1

Fundamentals of Magnetic Devices

1.1 Introduction

Many electronic circuits require the use of inductors and transformers [1–60]. These are usually the largest, heaviest, and most expensive components in a circuit. They are defined by their electromagnetic (EM) behavior. The main feature of an inductor is its ability to store magnetic energy in the form of a magnetic field. The important feature of a transformer is its ability to couple magnetic fluxes of different windings and transfer AC energy from the input to the output through the magnetic field. The amount of energy transferred is determined by the operating frequency, flux density, and temperature. Transformers are used to change the AC voltage and current levels as well as to provide DC isolation while transmitting AC signals. They can combine energy from many AC sources by the addition of the magnetic flux and deliver the energy from all the inputs to one or multiple outputs simultaneously. The magnetic components are very important in power electronics and other areas of electrical engineering. Power losses in inductors and transformers are due to DC current flow, AC current flow, and associated skin and proximity effects in windings, as well as due to eddy currents and hysteresis in magnetic cores. In addition, there are dielectric losses in materials used to insulate the core and the windings. Failure mechanisms in magnetic components are mostly due to excessive temperature rise. Therefore, these devices should satisfy both magnetic requirements and thermal limitations.

In this chapter, fundamental physical phenomena and fundamental physics laws of electromagnetism, quantities, and units of the magnetic theory are reviewed. Magnetic relationships are given and an equation for the inductance is derived. The nature is governed by a set of laws. A subset of these laws are the physics EM laws. The origin of the magnetic field is discussed. It is shown that moving charges are sources of the magnetic field. Hysteresis and eddy-current losses are studied. There are two kinds of eddy-current effects: skin effect and proximity effect. Both of these effects cause nonuniform distribution of the current density in conductors and increase the conductor AC resistance at high frequencies. A classification of winding and core losses is given.

High-Frequency Magnetic Components, Second Edition. Marian K. Kazimierczuk.

Companion Website: www.wiley.com/go/kazimierczuk_High2e

1.2 Fields

A *field* is defined as a spatial distribution of a quantity everywhere in a region. There are two categories of fields: scalar fields and vector fields. A *scalar field* is a set of scalars assigned at individual points in space. A scalar quantity has a magnitude only. Examples of scalar fields are time, temperature, humidity, pressure, mass, sound intensity, altitude of a terrain, energy, power density, electrical charge density, and electrical potential. The scalar field may be described by a real or a complex function. The intensity of a scalar field may be represented graphically by different colors or undirected field lines. A higher density of the field lines indicates a stronger field in the area.

A *vector field* is a set of vectors assigned at every point in space. A vector quantity has both magnitude and direction. Examples of vector fields are velocity **v**, the Earth's gravitational force field **F**, electric current density field **J**, magnetic field intensity **H**, and magnetic flux density **B**. The vector field may be represented graphically by directed field lines. The density of field lines indicates the field intensity, and the direction of field lines indicates the direction of the vector at each point. In general, fields are functions of position and time, for example, $\rho_v(x, y, z, t)$. The rate of change of a scalar field with distance is a vector.

1.3 Magnetic Relationships

The magnetic field is characterized by magnetomotive force (MMF) $\mathcal{F}$, magnetic field intensity **H**, magnetic flux density **B**, magnetic flux ϕ, and magnetic flux linkage λ.

1.3.1 Magnetomotive Force

An inductor with N turns carrying an AC current i produces the *MMF*, which is also called the *magnetomotance*. The MMF is given by

$$\mathcal{F} = Ni \text{ (A} \cdot \text{turns)}. \tag{1.1}$$

Its descriptive unit is ampere-turns (A·t). However, the approved SI unit of the MMF is the ampere (A), where 1 A = 6.25×10^{18} electrons/s. The MMF is a *source* in magnetic circuits. The magnetic flux ϕ is forced to flow in a magnetic circuit by the MMF $\mathcal{F} = Ni$, driving a magnetic circuit. Every time another complete turn with the current i is added, the result of the integration increases by the current i.

The MMF between any two points P_1 and P_2 produced by a magnetic field **H** is determined by a line integral of the magnetic field intensity **H** present between these two points

$$\mathcal{F} = \int_{P_1}^{P_2} \mathbf{H} \cdot d\mathbf{l} = \int_{P_1}^{P_2} H \cos\theta dl, \tag{1.2}$$

where $d\mathbf{l}$ is the incremental vector at a point located on the path l and $\mathbf{H} \cdot d\mathbf{l} = (H \cos\theta)dl = H_l dl = H(dl \cos\theta)$. The MMF depends only on the endpoints, and it is independent of the path between points P_1 and P_2. Any path can be chosen. If the path is broken up into segments parallel and perpendicular to H, only parallel segments contribute to $\mathcal{F}$. The contributions from the perpendicular segments are zero.

For a uniform magnetic field and parallel to path l, the MMF is given by

$$\mathcal{F} = Hl. \tag{1.3}$$

Thus,

$$\mathcal{F} = Hl = Ni. \tag{1.4}$$

The MMF forces a magnetic flux ϕ to flow.

The MMF is analogous to the electromotive force (EMF) V. It is a potential difference between any two points P_1 and P_2. field **E** between any two points P_1 and P_2 is equal to the line integral of the electric field **E** between these two points along any path

$$V = V_{P2} - P_{P2} = \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} = \int_{P_1}^{P_2} E \cos\, \theta dl. \tag{1.5}$$

The result is independent of the integration path. For a uniform electric field E and parallel to path l, the EMF is

$$V = El. \tag{1.6}$$

The EMF forces a current $i = V/R$ to flow. It is the work per unit charge (J/C).

1.3.2 Magnetic Field Intensity

The *magnetic field intensity* (or *magnetic field strength*) is defined as the MMF $\mathcal{F}$ per unit length

$$H = \frac{\mathcal{F}}{l} = \frac{Ni}{l} = \left(\frac{N}{l}\right) i \text{ (A/m)}, \tag{1.7}$$

where l is the inductor length and N is the number of turns. Magnetic fields are produced by moving charges. Therefore, magnetic field intensity H is directly proportional to the amount of current i and the number of turns per unit length N/l. If a conductor conducts current i (which a moving charge), it produces a magnetic field H. Thus, the source of the magnetic field H is a conductor carrying a current i. The magnetic field intensity **H** is a vector field. It is described by a magnitude and a direction at any given point. The lines of magnetic field H always form closed loops. By Ampère's law, the magnetic field produced by a straight conductor carrying current i is given by

$$\mathbf{H}(r) = \frac{i}{2\pi r}\mathbf{a}_\phi. \tag{1.8}$$

The magnetic field intensity H is directly proportional to current i and inversely proportional to the radial distance from the conductor r. The Earth's magnetic field intensity is approximately 50 μT.

1.3.3 Magnetic Flux

The amount of the *magnetic flux* passing through an open surface S is determined by a surface integral of the magnetic flux density **B**

$$\phi = \int\int_S \mathbf{B} \cdot d\mathbf{S} = \int\int_S \mathbf{B} \cdot \mathbf{n} dS \text{ (Wb)}, \tag{1.9}$$

where **n** is the unit vector normal to the incremental surface area dS at a given position, $d\mathbf{S} = \mathbf{n}dS$ is the incremental surface vector normal to the local surface dS at a given position, and $d\phi = \mathbf{B} \cdot d\mathbf{S} = \mathbf{S} \cdot \mathbf{n}dS$. The magnetic flux is a scalar. The unit of the magnetic flux is Weber.

If the magnetic flux density B is uniform and forms an angle θ_B with the vector perpendicular to the surface S, the amount of the magnetic flux passing through the surface S is

$$\phi = \mathbf{B} \cdot \mathbf{S} = BS \cos\, \theta_B \text{ (Wb)}. \tag{1.10}$$

If the magnetic flux density B is uniform and perpendicular to the surface S, the angle between vectors **B** and $d\mathbf{S}$ is $\theta_B = 0^\circ$ and the amount of the magnetic flux passing through the surface S is

$$\phi = \mathbf{B} \cdot \mathbf{S} = BS \cos\, 0^\circ = BS \text{ (Wb)}. \tag{1.11}$$

If the magnetic flux density B is parallel to the surface S, the angle between vectors **B** and $d\mathbf{S}$ is $\theta_B = 90^\circ$ and the amount of the magnetic flux passing through the surface S is

$$\phi = SB \cos\, 90^\circ = 0. \tag{1.12}$$

For an inductor, the amount of the magnetic flux ϕ may be increased by increasing the surface area of a single turn A, the number of turns in the layer N_{tl}, and the number of layers N_l. Hence, $S = N_{tl}N_lA = NA$, where $N = N_{tl}N_l$ is the total number of turns.

The direction of a magnetic flux density B is determined by the right-hand rule (RHR). This rule states that if the fingers of the right hand encircle a coil in the direction of the current i, the thumb indicates the direction of the magnetic flux density B produced by the current i, or if the fingers of the right hand encircle a conductor in the direction of the magnetic flux density B, the thumb indicated the direction of the current i. The magnetic flux lines are always continuous and closed loops.

1.3.4 Magnetic Flux Density

The *magnetic flux density*, or *induction*, is the magnetic flux per unit area given by

$$B = \frac{\phi}{S} \text{ (T)}. \tag{1.13}$$

The unit of magnetic flux density B is Tesla. The magnetic flux density is a vector field and it can be represented by magnetic lines. The density of the magnetic lines indicates the magnetic flux density B, and the direction of the magnetic lines indicates the direction of the magnetic flux density at a given point. Every magnet has two poles: south and north. Magnetic monopoles do not exist. Magnetic lines always flow from south to north pole inside the magnet, and from north to south pole outside the magnet.

The relationship between the magnetic flux density B and the magnetic field intensity H is given by

$$B = \mu H = \mu_r\mu_0 H = \frac{\mu N i}{l_c} = \frac{\mu\mathcal{F}}{l_c} < B_s \text{ (T)}, \tag{1.14}$$

where the permeability of free space is

$$\mu_0 = 4\pi \times 10^{-7} \text{ (H/m)}, \tag{1.15}$$

$\mu = \mu_r\mu_0$ is the permeability, $\mu_r = \mu/\mu_0$ is the relative permeability (i.e., relative to that of free space), and l_c is the length of the core. Physical constants are given in Appendix A. For free space, insulators, and nonmagnetic materials, $\mu_r = 1$. For diamagnetics such as copper, lead, silver, and gold, $\mu_r \approx 1 - 10^{-5} \approx 1$. However, for ferromagnetic materials such as iron, cobalt, nickel, and their alloys, $\mu_r > 1$ and it can be as high as 100 000. The permeability is the measure of the ability of a material to conduct magnetic flux ϕ. It describes how easily a material can be magnetized. For a large value of μ_r, a small current i produces a large magnetic flux density B. The magnetic flux ϕ takes the path of the highest permeability.

The magnetic flux density field is a vector field. For example, the vector of the magnetic flux density produced by a straight conductor carrying current i is given by

$$\mathbf{B(r)} = \mu\mathbf{H}(r) = \frac{\mu i}{2\pi r}\mathbf{a}_\phi. \tag{1.16}$$

For ferromagnetic materials, the relationship between B and H is nonlinear because the relative permeability μ_r depends on the magnetic field intensity H. Figure 1.1 shows simplified plots of the magnetic flux density B as a function of the magnetic field intensity H for air-core inductors (straight line) and for ferromagnetic core inductors. The straight line describes the air-core inductor and has a slope μ_0 for all values of H. These inductors are linear. The piecewise linear approximation corresponds to the ferromagnetic core inductors, where B_s is the saturation magnetic flux density and $H_s = B_s/(\mu_r\mu_0)$ is the magnetic field intensity corresponding to B_s. At low values of the magnetic flux density $B < B_s$, the relative permeability μ_r is high and the slope of the B–H curve $\mu_r\mu_0$ is also high. For $B > B_s$, the core saturates and $\mu_r = 1$, reducing the slope of the B–H curve to μ_0.

The total peak magnetic flux density B_{pk}, which in general consists of both the DC component B_{DC} and the amplitude of AC component B_m, should be lower than the saturation flux density B_s of a magnetic core at the highest operating temperature T_{max}

$$B_{pk} = B_{DC(max)} + B_{m(max)} \leq B_s. \tag{1.17}$$

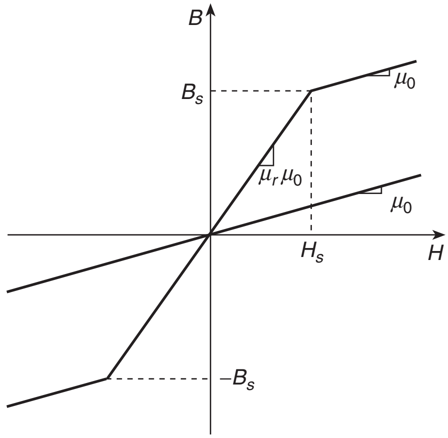

Figure 1.1 Simplified plots of magnetic flux density B as a function of magnetic field intensity H for air-core inductors (straight line) and ferromagnetic core inductors (piecewise linear approximation)

The DC component of the magnetic flux density B_{DC} is caused by the DC component of the inductor current I_L

$$B_{DC} = \frac{\mu_r \mu_0 N I_L}{l_c}. \tag{1.18}$$

The amplitude of the AC component of the magnetic flux density B_m corresponds to the amplitude of the AC component of the inductor current I_m

$$B_m = \frac{\mu_r \mu_0 N I_m}{l_c}. \tag{1.19}$$

Hence, the peak value of the magnetic flux density can be written as

$$B_{pk} = \frac{\mu_r \mu_0 N I_L}{l_c} + \frac{\mu_r \mu_0 N I_m}{l_c} = \frac{\mu_r \mu_0 N (I_L + I_m)}{l_c} = \frac{\mu_r \mu_0 N I_{Lpk}}{l_c} \le B_s \tag{1.20}$$

where $I_{Lpk} = I_L + I_m$. The saturation flux density B_s decreases with temperature. For ferrites, B_s may decrease by a factor of 2 as the temperature increases from $20\,^\circ$C to $90\,^\circ$C. The amplitude of the magnetic flux density B_m is limited either by core saturation or by core losses.

1.3.5 Magnetic Flux Linkage

The *magnetic flux linkage* is the sum of the flux enclosed by each turn of the wire wound around the core

$$\lambda = N \int\int_S \mathbf{B} \cdot d\lambda S = \int v dt. \tag{1.21}$$

For the uniform magnetic flux density, the magnetic flux linkage is the magnetic flux linking N turns and is described by

$$\lambda = N\phi = NA_c B = A_{eff} B = NA_c \mu H = \frac{\mu A_c N^2 i}{l_c} = \frac{N^2}{\mathcal{R}} i = Li \text{ (V} \cdot \text{s)} \tag{1.22}$$

where $\mathcal{R}$ is the core reluctance and $A_{eff} = NA_c$ is the effective area through which the magnetic flux ϕ passes. Equation (1.22) is analogous to Ohm's law $v = Ri$ and the equation for the capacitor charge $Q = Cv$. The unit of the flux linkage is Wb·turn. For sinusoidal waveforms, the relationship among the amplitudes is

$$\lambda_m = N\phi_m = NA_c B_m = NA_c \mu H_m = \frac{\mu_r \mu_0 A_c N^2 I_m}{l_c}. \tag{1.23}$$

The change in the magnetic linkage can be expressed as

$$\Delta\lambda = \int_{t_1}^{t_2} v dt = \lambda(t_2) - \lambda(t_1). \tag{1.24}$$

1.4 Magnetic Circuits

1.4.1 Reluctance

The *reluctance* $\mathcal{R}$ is the resistance of the core to the flow of the magnetic flux ϕ. It opposes the magnetic flux flow, in the same way as the resistance opposes the electric current flow. An element of a magnetic circuit can be called a *reluctor*. The concept of the reluctance is illustrated in Fig. 1.2. The reluctance of a linear, isotropic, and homogeneous magnetic material is given by

$$\mathcal{R} = \frac{1}{\mathcal{P}} = \frac{l_c}{\mu A_c} = \frac{l_c}{\mu_0 \mu_r A_c} \text{ (A} \cdot \text{turns/Wb)} \quad \text{or} \quad \text{(turns/H)}, \tag{1.25}$$

where A_c is the cross-sectional area of the core (i.e., the area through which the magnetic flux flows) and l_c is the mean magnetic path length (MPL), which is the mean length of the closed path that the magnetic flux flows around a magnetic circuit. The reluctance is directly proportional to the length of the magnetic path l_c and is inversely proportional to the cross-sectional area A_c through which the magnetic flux ϕ flows. The *permeance* of a basic magnetic circuit element is

$$\mathcal{P} = \frac{1}{\mathcal{R}} = \frac{\mu A_c}{l_c} = \frac{\mu_0 \mu_r A_c}{l_c} \quad \text{(Wb/A} \cdot \text{turns)} \quad \text{or} \quad \text{(H/turns)}\,. \tag{1.26}$$

When the number of turns $N = 1$, $L = \mathcal{P}$. The reluctance is the magnetic resistance because it opposes the establishment and the flow of a magnetic flux ϕ in a medium. A poor conductor of the magnetic flux has a high reluctance and a low permeance. Magnetic Ohm's law is expressed as

$$\phi = \frac{\mathcal{F}}{\mathcal{R}} = \mathcal{P}\mathcal{F} = \frac{Ni}{\mathcal{R}} = \frac{\mu A_c N i}{l_c} = \frac{\mu_{rc}\mu_0 A_c N i}{l_c} \text{ (Wb)}. \tag{1.27}$$

Magnetic flux always takes the path with the highest permeability μ.

In general, the magnetic circuit is the space in which the magnetic flux flows around the coil(s). Figure 1.3 shows an example of a magnetic circuit. The reluctance in magnetic circuits is analogous to the resistance R in electric circuits. Likewise, the permeance in magnetic circuits is analogous to the conductance in electric circuits. Therefore, magnetic circuits described by the equation $\phi = \mathcal{F}/\mathcal{R}$ can be solved in a similar manner as electric circuits described by Ohm's law $I = V/R = GV = (\sigma A/l)V$, where ϕ, $\mathcal{F}$, $\mathcal{R}$, $\mathcal{P}$, B, λ, and μ, correspond to I, V, R, G, J, Q, and σ, respectively. For example, the reluctances can be connected in series or in parallel. In addition, the reluctance $\mathcal{R} = l_c/\mu A_c$ is analogous to the electric resistance $R = l/\sigma A$ and the magnetic flux density $B = \phi/A_c$ is analogous to the current density $J = I/A$. Table 1.1 lists analogous magnetic and electric quantities.

