Let $a, b \in \mathbb{R}$. Consider the function $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$T(x,y) = (\operatorname{Re}[(a+ib)(x+iy)], \operatorname{Im}[(a+ib)(x+iy)]).$$
   Prove that T is a linear transformation on $\mathbb{R}^2$, and determine a 2×2 matrix form for T. What does T represent?

8. For ordered pairs of real numbers (a,b) and (c,d), what drawbacks are there to defining multiplication of complex numbers by $(a,b)(c,d) = (ac, bd)$?

1.3 The Geometry of the Complex Plane

The real number line is the geometric realization of the set of real numbers and accordingly is a useful tool for conceptualization. Since complex numbers are defined to be ordered pairs of real numbers, it is only natural to visualize the set of complex numbers as the points in the Cartesian coordinate plane $\mathbb{R}^2$. This geometric interpretation is essential to the analysis of complex functions.

1.3.1 Definition. When its points are considered to be complex numbers, the Cartesian coordinate plane is referred to as the *complex plane* $\mathbb{C}$. The x- and y-axes in the plane are called the *real and imaginary axes*, respectively, in $\mathbb{C}$.

Because addition of complex numbers mirrors addition of vectors in $\mathbb{R}^2$, we use vectors to geometrically interpret addition in terms of parallelograms. Continuing this line of thought, we see that the value $|z|$, as the distance from the point $z \in \mathbb{C}$ to 0, is the length (or magnitude) of the vector z. If $z, w \in \mathbb{C}$, then $z - w$, in vector form, is the vector pointing from w to z. Therefore $|z - w|$ is equal to the distance between z and w. Lastly, we note that the operation of complex conjugation is realized geometrically as reflection in the real axis. See Figure 1.1.

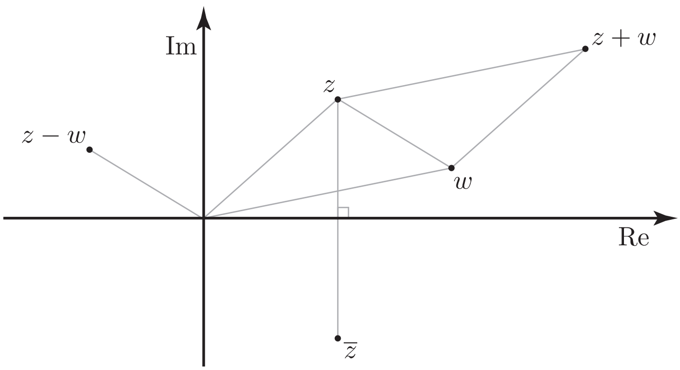

Figure 1.1 $z + w$, $z - w$, and $\overline{z}$ for some $z, w \in \mathbb{C}$

Another geometric consequence of the parallelogram interpretation is the triangle inequality, which gives that if $z, w \in \mathbb{C}$, then the distance from 0 to $z + w$ is never greater than the sum of the distances from 0 to z and 0 to w. We prove it as follows.

1.3.2 Triangle Inequality. *If $z, w \in \mathbb{C}$, then*

$$|z + w| \leq |z| + |w|. \tag{1.3.1}$$

Proof. We use Exercises 4 and 5 of Section 1.2 to calculate

$$\begin{aligned} |z + w|^2 &= |z|^2 + 2 \operatorname{Re} z\overline{w} + |w|^2 \\ &\leq |z|^2 + 2|z\overline{w}| + |w|^2 \end{aligned}$$

$$= |z|^2 + 2|z||w| + |w|^2$$
$$= (|z| + |w|)^2.$$

Taking the square root of both sides completes the proof. $\square$

If $a \in \mathbb{C}$ and $r > 0$, then the circle in $\mathbb{C}$ centered at a of radius r is the set of all points whose distance from a is r. From our above observation, this circle can be described in set notation by $\{z \in \mathbb{C} : |z - a| = r\}$. The inside of a circle, called a *disk*, is a commonly used object and is denoted by

$$D(a; r) = \{z \in \mathbb{C} : |z - a| < r\}. \tag{1.3.2}$$

The most prominently used disk in the plane is the unit disk – the disk centered at 0 of radius 1. For this, we use the special symbol

$$\mathbb{D} = D(0; 1). \tag{1.3.3}$$

1.3.3 Example. Let us consider the geometry of the set

$$E = \{z \in \mathbb{C} : |1 + iz| < 2\}$$

in two ways. First, write $z = x + iy$ to see that the condition defining E is equivalent to $|1 - y + ix| < 2$ or $\sqrt{x^2 + (y - 1)^2} < 2$. This describes all planar points of distance less than 2 from $(0, 1) = i$. Hence $E = D(i; 2)$.

In this circumstance, there is an advantage to eschewing real and imaginary parts. Note that

$$|1 + iz| = |i(-i + z)| = |i||z - i| = |z - i|.$$

Thus $E = \{z \in \mathbb{C} : |z - i| < 2\} = D(i; 2)$.

Summary and Notes for Section 1.3.

Since the complex numbers are ordered pairs of real numbers, the set $\mathbb{C}$ of complex numbers is geometrically realized as the plane $\mathbb{R}^2$. The addition and modulus of complex numbers parallel the addition and magnitude of planar vectors. The triangle inequality gives an important bound on sums.

In 1797, a Norwegian surveyor named Caspar Wessel was the first of many to consider the geometric interpretation of the complex numbers, but his work was largely unknown as was that of Jean-Robert Argand in 1806. (The complex plane is often referred to as the *Argand plane*.) The work of Carl Friedrich Gauss in the first half of the 19$^{\text{th}}$ century brought the concept to the masses.

Exercises for Section 1.3.

1. Geometrically illustrate the parallelogram rule for the complex numbers $2+i$ and $1+3i$.
2. Geometrically illustrate the relationship between the complex number $-1 + 2i$ and its conjugate.
3. Describe the following sets geometrically. Sketch each.

(a) $\{z \in \mathbb{C} : |z - 1 + i| = 2\}$
(b) $\{z \in \mathbb{C} : |z - 1|^2 + |z + 1|^2 \leq 6\}$
(c) $\{z \in \mathbb{C} : \operatorname{Im} z > \operatorname{Re} z\}$
(d) $\{z \in \mathbb{C} : \operatorname{Re}(iz + 1) < 0\}$

4. Provide a geometric description of complex multiplication for nonzero $z, w \in \mathbb{C}$. It is helpful to write $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\varphi + i\sin\varphi)$ and use trigonometric identities.
5. For which pairs of complex numbers is equality attained in the triangle inequality? Prove your answer.
6. $\triangleright$ Prove that for all $z, w \in \mathbb{C}$, $||z| - |w|| \leq |z - w|$.
7. Show that for all $z \in \mathbb{C}$, $|\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}\,|z|$.
8. This exercise concerns lines in $\mathbb{C}$.

(a) Let $a, b \in \mathbb{C}$ with $b \neq 0$. Prove that the set

$$L = \left\{ z \in \mathbb{C} : \operatorname{Im}\left(\frac{z - a}{b}\right) = 0 \right\}$$

is a line in $\mathbb{C}$. Explain the role of a and b in the geometry of L. (*Hint*: Recall the vector form of a line from multivariable calculus.)

(b) Let C be a circle in $\mathbb{C}$ with center c and radius $r > 0$. If a lies on C, write the line tangent to C at a in the form given in part (a).

1.4 The Topology of the Complex Plane

The *topology* of a certain space (in our case $\mathbb{C}$) gives a useful alternative to traditional geometry to describe relationships between points and sets. The key concepts of limits and continuity from calculus are tied to the topology of the real line, as we will see is also true in the plane. That connection just scratches the surface of how powerful a tool we will find planar topology to be for analyzing functions.