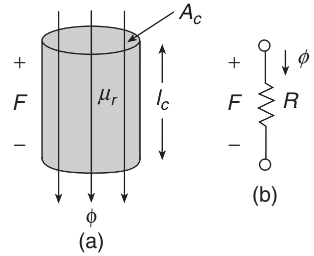

Figure 1.2 Reluctance. (a) Basic magnetic circuit element conducting magnetic flux ϕ. (b) Equivalent magnetic circuit

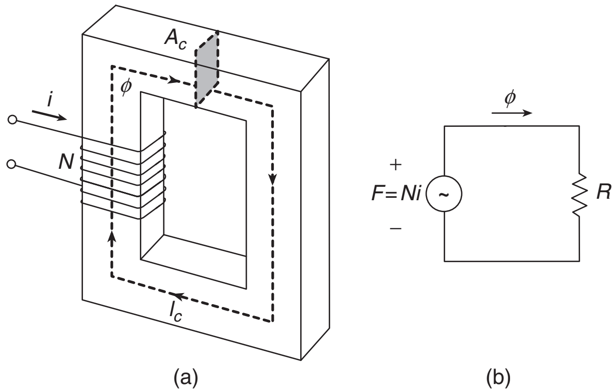

Figure 1.3 Magnetic circuit. (a) An inductor composed of a core and a winding. (b) Equivalent magnetic circuit

Table 1.1 Analogy between magnetic and electric quantities

Magnetic quantity	Electric quantity
$\mathcal{F} = Ni$	V
$\mathcal{F} = Hl$	$V = El$
ϕ	I
H	E
B	J
$\mathcal{R}$	R
$\mathcal{P}$	G
λ	Q
μ	ϵ
L	C
$\phi = \frac{\mathcal{F}}{\mathcal{R}}$	$I = \frac{V}{R}$
$B = \frac{\phi}{A}$	$J = \frac{I}{A}$
$H = \frac{\mathcal{F}}{l} = \frac{Ni}{l}$	$E = \frac{V}{l}$
$\mathcal{R} = \frac{l}{\mu A}$	$R = \frac{l}{\sigma A}$
$\mathbf{B} = \mu\mathbf{H}$	$\mathbf{D} = \epsilon\mathbf{E}$
$\lambda = Li$	$Q = Cv$
$i = \frac{d\lambda}{dt}$	$v = \frac{dQ}{dt}$
$v = L\frac{di}{dt}$	$i = C\frac{dv}{dt}$
$d\mathbf{B} = \frac{\mu(Id\mathbf{l}\times\mathbf{a}_R)}{4\pi R^2}$	$\mathbf{F} = \frac{Q_1 Q_2 \mathbf{a}_R}{2\pi\epsilon R^2}$
$w_m = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$	$w_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}$
$w_m = \frac{1}{2}\mu H^2$	$w_e = \frac{1}{2}\epsilon E^2$
$W_m = \frac{1}{2}i\lambda$	$W_e = \frac{1}{2}vQ$
$W_m = \frac{1}{2}Li^2$	$W_e = \frac{1}{2}Cv^2$

1.4.2 Magnetic KVL

Physical structures, which are made of magnetic devices, such as inductors and transformers, can be analyzed just like electric circuits. The magnetic law, analogous to Kirchhoff's voltage law (KVL), states that the sum of the MMFs $\sum_{k=1}^{n} \mathcal{F}_k$ and the magnetic potential differences $\sum_{k=1}^{m} \mathcal{R}_k \phi_k$ around the closed magnetic loop is zero

$$\sum_{k=1}^{n} \mathcal{F}_k - \sum_{k=1}^{m} \mathcal{R}_k \phi_k = 0. \tag{1.28}$$

For instance, an inductor with a simple core having an air gap as illustrated in Fig. 1.4 is given by

$$Ni = \mathcal{F} = \mathcal{F}_c + \mathcal{F}_g = \phi(\mathcal{R}_c + \mathcal{R}_g), \tag{1.29}$$

where the reluctance of the core is

$$\mathcal{R}_c = \frac{l_c}{\mu_{rc}\mu_0 A_c} \tag{1.30}$$

the reluctance of the air gap is

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_c} \tag{1.31}$$

and it is assumed that $\phi_c = \phi_g = \phi$. This means that the fringing flux in neglected. If $\mu_r \gg 1$, the magnetic flux is confined to the magnetic material, reducing the leakage flux. The ratio of the air-gap reluctance to the core reluctance is

$$\frac{\mathcal{R}_g}{\mathcal{R}_c} = \mu_{rc}\frac{l_g}{l_c}. \tag{1.32}$$

The reluctance of the air gap $\mathcal{R}_g$ is much higher than the reluctance of the core $\mathcal{R}_c$ if $\mu_{rc} \gg l_c/l_g$.

The magnetic potential difference between points a and b is

$$\mathcal{F}_{ab} = \int_a^b \mathbf{H} \cdot d\mathbf{l} = \mathcal{R}_{ab}\phi, \tag{1.33}$$

where $\mathcal{R}_{ab}$ is the reluctance between points a and b.

1.4.3 Magnetic Flux Continuity

The continuity of the magnetic flux law states that the net magnetic flux through any closed surface is always zero

$$\phi = \oint\!\!\oint_S BdS = 0 \tag{1.34}$$

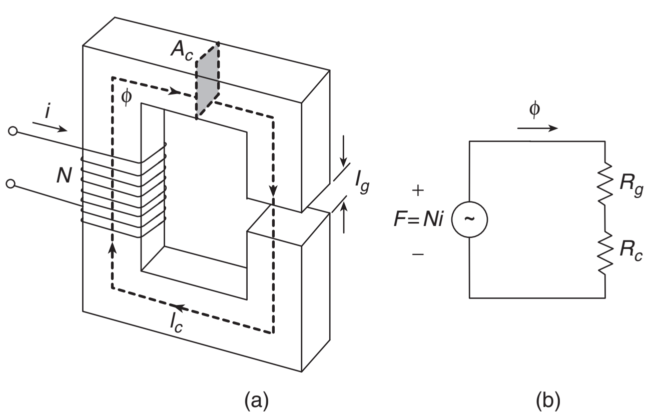

Figure 1.4 Magnetic circuit illustrating the magnetic KVL. (a) An inductor composed of a core with an air gap and a winding. (b) Equivalent magnetic circuit

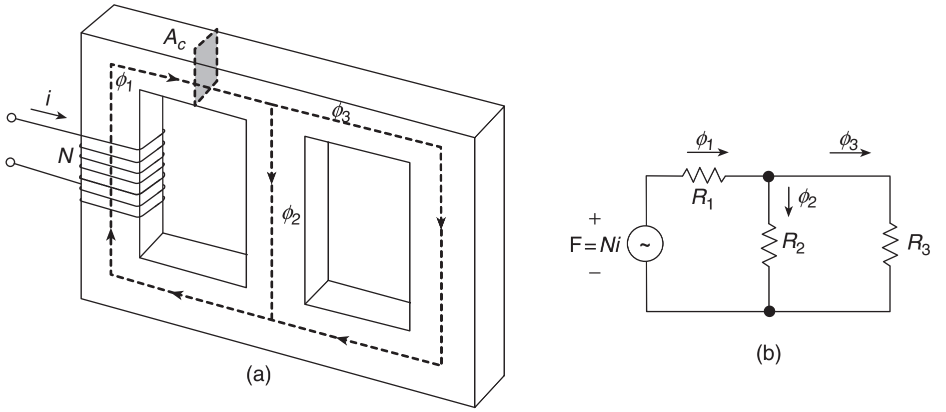

Figure 1.5 Magnetic circuit illustrating the continuity of the magnetic flux for EE core. (a) An inductor composed of a core and a winding. (b) Equivalent magnetic circuit

or the net magnetic flux entering and exiting the node is zero

$$\sum_{k=1}^{n} \phi_k = \sum_{k=1}^{n} S_k B_k = 0. \tag{1.35}$$

This law is analogous to Kirchhoff's current law (KCL) introduced by Gauss and can be called Kirchhoff's flux law (KFL). Figure 1.5 illustrates the continuity of the magnetic flux law. For example, when three core legs meet at a node,

$$\phi_1 = \phi_2 + \phi_3, \tag{1.36}$$

which can be expressed by

$$\frac{\mathcal{F}_1}{\mathcal{R}_1} = \frac{\mathcal{F}_2}{\mathcal{R}_2} + \frac{\mathcal{F}_3}{\mathcal{R}_3}. \tag{1.37}$$

If all the three legs of the core have windings, then we have

$$\frac{N_1 i_1}{\mathcal{R}_1} = \frac{N_2 i_2}{\mathcal{R}_2} + \frac{N_3 i_3}{\mathcal{R}_3}. \tag{1.38}$$

Usually, most of the magnetic flux is confined inside an inductor, for example, for an inductor with a toroidal core. The magnetic flux outside an inductor is called the *leakage flux*.

1.5 Magnetic Laws

1.5.1 Ampère's Law

Ampère[1] discovered the relationship between current and the magnetic field intensity. *Ampère's law* relates the magnetic field intensity H inside a closed loop to the current passing through the loop. A magnetic field can be produced by a current and a current can be produced by a magnetic field. Ampère's law is illustrated in Fig. 1.6. A magnetic field is present around a current-carrying conductor or conductors. The integral form of Ampère's circuital law, or simply Ampère's law, (1826) describes the relationship between the (conduction, convection, and/or displacement) current and the magnetic field produced by this current. It states that the closed line integral of the magnetic field intensity **H**

[1] André-Marie Ampère (1775–1836) was a French physicist and mathematician, who is the father of electrodynamics.

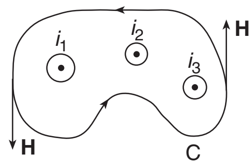

Figure 1.6 Illustration of Ampère's law

around a closed path (Amperian contour) C (2D or 3D) is equal to the total current i_{enc} enclosed by that path and passing through the interior of the closed path bounding the open surface S

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = \int\int_S \mathbf{J}\cdot d\mathbf{S} = \sum_{n=1}^{N} i_n = i_1 + i_2 + \ldots + i_N = i_{enc}, \quad (1.39)$$

where $d\mathbf{l}$ is the vector length element pointing in the direction of the Amperian path C and $\mathbf{J}$ is the conduction (or drift) and convection current density. The current i_{enc} enclosed by the path C is given by the surface integral of the normal component $\mathbf{J}$ over the open surface S. The surface integral of the current density $\mathbf{J}$ is equal to the current I flowing through the surface S. In other words, the integrated magnetic field intensity around a closed loop C is equal to the electric current passing through the loop. The surface integral of $\mathbf{J}$ is the current flowing through the open surface S. The conduction current is caused by the movement of electrons originating from the outermost shells of atoms. When conduction current flows, the atoms of medium normally do not move. The convection current is caused by the movement of electrically charged medium.

For example, consider a long, straight, round conductor that carries current I. The line integral about a circular path of radius r centered on the axis of the round wire is equal to the product of the circumference and the magnetic field intensity H_ϕ

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = 2\pi r H_\phi = I, \quad (1.40)$$

yielding the magnetic field intensity

$$H_\phi = \frac{I}{2\pi r}. \quad (1.41)$$

Thus, the magnetic field decreases in the radial direction away from the conductor.

For an inductor with N turns, Ampère's law is

$$\oint_C \mathbf{H}\cdot d\mathbf{l} = Ni. \quad (1.42)$$

Ampère's law in the discrete form can be expressed as

$$\sum_{k=1}^{n} H_k l_k = \sum_{k=1}^{m} N_k i_k. \quad (1.43)$$

For example, Ampère's law for an inductor with an air gap is given by

$$H_c l_c + H_g l_g = Ni. \quad (1.44)$$

If the current density J is uniform and perpendicular to the surface S,

$$HC = SJ. \quad (1.45)$$

The current density J in winding conductors of magnetic components used in power electronics is usually in the range 0.1–10 A/mm^2. The displacement current is neglected in (1.39). The generalized Ampère's law by adding the displacement current constitutes one of Maxwell's equations. This is known as Maxwell's correction to Ampère's law.

Ampère's law is useful when there is a high degree of symmetry in the arrangement of conductors and it can be easily applied in problems with symmetrical current distribution. For example, the magnetic field produced by an infinitely long wire conducting a current I outside the wire is

$$\mathbf{B} = \frac{I}{2\pi r}\mathbf{a}_\phi \text{ (A/m)}. \tag{1.46}$$

Ampère's law is a special case of Biot–Savart's law.

Example 1.1

An infinitely long round solid straight wire of radius r_o carries sinusoidal current $i = I_m \cos \omega t$ in steady state at low frequencies (with no skin effect). Determine the waveforms of the magnetic field intensity $H(r,t)$, magnetic flux density $B(r,t)$, and magnetic flux $\phi(r,t)$ inside and outside the wire.

Solution: At low frequencies, the skin effect can be neglected and the current is uniformly distributed over the cross section of the wire, as shown in Fig. 1.7. To determine the magnetic field intensity $H(r,t)$ everywhere, two Amperian contours C_1 and C_2 are required, one inside the conductor for $r \le r_o$ and the other outside the conductor for $r > r_o$.

The Magnetic Field Intensity Inside the Wire. The current in the conductor induces a concentric magnetic field intensity both inside and outside the conductor. The current density inside the conductor is uniform. The vector of the current density amplitude inside the conductor is assumed to be parallel to the conductor axis and is given by

$$\mathbf{J}_m = J_{mz}\mathbf{a}_z. \tag{1.47}$$

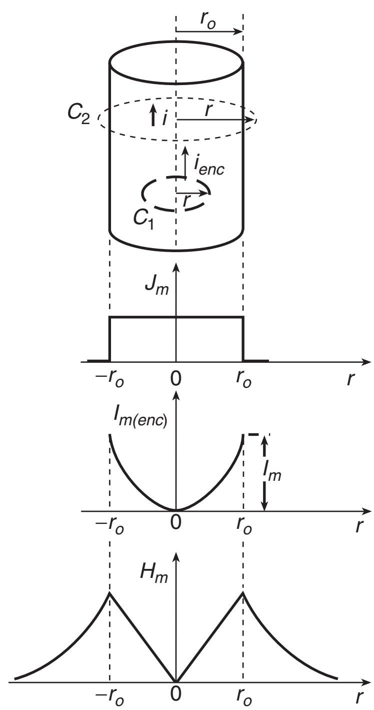

Figure 1.7 Cross section of an infinitely long round straight wire carrying a sinusoidal $i = I_m \cos \omega t$ and amplitudes of current density J_m, enclosed current $I_{m(enc)}$, and magnetic field intensity H_m as a function of the radial distance r from the wire center at low frequencies, that is, when the skin effect can be neglected ($\delta > r_o$)

Consider a radial contour C_1 inside the conductor. The current flowing through the area enclosed by the cylindrical shell of radius r at low frequencies is given by

$$i_{enc} = I_{m(enc)} \cos \omega t, \tag{1.48}$$

where $I_{m(enc)}$ is the amplitude of the current enclosed by the shell of radius r. Hence, the amplitude of the current density at a radius r is

$$J_m(r) = \frac{I_{m(enc)}}{\pi r^2} \quad \text{for} \quad 0 \le r \le r_o \tag{1.49}$$

and the amplitude of the current density at the wire surface $r = r_o$ is

$$J_m(r_o) = \frac{I_m}{\pi r_o^2}. \tag{1.50}$$

The current density is uniform at low frequencies (where the skin effect can be neglected), that is, $J_m(r) = J_m(r_o)$, yielding the amplitude of the enclosed current

$$I_{m(enc)} = I_m \left(\frac{\pi r^2}{\pi r_o^2} \right) = I_m \left(\frac{A_r}{A_{ro}} \right) = I_m \left(\frac{r}{r_o} \right)^2 \quad \text{for} \quad 0 \le r \le r_o, \tag{1.51}$$

where $A_r = \pi r^2$ and $A_{ro} = \pi r_o^2$. Figure 1.7 shows a plot of $I_{m(enc)}$ as a function of the radial distance from the conductor center r. The vector of the magnetic flux density is

$$\mathbf{H} = \mathbf{H}_\phi = H(r)\mathbf{a}_\phi. \tag{1.52}$$

From Ampère's law,

$$I_{m(enc)} = \oint_{C_1} \mathbf{H} \cdot d\mathbf{l} = H_m(r) \oint_{C_1} dl = 2\pi r H_m(r) \quad \text{for} \quad 0 \le r \le r_o, \tag{1.53}$$

where $C_1 = 2\pi r$ for $r \le r_o$. Equating the right-hand sides of (1.51) and (1.53), the amplitude of the magnetic field intensity inside the wire at low frequencies is obtained

$$H_m(r) = I_m \left(\frac{r}{r_o} \right)^2 \frac{1}{2\pi r} = I_m \frac{r}{2\pi r_o^2} \quad \text{for} \quad 0 \le r \le r_o. \tag{1.54}$$

Figure 1.7 shows a plot of the amplitude of the magnetic field intensity H_m as a function of r. The amplitude of the magnetic field intensity H_m is zero at the wire center because the enclosed current is zero. The waveform of the magnetic field inside the wire at low frequencies

$$H(r, t) = I_m \frac{r}{2\pi r_o^2} \cos \omega t \quad \text{for} \quad r \le r_o. \tag{1.55}$$

Thus, the amplitude of the magnetic field intensity H_m inside the wire at radius r is determined solely by the amplitude of the current inside the radius r. The maximum amplitude of the magnetic field intensity occurs on the conductor surface

$$H_{m(max)} = H_m(r_o) = \frac{I_m}{2\pi r_o}. \tag{1.56}$$

The amplitude of the magnetic flux density inside the wire at low frequencies is

$$B_m(r) = \mu_0 H_m(r) = \mu_0 I_m \left(\frac{r}{r_o} \right)^2 \frac{1}{2\pi r} = \mu_0 I_m \frac{r}{2\pi r_o^2} \quad \text{for} \quad 0 \le r \le r_o. \tag{1.57}$$

The amplitude of the magnetic flux inside the wire at low frequencies is

$$\phi_m(r) = AB_m(r) = \mu_0 A_r H_m(r) = \mu_0 I_m \frac{r(\pi r^2)}{2\pi r_o^2} = \mu_0 I_m \frac{r^3}{2r_o^2} \quad \text{for} \quad 0 \le r \le r_o. \tag{1.58}$$

The waveform of the magnetic flux is

$$\phi(x, t) = \phi \cos \omega t = \mu_0 I_m \frac{r^3}{2r_o^2} \cos \omega t \quad \text{for} \quad 0 \le r \le r_o. \tag{1.59}$$

The Magnetic Field Intensity Outside the Wire. Consider a radial contour C_2 outside the conductor. The entire current $i = I_m \cos \omega t$ is enclosed by a path of radius $r \geq r_o$. From Ampère's law, the amplitude of the entire current i is

$$I_m = \oint_{C_2} \mathbf{H} \cdot d\mathbf{l} = H_m(r) \oint_{C_2} dl = 2\pi r H_m(r) \quad \text{for} \quad r \geq r_o, \tag{1.60}$$

where $C_2 = 2\pi r$ with $r \geq r_o$. The amplitude of the near-magnetic field intensity outside the conductor at any frequency is given by the expression

$$H_m(r) = \frac{I_m}{2\pi r} \quad \text{for} \quad r \geq r_o \tag{1.61}$$

and the waveform of this field is

$$H(r,t) = \frac{I_m}{2\pi r} \cos \omega t \quad \text{for} \quad r \geq r_o. \tag{1.62}$$

The amplitude of the magnetic field intensity increases linearly with r inside the wire from 0 to $H_m(r_o) = I_m/(2\pi r_o)$ at low frequencies. The amplitude of the magnetic field intensity is inversely proportional to r outside the wire at any frequency.

The waveform of the magnetic flux density is

$$B(r,t) = \mu_0 H(r,t) = \frac{\mu_0 I_m}{2\pi r} \cos \omega t \quad \text{for} \quad r \geq r_o. \tag{1.63}$$

The waveform of the magnetic flux enclosed by a cylinder of radius $r > r_o$ is

$$\phi(x,t) = A_w B(r,t) = A_w \mu_0 H(r,t) = \frac{r_o^2 \mu_0 I_m}{2r} \cos \omega t \quad \text{for} \quad r \geq r_o. \tag{1.64}$$

Example 1.2

Toroidal Inductor. Consider an inductor with a toroidal core of inner radius a and outer radius b. Find the magnetic field inside the core and in the region exterior to the torus core.

Solution: Consider the circle C of radius $a \leq r \leq b$. The magnitude of the magnetic field is constant on this circle and is tangent to it. Therefore, $\mathbf{B} \cdot l\mathbf{l} = Bdl$. From the Ampère's law, the magnetic field density in a toroidal core (torus) is

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = B \oint_C dl = B(2\pi r) = \mu_r \mu_0 NI \quad \text{for} \quad a \leq r \leq b \tag{1.65}$$

where r is the distance from the torus center to a point inside the torus. Hence,

$$B = \frac{\mu_r \mu_0 NI}{2\pi r} \quad \text{for} \quad a \leq r \leq b. \tag{1.66}$$

For an ideal toroid in which the turns are closely spaced, the external magnetic field is zero. For an Amperian contour with radius $r < a$, there is no current flowing through the contour surface, and therefore $\mathbf{H} = 0$ for $r < a$. For an Amperian contour C with radius $r > b$, the net current flowing through its surface is zero because an equal number of current paths cross the contour surface in both directions, and therefore $\mathbf{H} = 0$ for $r > b$.