We begin by observing that with respect to a given subset of $\mathbb{C}$, each point of $\mathbb{C}$ is of one of three types.

1.4.1 Definition. Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. Then a is an *interior point* of A if A contains a disk centered at a, a is an *exterior point* of A if it is an interior point of the complement $\mathbb{C} \setminus A$, and a is a *boundary point* of A if it is neither an interior point of A nor an exterior point of A. (See Figure 1.2.)

These points form the following sets.

1.4.2 Definition. Let $A \subseteq \mathbb{C}$. The set of interior points of A is called the *interior* of A and is denoted A°. The set of boundary points of A is called the *boundary* of A and is denoted ∂A.

Note that the set of exterior points of A is the interior of $\mathbb{C} \setminus A$, and so we need not define a new symbol for this set. We have that $\mathbb{C}$ can be decomposed into the disjoint union

$$\mathbb{C} = A^\circ \cup \partial A \cup (\mathbb{C} \setminus A)^\circ.$$

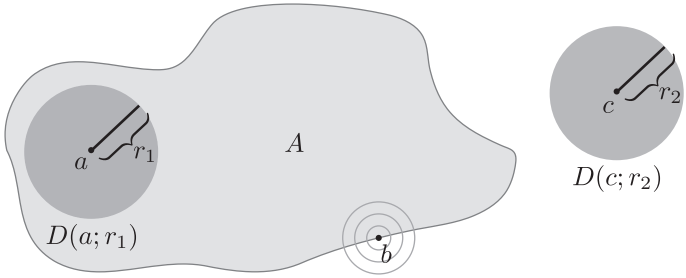

Figure 1.2 a is an interior point of A, b is a boundary point of A, and c is an exterior point of A

The set A contains all of its interior points, none of its exterior points, and none, some, or all of its boundary points. The extremal cases are special.

1.4.3 Definition. Let $A \subseteq \mathbb{C}$. If $\partial A \cap A = \varnothing$, then A is *open*. If $\partial A \subseteq A$, then A is *closed*.

The following properties can be deduced from the above definitions. Their proofs are left as an exercise.

1.4.4 Theorem. *The following hold for* $A \subseteq \mathbb{C}$.

(a) *The set* A *is closed if and only if its complement* $\mathbb{C} \setminus A$ *is open.*

(b) *The set* A *is open if and only if for every* $a \in A$*, there exists* $r > 0$ *such that* $D(a; r) \subseteq A$.

(c) *A point* $a \in \mathbb{C}$ *is in* ∂A *if and only if* $D(a; r) \cap A \neq \varnothing$ *and* $D(a; r) \setminus A \neq \varnothing$ *for all* $r > 0$.

1.4.5 Example. We study the disk $D(a; r)$ for some $a \in \mathbb{C}$ and $r > 0$. If $z_0 \in D(a; r)$, then let $\rho = r - |z_0 - a|$. Then $0 < \rho \leq r$. For all $z \in D(z_0; \rho)$,

$$|z - a| = |z - z_0 + z_0 - a| \leq |z - z_0| + |z_0 - a| < \rho + (r - \rho) = r$$

by the triangle inequality, showing $z \in D(a; r)$. Therefore $D(z_0; \rho) \subseteq D(a; r)$. This implies that $D(a; r)$ is an open set.

It is left as an exercise (using an argument quite similar to the one just presented) to show that the exterior points of $D(a; r)$ form the set $\{z \in \mathbb{C} : |z - a| > r\}$. Therefore the boundary of the disk is

$$\partial D(a; r) = \{z \in \mathbb{C} : |z - a| = r\}, \tag{1.4.1}$$

which is exactly the circle of radius r centered at a.

1.4.6 Definition. Given any set $A \subseteq \mathbb{C}$, the set

$$\overline{A} = A \cup \partial A \tag{1.4.2}$$

is called the *closure* of A.

Many of the properties of the closure are addressed in the exercises.

1.4.7 Example. If $D(a;r)$ is the disk in Example 1.4.5, then its closure is the *closed disk*

$$\overline{D}(a;r) = \overline{D(a;r)} = \{z \in \mathbb{C} : |z-a| \leq r\}. \tag{1.4.3}$$

1.4.8 Definition. Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. We say that a is a *limit point* of A provided that $D(a;r) \cap A \setminus \{a\} \neq \varnothing$ for every $r > 0$.

In other words, a is a limit point of A if every disk centered at a intersects A at a point *other than* a. This leads to another useful characterization of closed sets.

1.4.9 Theorem. *A set $E \subseteq \mathbb{C}$ is closed if and only if E contains all of its limit points.*

Proof. Suppose that E is closed and that a is a limit point of E. Were a an exterior point of E, we would have $D(a;r) \subseteq \mathbb{C} \setminus E$ for some $r > 0$. Since this contradicts that a is a limit point of E, it must be that a is an interior point or boundary point of E. Either way, $a \in E$.

Conversely, assume that E contains all of its limit points. Suppose that $a \in \partial E \setminus E$. For any $r > 0$, $D(a;r) \cap E \neq \varnothing$ by Theorem 1.4.4. Since $a \notin E$, $D(a;r) \cap E \setminus \{a\} \neq \varnothing$ for all $r > 0$, and thus a is a limit point of E. This shows that $a \in E$, a contradiction. Thus $\partial E \subseteq E$, and hence E is closed. □

We continue with two more definitions.

1.4.10 Definition. A set $A \subseteq \mathbb{C}$ is *bounded* if $A \subseteq D(0;R)$ for some $R > 0$.

1.4.11 Definition. A set $K \subseteq \mathbb{C}$ is *compact* if K is closed and bounded.

One must be careful not to be misled by the simplicity of the above definition and underestimate the importance of compact sets to the study of analysis. In fact, many properties of complex functions depend on compactness.

A reader with some previous exposure to topological concepts may have seen compactness defined in terms of "open covers." This definition is of great importance to the study of topology, but does not serve our purpose in this text. That our definition is equivalent is the content of the Heine–Borel theorem. An outline of the proof of this theorem is included in the exercises.

We now consider our final topological concept.

1.4.12 Definition. Nonempty sets $A, B \subseteq \mathbb{C}$ are *separated* if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. A nonempty set $E \subseteq \mathbb{C}$ is *connected* if E is not equal to the union of separated sets. Otherwise, E is *disconnected*.

While this definition may seem complicated, it should bring the reader comfort that the intuitive notion of connectedness matches the rigorous definition. The following observation is key to a method of detecting connectedness that will be sufficient for most circumstances we will encounter.

1.4.13 Lemma. *Let $A, B \subseteq \mathbb{C}$ be separated sets, $a \in A$, $b \in B$, and L be the line segment with endpoints a and b. Then $L \nsubseteq A \cup B$.*

Proof. Suppose $L \subseteq A \cup B$. If $u = (b-a)/|b-a|$, then $L = \{a + tu : 0 \le t \le |b-a|\}$. Set

$$t_0 = \sup\{t \in [0, |b-a|] : a + tu \in A\}, \qquad c = a + t_0 u \in L.$$

(See Appendix B for properties of the supremum.)

If $c \in A$, then $c \notin \overline{B}$, and hence $D(c;r) \cap B = \varnothing$ for some $r > 0$. Furthermore, $t_0 < |b-a|$, and if $t_0 < t < \min\{t_0 + r, |b-a|\}$, then $a + tu \in L \setminus A \subseteq B$. But $|(a+tu) - c| = t - t_0 < r$, a contradiction.