1.5.2 Faraday's Law

A time-varying current produces a magnetic field, and a time-varying magnetic field can produce an electric current. In 1820, a Danish scientist Oersted[2] showed that a current-carrying conductor

[2]Hans Christian Oersted (1777–1851) was a Danish physicist and chemist, who discovered that an electric current produces a magnetic field. This discovery established the connection between electricity and magnetism, leading to the origination of science of electromagnetism.

produces a magnetic field, which can affect a compass magnetic needle. He connected electricity and magnetism. Ampère measured that this magnetic field intensity is linearly related to the current, which produces it. In 1831, the English experimentalist Michael Faraday[3] discovered that a current can be produced by an alternating magnetic field and that a time-varying magnetic field can induce a voltage, or an EMF, in an adjacent circuit. This voltage is proportional to the rate of change of magnetic flux linkage λ, or magnetic flux ϕ, or current i, producing the magnetic field.

Faraday's law (1831), also known as *Faraday's law of induction*, states that a time-varying magnetic flux $\phi(t)$ passing through a closed stationary loop, such as an inductor turn, generates a voltage $v(t)$ in the loop and for a linear inductor is expressed by

$$v(t) = \frac{d\lambda}{dt} = \frac{d(N\phi)}{dt} = N\frac{d\phi}{dt} = N\frac{d(AB)}{dt} = NA\frac{dB}{dt} = NA\mu\frac{dH}{dt} = \frac{\mu AN^2}{l}\frac{di}{dt}$$
$$= N\frac{d}{dt}\left(\frac{\mathcal{F}}{\mathcal{R}}\right) = N\frac{d}{dt}\left(\frac{Ni}{\mathcal{R}}\right) = \frac{N^2}{\mathcal{R}}\frac{di}{dt} = \mathcal{P}N^2\frac{di}{dt} = L\frac{di}{dt}. \tag{1.67}$$

This voltage, in turn, may produce a current $i(t)$. The voltage $v(t)$ is proportional to the rate of change of the magnetic linkage $d\lambda/dt$, or to the rate of change of the magnetic flux density dB/dt and the effective area NA through which the flux is passing. The inductance L relates the induced voltage $v(t)$ to the current $i(t)$. The voltage $v(t)$ across the terminals of an inductor L is proportional to the time rate of change of the current $i(t)$ in the inductor and the inductance L. If the inductor current is constant, the voltage across an ideal inductor is zero. The inductor behaves as a short circuit for DC current. The inductor current cannot change instantaneously. Figure 1.8 shows an equivalent circuit of an ideal inductor. The inductor is replaced by a dependent voltage source controlled by di/dt.

The voltage between the terminals of a single turn of an inductor is

$$v_T(t) = \frac{d\phi(t)}{dt}. \tag{1.68}$$

Hence, the total voltage across the inductor consisting of N identical turns is

$$v_L(t) = Nv_T(t) = N\frac{d\phi(t)}{dt} = \frac{d\lambda(t)}{dt}. \tag{1.69}$$

Since $v = Ldi/dt$,

$$di = \frac{1}{L}vdt \tag{1.70}$$

yielding the current in an inductor

$$i(t) = \int_0^t idt + i(0) = \frac{1}{L}\int_0^t vdt + i(0) = \frac{1}{\omega L}\int_0^{\omega t} vd(\omega t) + i(0). \tag{1.71}$$

For sinusoidal waveforms, the derivative d/dt can be replaced by $j\omega$ and differential equations may be replaced by algebraic equations. A phasor is a complex representation of the magnitude, phase, and space of a sinusoidal waveform. The phasor is not dependent on time. A graphical representation

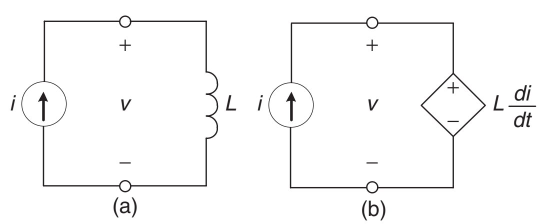

Figure 1.8 Equivalent circuit of an ideal inductor. (a) Inductor. (b) Equivalent circuit of an inductor in the form of dependent voltage source controlled by the rate of change of the inductor current di/dt

[3] Michael Faraday (1791–1867) was an English physicist and chemist, who discovered electromagnetic induction and invented the method of generating electricity.

of a phasor is known as a phasor diagram. Faraday's law in phasor form can be expressed as

$$\mathbf{V}_{Lm} = j\omega\boldsymbol{\lambda}_{\mathbf{m}} = j\omega L\mathbf{I}_{Lm} = \omega L I_{Lm} e^{j90^\circ}. \tag{1.72}$$

The sinusoidal inductor current legs the sinusoidal inductor voltage by 90°.

The impedance of a lossless inductive component in terms of phasors of sinusoidal inductor current $\mathbf{I}_{Lm}$ and voltage $\mathbf{V}_{Lm} = j\omega\boldsymbol{\lambda}_{\mathbf{m}}$ is

$$\mathbf{Z}_L = \frac{\mathbf{V}_m}{\mathbf{I}_{Lm}} = \frac{j\omega\lambda_m}{I_m} = j\omega L, \tag{1.73}$$

where $L = \lambda_m / I_m$. The impedance of lossy inductive components in terms of phasors is

$$\mathbf{Z}_L = \frac{\mathbf{V}_m}{\mathbf{I}_{Lm}} = R + j\omega L. \tag{1.74}$$

For nonlinear, time-varying inductors, the relationships are

$$\lambda(t) = L(i)i(t) \tag{1.75}$$

and

$$\begin{aligned} v(t) &= \frac{d\lambda(t)}{dt} = L(i)\frac{di(t)}{dt} + i(t)\frac{dL(i)}{dt} = L(i)\frac{di(t)}{dt} + i(t)\frac{dL(i)}{di}\frac{di(t)}{dt} \\ &= \left[L(i) + i(t)\frac{dL(i)}{di}\right]\frac{di(t)}{dt} = L_{eq}\frac{di(t)}{dt}, \end{aligned} \tag{1.76}$$

where

$$L_{eq} = L(i) + i(t)\frac{dL(i)}{di}. \tag{1.77}$$

In summary, a time-varying electric current $i(t)$ produces magnetic fields $H(t)$, $\phi(t)$, and $\lambda(t)$ by Ampère's law. In turn, the magnetic field produces a voltage $v(t)$ by Faraday's law. This process can be reversed. A voltage $v(t)$ produces a magnetic fields $H(t)$, $\phi(t)$, and $\lambda(t)$, which produced electric current $i(t)$.

1.5.3 Lenz's Law

Lenz[4] discovered the relationship between the direction of the induced current and the change in the magnetic flux. *Lenz's law* (1834) states that the EMF $v(t) = -Nd\phi(t)/dt$ induced by an applied time-varying magnetic flux $\phi_a(t)$ has such a direction that induces current $i_E(t)$ in the closed loop, which in turn induces a magnetic flux $\phi_i(t)$ that tends to oppose the change in the applied flux $\phi_a(t)$, as illustrated in Fig. 1.9. If the applied magnetic flux $\phi_a(t)$ increases, the induced current $i_E(t)$

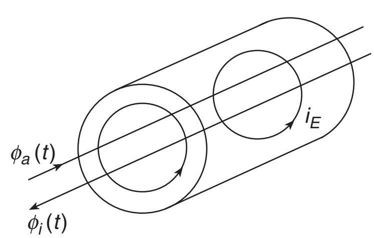

Figure 1.9 Illustration of Lenz's law generating eddy currents. The applied time-varying magnetic flux $\phi_a(t)$ induces eddy current $i_E(t)$, which in turn generates induced flux $\phi_i(t)$ that opposes changes in the applied flux $\phi_a(t)$

[4]Heinrich Friedrich Emil Lenz (1804–1865) was a Russian physicist of German ethnicity born in Estonia, who made a contribution to electromagnetism in the form of his law.

produces an opposing flux $\phi_{(t)}$. If the applied magnetic flux $\phi_a(t)$ decreases, the induced current $i_E(t)$ produces an aiding flux $\phi_i(t)$. The induced magnetic flux ϕ_i always opposes the inducing (applied) magnetic flux ϕ_a. If $\phi_a(t)$ increases, the induced current produces an opposing flux $\phi_i(t)$. If $\phi_a(t)$ decreases, the induced current produces an aiding magnetic flux $\phi_i(t)$. The direction of the induced current $i_E(t)$ with respect of the induced magnetic field $\phi_i(t)$ is determined by the RHR.

If a time-varying magnetic field is applied to a conducting loop (e.g., an inductor turn), a current is induced in such a direction as to oppose the change in the magnetic flux enclosed by the loop. The induced currents flowing in closed loops are called *eddy currents*. Eddy currents occur when a conductor is subjected to time-varying magnetic field(s). In accordance with Lenz's law, the eddy currents produce their own magnetic field(s) to oppose the original field.

The effects of eddy currents on winding conductors and magnetic cores are nonuniform current distribution, increased effective resistance, increased power loss, and reduced internal inductance. If the resistivity of a conductor was zero (as in a perfect conductor), eddy-current loops would be generated with such a magnitude and phase to exactly cancel the applied magnetic field. A perfect conductor would oppose any change in externally applied magnetic field. Circulating eddy currents would be induced to oppose any buildup of the magnetic field in the conductor. In general, nature opposes to everything we want to do.

1.5.4 Volt–Second Balance

Faraday's law is $v_L = d\lambda/dt$, yielding $d\lambda = v_L dt$. Hence,

$$\lambda(t) = \int d\lambda = \int v_L dt. \tag{1.78}$$

For periodic waveforms in steady state,

$$\int_0^T v_L(t)dt = \lambda(t)\Big|_0^T = \lambda(T) - \lambda(0) = 0. \tag{1.79}$$

This equation is called a *volt–second balance*, which states that the total area enclosed by the inductor voltage waveform v_L is zero for steady state. As a result, the area enclosed by the inductor voltage waveform v_L above zero must be equal to the area enclosed by the inductor voltage waveform v_L below zero for steady state. The volt–second balance can be expressed by

$$\int_0^{t_o} v_L(t)dt + \int_{t_o}^T v_L(t)dt = 0 \tag{1.80}$$

which gives

$$\int_0^{t_o} v_L(t)dt = -\int_{t_o}^T v_L(t)dt. \tag{1.81}$$

This can be written as $A^+ = A^-$.

1.5.5 Ohm's Law

Materials resist the flow of electric charge. The physical property of materials to resist current flow is known as *resistivity*. Therefore, a sample of a material resists the flow of electric current. This property is known as *resistance*. Ohm[5] discovered that the voltage across a resistor is directly proportional to its current and is constant, called resistance. Microscopic Ohm's law describes the relationship between the conduction current density **J** and the electric field intensity **E**. The conduction current is caused by the movement of electrons. Conductors exhibit the presence of many free (conduction or

[5]Georg Simon Ohm (1787–1854) was a German physicist and mathematician, who discovered the relationship between voltage and current for a resistor.

valence) electrons, from the outermost atom shells of a conducting medium. These free electrons are in random constant motion in different directions in a zigzag fashion due to thermal excitation. The average electron thermal energy per one degree of freedom is $E_{T1} = kT/2$ and the average thermal energy of an electron in three dimensions is $E_{T3} = 3E_{T1} = 3kT/2$. At the collision, the electron kinetic energy is equal to the thermal energy

$$\frac{1}{2}m_e v_{th}^2 = \frac{3}{2}kT, \tag{1.82}$$

where $k = 10^{-23}$ J/K is the Boltzmann's constant and $m_e = 9.1095 \times 10^{-31}$ kg is the rest mass of a free electron. The thermal velocity of electrons between collisions is

$$v_{th} = \sqrt{\frac{3kT}{m_e}} \approx 1.12238 \times 10^7 \text{ cm/s} = 112.238 \times 10^5 \text{ m/s} = 112.238 \text{ km/s}. \tag{1.83}$$

In good conductors, mobile free electrons drift through a lattice of positive ions encountering frequent collisions with the atomic lattice. If the electric field **E** in a conductor is zero, the net charge movement over a large volume (compared with atomic dimensions) is zero, resulting in zero net current. If an electric field **E** is applied to a conductor, a Coulomb's force **F** is exerted on an electron with charge $-q$

$$\mathbf{F} = -q\mathbf{E}. \tag{1.84}$$

According to Newton's second law, the acceleration of electrons between collisions is

$$a = \frac{F}{m_e} = \frac{-qE}{m_e}, \tag{1.85}$$

where $m_e = 9.11 \times 10^{-31}$ kg is the mass of electron. If the electric field intensity E is constant, then the average drift velocity of electrons increases linearly with time

$$v_d = at = -\frac{qEt}{m_e}. \tag{1.86}$$

The average drift velocity is directly proportional to the electric field intensity E for low values of E and saturates at high value of E. Electrons are involved in collisions with thermally vibrating lattice structure and other electrons. As the electron accelerates due to electric field, the velocity increases. When the electron collides with an atom, it loses most or all of its energy. Then, the electron begins to accelerate due to electric field E and gains energy until a new collision. The average position change x_{avg} of a group of N electrons in time interval Δt is called the *drift velocity*

$$v_d = \frac{x_{avg}}{\Delta t} = \frac{x_1 + x_2 + x_3 + \ldots + x_N}{N\Delta t} = \frac{v_1 + v_2 + v_3 + \ldots + v_N}{N}. \tag{1.87}$$

The drift velocity of electrons $\mathbf{v}_d$ has the opposite direction to that of the applied electric field **E**. By Newton's law, the average change in the momentum of a free electron is equal to the applied force

$$\mathbf{F} = \frac{m_e \mathbf{v}_d}{\tau_c}, \tag{1.88}$$

where the mean time between the successive collisions of electrons with atom lattice, called the *relaxation time*, is given by

$$\tau_c = \frac{l_n}{v_d} \tag{1.89}$$

in which l_n is length of the mean free path of electrons between collisions. Equating the right-hand sides of (1.84) and (1.88), we obtain

$$\frac{m_e \mathbf{v}_d}{\tau_c} = -q\mathbf{E} \tag{1.90}$$

yielding the average drift velocity of electrons

$$\mathbf{v}_d = -\frac{q\tau_c}{m_e}\mathbf{E} = -\mu_n \mathbf{E} \tag{1.91}$$

where the mobility of electrons in a conductor is

$$\mu_n = \frac{q\tau_c}{m_e} = \frac{q l_n}{m_e v_d}. \tag{1.92}$$

The volume charge density in a conductor is

$$\rho_v = -nq, \tag{1.93}$$

where n is the concentration of free (conduction or valence) electrons in a conductor, which is equal to the number of conduction electrons per unit volume of a conductor. The resulting flow of electrons is known as the conduction (or drift) current. The conduction (drift) current density, corresponding to the motion of charge forced by electric field E, is given by

$$\mathbf{J} = \frac{\mathbf{I}}{A} = \rho_v \mathbf{v}_d = -nq\mathbf{v}_d = -nq\mu_n \mathbf{E} = \frac{nq^2\tau_c}{m_e}\mathbf{E} = \sigma\mathbf{E} = \frac{\mathbf{E}}{\rho}, \tag{1.94}$$

where the conductivity of a conductor is

$$\sigma = nq\mu_n = nq\frac{q\tau_c}{m_e} = \frac{nq^2\tau_c}{m_e} = \frac{nq^2 v_d}{m_e l_n} \tag{1.95}$$

and the resistivity of a conductor is

$$\rho = \frac{1}{\sigma} = \frac{1}{nq\mu_n} = \frac{m_e}{nq^2\tau_c} = \frac{m_e l_n}{nq^2 v_d}. \tag{1.96}$$

Hence, the point (or microscopic) form of Ohm's law (1827) for conducting materials is

$$\mathbf{E} = \rho\mathbf{J} = \frac{\mathbf{J}}{\sigma}. \tag{1.97}$$

The typical value of mobility of electrons in copper is $\mu_n = 0.0032\ \mathrm{m^2/V\cdot s}$. At $E = 1$ V/m, the average drift velocity of electrons in copper is $v_d = 0.32$ cm/s. The thermal velocity of electrons between collisions is $v_{th} = 1.12 \times 10^7$ cm/s. Due to collisions of electrons with atomic lattice and the resulting loss of energy, the velocity of individual electrons in the direction opposite to the electric field $\mathbf{E}$ is much lower than the thermal velocity. The average drift velocity is much lower than the thermal velocity by two orders on magnitude. The average time interval between collisions of electrons is called the *relaxation time* and its typical value for copper is $\tau_c = 3.64 \times 10^{-14}$ s $= 36.4$ fs. The convection current and the displacement current do not obey Ohm's law, whereas the conduction current does it.

To illustrate Ohm's law, consider a straight round conductor of radius r_o and resistivity ρ carrying a DC current I. The current is evenly distributed in the conductor. Thus, the current density is

$$\mathbf{J} = \frac{\mathbf{I}}{A_w} = \frac{I}{\pi r_o^2}\mathbf{a}_z. \tag{1.98}$$

According to Ohm's law, the electric field intensity in the conductor is

$$\mathbf{E} = \rho\mathbf{J} = \rho\frac{I}{\pi r_o^2}\mathbf{a}_z. \tag{1.99}$$

1.5.6 Biot–Savart's Law

Hans Oersted discovered in 1819 that *currents produce magnetic fields* that form closed loops around conductors (e.g., wires). Moving charges are sources of the magnetic field. Jean Biot and Félix Savart arrived in 1820 at a mathematical relationship between the magnetic field $\mathbf{H}$ at any point P of space and the current I that generates $\mathbf{H}$. Current I is a source of magnetic field intensity $\mathbf{H}$. The Biot–Savart's law allows us to calculate the differential magnetic field intensity $d\mathbf{H}$ produced by a small current element $Id\mathbf{l}$. Figure 1.10 illustrates the Biot–Savart's law. The differential form of the Biot–Savart's law is given by

$$d\mathbf{H} = \frac{I}{4\pi}\frac{d\mathbf{l} \times \mathbf{a}_R}{R^2} \tag{1.100}$$

where $d\mathbf{l}$ is the current element equal to a differential length of a conductor carrying electric current I and points in the direction of the current I, and $\mathbf{R} = R\mathbf{a}_R$ is the distance vector between $d\mathbf{l}$ and an observation point P with field H. The vector $d\mathbf{H}$ is perpendicular to both $d\mathbf{l}$ and to the unit vector $\mathbf{a}_R$

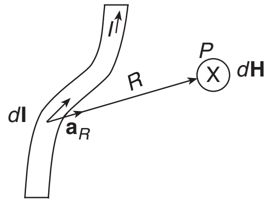

Figure 1.10 Magnetic field $d\mathbf{H}$ produced by a small current element $Id\mathbf{l}$

directed from $d\mathbf{l}$ to P. The magnitude of $d\mathbf{H}$ is inversely proportional to R^2, where R is the distance from $d\mathbf{l}$ to P. The magnitude of $d\mathbf{H}$ is proportional to $\sin\theta$, where θ is the angle between the vectors $d\mathbf{l}$ and $\mathbf{a}_R$. The Biot–Savart's law is analogous to Coulomb's law that relates the electric field E to an isolated point charge Q, which is a source of radial electric field $E = Q/(4\pi\epsilon R^2)$.

The total magnetic field $\mathbf{H}$ induced by a current I is given by the integral form of the Biot–Savart's law

$$\mathbf{H} = \frac{I}{4\pi}\int_l \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2} \quad \text{(A/m)}. \tag{1.101}$$

The integral must be taken over the entire current distribution.

1.5.7 Maxwell's Equations

Maxwell[6] assembled the laws of Faraday, Ampère, and Gauss (for both electric and magnetic fields) into a set of four equations to produce a unified EM theory. Maxwell's equations (1865), together with the law of conservation of charge (the continuity equation), form a foundation of a unified and coherent theory of electricity and magnetism. They couple electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, current density $\mathbf{J}$, and charge density $\boldsymbol{\rho}_v$. These equations provide the qualitative and quantitative description of static and dynamic EM fields. They can be used to explain and predict electromagnetic phenomena. In particular, they govern the behavior of EM waves.