If $c \in B$, then $c \notin \overline{A}$, and so $D(c;r) \cap A = \varnothing$ for some $r > 0$. But $t_0 > 0$, and there must exist $\max\{t_0 - r, 0\} < t < t_0$ such that $a + tu \in A$. (See Theorem B.4.) But $|(a+tu) - c| = t_0 - t < r$, a contradiction. Hence $L \nsubseteq A \cup B$. $\square$

We now see that if a nonempty set $E \subseteq \mathbb{C}$ is such that the line segment connecting two arbitrary points in E lies in E, then E is connected. *For instance, all open and closed disks are connected, as are all lines, rays, and line segments.* This can be taken a step further. The proof of the following theorem and related results are considered in the exercises. See Figure 1.3.

1.4.14 Theorem. *Let $E \subseteq \mathbb{C}$ be nonempty. If for all $a, b \in E$, there are $a_0, \ldots, a_n \in E$ such that $a_0 = a$, $a_n = b$, and for each $k = 1, \ldots, n$, the line segment with endpoints a_{k-1} and a_k lies in E, then E is connected.*

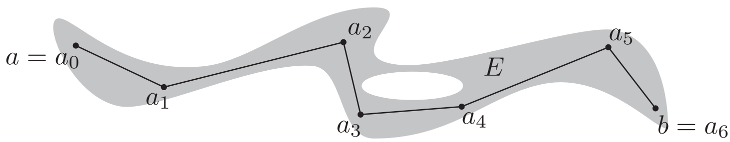

Figure 1.3 An illustration of Theorem 1.4.14 with $n = 6$

In our upcoming work, the most important connected sets are also open and warrant their own name.

1.4.15 Definition. A connected open subset of $\mathbb{C}$ is called a *domain*.

1.4.16 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a nonempty open set. Then Ω is a domain if and only if it is not equal to the union of disjoint nonempty open sets.*

Proof. It is equivalent to show that Ω is disconnected if and only if $\Omega = A \cup B$, where both A and B are open, nonempty, and $A \cap B = \varnothing$.

If Ω is disconnected, then let $\Omega = A \cup B$, where A and B are separated. Let $a \in A$. Since a is an exterior point of B, there is $r_1 > 0$ such that $D(a;r_1) \cap B = \varnothing$. Since $a \in \Omega$, there is $r_2 > 0$ such that $D(a;r_2) \subseteq \Omega$. If $r = \min\{r_1, r_2\} > 0$, then $D(a;r) \subseteq \Omega \setminus B = A$, showing A is open. A symmetric argument shows B is open.

Conversely, suppose $\Omega = A \cup B$ for disjoint nonempty open sets A, B. Every point of A is an exterior point of B and hence $A \cap \overline{B} = \varnothing$. Similarly, $\overline{A} \cap B = \varnothing$, showing A and B are separated. Thus Ω is disconnected. $\square$

We conclude this section with one more definition.

1.4.17 Definition. A maximal connected subset of a set $E \subseteq \mathbb{C}$ is called a *component* of E.

This means that $A \subseteq E$ is a component of E if A is connected and for any connected set $B \subseteq E$ such that $A \subseteq B$, it must be that $A = B$. It is an exercise to show that every point of E lies in a component of E (and hence components exist).

1.4.18 Example. We know that intervals in $\mathbb{R}$ are connected. If $E = (-1, 0) \cup (0, 1)$, then E is disconnected using the separated sets $A = (-1, 0)$ and $B = (0, 1)$. In fact, A and B are components of E.

One may easily decompose any set with at least two elements into the union of two nonempty disjoint subsets, showing the importance of $\overline{A}$ and $\overline{B}$ in Definition 1.4.12. Observing that $\overline{A} \cap \overline{B} = \{0\} \neq \varnothing$ in this example shows why only one closure is considered at a time.

1.4.19 Example. We conclude by analyzing a set with regard to all concepts introduced in this section. Filling in the details of the statements made is left as an exercise. Let

$$E = \{z \in \mathbb{C} : |\operatorname{Im} z| < |\operatorname{Re} z|\}.$$

(See Figure 1.4.) Each point $a \in E$ is an interior point of E, and hence E is open. Indeed, one may show that $D(a; r) \subseteq E$, where $r = (|\operatorname{Re} a| - |\operatorname{Im} a|)/2$, using the triangle inequality. Likewise, if $a \in \mathbb{C}$ is such that $|\operatorname{Re} a| < |\operatorname{Im} a|$, then $D(a; r) \subseteq \mathbb{C} \setminus E$ if $r = (|\operatorname{Im} a| - |\operatorname{Re} a|)/2$, showing a is an exterior point of E. We also have that ∂E consists of those $a \in \mathbb{C}$ for which $|\operatorname{Re} a| = |\operatorname{Im} a|$ since for such a and $r > 0$, at least one of $a \pm r/2$ lies in E and a lies in $\mathbb{C} \setminus E$. We conclude from this reasoning that the limit points of E are precisely the points in $\overline{E}$. Since $\partial E \not\subseteq E$, E is not closed. Moreover, E is not bounded, as $(0, \infty) \subseteq E$. Hence E fails both conditions required of compactness. Lastly, we note that $A = \{z \in E : \operatorname{Re} z < 0\}$ and $B = \{z \in E : \operatorname{Re} z > 0\}$ are connected because any pair of points in one set is connected by a line segment contained within that set. The above logic shows A and B are open, and hence E is disconnected by Theorem 1.4.16. It follows that A and B are connected components of E and are each domains.

Summary and Notes for Section 1.4.

We have defined basic topological concepts such as open, closed, compact, bounded, and connected sets in the complex plane $\mathbb{C}$.

The field of topology is vast and deep and comes in many flavors. Point set topology is the study of abstract topological spaces where open sets are defined by a set of axioms. Despite its importance, it is a relatively young area, only coming into its own in the early 20$^{\text{th}}$ century. This particular flavor of topology was motivated

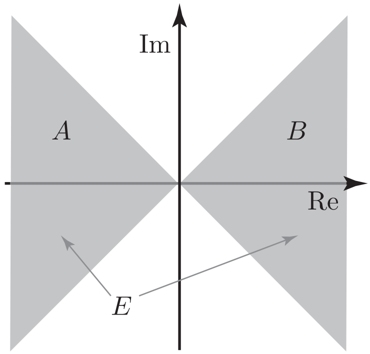

Figure 1.4 The set E and its components A and B from Example 1.4.19

by problems in abstract analysis. Indeed, John L. Kelley, in the preface to his classic book on general topology [14], wrote, “I have, with difficulty, been prevented by my friends from labeling [this book]: What Every Young Analyst Should Know.”

Exercises for Section 1.4.

1. For each of the following sets E, determine whether E is open, closed, bounded, compact, connected, a domain. In addition, identify E°, ∂E, $\overline{E}$, the collection of limit points of E, and the components of E. Do not include proofs.

   (a) $E = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

   (b) $E = \{z \in \mathbb{C} : 0 < |z| < 1\}$

   (c) $E = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1,\ |z| > 2\}$

   (d) $E = \bigcup_{n \in \mathbb{Z}} D(n; 1/2)$

   (e) $E = \{z \in \mathbb{C} : \operatorname{Re} z \neq |z|\}$

   (f) $E = \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}\}$

   (g) $E = \{1/n + i/m : n, m \in \mathbb{N}\}$

2. Justify the statements made in Example 1.4.19.

3. Let $a \in \mathbb{C}$ and $r > 0$. Show that the set of exterior points of the disk $D(a; r)$ is $\{z \in \mathbb{C} : |z - a| > r\}$.

4. Prove Theorem 1.4.4.

5. Prove the following for a set $E \subseteq \mathbb{C}$.

   (a) The set $\overline{E}$ is a closed set.

   (b) The set E is closed if and only if $E = \overline{E}$.

6. Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. Prove that a is a limit point of A if and only if $D(a; r) \cap A$ is infinite for all $r > 0$.

7. Let $A \subseteq \mathbb{C}$. Show that if $B \subseteq A$, then $\overline{B} \subseteq \overline{A}$.

8. $\triangleright$ Prove that a set $A \subseteq \mathbb{C}$ is open if and only if for every $a \in A$, there is a closed disk $\overline{D}(a; r) \subseteq A$ for some $r > 0$.

9. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be a collection of open subsets of $\mathbb{C}$. (Here, I is an index set. See Appendix A.)
   (a) Prove that $\bigcup_{\alpha \in I} U_\alpha$ is open.
   (b) Prove that $\bigcap_{\alpha \in I} U_\alpha$ is open if I is finite.
   (c) Give an example of an infinite collection $\mathcal{U}$ where the intersection in part (b) is not open.
10. Let $\mathcal{E} = \{E_\alpha : \alpha \in I\}$ be a collection of closed subsets of $\mathbb{C}$, where I is an index set.
    (a) Prove that $\bigcap_{\alpha \in I} E_\alpha$ is closed.
    (b) Prove that $\bigcup_{\alpha \in I} E_\alpha$ is closed if I is finite.
    (c) Give an example of an infinite collection $\mathcal{E}$ where the union in part (b) is not closed.
11. Find an example of a disconnected set whose closure is connected.
12. Suppose that $E, F \subseteq \mathbb{C}$ are connected. Is $E \cap F$ necessarily connected if $E \cap F \neq \varnothing$? Provide a proof or a counterexample.
13. Use the following steps to show that if $E \subseteq \mathbb{C}$, then each $a \in E$ lies in a component of E.
    (a) Suppose I is an index set and $E_\alpha \subseteq \mathbb{C}$ is connected for each $\alpha \in I$. Show that if $\bigcap_{\alpha \in I} E_\alpha \neq \varnothing$, then $\bigcup_{\alpha \in I} E_\alpha$ is connected.
    (b) Let F be the union of all connected subsets of E containing a. Show that F is a component.
14. Show that every component of an open set is a domain. (*Hint*: Part (a) of Exercise 13 is helpful.)
15. Which subsets of $\mathbb{C}$ are both open and closed? Prove your answer.
16. Prove Theorem 1.4.14. Show that the converse holds if E is open using the following strategy: Let $a \in E$ and A consist of all $b \in E$ such that a and b are connected by a sequence of line segments as in the statement of Theorem 1.4.14. Show A and $E \setminus A$ are open.
17. Let $U \subseteq \mathbb{C}$ be a nonempty open set. Prove that U has an *exhaustion* by compact sets. That is, show that there are compact sets $K_n \subseteq U$ for each $n \in \mathbb{N}$ such that $K_n \subseteq K_{n+1}$ for every n and $U = \bigcup_{n=1}^{\infty} K_n$.
18. In this exercise, we consider the abstract topological definition of compactness. Let $K \subseteq \mathbb{C}$. An *open cover* of K is a collection $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of open subsets of $\mathbb{C}$ such that $K \subseteq \bigcup_{\alpha \in I} U_\alpha$. A subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\} \subseteq \mathcal{U}$ is said to be a *finite subcover* of K if $K \subseteq \bigcup_{k=1}^{n} U_{\alpha_k}$. The *Heine–Borel theorem* states that a set K is compact if and only if every open cover of K contains a finite subcover of K. Prove the Heine–Borel theorem using the following steps.
    (a) If every open cover of K contains a finite subcover of K, show that K is bounded.
    (b) If every open cover of K contains a finite subcover of K, show that K is closed.
    (c) Let $S \subseteq \mathbb{C}$ be a closed square with sides parallel to the axes. Show that every open cover of S contains a finite subcover of S. (*Hint*: Look ahead to the subdivision argument in the proof of the Bolzano–Weierstrass theorem [Theorem 1.6.14] and use something similar in a proof by contradiction.)

(d) Show that if K is compact, then every open cover of K contains a finite subcover of K by choosing a square S that contains K and extending the open cover of K to an open cover of S by adding the set $\mathbb{C} \setminus K$.

1.5 The Extended Complex Plane

In analysis, we frequently need to deal with infinite limits. Even in calculus, one sees limits approaching $\pm\infty$. Unfortunately, there is a multitude of directions in $\mathbb{C}$ in which a limit can approach "infinity." To escape this problem, we use a clever topological identification that will relate the complex plane to a sphere.

As is learned in multivariable calculus, $\mathbb{R}^3$ is the set (space) of all points of the form (x_1, x_2, x_3), where $x_1, x_2, x_3 \in \mathbb{R}$. The *unit sphere* in $\mathbb{R}^3$ is the set of all points of distance 1 from the origin; in other words, it is the set

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.$$

We put the elements of $\mathbb{C}$ into a one-to-one correspondence with the elements of S with the exception of the "north pole" $N = (0, 0, 1)$. To do so, we first associate the plane $\mathbb{C}$ to the plane in $\mathbb{R}^3$ described by the equation $x_3 = 0$. This is done in the natural way by writing $z \in \mathbb{C}$ as $x_1 + ix_2$ and identifying this with the point $(x_1, x_2, 0)$. If we let L be the line in $\mathbb{R}^3$ through the points $(x_1, x_2, 0)$ and N, then $L \cap S \setminus \{N\}$ contains exactly one point, which we call Z. This establishes the desired one-to-one correspondence between $z \in \mathbb{C}$ and $Z \in S \setminus \{N\}$. Topologically, this correspondence "wraps" the plane onto the sphere, leaving only the north pole uncovered. Note that $\mathbb{D}$ is sent to the "lower hemisphere," $\partial\mathbb{D}$ is fixed, and $\mathbb{C} \setminus \overline{\mathbb{D}}$ is sent to the "upper hemisphere." See Figure 1.5.

1.5.1 Definition. The one-to-one correspondence described above is called the *stereographic projection* of $\mathbb{C}$ onto $S \setminus \{N\}$.

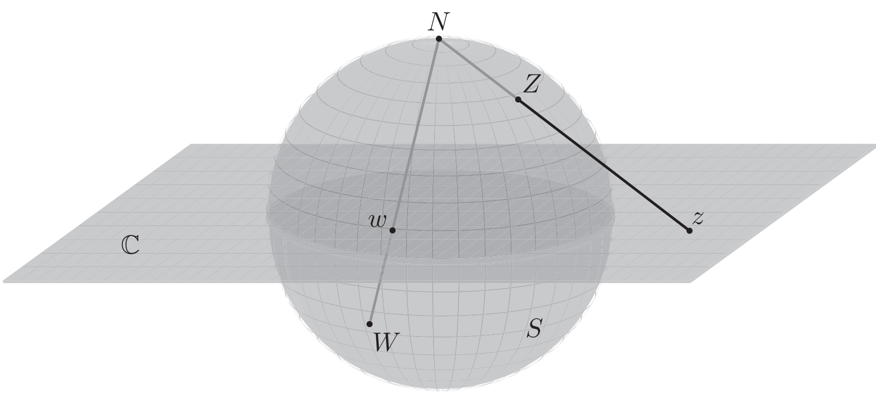

Figure 1.5 Stereographic projection of $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ onto Z and $w \in \mathbb{D}$ onto W

As the north pole is the lone point on the sphere left uncovered by the stereographic projection, it is natural to identify this point with ∞.