Maxwell's equations in differential (point or microscopic) forms in the time domain at any point in space and at any time are given by

$$\nabla \times \mathcal{H} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t} = \boldsymbol{\sigma}\mathcal{E} + \epsilon\frac{\partial \mathcal{E}}{\partial t} \quad \text{(Ampere's law)}, \tag{1.102}$$

$$\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} = -\mu\frac{\partial \mathcal{H}}{\partial t} \quad \text{(Faraday's law)}, \tag{1.103}$$

$$\nabla \cdot \mathcal{D} = \rho_v \quad \text{(Gauss's law)}, \tag{1.104}$$

and

$$\nabla \cdot \mathcal{B} = 0 \quad \text{(Gauss's magnetic law)}, \tag{1.105}$$

where $J_D = \partial\mathcal{D}/\partial t$ is the *displacement current density*. The conductive current density (corresponding to the motion of charge) $\mathcal{J}$ and the displacement current density J_D are sources of EM fields $\mathcal{H}$, $\mathcal{E}$, $\mathcal{B} = \mu\mathcal{H}$, and the volume charge density ρ_v is a source of the electric fields $\mathcal{E}$ and $\mathcal{D} = \epsilon\mathcal{E}$, where μ is permeability and ϵ is the permittivity of a material. Maxwell's equations include two Gauss's[7] laws. Gauss's law states that charge is a source of electric field. In contrast, Gauss's magnetic law states that magnetic field is sourceless (divergenceless), that is, there are no magnetic sources or sinks. This law also indicates that magnetic flux lines close upon themselves. Two Maxwell's equations

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[6]James Clerk Maxwell (1831–1879) was a Scottish physicist and mathematician, who mathematically unified Faraday's, Ampère's, and Gauss's laws. "Maxwell's equations" are foundations of EM fields and waves.
[7]Karl Fredrich Gauss (1777–1855) was a German mathematician and physicist.

are partial differential equations because magnetic and electric fields, current, and charge may vary simultaneously with space and time.

Neglecting the generation and recombination of carrier charges like in semiconductors, the *continuity equation* or the *law of local conservation of electric charge* must be satisfied at all times

$$\nabla \cdot \mathcal{J} = -\frac{\partial \rho_v}{\partial t}. \tag{1.106}$$

This law states that the time rate of change of electric charge ρ_v is a source of electric current density field J. This means that the current density is continuous and charge can be neither created nor destroyed. It can only be transferred. The continuity equation is a point form of KCL known in circuit theory. The script letters are used to designate instantaneous field quantities, which are functions of position and time, for example, $\mathcal{E}(x, y, z, t)$. Maxwell's equations are the cornerstone of electrodynamics. A time-varying magnetic field is always accompanied by an electric field, and a time-varying electric field is always accompanied by a magnetic field. For example, a radio antenna generates radiofrequency (RF) waves that consist of both the electric and magnetic fields. The divergence of **B** equal to zero indicates that magnetic charges do not exist in the nature. It is a magnetic flux continuity law. Maxwell's equations also indicate that conductive and/or displacement current is a source of magnetic field, and charge is a source of electric field.

The *divergence* of the electric field intensity **E** at a point is the net outward electric field flow per unit volume over a closed incremental surface S and is defined as

$$\nabla \cdot \mathbf{E} = \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{S}}{\Delta V} = \frac{\rho_v}{\epsilon}. \tag{1.107}$$

where S is the closed surface, which encloses the volume V, and ϵ is the permittivity of a medium. The closed surface integral $\oint_S \mathbf{E} \cdot d\mathbf{S}$ is the flux of vector **E** outflowing from the volume V. In the limit, the volume V shrinks to a point. Electric fields **E** and $\mathbf{D} = \epsilon \mathbf{E}$ are *source fields* or *sink fields* because the divergence of these fields is not equal to zero ($\nabla \cdot \mathbf{E} \neq 0$ and $\nabla \cdot \mathbf{D} \neq 0$).

In general, a curl-free vector field is called *irrotational*, or a *conservative*, or a *potential* field. Electrostatic fields **D** and **E** are *irrotational* because their curl is equal to zero. If a scalar source (in the form of a charge) of the field **E** is present at a point P, then divergence of **E** is nonzero. Therefore, the vector field whose divergence is nonzero is called a source field. If $\nabla \cdot \mathbf{E} > 0$, the field is a source field. If $\nabla \cdot \mathbf{E} < 0$, the field is a sink field. If $\nabla \cdot \mathbf{E} = 0$, the field is sourceless. A positive charge Q is a source of an electric field **E**, and a negative charge Q is a sink of an electric field **E**.

The *curl* of the magnetic field density at a point is the circulation of **B** per unit area and is defined as

$$\nabla \times \mathbf{B} = \left[\lim_{\Delta S \to 0} \frac{\oint_C \mathbf{B} \cdot d\mathbf{l}}{\Delta S} \right]_{max} = \frac{\Delta I}{\Delta S}, \tag{1.108}$$

where the area ΔS of the contour C is oriented so that the circulation is maximum. In the limit as ΔS shrinks to zero around a point P, the curl of **B** is obtained. Magnetic fields **H** and $\mathbf{B} = \mu \mathbf{H}$ are *rotational* and *sourceless*. They are rotational because their curl is not equal to zero ($\nabla \times \mathbf{B} \neq 0$). They are sourceless because their divergence is equal to zero ($\nabla \cdot \mathbf{H} = 0$). It is worth noting that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.109}$$

The curl of **H** has a nonzero value whenever current is present.

The generalized Ampère's law given by (1.102) states that both conductive and displacement currents induce magnetic field. In other words, a time-varying electric field will give rise to a magnetic field, even in the absence of a conduction (drift) current flow. Maxwell added the displacement current to the Ampère's equation, making (1.109) general.

Gauss's law given by (1.104) states that the net outflow of the electric flux density at any point in space is equal to the charge density at that point. The electric flux starts from a charge and ends on a charge. This means that the electric field is a *divergent* field or *source* field. A positive divergence at a point indicates the presence of a positive charge at that point (i.e., a positive charge is a flux source). Conversely, the negative divergence at a point indicated the presence of a negative charge at that point (i.e., a negative charge is a flux sink).

Equation (1.105) states that the magnetic flux lines always form closed paths, that is, they close on themselves. This means that the magnetic field is a divergenceless field. This law implies that there is no isolated magnetic charges. The magnetic field is sourceless.

If $\mathbf{J} = 0$, Maxwell's equation in (1.102) becomes

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon_r \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.110}$$

This equation states that a time-varying electric field induces a changing magnetic field without electric conduction and convection currents, and a changing magnetic field induces a changing electric field. There would be no radiation and propagation of EM waves without the displacement current. In particular, there would be no wireless communications.

Maxwell's equations in integral (or macroscopic) forms are as follows:

$$V = EMF = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{d\phi}{dt} \quad \text{(Faraday's law)}, \tag{1.111}$$

and

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int\int_S \mathbf{J} \cdot d\mathbf{S} + \int\int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} = i_{enc} + \frac{d\phi_E}{dt} \quad \text{(Ampere–Maxwell's law)}, \tag{1.112}$$

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = \int\int\int_V \rho_v dV = Q_{enc} \quad \text{(Gauss's electric law)}, \tag{1.113}$$

$$\oiint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{(Gauss's magnetic law)}. \tag{1.114}$$

The current density $\mathbf{J}$ may consist of a conduction (or drift) current $\mathbf{J}_c = \sigma \mathbf{E}$ caused by the presence of free electrons and an electric field $\mathbf{E}$ in a conducting medium, the diffusion current density $\mathbf{J}_{diff}$ caused by the gradient of charge carrier concentration, as well as a convection current density $\mathbf{J}_{conv} = \rho_v \mathbf{v}$ due to the motion of free-charge distribution (i.e., the movement of electrically charged medium).

Faraday's law of induction given by (1.111) describes the creation of an electric field by a changing magnetic flux. The EMF, which is equal to the line integral of the electric field $\mathbf{E}$ around any closed path C, is equal to the rate of change of magnetic flux through any surface area S bounded by that path. As a result, for instance, the current is induced in a conducting loop placed in a time-varying magnetic field.

The generalized Ampère's circuital law given by (1.112) describes how a magnetic field can be produced by both an electric current and/or a time-varying electric flux ϕ_E. It states that the line integral of magnetic field $\mathbf{H}$ around any closed path is the sum of the net current through that path and the rate of change of electric flux through any surface bounded by that path.

Gauss's law in the integral form for electric field given by (1.113) states that the total electric flux through any closed surface S is equal to the net charge Q inside that surface.

Gauss's law in the integral form for magnetic field given by (1.114) states that the net magnetic flux through any closed surface is always zero. This means that the number of magnetic field lines that enter a closed volume is equal to the number of magnetic field lines that leave that volume. Magnetic field lines are continuous with no starting or end points. There are no magnetic sources or sinks. A magnetic monopole does not exist. Equation (1.114) also means that there are no magnetic charges.

The phasor technique is a useful mathematical tool for solving problems in linear systems that involve periodic sinusoidal or periodic nonsinusoidal waveforms in steady state, where the amplitude A_m frequency ω and phase ϕ are time-invariant. In this case, complex algebra can be used as a mathematical tool. Periodic nonsinusoidal waveforms, such as a rectangular wave, can be expanded into a Fourier series of sinusoidal components, which is a superposition of harmonic sinusoids. If the excitation is a sinusoidal function of time, the steady-state waveforms described in the time domain can be represented by phasors (complex amplitudes), the trigonometric equations are replaced by algebraic equations, and linear integro-differential equations become linear algebraic equations with no sinusoidal functions, which are easy to solve. Differentiation in the time domain is equivalent to multiplication by $j\omega$ in the phasor domain, and integration in the time domain is equivalent to division by $j\omega$ in the phasor domain. The solutions in the phasor domain can be converted back into the time domain. The sinusoidal current $i(t) = I_m \cos(\omega t + \phi)$ can be represented as $i(t) = Re\{I_m e^{j(\omega t + \phi)}\} = Re\{I_m e^{j\phi} e^{j\omega t}\} = Re\{\mathbf{I}_m e^{j\omega t}\}$, where the complex amplitude $\mathbf{I}_m = I_m e^{j\phi}$ is called a phasor.

The electric field intensity for one-dimensional case in the time domain is given by

$$\mathcal{E}(x,t) = E_m(0)e^{-\frac{x}{\delta}} \cos\left(\omega t - \frac{x}{\delta} + \phi_o\right) = Re\{\mathbf{E}(x)e^{j\omega t}\}, \tag{1.115}$$

where δ is the skin depth of a conductor and ϕ_o is the phase of the electric field. The phasor of the sinusoidal (harmonic) electric field intensity is

$$\mathbf{E}(x) = E_m(0)e^{-\frac{x}{\delta w}} e^{-j\frac{x}{\delta}} e^{j\phi_o}. \tag{1.116}$$

Similarly, the sinusoidal magnetic field intensity is

$$\mathcal{H}(x,t) = H_m(0)e^{-\frac{x}{\delta}} \cos\left(\omega t - \frac{x}{\delta} + \theta_o\right) = Re\{\mathbf{H}(x)e^{j\omega t}\}, \tag{1.117}$$

where θ_o is the phase of the magnetic field and the phasor of the magnetic field intensity is

$$\mathbf{H}(x) = H_m(0)e^{-\frac{x}{\delta}} e^{-j\frac{x}{\delta}} e^{j\theta_o}. \tag{1.118}$$

Substituting the electric and magnetic field intensities into Maxwell's equation in the time domain, we obtain

$$\nabla \times Re\{\mathbf{E}(x)e^{j\omega t}\} = -\frac{\partial}{\partial t} Re\{\mu\mathbf{H}(x)e^{j\omega t}\}, \tag{1.119}$$

which becomes

$$Re\{\nabla \times \mathbf{E}(x)e^{j\omega t}\} = Re\{-j\omega\mu\mathbf{H}(x)e^{j\omega t}\}. \tag{1.120}$$

Thus, $\frac{\partial}{\partial t}$ in Maxwell's equations in the time domain can be replaced by $j\omega$ to obtain Maxwell's equations for sinusoidal field waveforms in phasor forms

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} = -j\omega\mathbf{B}, \tag{1.121}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} = \mathbf{J} + j\omega\epsilon\mathbf{E} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} = (\sigma + j\omega\epsilon)\mathbf{E}, \tag{1.122}$$

$$\nabla \cdot \mathbf{D} = \rho_v, \tag{1.123}$$

and

$$\nabla \cdot \mathbf{B} = 0. \tag{1.124}$$

The *constitutive equations* or *material equations* for linear and isotropic materials are

$$\mathbf{B} = \mu\mathbf{H} \tag{1.125}$$

$$\mathbf{D} = \epsilon\mathbf{E} \tag{1.126}$$

and

$$\mathbf{J} = \sigma\mathbf{E}, \tag{1.127}$$

where **D** is the electric flux density.

In general, the complex propagation constant is given by

$$\begin{aligned}\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} &= \omega\sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{\frac{1}{2}} + j\omega\sqrt{\frac{\epsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]^{\frac{1}{2}} \\ &= \alpha + j\beta = \frac{1}{\delta} + j\beta,\end{aligned} \tag{1.128}$$

where $\alpha = Re\{\gamma\}$ is the attenuation constant and $\beta = Im\{\gamma\}$ is the phase constant. The skin depth is

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]^{\frac{1}{2}}}. \tag{1.129}$$

Figure 1.11 shows a plot of skin depth δ as a function of frequency f for copper. The plot is made using MATLAB®. The quantities $\rho = 1/\sigma$, μ, and ϵ describe the electrical properties of materials. The quantities ω, ρ, μ, and ϵ determine whether a material behaves more like a conductor or more like a dielectric.

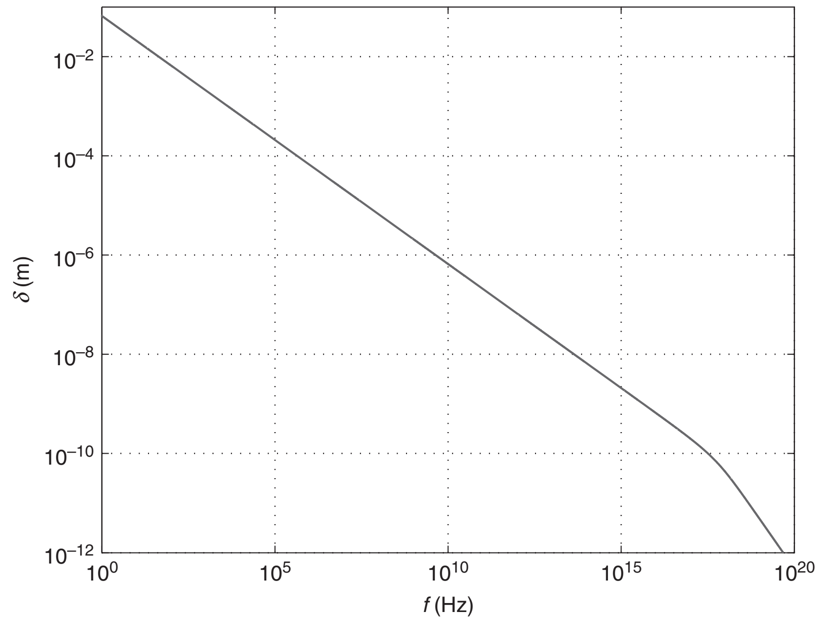

Figure 1.11 Skin depth δ as a function of frequency f

For good conductors, $\sigma \gg \omega\epsilon$, that is, $\sigma/(\omega\epsilon) \gg 1$ or $\omega\epsilon\rho \ll 1$, the complex propagation constant simplifies to the form

$$\gamma \approx \sqrt{\frac{\omega\mu\sigma}{2}} + j\sqrt{\frac{\omega\mu\sigma}{2}} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{j\omega\mu\sigma} = \alpha + j\beta, \tag{1.130}$$

where $(1+j)/\sqrt{2} = e^{45^\circ} = \sqrt{j}$. The skin depth for good conductors is

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{2\rho}{\omega\mu}} = \frac{1}{\sqrt{\pi\mu\sigma f}} = \sqrt{\frac{\rho}{\pi\mu f}}. \tag{1.131}$$

The wavelength for good conductors is

$$\lambda = \frac{2\pi}{\beta} = 2\sqrt{\frac{\pi}{\mu\sigma f}}. \tag{1.132}$$

The propagation speed or phase velocity for good conductors is

$$v_p = \lambda f = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}. \tag{1.133}$$

Under the condition of $\omega\epsilon\rho \ll 1$, the system is magnetoquasistatic. This is the case if and only if the left-hand side of the inequality is no more than two orders of magnitude less than unity, that is, $\omega\epsilon\rho \ll 1/100$.

For copper windings, $\rho_{Cu} = 17.24$ nΩm at $T = 20\,^\circ$C and $\epsilon = \epsilon_0 = 10^{-9}/(36\pi) = 8.854 \times 10^{-12}$ F/m, the maximum frequency for magnetoquasistatic operation is

$$f_{max} = \frac{1}{2\pi \times 100 \times \epsilon_0 \rho_{Cu}} = \frac{36\pi}{2\pi \times 100 \times 10^{-9} \times 17.24 \times 10^{-9}} = 10.44 \times 10^{15}$$
$$= 10.44 \text{ PHz}. \tag{1.134}$$

For $\sigma/(\epsilon\omega) \ll 1$, the conductor becomes a dielectric. The skin depth is given by

$$\delta = \frac{1}{\omega\sqrt{\mu\epsilon}}. \tag{1.135}$$

A summary of Maxwell's equations is given in Appendix B.

1.5.8 Maxwell's Equations for Good Conductors

In general, Maxwell's equation in phasor form, which is the differential form of Ampère's equation, together with Ohm's law ($\mathbf{J} = \sigma\mathbf{E}$) is given by

$$\nabla \times H = \mathbf{J} + j\omega\mathbf{D} = \sigma\mathbf{E} + j\omega\epsilon\mathbf{E} = (\sigma + j\omega\epsilon)\mathbf{E}, \tag{1.136}$$

where σ is the conductivity of a medium. For good conductors, the displacement current is negligible in comparison with conduction (drift) current. Since conduction current density $\mathbf{J} = \sigma\mathbf{E}$ dominates the displacement current density $\mathbf{J}_d = j\omega\mathbf{D} = j\omega\epsilon\mathbf{E}$, that is, $\mathbf{J} \gg \mathbf{J}_\mathrm{D}$, the following inequality is satisfied

$$\sigma \gg \omega\epsilon, \tag{1.137}$$

which becomes $\sigma/(\omega\epsilon) \gg 1$ or $\omega\rho\epsilon \ll 1$. For copper, $J = J_D$ when $\sigma = \omega\epsilon_0$ at $f = 1/(2\pi\epsilon_0\rho) = 1.00441 \times 10^{18} = 1.0441$ EHz. Also, $J = 100J_D$ when $\sigma \geq 100\omega\epsilon$ for frequencies $f \leq 10^{16}$ Hz = 10 PHz.

Since $J_D = j\omega D = 0$, Maxwell's equation for good conductors (which is Ampère's law) becomes

$$\nabla \times \mathbf{H} \approx \mathbf{J} = \sigma\mathbf{E}. \tag{1.138}$$

It states that the maximum circulation of $\mathbf{H}$ per unit area as the area shrinks to zero (called the curl of $\mathbf{H}$) is equal to the current density J.

For sinusoidal waveforms, Maxwell's equation in phasor form for good conductors, which is the differential (microscopic) form of Faraday's law, is expressed as

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} = -j\omega\mu\mathbf{H}. \tag{1.139}$$

Using Ohm's law $\mathbf{E} = \mathbf{J}/\sigma$, we obtain

$$\nabla \times \frac{\mathbf{J}}{\sigma} = -j\omega\mu\mathbf{H} \tag{1.140}$$

producing another form of Maxwell's equation

$$\nabla \times J = -j\omega\mu\sigma\mathbf{H}. \tag{1.141}$$

Assuming that σ and μ are homogeneous, taking the curl on both sides of the above equation and substituting into Maxwell's equation,

$$\nabla \times (\nabla \times \mathbf{J}) = -j\omega\mu\sigma\nabla \times \mathbf{H} = -j\omega\mu\sigma\mathbf{J}. \tag{1.142}$$

Expanding the left-hand side,

$$\nabla(\nabla \cdot \mathbf{J}) - \nabla^2\mathbf{J} = -j\omega\mu\sigma\mathbf{J}, \tag{1.143}$$

where the law of conservation of charge states that charge can be neither created nor destroyed and its point (microscopic) form is expressed by $\nabla \cdot \mathbf{J} = 0$. It is a point form of Kirchhoff's current law. The conduction (or drift) current density $\mathbf{J}$ in good conductors must satisfy the following second-order partial differential equation

$$\nabla^2\mathbf{J} = j\omega\mu\sigma\mathbf{J} = \gamma^2\mathbf{J}, \tag{1.144}$$

where $\gamma^2 = j\omega\mu\sigma$.

For good conductors,

$$\nabla \cdot (\nabla \times \mathbf{H}) = (\sigma + j\omega\epsilon)(\nabla \cdot \mathbf{E}) = 0. \tag{1.145}$$

Hence, Maxwell's equation for good conductors becomes

$$\nabla \cdot \mathbf{D} = \rho_v = 0. \tag{1.146}$$

1.5.9 Poynting's Vector

Poynting[8] developed the mathematical description of the magnitude and the direction of EM energy density transmission. The *instantaneous Poynting vector* (1883) at a given point describes the EM

[8]John Henry Poynting (1852–1914) was an English physicist (Maxwell's student), who described the magnitude and the direction of EM energy flow.

power flux surface density of EM wave

$$\mathcal{S} = \mathcal{E} \times \mathcal{H} \text{ (W/m}^2). \tag{1.147}$$

The Poynting vector represents the density and the direction of power flow of electromagnetic fields at any point in space, that is, it is the rate at which energy flows through a unit surface area perpendicular to the direction of wave propagation. The direction of vector $\mathcal{S}$ is normal to both $\mathcal{E}$ and $\mathcal{H}$. The cross product $\mathbf{E} \times \mathbf{H}$ is *pointing* in the direction of power flow, that is, in the direction of wave propagation. The vector $\mathcal{S}$ represents an instantaneous surface power density. Since the unit of $\mathcal{E}$ is V/m and the unit of $\mathcal{H}$ is A/m, the unit of $\mathcal{S}$ is (V/m) $\times$ (A/m) $=$ VA/m^2 $=$ W/m^2.

For time-harmonic fields, the *complex Poynting vector* is

$$\mathbf{S}_c = \mathbf{E} \times \mathbf{H}^* \quad \text{(W/m}^2). \tag{1.148}$$

The time-average power density, defined as the power density averaged over one period of the sinusoidal excitation, is given by the *time-average Poynting vector*

$$\mathbf{S}_{av} = \frac{1}{2} Re\{\mathbf{E} \times \mathbf{H}^*\} \quad \text{(W/m}^2). \tag{1.149}$$

The amount of time-average power passing through a surface S is

$$P_{av} = \int\int_S \mathbf{S}_{av} \cdot d\mathbf{S} = \int\int_S \mathbf{S}_{av} \cdot \mathbf{n} dA = \frac{1}{2} Re \left\{ \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} \right\} \text{ (W).} \tag{1.150}$$

where $d\mathbf{S} = \mathbf{a}_n dA$, $\mathbf{a}_n$ is the unity vector normal to surface S, and dA is the differential surface. The surface integral of $\mathbf{S}_{av}$ describes the total power generated or dissipated inside the enclosed surface S.

For a linear, isotropic, and time-invariant medium of volume V enclosed in a closed surface S, the Poynting theorem relates the following energies: (i) the delivered energy,(ii) the dissipated energy, (iii) the magnetic stored energy, and (iv) the electric stored energy. This theorem describes the principle of conservation of energy. The integral form of the Poynting theorem is given by

$$\begin{aligned} \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} &= -\int\int\int_V \mathbf{J} \cdot \mathbf{E} dV - \frac{\partial}{\partial t} \int\int\int_V \left(\frac{1}{2} \mathbf{B} \cdot \mathbf{H} + \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \right) dV \\ &= -\int\int\int_V \mathbf{J} \cdot \mathbf{E} dV - \frac{\partial}{\partial t} \int\int\int_V \left(\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \right) dV. \end{aligned} \tag{1.151}$$

For sinusoidal field waveforms,

$$\begin{aligned} \oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} &= -\int\int\int_V [\mathbf{E} \cdot \mathbf{J}^* + j\omega(\mathbf{H}^* \cdot \mathbf{B} + \mathbf{E}^* \cdot \mathbf{D})] dV \\ &= -\frac{1}{2} \int\int\int_V \rho |J|^2 dV - j\omega \int\int\int_V \left(\frac{1}{2} \mu H^2 + \frac{1}{2} \epsilon E^2 \right) dV \\ &= -\int\int\int_V p_D dV - \frac{\partial}{d\partial} \int\int\int_V (w_m + w_e) dV, \end{aligned} \tag{1.152}$$

where the asterisk * in the phasor superscript indicates a complex conjugate quantity, $p_D = \frac{1}{2} \rho |J|^2$ is the ohmic power loss density (Joule's law), $w_m = \frac{1}{2} \mu |H|^2$ is the magnetic energy density stored in the magnetic field, and $w_e = \frac{1}{2} \epsilon |E|^2$ is the electric energy density stored in the electric field. The first term on the right-hand side of (1.152) represents the ohmic power dissipated as heat in the volume V (Joule's law) as a result of the flow of conduction current density $\mathbf{J} = \sigma \mathbf{E}$ due to the presence of the electric field $\mathbf{E}$ (Ohm's law). This power exits the volume V through its surface S. The second and third terms represent the time rate of change of the magnetic and electric energies stored in the magnetic and electric fields, respectively. The left-hand side of (1.152) describes the total power leaving the closed surface S. The Poynting theorem describes the principle of conservation of energy. It states that the total power flow out of a closed surface S at any time instant is equal to the sum of the ohmic power dissipated within the enclosed volume V and the rates of decrease of the stored magnetic and electric energies. If there are sources inside the volume V, the dot product $\mathbf{J} \cdot \mathbf{E}$ has the minus sign and represents the power density added to the volume V by these sources.

For steady state, the *complex power* flowing into a volume V surrounded by a closed surface S is given by

$$P = \frac{1}{2}\oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{S} = P_D + 2j\omega(W_m - W_e) \text{ (W)}, \tag{1.153}$$

where the time-average real power dissipated in the volume V is given by Joule's law as

$$P_D = \frac{1}{2}\int\int\int_V \mathbf{E} \cdot \mathbf{J}^* dV = \frac{1}{2}\int\int\int_V \rho|\mathbf{J}|^2 dV = \frac{1}{2}\int\int\int_V \sigma|\mathbf{E}|^2 dV. \tag{1.154}$$

If $W_m > W_c$, the device inside the volume V is inductive. If $W_c > W_c$, the device inside the volume V is capacitive. If $W_m = W_c$, the device inside the volume V operates at self-resonant frequency (SRF).

For harmonic fields, the instantaneous magnetic energy density in an isotropic medium is

$$w_m(t) = \frac{1}{2}Re\{\mathbf{B}_m e^{j\omega t}\} \cdot Re\{\mathbf{H}_m e^{j\omega t}\} = \frac{1}{2}B_m H_m \cos^2\omega t = \frac{B_m}{2\mu}\cos^2\omega t = \frac{1}{2}\mu H_m^2 \cos^2\omega t. \tag{1.155}$$

Hence, the time-average magnetic energy density is

$$w_{m(av)} = \frac{1}{2}Re\{\mathbf{H} \cdot \mathbf{B}^*\} = \frac{1}{4}H_m B_m = \frac{1}{4}\mu H_m^2 = \frac{B_m^2}{4\mu}. \tag{1.156}$$

1.5.10 Joule's Law

Joule's law (1841) states that the rate of heat dissipation in a conductor is proportional to the square of the current through it and the conductor resistance. The power dissipated in a conductor is $P = RI^2$ and the energy dissipated in a conductor during time interval Δt is $W = P\Delta t$. This law can be extended to distributed systems.

Let us consider the power dissipated in a conductor caused by the movement of electrons forced by electric field $\mathbf{E}$. The charge density of free electrons is ρ_v. The electron charge in a small conductor volume ΔV is given by

$$q = \rho_v \Delta V. \tag{1.157}$$

The electric force exerted on the charge q by the electric field $\mathbf{E}$ is

$$\mathbf{F} = q\mathbf{E} = \mathbf{E}\rho_v \Delta V. \tag{1.158}$$

The incremental amount of energy (or work) ΔW done by the electric force $\mathbf{F}$ in moving the charge q by an incremental distance $\Delta\mathbf{l}$ is

$$\Delta W = \mathbf{F} \cdot \Delta\mathbf{l} = q\mathbf{E} \cdot \Delta\mathbf{l} = \mathbf{E} \cdot \Delta\mathbf{l}\rho_v \Delta V. \tag{1.159}$$

The power used to perform the work ΔW in time interval Δt is given by

$$\Delta P = \frac{\Delta W}{\Delta t} = \frac{\mathbf{F} \cdot \Delta\mathbf{l}}{\Delta t} = q\mathbf{F} \cdot \mathbf{v}_d = \mathbf{E} \cdot (\rho_v \mathbf{v}_d)\Delta V = \mathbf{E} \cdot \mathbf{J}\Delta V = \frac{\mathbf{E} \cdot \mathbf{E}}{\rho}\Delta V = \rho\mathbf{J} \cdot \mathbf{J}\Delta V, \tag{1.160}$$

where $\mathbf{v}_d = \Delta l/\Delta t$ is the electron drift velocity. The power loss density describing the time rate at which energy is converted into heat per unit volume of a conductor is given by

$$p_D = \frac{\Delta P}{\Delta V} = \frac{\Delta W/\Delta t}{\Delta V} = \mathbf{E} \cdot \mathbf{J} = \frac{\mathbf{E} \cdot \mathbf{E}}{\rho} = \rho\mathbf{J} \cdot \mathbf{J} \text{ (W/m}^3\text{)}. \tag{1.161}$$

The power loss in the conductor converted into heat is

$$P = \int\int\int_V p_D dV = \int\int\int_V \mathbf{E} \cdot \mathbf{J} dV = \int\int\int_V \frac{\mathbf{E} \cdot \mathbf{E}}{\rho} dV = \int\int\int_V \rho\mathbf{J} \cdot \mathbf{J} dV \text{ (W)}. \tag{1.162}$$

For sinusoidal field waveforms, using Ohm's law, the power dissipated per unit volume is given by the point Joule's law

$$p_D = \mathbf{J} \cdot \mathbf{E}^* = \rho\mathbf{J} \cdot \mathbf{J}^* = \rho|J|^2 = \frac{\mathbf{E} \cdot \mathbf{E}^*}{\rho} = \frac{|E|^2}{\rho}. \tag{1.163}$$

The power dissipated in a conductor of volume V and resistivity ρ as thermal energy (i.e., heat) is given by the integral form of Joule's law

$$P_D = \int\int\int_V p_D dV = \int\int\int_V \mathbf{J}\cdot\mathbf{E}^* dV = \int\int\int_V \rho\mathbf{J}\cdot\mathbf{J}^* dV = \int\int\int_V \rho|J|^2 dV. \tag{1.164}$$

For a linear conductor carrying a multiple-harmonic inductor current waveform, the current density and electric fields can be expanded into Fourier series and the power loss density may be expressed by

$$p_D = \sum_{n=1}^{\infty}\mathbf{J}_n\cdot\mathbf{E}_n^* = \sum_{n=1}^{\infty}\rho\mathbf{J}_n\cdot\mathbf{J}_n^* = \sum_{n=1}^{\infty}\rho|J_n|^2 = \sum_{n=1}^{\infty}\frac{\mathbf{E}_n\cdot\mathbf{E}_n^*}{\rho} = \sum_{n=1}^{\infty}\frac{|E_n|^2}{\rho} \tag{1.165}$$

and the total power loss is

$$P_D = \sum_{n=1}^{\infty}\int\int\int_V p_{Dn} dV = \sum_{n=1}^{\infty}\int\int\int_V \mathbf{J}_n\cdot\mathbf{E}_n^* dV = \sum_{n=1}^{\infty}\int\int\int_V \rho\mathbf{J}_n\cdot\mathbf{J}_n^* dV$$
$$= \sum_{n=1}^{\infty}\int\int\int_V \rho|J_n|^2 dV = \sum_{n=1}^{\infty}\int\int\int_V \frac{\mathbf{E}_n\cdot\mathbf{E}_n^*}{\rho} = \sum_{n=1}^{\infty}\int\int\int_V \frac{|E_n|^2}{\rho}, \tag{1.166}$$

where J_n and E_n are the amplitudes of current density and electric field intensity at nth harmonic, respectively.

The current density in a conductor in the time domain in steady state for one-dimensional case is described by

$$Re\{\mathbf{J}(x)e^{j\omega t}\} = J(x,t) = J_m(0)e^{-\frac{x}{\delta_w}}\cos\left(\omega t - \frac{x}{\delta_w} + \phi_o\right), \tag{1.167}$$

where δ_w is the skin depth and ϕ_o is the initial phase. It is assumed that the current amplitude varies only in the x-direction. From Ohm's law,

$$E(x,t) = \rho J(x,t) = \rho J_m(0)e^{-\frac{x}{\delta_w}}\cos\left(\omega t - \frac{x}{\delta_w} + \phi_o\right), \tag{1.168}$$

where $E_m(0) = \rho J_m(0)$. Assuming that ρ is a real number, the phase shift between $J(x,t)$ and $E(x,t)$ is zero. The instantaneous power density at a point is given by

$$p(x,t) = J(x,t)E(x,t) = J_m(0)E_m(0)e^{-\frac{2x}{\delta_w}}\cos^2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right)$$
$$= \rho J_m^2(0)e^{-\frac{2x}{\delta_w}}\cos^2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right)$$
$$= \frac{J_m(0)E_m(0)}{2}e^{-\frac{2x}{\delta_w}} + \frac{J_m(0)E_m(0)}{2}e^{-\frac{2x}{\delta_w}}\cos 2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right)$$
$$= \frac{\rho J_m^2(0)}{2}e^{-\frac{2x}{\delta_w}} + \frac{\rho J_m^2(0)}{2}e^{-\frac{2x}{\delta_w}}\cos 2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right)$$
$$= p_D(x) + p_D(x)\cos 2\left(\omega t - \frac{x}{\delta_w} + \phi_o\right), \tag{1.169}$$

where $\cos^2 z = 1/2 + 1/2\cos 2z$. The first term in the above equation represents the time-average real power density dissipated in a conductor at a point, and the second term represents the AC component of the instantaneous real power density dissipated in a conductor as heat at a point. The time-average real power density dissipated in a conductor at a point is

$$p_D(x) = \frac{1}{T}\int_0^T p(x,t)dt = \frac{1}{2\pi}\int_0^{2\pi} p(x,\omega t)d(\omega t) = \frac{J_m(0)E_m(0)}{2}e^{-\frac{2x}{\delta_w}} = \frac{\rho J_m^2(0)}{2}e^{-\frac{2x}{\delta_w}}, \tag{1.170}$$

where T is the period. The total time-average power dissipated as heat in a conductor of volume V is

$$P_D = \int\int\int_V p_D(x)dV = \frac{1}{2}\int\int\int_V J_m(0)E_m(0)e^{-\frac{2x}{\delta_w}}dxdydz$$

$$= \frac{1}{2} \int \int \int_V \rho J_m^2(0) e^{-\frac{2x}{\delta_w}} dxdydz. \tag{1.171}$$

When EM fields are sinusoidal, phasors are described in space as follows: $\mathbf{H}(\mathbf{r}) = \mathbf{H}(x, y, z)$, $\mathbf{E}(\mathbf{r}) = \mathbf{E}(x, y, z)$, and $\mathbf{J}(\mathbf{r}) = \mathbf{J}(x, y, z)$. The instantaneous point (local) power density is

$$p(\mathbf{r}, t) = Re\{\mathbf{J}(\mathbf{r}, t\} \cdot Re\{\mathbf{E}(\mathbf{r}, t\} = \frac{1}{4}[\mathbf{J}(\mathbf{r}, t) + \mathbf{J}^*(\mathbf{r}, t)][\mathbf{E}(\mathbf{r}, t) + \mathbf{E}^*(\mathbf{r}, t)]$$
$$= \frac{1}{4}[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})e^{2j\omega t} + \mathbf{J}^*(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) + \mathbf{J}^*(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r})e^{-2j\omega t}]$$
$$= \frac{1}{2} Re[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})e^{2j\omega t}]. \tag{1.172}$$

The time-average real power density dissipated in a conductor at a point $\mathbf{r}$ is

$$p_D(\mathbf{r}) = \frac{1}{T} \int_0^T p(\mathbf{r}, t) dt = \frac{1}{2} Re[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r})]. \tag{1.173}$$

The time-average power dissipated as heat in the conductor of volume V is given by

$$P_D = \int \int \int_V p_D(\mathbf{r}) dV = \frac{1}{2} Re \int \int \int_V \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^*(\mathbf{r}) dV$$
$$= \frac{1}{2} \int \int \int_V \rho \mathbf{J}(\mathbf{r}) \cdot \mathbf{J}^*(\mathbf{r}) dV = \frac{1}{2} \int \int \int_V \rho |\mathbf{J}(\mathbf{r})|^2 dV. \tag{1.174}$$

The current density in phasor form for one-dimensional case is given by

$$\mathbf{J}(x) = J_m(0) e^{-\frac{x}{\delta_w}} e^{-j\frac{x}{\delta_w}} e^{j\phi_o} = J_m(x) e^{j(\phi_o - \frac{x}{\delta_w})}, \tag{1.175}$$

where the amplitude is

$$J_m(x) = J_m(0) e^{-\frac{x}{\delta_w}}. \tag{1.176}$$

The time-average point power density for sinusoidal waveforms is given by point Joule's law in phasor form

$$P_D(x) = \frac{1}{2} Re(\mathbf{J} \cdot \mathbf{E}^*) = \frac{1}{2} \rho \mathbf{J} \cdot \mathbf{J}^* = \frac{1}{2} \rho |J(x)|^2 = \frac{1}{2} \rho J_m^2(0) e^{-\frac{2x}{\delta_w}}. \tag{1.177}$$

For periodic waveforms, the time-average real power dissipated in a conductor of volume V and resistivity ρ due to conversion of EM energy to thermal energy (heat) is given by Joule's law in phasor form

$$P_D = \frac{1}{2} Re \int \int \int_V \mathbf{J} \cdot \mathbf{E}^* dV = \frac{1}{2} \int \int \int_V \rho \mathbf{J} \cdot \mathbf{J}^* dV = \frac{1}{2} \int \int \int_V \rho |J|^2 dV, \tag{1.178}$$

where J and E are the amplitudes of the current density and the electric field intensity, respectively. The time-average power loss density P_v is defined as the total time-average power loss P_D per unit volume

$$P_v = \frac{P_D}{V}, \tag{1.179}$$

where V is the volume carrying the current.