1.5.2 Definition. The *extended complex plane* is the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, where ∞ inherits all of the natural properties from the north pole of the sphere through the stereographic projection. Because of this correspondence, the extended complex plane is often called the *Riemann sphere.*

This method of adding the point ∞ to $\mathbb{C}$ is alternatively referred to as the *one-point compactification of* $\mathbb{C}$.

Summary and Notes for Section 1.5.

Through the stereographic projection, we can identify the points of the plane $\mathbb{C}$ with the points of a sphere without its "north pole." This then allows the north pole to be identified with ∞. Unlike on the real line, where we consider signed infinities, the complex ∞ is a single geometric point, and the full sphere is called the extended complex plane, denoted by $\mathbb{C}_\infty$.

The stereographic projection is certainly not a new idea, and it was not motivated by complex analysis. Ancient Greeks knew of the projection, and it was used for centuries for making both celestial and geographic maps.

Exercises for Section 1.5.

1. Which sets in S correspond to the real and imaginary axes in $\mathbb{C}$ under the stereographic projection?

2. Let $z = x + iy \in \mathbb{C}$ be given. Calculate the coordinates of the point $Z = (x_1, x_2, x_3) \in S$ corresponding to z under the stereographic projection.

3. Let $Z = (x_1, x_2, x_3) \in S$ be given. Calculate the complex number z corresponding to Z under the stereographic projection.

4. For any $z, w \in \mathbb{C}_\infty$ define the *spherical distance* between z and w, denoted $d(z, w)$, to be the distance in $\mathbb{R}^3$ between the points $Z, W \in S$ corresponding to z and w under the stereographic projection. Use Exercises 2 and 3 to prove the following.

   (a) If $z, w \in \mathbb{C}$, then
   $$d(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}}.$$

   (b) If $z \in \mathbb{C}$, then
   $$d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

5. $\triangleright$ Show that circles on S correspond to circles and lines in $\mathbb{C}$. (*Hint*: Recall the following facts from analytic geometry: Every circle in the xy-plane can be expressed by an equation of the form $x^2 + y^2 + ax + by + c = 0$ for some $a, b, c \in \mathbb{R}$. Every circle on S is the intersection of S with a plane in $\mathbb{R}^3$ described by the equation $Ax_1 + Bx_2 + Cx_3 = D$ for some $A, B, C, D \in \mathbb{R}$. Then use Exercise 2.)

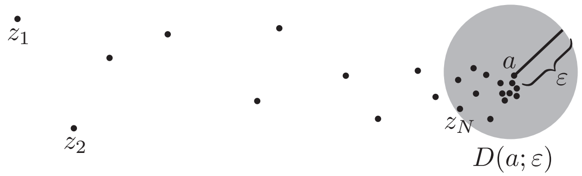

Figure 1.6 $z_n \to a$

1.6 Complex Sequences

Sequences are a fundamental tool in analysis for the purpose of approximation. They are closely related to the topology of $\mathbb{C}$ and essential to the study of series, which lie at the core of function theory.

1.6.1 Definition. A function of the form $z\colon \{m, m+1, \dots\} \to \mathbb{C}$ for some $m \in \mathbb{Z}$ is called a *sequence* of complex numbers.

The function value $z(n)$ for some $n \geq m$ is denoted by z_n. This allows a sequence to be described, in a more familiar manner, by a list $\{z_m, z_{m+1}, \dots\}$ or $\{z_n\}_{n=m}^{\infty}$. The subscripts are the *indices* of the sequence. Although indices can begin at any integer m, we will typically consider sequences with $m = 1$ for simplicity. This does not affect generality, however, since any sequence can be *reindexed*. In other words, if a sequence begins at $n = m$ for some $m \in \mathbb{Z}$, then we could replace each index n by $n - m + 1$ so that the sequence begins with the index $n = 1$ or vice versa. In most circumstances, such a maneuver is only used to provide some sort of algebraic simplification.

1.6.2 Definition. A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers *converges* to a number $a \in \mathbb{C}$, called the *limit* of $\{z_n\}$, provided that for any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|z_n - a| < \varepsilon$ whenever $n \geq N$. A sequence *diverges* if it fails to converge.

Convergence is typically denoted by writing $z_n \to a$ as $n \to \infty$ (or just $z_n \to a$ when the context is clear) or by the expression

$$\lim_{n\to\infty} z_n = a. \tag{1.6.1}$$

Geometrically, $z_n \to a$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $z_n \in D(a;\varepsilon)$. See Figure 1.6.

Because of our understanding of ∞ from Section 1.5, we can deal simply with infinite limits, a special type of divergence.

1.6.3 Definition. The sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers *diverges to* ∞, denoted by $z_n \to \infty$ (as $n \to \infty$) or $\lim_{n\to\infty} z_n = \infty$, if given any $R > 0$, there is $N \in \mathbb{N}$ such that $|z_n| > R$ for all $n \geq N$.

Exercise 4 of Section 1.5 can be used to show that the set $\{z \in \mathbb{C} : |z| > R\}$ corresponds to the set of points on the sphere S of distance less than $\varepsilon = 2/\sqrt{1+R^2}$

from the north pole N under the stereographic projection, and therefore this definition of an infinite limit naturally corresponds to a sequence on S converging to N.

We now consider some intuitive properties of sequences.

1.6.4 Theorem. *A convergent sequence has a unique limit.*

Proof. Suppose that $\{z_n\}_{n=1}^{\infty}$ converges to both a and b in $\mathbb{C}$. If $a \neq b$, then let $\varepsilon = |a-b|/2 > 0$. For some $N_1, N_2 \in \mathbb{N}$, $n \geq N_1$ implies $|z_n - a| < \varepsilon$ and $n \geq N_2$ implies $|z_n - b| < \varepsilon$. But if $n \geq \max\{N_1, N_2\}$, then by the triangle inequality,

$$2\varepsilon = |a-b| = |a - z_n + z_n - b| \leq |a - z_n| + |z_n - b| < 2\varepsilon,$$

a contradiction. Thus $a = b$. $\square$

1.6.5 Definition. A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is *bounded* if there exists $R > 0$ such that $|z_n| \leq R$ for all $n \in \mathbb{N}$.

1.6.6 Theorem. *If a sequence converges, then it is bounded.*

Proof. Suppose that the sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers converges to some $a \in \mathbb{C}$. Then for some $N \in \mathbb{N}$, $|z_n - a| < 1$ whenever $n \geq N$. Fix

$$R = \max\{|z_1|, \ldots, |z_N|, 1 + |a|\}.$$

Clearly, if $n \leq N$, then $|z_n| \leq R$. If $n > N$, then by the triangle inequality,

$$|z_n| = |z_n - a + a| \leq |z_n - a| + |a| < 1 + |a| \leq R.$$

Hence the sequence is bounded. $\square$

Sequences of real numbers are familiar from calculus. The following theorem shows how our study of complex sequences can rely on their real counterparts.

1.6.7 Theorem. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. For each $n \in \mathbb{N}$, write $z_n = x_n + iy_n$. Then $\{z_n\}$ converges if and only if both $\{x_n\}$ and $\{y_n\}$ converge. In this case,*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n. \tag{1.6.2}$$

Proof. Suppose $x_n \to a$ and $y_n \to b$ for some $a, b \in \mathbb{R}$. Let $\varepsilon > 0$. There are $N_1, N_2 \in \mathbb{N}$ such that $|x_n - a| < \varepsilon/2$ for all $n \geq N_1$ and $|y_n - b| < \varepsilon/2$ for all $n \geq N_2$. If $N = \max\{N_1, N_2\}$, then for all $n \geq N$,

$$|z_n - (a+ib)| = |x_n - a + i(y_n - b)| \leq |x_n - a| + |y_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

using the triangle inequality. This shows $\{z_n\}$ converges to $a + ib$, verifying (1.6.2).

Conversely, suppose $z_n \to c$ for some $c \in \mathbb{C}$. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $|z_n - c| < \varepsilon$ for all $n \geq N$. But for such n,

$$|x_n - \operatorname{Re} c| = |\operatorname{Re}(z_n - c)| \leq |z_n - c| < \varepsilon$$

using Exercise 4 from Section 1.2. Hence $\{x_n\}$ converges. A similar argument gives that $\{y_n\}$ converges. $\square$

We know from calculus that convergent real sequences satisfy the following algebra rules. The complex versions can be proved by resorting to real and imaginary parts and using Theorem 1.6.7 and are left as exercises.

1.6.8 Theorem. *Suppose that $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ are sequences of complex numbers such that $z_n \to a$ and $w_n \to b$ for some $a, b \in \mathbb{C}$ as $n \to \infty$. Furthermore, let $c \in \mathbb{C}$. Then*

(a) $\lim_{n\to\infty} cz_n = ca$,

(b) $\lim_{n\to\infty}(z_n + w_n) = a + b$,

(c) $\lim_{n\to\infty} z_n w_n = ab$, *and*

(d) $\lim_{n\to\infty} z_n/w_n = a/b$ *if* $w_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $b \neq 0$.

From calculus, we have techniques, such as l'Hôpital's rule, for dealing with limits of real sequences. Theorem 1.6.7 gives us one route to apply these real techniques to find limits of complex sequences. Another method, relying on the modulus, is the following.

1.6.9 Theorem. *Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, $\{c_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers, and $a \in \mathbb{C}$.*

(a) *If $c_n \to 0$ and $|z_n - a| \leq c_n$ for all $n \in \mathbb{N}$, then $z_n \to a$.*

(b) *If $c_n \to \infty$ and $|z_n| \geq c_n$ for all $n \in \mathbb{N}$, then $z_n \to \infty$.*

We leave the proof of this theorem as an exercise but consider the following example of its helpfulness.

1.6.10 Example. Consider the sequence of complex numbers

$$\left\{\frac{n+i}{(1-2i)^n}\right\}_{n=1}^{\infty}.$$

By the triangle inequality, we have

$$\left|\frac{n+i}{(1-2i)^n}\right| \leq \frac{n+1}{|1-2i|^n} = \frac{n+1}{5^{n/2}}.$$

An application of l'Hôpital's rule shows that the right-hand side converges to 0. Therefore our sequence converges to 0 by Theorem 1.6.9. Note that we do not have a version of l'Hôpital's rule for complex sequences to directly apply in these circumstances. (Nor will we.)

1.6.11 Definition. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of positive integers. The sequence $\{z_{n_k}\}_{k=1}^{\infty}$ is called a *subsequence* of $\{z_n\}$.

In looser terms, a subsequence of a sequence is a sequence formed by taking terms of the original sequence, in order.

1.6.12 Theorem. *A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers converges to a number $a \in \mathbb{C}$ if and only if every subsequence of $\{z_n\}$ converges to a.*

Proof. Suppose that $z_n \to a$ as $n \to \infty$ and that $\{z_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{z_n\}$. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - a| < \varepsilon$ whenever $n \geq N$. Since $\{n_k\}_{k=1}^{\infty}$ is strictly increasing, $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore $|z_{n_k} - a| < \varepsilon$ whenever $k \geq N$, and hence $z_{n_k} \to a$ as $k \to \infty$.

The converse follows because a sequence is a subsequence of itself. □

We now present some vital connections between sequences and the topology of $\mathbb{C}$. The first classifies closed sets in terms of sequences.

1.6.13 Theorem. *A set $E \subseteq \mathbb{C}$ is closed if and only if every convergent sequence of elements of E has its limit in E.*

Proof. Suppose that E is closed. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of elements of E that converges to some $a \in \mathbb{C}$. If $a \notin E$, then a is an exterior point of E. There exists $\varepsilon > 0$ such that $D(a;\varepsilon) \cap E = \varnothing$. It follows that $|z_n - a| \geq \varepsilon$ for all $n \in \mathbb{N}$, a contradiction. Thus $a \in E$.

Conversely, suppose that every convergent sequence in E converges to a point in E. Let $a \in \partial E$. Then for each $n \in \mathbb{N}$, there exists $z_n \in D(a;1/n) \cap E$ by Theorem 1.4.4. Since $|z_n - a| < 1/n$ for all $n \in \mathbb{N}$, $z_n \to a$ by Theorem 1.6.9. Therefore $a \in E$, showing E is closed. □

We now come to an important theorem that relates compactness to sequences. Its power is in the impact that a set being closed and bounded has on sequences within the set.

1.6.14 Bolzano–Weierstrass Theorem. *A set $K \subseteq \mathbb{C}$ is compact if and only if every sequence of elements of K has a subsequence that converges to an element of K.*

Proof. Suppose that K is compact, and hence bounded, and let $\{z_n\}_{n=1}^{\infty}$ be a sequence in K. There is a square $S = \{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b,\ c \leq \operatorname{Im} z \leq d\}$, where $a, b, c, d \in \mathbb{R}$ and $b - a = d - c = \alpha$ for some $\alpha > 0$, such that $K \subseteq S$. If S is divided symmetrically into four closed subsquares, then (at least) one subsquare contains infinitely many terms of $\{z_n\}$. Call this subsquare S_1, and let $n_1 \in \mathbb{N}$ be such that $z_{n_1} \in S_1$. Note that $S_1 = \{z \in \mathbb{C} : a_1 \leq \operatorname{Re} z \leq b_1,\ c_1 \leq \operatorname{Im} z \leq d_1\}$, where $b_1 - a_1 = d_1 - c_1 = \alpha/2$.

Now continue this process inductively to generate a collection of closed squares $S_k = \{z \in \mathbb{C} : a_k \leq \operatorname{Re} z \leq b_k,\ c_k \leq \operatorname{Im} z \leq d_k\}$, where $b_k - a_k = d_k - c_k = \alpha/2^k$, $S_k \subseteq S_{k-1}$ for all $k \geq 2$, and each S_k contains infinitely many terms of $\{z_n\}$. (See Figure 1.7.) At each step, we choose $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $z_{n_k} \in S_k$. Evidently, $a_k < b_j$ for all $j, k \in \mathbb{N}$. Set $x_0 = \sup\{a_k : k \in \mathbb{N}\}$. Then $a_k \leq x_0 \leq b_k$ for all k. Likewise, set $y_0 = \sup\{c_k : k \in \mathbb{N}\}$ so that $c_k \leq y_0 \leq d_k$ for all k. If $z_0 = x_0 + iy_0$, then $z_0 \in S_k$ for all k.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\alpha/2^N < \varepsilon/\sqrt{2}$. Then the distance between any two points in S_N is less than ε, proving $S_N \subseteq D(z_0; \varepsilon)$. It immediately follows that $|z_{n_k} - z_0| < \varepsilon$ for all $k \geq N$, showing $z_{n_k} \to z_0$. Because $\{z_{n_k}\}_{k=1}^{\infty} \subseteq K$ and K is closed, $z_0 \in K$ by Theorem 1.6.13. Thus $\{z_n\}$ has a subsequence that converges to an element of K.