Since $\mathbf{B} = \mu\mathbf{H}$, the point (local) magnetic energy density for sinusoidal waveforms is given by

$$w_m(x) = \frac{1}{2} \mathbf{B} \cdot \mathbf{H}^* = \frac{1}{2} \mu \mathbf{H} \cdot \mathbf{H}^* = \frac{1}{2} \mu |H(x)|^2. \tag{1.180}$$

The maximum magnetic energy stored in inductor L is given by

$$W_m = \int i v dt = \int i L \frac{di}{dt} dt = L \int_0^{I_m} i di = \frac{1}{2} L I_m^2 = \frac{1}{2} \frac{N^2}{\mathcal{R}} I_m^2 = \frac{\mathcal{F}_m^2}{2\mathcal{R}}$$
$$= \frac{1}{2} \left(\frac{\mu N^2 S}{l} \right) \left(\frac{Bl}{\mu N} \right)^2 = \frac{1}{2} \int \int \int_V \mu |H(x)|^2 dV, \tag{1.181}$$

where V is the volume of the interior of the inductor and $v = Ldi/dt$. The magnetic energy density w_m is defined as the magnetic energy W_m per unit volume

$$w_m = \frac{W_m}{V}. \tag{1.182}$$

The time-average local magnetic energy density is given by

$$w_m(x) = \frac{1}{2}\mathbf{B} \cdot \mathbf{H}^* = \frac{1}{2}\mu\mathbf{H} \cdot \mathbf{H}^* = \frac{1}{2}\mu|H(x)|^2 \text{ (J/m}^3\text{)}. \tag{1.183}$$

The total time-average magnetic energy is

$$W_m = \frac{1}{2}\int\int\int_V \mathbf{B(x)} \cdot \mathbf{H}^*(\mathrm{x})dV = \frac{1}{2}\int\int\int_V \mu|H(x)|^2 dV \text{ (J)}. \tag{1.184}$$

The time-average magnetic energy is the average energy per unit time over a period of time and is

$$W_m = \frac{1}{4}L|I|^2 = \frac{1}{4}\int\int\int_V \mathbf{B(x)} \cdot \mathbf{H}^*(\mathrm{x})dV = \frac{1}{4}\int\int\int_V \mu|H(x)|^2 dV \text{ (J)}. \tag{1.185}$$

1.6 Eddy Currents

Eddy currents were discovered by Foucault in 1851. Therefore, they are also called Foucault's currents. Figure 1.12 illustrates the eddy current $i_E(t)$ of density $J_e(t)$ induced by a time-varying magnetic field $H(t)$. Eddy currents circulate in closed paths. In a conductor, the induced magnetic field $H(t)$ may be caused by the Conductor's own AC current or by the AC current flowing in adjacent conductors.

In accordance with Faraday's law, a time-varying magnetic field $H(t) = B(t)/\mu$ induces an electric field E

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.186}$$

According to Ohm's law, the electric field induces an eddy current

$$\mathbf{J}_e = \sigma\mathbf{E} = \frac{\mathbf{E}}{\rho}. \tag{1.187}$$

According to Ampère's law, eddy currents induce a magnetic field $H(t)$, as shown in Fig. 1.12. These currents are similar to a current flowing in turns of a multilayer solenoid, and therefore produce a magnetic field. According to Lenz's law, the induced magnetic field opposes the applied magnetic field.

The magnetic field induces an EMF $v(t) = d\phi/dt = A(dB/dt) = A\mu(dH/dt)$ in a conducting material of conductivity σ, which in turn produces eddy currents $i_E(t)$. The flow of eddy currents causes power losses in winding conductors and magnetic cores.

The applied time-varying magnetic field $H_a(t)$ induces the electric field $E(t)$, which induces eddy currents $i_E(t)$, and these currents generate a time-varying magnetic field $H(t)$ that opposes the original

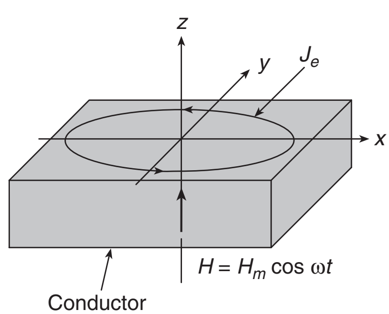

Figure 1.12 Eddy current

applied magnetic field $H_a(t)$, according to Lenz's law. The direction of the induced eddy currents $i_E(t)$ with respect to the induced magnetic field $H(t)$ is determined by the RHR, as shown in Fig. 1.12. The opposing magnetic flux can be found using Ampère's law

$$\nabla \times \mathbf{H} = \mathbf{J}_a + \mathbf{J}_e, \tag{1.188}$$

where $\mathbf{J}_a$ is the applied current density and $\mathbf{J}_e$ is the eddy current density. When the applied current $\mathbf{J}_a$ is zero and the magnetic field is generated by adjacent conductors, Ampère's law becomes

$$\nabla \times \mathbf{H} = \mathbf{J}_e. \tag{1.189}$$

The eddy-current density in a conductor of conductivity $\sigma = 1/\rho$ can be described by

$$\mathbf{J}_e = \sigma \mathbf{E} = \frac{\mathbf{E}}{\rho} \tag{1.190}$$

For sinusoidal waveforms, the phasor of the eddy-current density is given by

$$\mathbf{J}_e = -j\omega\sigma\mathbf{A}, \tag{1.191}$$

where $\mathbf{A}$ is the phasor of the magnetic vector potential. Eddy currents can be reduced using selecting high-resistivity materials (such as ferrites) or using thin plates, called laminations. These currents dissipate energy as heat in magnetic cores and winding conductors of inductors and transformers.

There are two effects associated with eddy currents: skin effect and proximity effect. The skin effect current density J_s is orthogonal to the proximity effect current density J_p. The winding power loss due to eddy currents is

$$P = \frac{1}{2\sigma}\int\int\int (J_s J_s^* + J_p J_p^*) dV. \tag{1.192}$$

Eddy currents are driven by a voltage induced in a conductor by the magnetic field. According to Faraday's law, the voltage induced in a conductor by the magnetic flux $\phi = AB = A\mu B$ is

$$v(t) = \frac{d\phi}{dt} = \frac{d(AB)}{dt} = A\frac{dB}{dt} = A\mu\frac{dH}{dt}. \tag{1.193}$$

Eddy currents flow in a plane perpendicular to the magnetic field density $\mathbf{B}$. From Ohm's law, the rms value of the eddy current in a conductor of resistance R is

$$I = \frac{V}{R}. \tag{1.194}$$

The power loss caused by sinusoidal eddy currents in a conductor of resistance R is

$$P_1 = \frac{V^2}{R} = \frac{A^2}{R}\left(\frac{dB}{dt}\right)^2. \tag{1.195}$$

Let us divide the conductor into two insulated parts so that the surface of one part $A_2 = A/2$. Hence, the resistance of one-half of the conductor is

$$R_2 = \frac{\rho l}{A_2} = \frac{2\rho l}{A} = 2R. \tag{1.196}$$

The voltage induced in a conductor by the magnetic flux $\phi = A_2 B$ is

$$V_2 = \frac{d\phi}{dt} = \frac{d(A_2 B)}{dt} = A_2\frac{dB}{dt} = \frac{A}{2}\frac{dB}{dt}. \tag{1.197}$$

The power loss caused by sinusoidal eddy currents in one-half of the conductor of resistance R_2 is

$$P_{2(1)} = \frac{V_2^2}{R_2} = \frac{1}{2R}\frac{A^2}{4}\left(\frac{dB}{dt}\right)^2 = \frac{A^2}{8R}\left(\frac{dB}{dt}\right)^2. \tag{1.198}$$

The power loss caused by sinusoidal eddy currents in both parts of the conductor is

$$P_2 = 2P_{2(1)} = \frac{A^2}{4R}\left(\frac{dB}{dt}\right)^2. \tag{1.199}$$

Let us divide the conductor into n parts so that $A_n = A/n$. Then, the power loss caused by sinusoidal eddy currents in all n parts of the conductor is

$$P_n = \frac{nA_n^2}{R_n}\left(\frac{dB}{dt}\right)^2 = \frac{n\left(\frac{A}{n}\right)^2}{nR}\left(\frac{dB}{dt}\right)^2 = \frac{A^2}{n^2R}\left(\frac{dB}{dt}\right)^2 = \frac{P_1}{n^2}. \tag{1.200}$$

Thus, the eddy-current power loss decreases by a factor of n^2 when the area of the conductor perpendicular to the magnetic flux is divided into n laminations, electrically insulated by the oxide. Laminations should be parallel to the magnetic field density **B**. Laminated magnetic cores are used to reduce eddy-current loss by reducing the magnitude of eddy currents.

Example 1.3

An infinitely long round solid straight wire of radius r_o carries sinusoidal current $i = I_m \cos \omega t$ in steady state at low frequencies (with no skin effect), as described in Example 1.1 and depicted in Fig. 1.7. Determine the eddy-current power loss, equivalent resistance, and optimum wire diameter.

Solution: The waveform of the induced voltage (EMF) is

$$V(x,t) = \frac{d\phi}{dt} = -\omega\mu_0 I_m \frac{r^3}{2r_o^2} \sin \omega t \quad \text{for} \quad 0 \le r \le r_o. \tag{1.201}$$

Hence, the amplitude of the induced voltage (EMF) inside the wire at low frequencies is

$$V_m(r) = \omega\phi_m(r) = \omega\mu_0 B_m(r) = \omega\mu_0 A H_m(r) = \omega\mu_0 I_m \frac{r^3}{2r_o^2} \quad \text{for} \quad 0 \le r \le r_o. \tag{1.202}$$

The cylindrical shell of radius r and thickness dr has a cross-sectional area $A_{sh} = l_w dr$ and the length of the current path $l_i = 2\pi r$. Hence, the resistance of the cylindrical shell is

$$R_{dr}(r) = \frac{\rho_w l_i}{A_{sh}} = \frac{2\pi r \rho_w}{l_w dr}. \tag{1.203}$$

The time-average power loss in the cylinder is

$$dP_e = \frac{V_m^2(r)}{2R_{dr}} = \left(\omega\mu_0 I_m \frac{r^3}{2r_o^2}\right)^2 \frac{l_w dr}{4\pi r \rho_w} = \frac{\omega^2\mu_0^2 I_m^2 l_w}{16\pi \rho_w r_o^4} r^5 dr \quad \text{for} \quad 0 \le r \le r_o. \tag{1.204}$$

The time-average eddy-current power loss inside the entire wire is

$$P_e = \int_0^{r_o} dP_e = \frac{\mu_0^2\omega^2 I_m^2 l_w}{16\pi \rho_w r_o^4} \int_0^{r_o} r^5 dr = \frac{\mu_0^2\omega^2 I_m^2 l_w r_o^2}{96\pi \rho_w} = \frac{\mu_0^2\omega^2 I_m^2 l_w d^2}{384\pi \rho_w}, \tag{1.205}$$

where $d = 2r_o$. The eddy-current power loss in terms of the equivalent eddy-current power loss resistance R_e is

$$P_e = \frac{1}{2} R_e I_m^2. \tag{1.206}$$

Thus,

$$\frac{1}{2} R_e I_m^2 = \frac{\mu_0^2\omega^2 I_m^2 l_w r_o^2}{96\pi \rho_w} = \frac{\mu_0^2\omega^2 I_m^2 l_w d^2}{384\pi \rho_w}. \tag{1.207}$$

Hence, the equivalent eddy-current resistance is

$$R_e = \frac{\mu_0^2\omega^2 l_w r_o^2}{48\pi \rho_w} = \frac{\mu_0^2\omega^2 l_w d^2}{192\pi \rho_w}. \tag{1.208}$$

The low-frequency resistance is

$$R_{wDC} = \frac{\rho_w l_w}{\pi r_o^2} = \frac{4\rho_w l_w}{\pi d^2}. \tag{1.209}$$

The total resistance of the wire is

$$R_w = R_{wDC} + R_e = \frac{\rho_w l_w}{\pi r_o^2} + \frac{\mu_0^2 \omega^2 l_w r_o^2}{48\pi \rho_w} = \frac{4\rho_w l_w}{\pi d^2} + \frac{\mu_0^2 \omega^2 l_w d^2}{192\pi \rho_w} = \frac{4\rho_w l_w}{\pi d^2}\left(1 + \frac{\mu_0^2 \omega^2 d^4}{768\rho_w^2}\right)$$

$$= \frac{\rho_w l_w}{\pi}\left(\frac{1}{r_o^2} + \frac{\pi^2 \mu_0^2 \omega^2 r_o^2}{48\rho_w^2}\right) = \frac{4\rho_w l_w}{\pi}\left(\frac{1}{d^2} + \frac{\mu_0^2 \omega^2 d^2}{768\rho_w^2}\right) = \frac{4\rho_w l_w}{\pi}\left(\frac{1}{d^2} + \frac{d^2}{192\delta_w^4}\right). \quad (1.210)$$

As the conductor diameter d is increased, the conductor DC resistance R_{DC} increases and the eddy-current resistance R_e decreases. Therefore, there is an optimum conductor diameter at which the total conductor resistance R_w takes on a minimum value. The derivative of the total wire resistance with respect to its diameter d is

$$\frac{dR_w}{dd} = \frac{4\rho_w l_w}{\pi}\left(-\frac{2}{d^3} + \frac{\mu_0^2 \omega^2 d}{384\rho_w^2}\right) = \frac{4\rho_w l_w}{\pi}\left(-\frac{2}{d^3} + \frac{d}{96\delta_w^4}\right) = 0. \quad (1.211)$$

Hence, the optimum conductor diameter is

$$\frac{d_{opt}}{\delta_w} = \sqrt[4]{192} \approx 3.722. \quad (1.212)$$

The AC-to-DC resistance ratio is

$$F_R = \frac{R_w}{R_{wDC}} = 1 + \frac{\mu_0^2 \omega^2 d^4}{96\pi \rho_w^2} = 1 + \frac{d^4}{192\delta_w^4}. \quad (1.213)$$

The AC-to-DC resistance ratio at $d = d_{opt}$ is

$$F_{Rv} = 1 + 1 = 2. \quad (1.214)$$

The minimum total resistance of the conductor is

$$R_{wmin} = F_{Rv} R_{wDC} = 2R_{wDC}. \quad (1.215)$$

1.7 Core Saturation

Many inductors are made up using magnetic cores. A magnetic core is a conductor of magnetic field H. For an inductor with a magnetic core of cross-sectional area A_c and a saturation magnetic flux density B_s, the magnetic flux at which the magnetic core begins to saturate is

$$\phi_s = A_c B_s, \quad (1.216)$$

resulting in the maximum value of the magnetic flux density

$$B_{pk} = \frac{\phi_{pk}}{A_c} = \frac{\phi_{DC(max)} + \phi_{AC(max)}}{A_c} < B_s. \quad (1.217)$$

The saturated magnetic flux density $B_s = \mu_0 H_s$ is nearly constant. Therefore, $v_L = NA_c dB(t)/dt \approx 0$ and the inductor behaves almost like a short circuit. To avoid core saturation, one has to reduce the maximum value of the magnetic flux ϕ_{pk} in the core or increase the core cross-sectional area A_c. A nonuniform magnetic flux distribution in ferrite cores creates localized magnetic saturation and hot spots.

The magnetic flux linkage at which the magnetic core begins to saturate is given by

$$\lambda_s = N\phi_s = NA_c B_s = LI_{m(max)}. \quad (1.218)$$

Thus,

$$N_{max} A_c B_{pk} = LI_{m(max)} \quad (1.219)$$

yielding the maximum number of turns

$$N_{max} = \frac{LI_{m(max)}}{A_c B_{max}}. \tag{1.220}$$

According to (1.7), the magnetic field intensity H is proportional to $\mathcal{F} = Ni$. Therefore, there is a maximum amplitude of the inductor current $I_{m(max)}$ at which the core saturates. Figure 1.13 shows plots of B as functions of H and i. The saturation flux density is

$$B_s = \mu H_s = \frac{\mu_{rc}\mu_0 N I_{m(max)}}{l_c}. \tag{1.221}$$

To avoid core saturation, the *ampere-turn limit* is given by

$$N_{max} I_{m(max)} = \frac{B_s l_c}{\mu_{rc}\mu_0} = B_s A_c \mathcal{R} = \frac{B_s l_c}{\mu_{rc}\mu_0}. \tag{1.222}$$

To avoid core saturation, one has to reduce the peak inductor current $I_{m(max)}$ or to reduce the number of turns N to satisfy the condition for all operating conditions

$$\frac{\mu_{rc}\mu_0 N I_m}{l_c} < B_s. \tag{1.223}$$

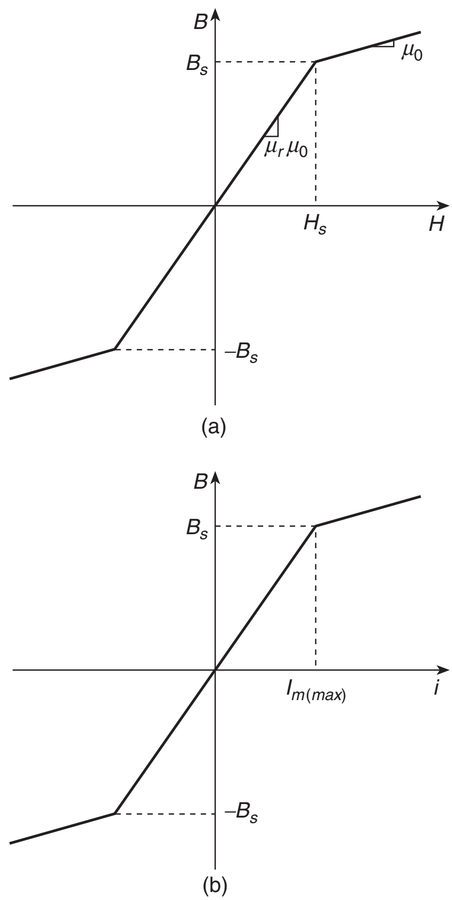

Figure 1.13 Magnetic flux density as functions of magnetic field intensity and inductor current. (a) Magnetic flux density B as a function of magnetic field intensity H. (b) Magnetic flux density B as a function of inductor current i at a fixed number of turns N

From Faraday's law, $d\lambda = v_L(t)dt$. Hence, the general relationship between the inductor voltage and the flux linkage is given by

$$\lambda(t) = \int_0^t v_L(t)dt + \lambda(0) = \frac{1}{\omega}\int_0^{\omega t} v_L(\omega t)d(\omega t) + \lambda(0). \tag{1.224}$$

For a transformer,

$$(N_1 i_1 + N_2 i_2 + \ldots)_{max} \le B_s A_c \mathcal{R} = \frac{B_s l_c}{\mu_{rc}\mu_0}. \tag{1.225}$$

It is important to avoid both local and global core saturation.

1.7.1 Core Saturation for Sinusoidal Inductor Voltage

Consider an inductor with a magnetic core of saturation flux density B_s. Figure 1.14 shows sinusoidal waveforms of the inductor voltage v_L and the magnetic flux linkage λ. The DC components of these waveforms are assumed to be zero. The inductor voltage is given by

$$v_L = V_{Lm} \sin \omega t. \tag{1.226}$$

The magnetic flux linkage is

$$\lambda(t) = \frac{1}{\omega}\int_0^{\omega t} v_L(\omega t)d(\omega t) + \lambda(0) = \frac{1}{\omega}\int_0^{\omega t} V_{Lm} \sin \omega t d(\omega t) + \lambda(0)$$
$$= \frac{V_{Lm}}{\omega}(1 - \cos \omega t) + \lambda(0) = \frac{V_{Lm}}{\omega} - \frac{V_{Lm}}{\omega}\cos \omega t + \lambda(0). \tag{1.227}$$

Thus, the peak-to-peak value of the magnetic flux linkage is

$$\Delta\lambda = \lambda(\pi) - \lambda(0) = \frac{2V_{Lm}}{\omega} = N\phi = NA_c B_m < NA_c B_s. \tag{1.228}$$

The initial value of the flux linkage is

$$\lambda(0) = -\frac{\Delta\lambda}{2} = -\frac{V_{Lm}}{\omega}. \tag{1.229}$$

The steady-state waveform of the magnetic flux linkage is given by

$$\lambda(t) = -\frac{V_{Lm}}{\omega}\cos \omega t = -\lambda_m \cos \omega t, \tag{1.230}$$

where the amplitude of the flux linkage is

$$\lambda_m = \frac{V_{Lm}}{\omega}. \tag{1.231}$$

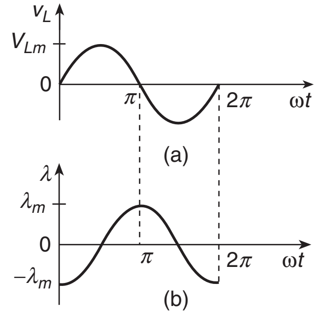

Figure 1.14 Waveforms of the square-wave inductor voltage and the corresponding magnetic flux linkage for sinusoidal inductor voltage. (a) Waveform of the inductor voltage v_L. (b) Waveform of the magnetic flux linkage λ

Thus, the amplitude of the magnetic flux linkage λ_m increases as the frequency f decreases. The minimum frequency f_{min} occurs when the amplitude of the magnetic flux linkage λ_m reaches the saturation value λ_s

$$\lambda_m = \lambda_s = \frac{V_{Lm(max)}}{\omega_{min}}. \tag{1.232}$$

The lowest frequency at which the inductor can operate without saturating the core is given by

$$f_{min} = \frac{V_{Lm(max)}}{2\pi\lambda_s} = \frac{V_{Lm(max)}}{2\pi N A_c B_s} = \frac{\sqrt{2}V_{Lrms(max)}}{2\pi N A_c B_s} = \frac{V_{Lrms(max)}}{K_f N A_c B_s} = \frac{V_{Lrms(max)}}{4.44 N A_c B_s}, \tag{1.233}$$

where the waveform factor for a sinusoidal inductor voltage is defined as the ratio of the rms value to the amplitude of the sinusoidal inductor voltage

$$K_f = \frac{V_{Lrms}}{V_{Lm}} = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2} = 4.44. \tag{1.234}$$

The minimum frequency f_{min} decreases, when N increases, A_c increases, B_s increases, and $V_{Lm(max)}$ decreases. As the temperature increases, B_s decreases. For ferrite cores, B_s may decrease by a factor of 2 as T increases from room temperature to 100 °C.

Another method to derive the minimum frequency is as follows. Assume that the initial condition is $\lambda(0) = -\lambda_s$. The magnetic flux linkage at core saturation is given by

$$\lambda_s = \frac{1}{\omega_{min}} \int_0^{\pi} v_L d(\omega t) + \lambda(0) = \frac{1}{\omega_{min}} \int_0^{\pi} V_{Lm} \sin\omega t d(\omega t) - \lambda_s = \frac{2V_{Lm}}{\omega_{min}} - \lambda_s, \tag{1.235}$$

resulting in

$$\lambda_s = \frac{\Delta\lambda_{max}}{2} = \frac{V_{Lm(max)}}{\omega_{min}}. \tag{1.236}$$

Hence, the lowest frequency at which the inductor can operate without saturating the core is given by

$$f_{min} = \frac{V_{Lm(max)}}{2\pi\lambda_s} = \frac{V_{Lm(max)}}{2\pi N A_c B_s}. \tag{1.237}$$

The maximum rms value of the sinusoidal voltage across an inductor is

$$V_{Lrms(max)} = \frac{\omega N A_c B_s}{\sqrt{2}} = \frac{2\pi f N A_c B_s}{\sqrt{2}} = \sqrt{2}\pi f N A_c B_s = K_f f N A_c B_s = 4.44 f N A_c B_s. \tag{1.238}$$

Example 1.4

A Philips 3F3 ferrite magnetic core material has the saturation flux density $B_s = 0.32$ T at temperature $T = 20$ °C. The core is expected to operate up to $T = 120$ °C. The core cross-sectional area is $A_c = 80 \times 10^{-6}$ m^2. The number of turns is $N = 10$. Find the maximum magnetic flux linkage and the minimum operating frequency for a sinusoidal voltage waveform with amplitude $V_{Lm} = 10$ V.

Solution: The saturation flux density of the ferrite magnetic core material is $B_s = 0.16$ T at temperature $T = 120$ °C. Hence, the saturation magnetic linkage is

$$\lambda_s = N\phi_s = N A_c B_s = 10 \times 80 \times 10^{-6} \times 0.16 = 128 \times 10^{-6} \text{ V} \cdot \text{s}. \tag{1.239}$$

The minimum frequency without core saturation is

$$f_{min} = \frac{V_{Lm(max)}}{2\pi\lambda_s} = \frac{10}{2\pi \times 128 \times 10^{-6}} = 12.434 \text{ kHz}. \tag{1.240}$$

1.7.2 Core Saturation for Square-Wave Inductor Voltage

If the inductor voltage waveform is a square wave $\pm V$, the magnetic flux linkage is a symmetrical triangular wave, as shown in Fig. 1.15. For the first half of the cycle,

$$v_L = V \quad \text{for} \quad 0 \leq t \leq \frac{T}{2} \tag{1.241}$$

and

$$\lambda(t) = \int_0^t v_L(t)dt + \lambda(0) = \int_0^t Vdt + \lambda(0) = Vt + \lambda(0) \quad \text{for} \quad 0 \leq t \leq \frac{T}{2}. \tag{1.242}$$

The flux linkage at $t = T/2$ is

$$\lambda\left(\frac{T}{2}\right) = \frac{VT}{2} + \lambda(0). \tag{1.243}$$

For the second half of the cycle,

$$v_L = -V \quad \text{for} \quad \frac{T}{2} \leq t \leq T \tag{1.244}$$

and

$$\lambda(t) = \int_{T/2}^t v_L(t)dt + \lambda\left(\frac{T}{2}\right) = \int_{T/2}^t (-V)dt + \lambda\left(\frac{T}{2}\right)$$
$$= -V\left(t - \frac{T}{2}\right) + \lambda\left(\frac{T}{2}\right) \quad \text{for} \quad \frac{T}{2} \leq t \leq T. \tag{1.245}$$

Hence, the peak-to-peak value of the magnetic flux linkage is

$$\Delta\lambda = \lambda\left(\frac{T}{2}\right) - \lambda(0) = \frac{VT}{2} + \lambda(0) - \lambda(0) = \frac{VT}{2} = \frac{V}{2f}, \tag{1.246}$$

$$-\lambda_m = \lambda(0) = -\frac{\Delta\lambda}{2} = -\frac{VT}{4} = -\frac{V}{4f}, \tag{1.247}$$

and

$$\lambda_m = \lambda\left(\frac{T}{2}\right) = \frac{\Delta\lambda}{2} = \frac{VT}{4} = \frac{V}{4f}. \tag{1.248}$$

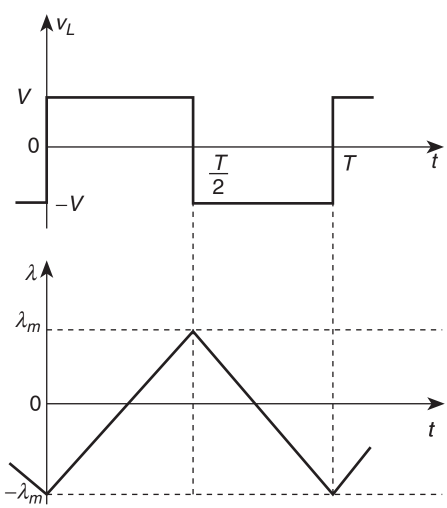

Figure 1.15 Waveforms of the square-wave inductor voltage and the corresponding magnetic flux linkage

The steady-state waveform of the magnetic linkage is

$$\lambda(t) = Vt - \frac{V}{4f} \quad \text{for} \quad 0 \le t \le \frac{T}{2} \tag{1.249}$$

and

$$\lambda(t) = -V\left(t - \frac{T}{2}\right) + \frac{V}{4f} \quad \text{for} \quad \frac{T}{2} \le t \le T. \tag{1.250}$$

The rms value of the square-wave inductor voltage is obtained as

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v_L^2 dt} = \sqrt{\frac{1}{T}\int_0^T V^2 dt} = V. \tag{1.251}$$

For the core saturation,

$$\lambda_s = \frac{\Delta\lambda_m}{2} = \frac{V_{max}}{4f_{min}} = N\phi_s = NA_cB_s, \tag{1.252}$$

where $V_{max} = V_{Lrms(max)}$ for the square-wave inductor voltage. The minimum frequency at which the core can be operated without saturation is given by

$$f_{min} = \frac{V_{max}}{4\lambda_s} = \frac{V_{max}}{4NA_cB_s} = \frac{V_{max}}{K_fNA_cB_s}, \tag{1.253}$$

where the waveform factor of the square-wave inductor voltage is

$$K_f = 4. \tag{1.254}$$

The maximum peak voltage of the square-wave inductor voltage at the operating frequency f is

$$V_{max} = V_{rms} = 4fNA_cB_s. \tag{1.255}$$

In general, the minimum core cross-sectional area is given by

$$A_c = \frac{V_{Lrms}}{K_ff_{min}NB_{pk}} = \frac{V_{Lrms}}{K_ff_{min}N(B_{DC} + B_m)}, \tag{1.256}$$

where

$$B_{pk} = B_{DC} + B_m \le B_s \quad \text{for} \quad T \le T_{max} \tag{1.257}$$

and K_f is the waveform coefficient of the inductor voltage. The peak value of the flux density B_{pk} must be lower than B_s at the maximum operating temperature T_{max} to avoid the core saturation. The amplitude of the AC component of the flux density B_m must be limited to avoid core saturation or to reduce core loss. As the amplitude of the AC component of the flux density B_m increases, the core loss also increases.

The saturation flux density B_s limits the maximum amplitude of the magnetic field intensity

$$H_s = H_{m(max)} = \frac{B_s}{\mu_{rc}\mu_0} = \frac{NI_{Lm(max)}}{l_c}. \tag{1.258}$$

The maximum amplitude of the current in the winding at which the core saturates is

$$I_{SAT} = I_{m(max)} = \frac{l_cB_s}{\mu_{rc}\mu_0N}. \tag{1.259}$$

As the amplitude of the inductor current I_{Lm} increases, the amplitude of the magnetic field H_m also increases. To avoid core saturation,

$$NI_{Lm(max)} < \frac{B_sl_c}{\mu_{rc}\mu_0}. \tag{1.260}$$

When a core with an air gap is used, both amplitudes H_m and I_{Lm} can be increased to

$$H_{m(max)} = \frac{B_s}{\mu_{re}\mu_0} \tag{1.261}$$

and

$$I_{SAT} = I_{Lm(max)} = \frac{l_cB_s}{N\mu_{re}\mu_0}, \tag{1.262}$$

where μ_{re} is the effective relative permeability of a gapped core.

1.7.3 Core Saturation for Rectangular Wave Inductor Voltage

Consider the situation, where the inductor voltage waveform is a rectangular wave whose high level is V_H and low level is $-V_L$, as depicted in Fig. 1.16. The magnetic flux linkage is an asymmetrical triangular wave, as shown in Fig. 1.16. For the first part of the cycle,

$$v_L = V_H \quad \text{for} \quad 0 \le t \le DT \tag{1.263}$$

and

$$\lambda(t) = \int_0^t v_L(t)dt + \lambda(0) = \int_0^t V_H dt + \lambda(0) = V_H t + \lambda(0) \quad \text{for} \quad 0 \le t \le DT, \tag{1.264}$$

where D is the duty cycle. The flux linkage at $t = DT$ is given by

$$\lambda(DT) = V_H DT + \lambda(0). \tag{1.265}$$

For the second part of the cycle,

$$v_L = -V_L \quad \text{for} \quad DT \le t \le T \tag{1.266}$$

and

$$\begin{aligned}\lambda(t) &= \int_{DT}^t v_L(t)dt + \lambda(DT) = \int_{DT}^t (-V_L)dt + \lambda(DT)\\ &= -V_L(t - DT) + \lambda(DT) \quad \text{for} \quad DT \le t \le T.\end{aligned} \tag{1.267}$$

Hence, the peak-to-peak value of the magnetic flux linkage is

$$\Delta\lambda = \lambda(DT) - \lambda(0) = V_H DT + \lambda(0) - \lambda(0) = V_H DT = \frac{DV_H}{f}. \tag{1.268}$$

The rms value of the inductor voltage is

$$V_{Lrms} = \sqrt{\frac{1}{T}\int_0^T v_L^2 dt} = \sqrt{\frac{1}{T}\left(\int_0^{DT} V_H^2 dt + \int_{DT}^T V_L^2 dt\right)} = \sqrt{DV_H^2 + (1-D)V_L^2}. \tag{1.269}$$

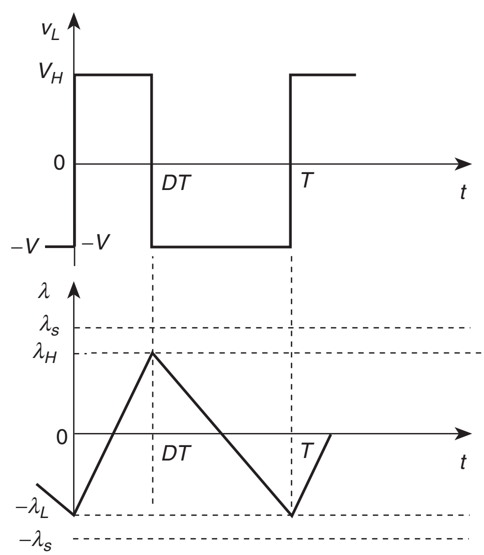

Figure 1.16 Waveforms of the rectangular inductor voltage and the corresponding magnetic flux linkage

Using the volt–second balance law, we obtain

$$V_H DT = V_L(1-D)T, \tag{1.270}$$

yielding

$$\frac{V_L}{V_H} = \frac{D}{1-D}. \tag{1.271}$$

Therefore,

$$V_{Lrms} = V_H\sqrt{\frac{D}{1-D}} = V_L\sqrt{\frac{1-D}{D}}. \tag{1.272}$$

The flux linkage at the beginning of the core saturation is

$$\lambda_s = \frac{\Delta\lambda_{max}}{2} = \frac{DV_H}{2f_{min}} = N\phi_s = NA_cB_s. \tag{1.273}$$

Hence, the minimum operating frequency is

$$f_{min} = \frac{DV_H}{2NA_cB_s} = \frac{V_{Lrms}}{NA_cB_s}\frac{\sqrt{D(1-D)}}{2} = \frac{V_{Lrms}}{K_f NA_cB_s}, \tag{1.274}$$

where the waveform coefficient is

$$K_f = \frac{2}{\sqrt{D(1-D)}}. \tag{1.275}$$

The minimum cross-sectional area is given by

$$A_c = \frac{V_{Lrms}}{K_{fmax} f B_s}. \tag{1.276}$$

Figure 1.17 shows a plot of K_f as a function of the duty cycle D. The minimum value of K_f occurs at $D = 0.5$. Figure 1.18 depicts $1/K_f$ as a function of D. The core cross-sectional area is proportional to $1/K_f$. The minimum value of the core cross-sectional area occurs at $D = 0.5$.

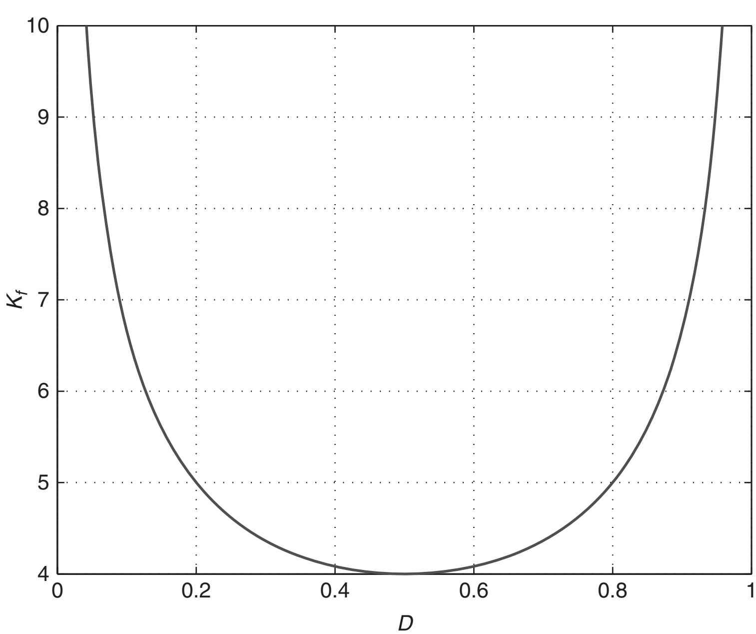

Figure 1.17 Waveform coefficient K_f as a function of duty cycle D

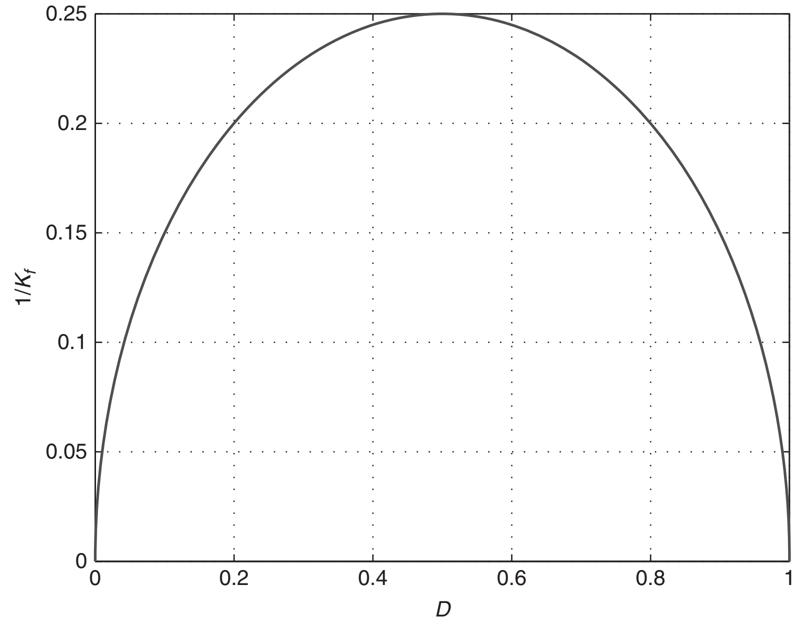

Figure 1.18 Coefficient $1/K_f$ as a function of duty cycle D

1.8 Inductance

1.8.1 Definitions of Inductance

An inductor is a two-terminal passive device that is able to store magnetic field and magnetic energy in this field. A coil is generally formed by winding a wire on a cylindrical former, called a bobbin. Any conductor has an inductance. The inductance depends on (i) winding geometry, (ii) core and bobbin geometry, (iii) permeability of the core material, and (iv) frequency. There are several methods to determine the inductance.

Magnetic Flux Linkage Method

The *inductance* (or *self-inductance*) for linear inductors is defined as the ratio of the total magnetic flux linkage λ to the time-varying (AC) current i producing the flux linkage

$$L = \frac{\lambda}{i}. \tag{1.277}$$

The inductance of a linear inductor is a proportionality constant in the expression $\lambda = Li$. An inductor is linear if its magnetic field is placed in a linear medium and the geometry of an inductor does not change.

The total inductance of an inductor is made up of two components: an *external inductance* L_{ext} and an *internal inductance* L_{int}

$$L = \frac{\lambda_{ext}}{i} + \frac{\lambda_{int}}{i} = L_{ext} + L_{int}. \tag{1.278}$$

The external inductance $L_{ext} = \lambda_{ext}/i$ is due to the external magnetic energy stored in the magnetic field outside the conductor. This inductance is usually independent of frequency. The internal

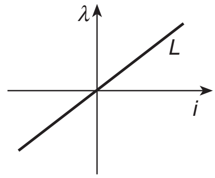

Figure 1.19 Magnetic flux linkage λ as a function of current i producing the flux linkage for linear inductors

inductance $L_{int} = \lambda_{int}/i$ is due to the magnetic energy stored in the internal magnetic field inside the conductor. This inductance depends on the frequency because the magnetic field intensity H distribution inside the conductor is a function of frequency due to skin effect. The internal inductance usually decreases with frequency.