Conversely, if every sequence of members of K has a subsequence converging to an element of K, then every convergent sequence of elements of K must converge to a member of K by Theorem 1.6.12. Thus K is closed by Theorem 1.6.13. Were K unbounded, for each $n \in \mathbb{N}$, there would exist $z_n \in K$ such that $|z_n| \geq n$. But any subsequence of $\{z_n\}_{n=1}^{\infty}$ would be unbounded and hence divergent by Theorem 1.6.6. Therefore K is compact. $\square$

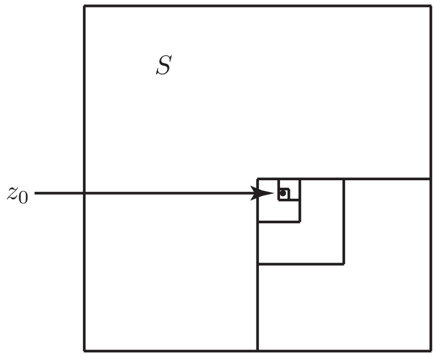

Figure 1.7 Possible inductive steps in the proof of Theorem 1.6.14

We immediately use the sequential notion of compactness to prove the following valuable theorem.

1.6.15 Theorem. *Suppose that for each $n \in \mathbb{N}$, $K_n \subseteq \mathbb{C}$ is a nonempty compact set and that $K_{n+1} \subseteq K_n$ for all n. Then*

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

Proof. For each $n \in \mathbb{N}$, let $z_n \in K_n$. Then $\{z_n\}_{n=1}^{\infty}$ is a sequence of elements of K_1 and hence there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ that converges to some $a \in K_1$ by the Bolzano–Weierstrass theorem.

Let $n \in \mathbb{N}$. Since $n_k \geq n$ when $k \geq n$, $\{z_{n_k}\}_{k=n}^{\infty}$ is a sequence in K_n that converges to a. Because K_n is closed, $a \in K_n$ by Theorem 1.6.13. But n was arbitrarily chosen, so $a \in \bigcap_{n=1}^{\infty} K_n$. $\square$

We conclude this section with some remarks about Cauchy sequences.

1.6.16 Definition. A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is a *Cauchy sequence* if for every $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ whenever $m, n \geq N$.

That every Cauchy sequence of real numbers is convergent is a characteristic called the completeness of $\mathbb{R}$. We have a similar result in $\mathbb{C}$. Its proof is left as an exercise.

1.6.17 Completeness of the Complex Numbers. *A sequence of complex numbers converges if and only if it is a Cauchy sequence.*

Summary and Notes for Section 1.6.

Sequences of complex numbers and their limits are defined in exactly the same way as their real counterparts are in calculus, and the algebra of convergent sequences holds as we would expect, as seen by considering the real and imaginary parts of their terms. Sequences are ubiquitous in analysis; they are a basic tool for developing complicated ideas.

Closed sets and compact sets can be characterized using sequences which makes these sets easier to study. In general topological spaces, these characterizations do not hold, which leads to calling certain sets sequentially closed or sequentially compact.

Exercises for Section 1.6.

1. Find limits of the following sequences, or explain why they diverge.

   (a) $\left\{ \dfrac{i2^n - n}{(3+i)^n} \right\}_{n=1}^{\infty}$

   (b) $\left\{ \dfrac{n + i^n}{n} \right\}_{n=1}^{\infty}$

   (c) $\left\{ \dfrac{1 - in^2}{n(n+1)} \right\}_{n=1}^{\infty}$

   (d) $\left\{ \dfrac{(1+i)^n}{n} \right\}_{n=1}^{\infty}$

2. Prove Theorem 1.6.8 by using the real and imaginary parts of $\{z_n\}$ and $\{w_n\}$ and Theorem 1.6.7.

3. Prove Theorem 1.6.8 using Definition 1.6.2. Theorem 1.6.6 is helpful.

4. Prove Theorem 1.6.9.

5. $\triangleright$ Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Show that if $z_n \to a$ for some $a \in \mathbb{C}$, then $|z_n| \to |a|$.

6. A sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers is *monotone* if it is either increasing or decreasing. The *monotone convergence theorem* from calculus states that a bounded monotone sequence must converge. Prove it.

7. $\triangleright$ Let $a \in \mathbb{C}$.

   (a) Prove that if $|a| < 1$, then $a^n \to 0$.

   (b) Prove that if $|a| > 1$, then $a^n \to \infty$.

8. Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be sequences of complex numbers.

(a) Show that if $\{z_n\}$ is bounded and $w_n \to \infty$, then $(z_n + w_n) \to \infty$.

(b) If $z_n \to \infty$ and $w_n \to \infty$, does it follow that $(z_n + w_n) \to \infty$?

9. $\triangleright$ Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Show that
$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

10. Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be sequences in $\mathbb{C}\backslash\{0\}$. If $\{1/z_n\}$ is bounded and $w_n \to 0$, show that $z_n/w_n \to \infty$.

11. $\triangleright$ Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. Prove that $a \in \overline{A}$ if and only if there is a sequence of elements of A converging to a.

12. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, and suppose that a is a limit point of $\{z_n : n \in \mathbb{N}\}$. Show that there is a subsequence of $\{z_n\}$ converging to a.

13. $\triangleright$ Let $a \in \mathbb{C}$ and $r > 0$. If $K \subseteq D(a;r)$ is compact, prove that there is some $\rho \in (0, r)$ such that $K \subseteq D(a;\rho)$.

14. Prove Theorem 1.6.17. (*Hint*: To show a Cauchy sequence converges, first show it is bounded, and then turn to the Bolzano–Weierstrass theorem.)

15. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, and let $z \in \mathbb{C}_\infty$. Prove that $z_n \to z$ as $n \to \infty$ if and only if $Z_n \to Z$ on the Riemann sphere S, where Z_n, Z correspond to z_n, z under the stereographic projection.

1.7 Complex Series

A series of complex numbers can intuitively be thought of as an infinite sum, but, in actuality, is a special kind of sequence. Our approach to studying function theory will rest squarely on series.

1.7.1 Definition. Let $m \in \mathbb{Z}$ and $\{z_n\}_{n=m}^{\infty}$ be a sequence of complex numbers. We define the *partial sums* of $\{z_n\}$ to be the complex numbers

$$s_N = \sum_{n=m}^{N} z_n = z_m + \cdots + z_N, \tag{1.7.1}$$

for each $N \in \mathbb{Z}$ such that $N \geq m$.

The partial sum sequence $\{s_N\}_{N=m}^{\infty}$ is well defined for any complex sequence.

1.7.2 Definition. If the sequence of partial sums in Definition 1.7.1 converges to some $s \in \mathbb{C}$, then we say that the *infinite series* $\sum_{n=m}^{\infty} z_n$ with *terms* $\{z_n\}_{n=m}^{\infty}$ *converges* to s. This is denoted by

$$s = \sum_{n=m}^{\infty} z_n = z_m + z_{m+1} + \cdots, \tag{1.7.2}$$

and we accordingly also refer to s as the *sum* of the series. Any series that fails to converge is said to *diverge*. If $s_N \to \infty$ as $N \to \infty$, then we say that the series *diverges to* ∞. This special type of divergence is written

$$\sum_{n=m}^{\infty} z_n = \infty.$$

Many series we consider will have beginning index $m = 1$ or $m = 0$. As with sequences, series can be reindexed, so there is no loss of generality in proving results for these specific cases.

The first theorem is a direct result of Theorem 1.6.4.

1.7.3 Theorem. *A convergent series has a unique sum.*

The next theorem establishes some rules concerning the algebra of series. The proof follows from the properties of the sequence of partial sums and Theorem 1.6.8 and is left as an exercise.

1.7.4 Theorem. *Let $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ be infinite series of complex numbers converging to a and b, respectively, and let $c \in \mathbb{C}$. Then*

(a) $\sum_{n=1}^{\infty} cz_n = ca$ *and*

(b) $\sum_{n=1}^{\infty}(z_n + w_n) = a + b$.

We refer the reader to Appendix B where the concept of a product of two series is considered.

The central problem when dealing with series is the determination of convergence. Often, we are only concerned with *whether or not* a series converges, not *to what* the series converges. The following is the first and simplest test. Its proof is left as an exercise.

1.7.5 Theorem. *Let $\sum_{n=1}^{\infty} z_n$ be a convergent series of complex numbers. Then $z_n \to 0$ as $n \to \infty$.*

The usefulness of the preceding theorem lies in the contrapositive. Divergence of a series can be detected by observing that its sequence of terms fails to converge to 0. The converse is not addressed; nothing is said about the behavior of the series when its terms do converge to 0.

1.7.6 Example. Consider the series $\sum_{n=0}^{\infty} z^n$, and take note of Exercise 7 in Section 1.6. (We adopt the traditional convention that $z^0 = 1$ for all $z \in \mathbb{C}$.) If $|z| \geq 1$, then $\{z^n\}$ fails to converge to 0, and hence the series is divergent by Theorem 1.7.5.

For $z \in \mathbb{D}$, consider the partial sums $s_N = \sum_{n=0}^{N} z^n$. Observe that

$$(1 - z)s_N = \sum_{n=0}^{N} z^n - \sum_{n=1}^{N+1} z^n = 1 - z^{N+1}.$$

Therefore

$$s_N = \frac{1 - z^{N+1}}{1 - z}.$$

Since $z^{N+1} \to 0$ as $N \to \infty$, it follows that

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1 - z} \tag{1.7.3}$$

for $z \in \mathbb{D}$.

This series is called the *geometric series*. Its elementary nature should not be underestimated; we will find it to be a useful tool on several upcoming occasions.

Detecting convergence of a series is usually a more difficult task than what we faced in Example 1.7.6. Here is a start. Its proof is standard from calculus and is left as an exercise.

1.7.7 Comparison Test. *Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of nonnegative real numbers such that $a_n \leq b_n$ for all $n \in \mathbb{N}$.*

(a) *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges and*

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

(b) *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

The comparison test only applies to series of nonnegative real numbers. This seems to address only a slight number of the series that we are likely to consider in complex analysis! The following concept helps to make a connection.

1.7.8 Definition. The series $\sum_{n=1}^{\infty} z_n$ of complex numbers is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

The phrase "absolutely convergent" seems to imply convergence. That is no accident.

1.7.9 Theorem. *If the series $\sum_{n=1}^{\infty} z_n$ of complex numbers is absolutely convergent, then it is convergent and*

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|. \tag{1.7.4}$$

Proof. Let $\varepsilon > 0$. Absolute convergence implies that there is $N \in \mathbb{N}$ such that

$$\left| \sum_{n=1}^{N} |z_n| - \sum_{n=1}^{\infty} |z_n| \right| = \sum_{n=N+1}^{\infty} |z_n| < \varepsilon.$$

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{k=1}^{\infty} z_k$. Then for any $n, m \geq N$ with $m < n$,

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} z_k \right| \leq \sum_{k=m+1}^{n} |z_k| < \varepsilon,$$

due (inductively) to the triangle inequality. This shows that $\{s_n\}$ is a Cauchy sequence, and is thus convergent. Hence $\sum_{n=1}^{\infty} z_n$ is convergent.

Now that we know that both sides of the (triangle) inequality

$$\left| \sum_{k=1}^{n} z_k \right| \leq \sum_{k=1}^{n} |z_k|$$

converge as $n \to \infty$, taking this limit and using limit inequalities gives (1.7.4). See Exercises 5 and 9 of Section 1.6 for the properties of sequences used here. □

1.7.10 Example. Consider the complex series

$$\sum_{n=0}^{\infty} \frac{(2i)^n - 1}{3^n + 2} = -\frac{1}{2} - \frac{1-2i}{5} - \frac{5}{11} + \cdots .$$

For each n, the triangle inequality gives

$$\left| \frac{(2i)^n - 1}{3^n + 2} \right| \leq \frac{2^n + 1}{3^n + 2} \leq \frac{2^n + 2^n}{3^n} = 2\left(\frac{2}{3}\right)^n .$$

The series $\sum_{n=0}^{\infty} 2(2/3)^n$ is seen to converge using the geometric series and Theorem 1.7.4. Therefore the given series converges absolutely by the comparison test.

1.7.11 Definition. Any series that is convergent but not absolutely convergent is called *conditionally convergent*.

While distinguishing between those series that converge absolutely and those that converge conditionally is a prevalent theme in calculus, absolute convergence will be our primary need. Many tests for convergence have been purposely postponed until the discussion of power series in Section 2.3. This will be the setting of most interest to us. The following exercises are more focused on developing the algebraic aspects of series, rather than addressing the convergence of specific series.

Summary and Notes for Section 1.7.

An infinite series of complex numbers is nothing more than a special type of sequence, the sequence of partial sums. We casually denote the sum (limit) of the series by the series itself which is terribly convenient and causes no confusion. Absolute convergence allows for comparison to series of nonnegative terms and their convergence tests learned in calculus. We will see that infinite series, in particular geometric series, are of essential importance in the study of function theory.

Infinite series (and sequences) have been considered for millennia, dating back to the ancient Greeks. The concept of convergence was historically quite fluid until the 19th century efforts to add rigor to analysis. Indeed, for centuries, infinite series were treated in a casual manner that seems a bit sloppy by today's standards.

Exercises for Section 1.7.

1. Determine the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{(n+i)(n+1+i)}.$$

(*Hint*: Use partial fractions and the sequence of partial sums.)

2. For each of the following series, list the first three terms of the series and determine whether or not the series converges. Can you find the sum of some of them?

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(1-2i)^{n+2}}{(4+i)^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(3-i)^n}{(1+2i)^{n+1}}$

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{3-(2i)^n}{4^{2n}}$

   (d) $\displaystyle\sum_{n=2}^{\infty} \frac{(3i)^n}{3+4^n}$

3. Prove Theorem 1.7.4.

4. Prove Theorem 1.7.5.

5. Prove Theorem 1.7.7.

6. Suppose that $\sum_{n=1}^{\infty} z_n$ is a convergent series of complex numbers. Prove that both of the real series $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ converge and

$$\sum_{n=1}^{\infty} \operatorname{Re} z_n = \operatorname{Re} \sum_{n=1}^{\infty} z_n, \qquad \sum_{n=1}^{\infty} \operatorname{Im} z_n = \operatorname{Im} \sum_{n=1}^{\infty} z_n.$$

Is the converse true?

7. Let $\{z_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be sequences of complex numbers.

   (a) Prove that if the series $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ are absolutely convergent, then $\sum_{n=1}^{\infty}(z_n + w_n)$ is absolutely convergent.

   (b) If $\sum_{n=1}^{\infty} z_n$ is absolutely convergent and $\sum_{n=1}^{\infty} w_n$ is conditionally convergent, what can be said about the convergence of $\sum_{n=1}^{\infty}(z_n + w_n)$?

   (c) If $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ are conditionally convergent, what can be said about the convergence of $\sum_{n=1}^{\infty}(z_n + w_n)$?