A single conductor carrying an AC current i is linked by its own magnetic flux. For linear inductors, the flux linkage λ is proportional to the current i, resulting in $\lambda = Li$. The inductance L is the slope of the λ-i characteristic, as illustrated in Fig. 1.19. This characteristic is analogous to the resistor characteristic $v = Ri$ or the capacitor characteristic $Q = Cv$. A circuit that is designed to have a self-inductance is called an inductor. An inductor has a self-inductance of 1 H if a current of 1 A produces a flux linkage of 1 V·s (or 1 Wb·turn).

A change in the current flowing through the inductor produces an induced EMF, called an electromotance, or voltage

$$v = \oint_C \mathbf{E} \cdot d\mathbf{l} = \frac{d\lambda}{dt} = L\frac{di}{dt}. \tag{1.279}$$

An inductor has a self-inductance of 1 H if the change in the inductor current at a rate of 1 A/s produces a voltage difference between its terminals of 1 V. The inductance L is a function of the number of turns N, core permeability μ_{rc}, core geometry, and frequency f.

The inductance can be defined as

$$L = \frac{\lambda}{i} = \frac{N}{i}\int\int_S \mathbf{B} \cdot d\mathbf{S}. \tag{1.280}$$

The magnetic field produced by a current-carrying conductor links itself. The associated inductance is called a self-inductance. In some cases, the magnetic flux links only a part of the current and the inductance is defined as

$$L = \frac{1}{i}\int\int_S \frac{i_{enc}}{i} d\phi. \tag{1.281}$$

The voltage across the inductance is

$$v_L = \frac{d\lambda}{dt} = N\frac{d\phi}{dt} = N\frac{d\phi}{di_L}\frac{di_L}{dt} = L\frac{di_L}{dt}. \tag{1.282}$$

The self-inductance L relates the voltage induced in an inductor v_L to the time-varying current i_L flowing through the same inductor.

Reluctance Method

The inductance of an inductor can be determined using the core reluctance $\mathcal{R}$ or the core permeance $\mathcal{P}$

$$L = \frac{\lambda}{i} = \frac{N\phi}{i} = \frac{NA_cB}{i} = \frac{NA_c\mu H}{i} = \frac{NA_c\mu Ni}{l_c i} = \frac{A_c\mu}{l_c}N^2$$
$$= \frac{N^2}{\mathcal{R}} = \mathcal{P}N^2 = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c}. \tag{1.283}$$

If $N = 1$, $L = \mathcal{P} = 1/\mathcal{R}$.

Biot–Savart's Law Method

Using (1.101), the total magnetic field density **B** induced by a current I is obtained

$$\mathbf{B} = \mu\mathbf{H} = \frac{I}{4\pi}\int_l \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2}. \tag{1.284}$$

The magnetic flux inside the core with a cross-sectional area A_c is

$$\phi = A_c B = \frac{IA_c}{4\pi}\int_l \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2}. \tag{1.285}$$

The magnetic flux linkage is

$$\lambda = N\phi = NA_c B = \frac{IA_c N}{4\pi}\int_l \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2}. \tag{1.286}$$

Hence, the inductance is

$$L = \frac{\lambda}{I} = \frac{A_c N}{4\pi}\int_l \frac{d\mathbf{l} \times \mathbf{a}_R}{R^2}. \tag{1.287}$$

Magnetic Energy Method

The instantaneous magnetic energy stored in a magnetic device is

$$W_m = \frac{1}{2}LI_m^2 = \frac{1}{2}\int\int\int_V (\mathbf{B}\cdot\mathbf{H}^*)dV, \tag{1.288}$$

yielding the inductance

$$L = \frac{2W_m}{I_m^2} = \frac{1}{I_m^2}\int\int\int_V (\mathbf{B}\cdot\mathbf{H}^*)dV, \tag{1.289}$$

where I_m is the amplitude of the current flowing in the closed path, and W_m is the energy stored in the magnetic field produced by the current flowing through the inductor

$$W_m = \frac{\mu}{2}\int\int\int_V H^2 dV = \frac{1}{2\mu}\int\int\int_V B^2 dV. \tag{1.290}$$

For a linear inductor, $B = \mu H$ and

$$W_m = \frac{1}{2}LI_m^2 = \frac{1}{2}\int\int\int_V \mu H^2 dV = \frac{1}{2\mu}\int\int\int_V B^2 dV, \tag{1.291}$$

resulting in

$$L = \frac{2W_m}{I_m^2} = \frac{1}{I_m^2}\int\int\int_V \mu H^2 dV = \frac{1}{\mu I_m^2}\int\int\int_V B^2 dV. \tag{1.292}$$

The magnetic energy method to determine the inductance is impractical in many situations because of the lack of finite volume over which to integrate the magnetic field.

Example 1.5

Internal Self-Inductance of Round Conductor Determine the internal self-inductance of a round solid conductor of radius r_o and length l at low frequencies.

Solution: The cylindrical coordinates (r, φ, z) will be used to solve this problem. Assume that a sinusoidal current $i = I_m \sin\omega t$ flows through the conductor. From Example 1.1, the magnetic field intensity inside the conductor with the current amplitude I_m at low frequencies is given by

$$H_m(r) = \frac{I_m}{2\pi r_o^2}r \quad \text{for} \quad 0 \le r \le r_o \tag{1.293}$$

and the magnetic flux density at low frequencies is

$$B_m(r) = \mu H_m(r) = \frac{\mu_r \mu_0 I_m}{2\pi r_o^2} r \quad \text{for} \quad 0 \le r \le r_o. \tag{1.294}$$

Note that $dy = r\tan(d\varphi) \approx rd\varphi$ and $dV = (dr)(dy)(dz) = (dr)(rd\varphi)(dz) = rdrd\varphi dz$. The internal self-inductance of round solid wire is

$$\begin{aligned} L_{int} &= \frac{1}{I_m^2}\int\int\int_V \mathbf{H}\cdot\mathbf{B}dV = \frac{1}{I_m^2}\int\int\int_V \mu H_m(r)^2 dV = \frac{1}{I_m^2}\int\int\int_V \frac{\mu I_m^2 r^2}{(2\pi r_o)^2}dV \\ &= \frac{\mu}{(2\pi r_o^2)^2}\int_0^l\int_0^{2\pi}\int_0^{r_o} r^2 rdrd\varphi dz = \frac{\mu}{(2\pi r_o^2)^2}\int_0^l dz\int_0^{2\pi} d\varphi\int_0^{r_o} r^3 dr = \frac{\mu l}{8\pi}. \end{aligned} \tag{1.295}$$

For copper conductors, $L_{int}/l = \mu_0/(8\pi) = 4\pi \times 10^{-7}/(8\pi) = 10^{-7}/2 = 50$ nH/m $= 0.5$ nH/cm. At high frequencies, the current density is not uniform, the magnetic field density does not increase linearly inside the conductor and is altered, the energy stored in the wire decreases, and the internal inductance decreases with frequency due to skin effect, as explained in Chapter 3.

The amplitude of the magnetic field intensity outside the conductor is

$$H_m(r) = \frac{I_m}{2\pi r} \quad \text{for} \quad r \ge r_o. \tag{1.296}$$

The external inductance is

$$\begin{aligned} L_{ext} &= \frac{1}{I_m^2}\int\int\int_V \mu H_m^2 dV = \frac{1}{I_m^2}\int\int\int_V \frac{\mu I_m^2}{4\pi^2 r^2}dV = \frac{\mu}{4\pi^2}\int_0^l dz\int_0^{2\pi}\int_{rO}^{r_1}\frac{dr}{r} \\ &= \frac{\mu l}{2\pi}(\ln r_1 - \ln r_o) = \frac{\mu l}{2\pi}\ln\left(\frac{r_1}{r_o}\right). \end{aligned} \tag{1.297}$$

As $r_1 \to \infty$, $L_{ext} \to \infty$.

Vector Magnetic Potential Method

The inductance can be determined using the vector magnetic potential **A**

$$L = \frac{1}{I_m^2}\int\int\int_V \mathbf{A}\cdot\mathbf{J}dV. \tag{1.298}$$

The vector magnetic potential is given by

$$\mathbf{A}\ (\mathbf{r}) = \frac{\mu}{4\pi}\int\int\int_V \frac{\mathbf{J}(\mathbf{r})}{R}dV. \tag{1.299}$$

Hence, the inductance is given by

$$L = \frac{1}{I^2}\int\int\int_V \left[\frac{\mu}{4\pi}\int\int\int_V \frac{\mathbf{J}(\mathbf{r})}{R}dV\right]\cdot\mathbf{J}(\mathbf{r})dV. \tag{1.300}$$

Small-Signal Inductance

For nonlinear inductors, $\lambda = f(i)$ is a nonlinear function. The small-signal (or incremental) inductance of a nonlinear inductor is defined as the ratio of the infinitesimal change in the flux linkage $d\lambda$ to the infinitesimal change in the current dl producing it at a given operating point $Q(I_{DC}, \lambda_{DC})$

$$L = \left.\frac{d\lambda}{di}\right|_Q. \tag{1.301}$$

Inductors with ferrous cores are nonlinear because the permeability depends on the applied magnetic field H. Figure 1.20 shows a plot of the magnetic flux linkage λ as a function of current i for nonlinear inductors. At low values of current, the core is not saturated and the relative permeability is high,

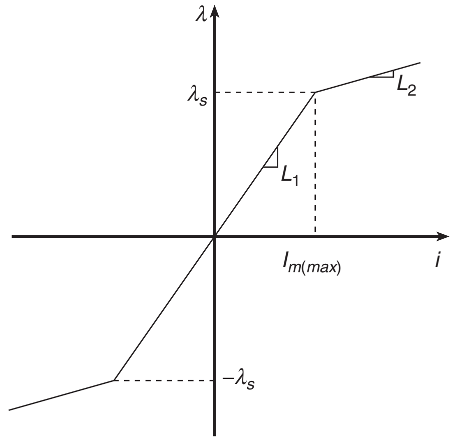

Figure 1.20 Magnetic flux linkage λ as a function of current i producing the flux linkage for nonlinear inductors

resulting in a high slope of the λ-i curve and a large inductance L_1. When the core saturates, the relative permeability μ_{rc} becomes equal to 1, the slope of the λ-i curve decreases, and the inductance decreases to a lower value L_2.

Example 1.6

An inductor is wound on a CC core (see Fig. 2.9) whose cross-sectional area is 2 cm × 2 cm, $l_c =$ 16 cm, core window is 3 cm × 3 cm, and $\mu_{rc} = 100$. The inductor has 10 turns. The core has no air gap. There is a magnetic flux in the core ϕ_c and a leakage flux ϕ_l in the air around the winding. Estimate the inductance using the reluctance method.

Solution: The total magnetic flux consists of the magnetic flux inside the core ϕ_c and the leakage magnetic flux ϕ_l

$$\phi = \phi_c + \phi_l. \tag{1.302}$$

The total reluctance $\mathcal{R}$ is equal to the parallel combination of the core reluctance $\mathcal{R}_c$ and the leakage reluctance $\mathcal{R}_l$. Hence, the inductance is

$$L = \frac{\lambda}{i} = \frac{N\phi}{i} = \frac{N(\phi_c + \phi_l)}{i} = N^2\left(\frac{1}{R_c} + \frac{1}{R_l}\right) = \frac{N^2}{\mathcal{R}} = N^2\left(\frac{\mu_{rc}\mu_0 A_c}{l_c} + \frac{\mu_0 A_l}{l_l}\right)$$

$$= \frac{\mu_{rc}\mu_0 A_c N^2}{l_c}\left[1 + \frac{1}{\mu_{rc}}\left(\frac{A_l}{A_c}\right)\left(\frac{l_c}{l_l}\right)\right] = L_c\left[1 + \frac{1}{\mu_{rc}}\left(\frac{A_l}{A_c}\right)\left(\frac{l_c}{l_l}\right)\right]. \tag{1.303}$$

Let $l_l = l_c/2$ and $A_l = 4A_c$. In this case, the inductance is given by

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c}\left[1 + \frac{1}{\mu_{rc}}\left(\frac{A_l}{A_c}\right)\left(\frac{l_c}{l_l}\right)\right]$$

$$= \frac{100 \times 4\pi \times 10^{-7} \times 4 \times 10^{-4} \times 10^2}{16 \times 10^{-2}}\left(1 + \frac{4 \times 2}{100}\right) = 3.1416 \times 10^{-6}(1 + 0.08)$$

$$= 33.929\ \mu\text{H}. \tag{1.304}$$

The inductance is increased by 8% due to the leakage magnetic flux. As μ_{rc} increases, the leakage flux effect on the inductance is reduced.

1.8.2 Inductance of Solenoid

A solenoid is made up of insulated wire wound in the shape of a cylinder to obtain N turns. Each turn acts like a magnetic dipole. A series connection of winding turns is used to increase the magnetic flux density B. The magnetic field lines are parallel to the solenoid axis. The magnetic field outside the solenoid is nearly zero. Neglecting the end effects, the amplitude of the magnetic flux density inside a long solenoid is uniform and it is given by

$$B_m = \mu H_m = \frac{\mu N I_m}{l_c}. \tag{1.305}$$

The amplitude of the magnetic flux inside the solenoid is

$$\phi_m = A_c B_m = \frac{\mu N I_m A_c}{l_c} = \frac{\pi \mu N r^2 I_m}{l_c}. \tag{1.306}$$

The amplitude of the flux linkage is

$$\lambda_m = N\phi_m = \frac{\mu N^2 I_m A_c}{l_c} = \frac{\pi \mu_{rc}\mu_0 N^2 r^2 I_m}{l_c} \tag{1.307}$$

where $A_c = \pi r^2$ is the core cross-sectional area of a round core and r is the mean coil radius. A long, tightly wound solenoid can be modeled by an equivalent current sheet that carries total current NI_m. The magnetic field for an infinitely long solenoid is uniform throughout the inside of the solenoid. The inductance of a long solenoid (theoretically, almost infinitely long) with a core and without an air gap in the core at low frequencies is

$$L_\infty = \frac{\lambda_m}{I_m} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = \frac{\pi \mu_{rc}\mu_0 r^2 N^2}{l_c} = \frac{N^2}{l_c/(\mu_{rc}\mu_0 A_c)} = \frac{N^2}{\mathcal{R}}$$
$$= \frac{\mu_{rc}\mu_0 A_c N^2}{Np} = \frac{\mu_{rc}\mu_0 A_c N}{p} \quad \text{for} \quad l_c \gg 2d, \tag{1.308}$$

where $l_c = Np$ is the mean core length and p is the winding pitch, equal to the distance between the centers of two adjacent conductors, μ_{rc} is the relative permeability of the core, and N is the total number of turns. The inductance L is proportional to the core relative permeability μ_{rc}, the square of the number of turns N^2, and the ratio of the core cross-sectional area to the MPL A_c/l_c.

The inductance of intermediate length-to-radius ratio is lower than that of an infinitely long round solenoid. As the inductor length increases, the magnetic field intensity $H = Ni/l_c$ decreases. Therefore, the inductance and the magnetic energy density $w_m = \frac{1}{2}BH = \frac{1}{2}\mu H^2$ also decrease. The inductance and the magnetic energy density stored in the magnetic field are inversely proportional to the length of the magnetic field. As r/l_c increases, L/L_∞ decreases, where L_∞ is the inductance of an infinitely long solenoid. For example, $K = L/L_\infty = 0.85$ for $r/l_c = 0.2$, $K = 0.74$ for $r/l_c = 0.4$, $K = 0.53$ for $r/l_c = 1$, $K = 0.2$ for $r/l_c = 5$, and $K = 0.12$ for $r/l_c = 10$. For r/l_c up to 2 or 3, a first-order approximation is

$$K = \frac{L}{L_\infty} \approx \frac{1}{1 + 0.9\frac{r}{l_c}}, \tag{1.309}$$

resulting in $L \approx KL_\infty = L_\infty/(1 + 0.9r/l_c)$. The inductance of a round single-layer solenoid of a finite length l_c with intermediate length-to-resistance ratio can be approximated by Wheeler's or Nagaoka formula [44], which is correct to within 1% for $r/l_c < 1.25$ or $l_c/(2r) > 0.4$

$$L = \frac{L_\infty}{1 + 0.9\frac{r}{l_c}} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c\left(1 + 0.9\frac{r}{l_c}\right)} = \frac{\pi\mu_{rc}\mu_0 r^2 N^2}{l_c\left(1 + 0.9\frac{r}{l_c}\right)} = \frac{\pi\mu_{rc}\mu_0 r^2 N^2}{l_c + 0.9r} \ \text{(H)}$$
$$= \frac{0.4\pi^2\mu_{rc} r^2 N^2}{l_c + 0.9r} \ (\mu\text{H}) \quad \text{for} \quad \frac{r}{l_c} < 1.25. \tag{1.310}$$

Figure 1.21 shows a plot of L/L_∞ as a function of r/l_c. As the ratio of the external diameter to the internal diameter decreases, the inductance also decreases.

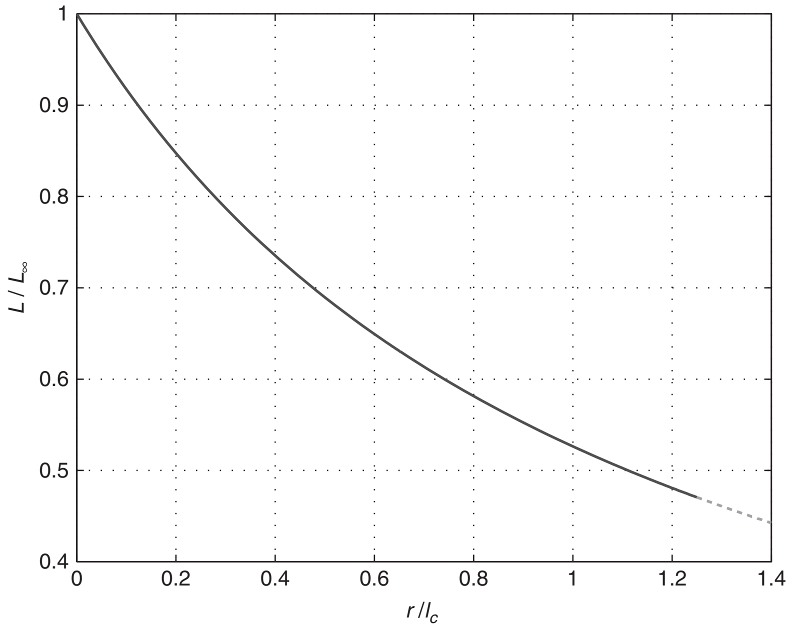

Figure 1.21 Plot of L/L_{∞} as a function of r/l_c

The inductance of a multilayer solenoid is given by

$$L = \frac{0.8\mu\pi r^2 N^2}{l_c + 0.9r + b}, \tag{1.311}$$

where b is the thickness of all layers (or coil build) and r is the average radius of the winding.

A more accurate equation for the inductance of a multilayer inductor is

$$L = \frac{\mu\pi r^2 N^2}{l_c} \frac{1}{1 + 0.9\left(\frac{r}{l_c}\right) + 0.32\left(\frac{b}{r}\right) + 0.84\left(\frac{b}{l_c}\right)}. \tag{1.312}$$

The inductance predicted by this equation is within 2% of the exact value.

The inductance of a short solenoid is given by

$$L = \mu r N^2 \left[\ln\left(\frac{8r}{a}\right) - 2\right] \tag{1.313}$$

where a is the wire radius and r is the outer radius of the solenoid and N is the number of turns. This equation is valid for $N = 1$.

Example 1.7

An air-core solenoid has $N = 20$, $l_c = 15$ cm, and $r = 3$ cm. Find the inductance.

Solution: The inductance of the solenoid is

$$L = \frac{\pi\mu_{rc}\mu_0 r^2 N^2}{l_c\left(1 + 0.9\frac{r}{l_c}\right)} = \frac{\pi \times 1 \times 4\pi \times 10^{-7} \times (3 \times 10^{-2})^2 \times 20^2}{15 \times 10^{-2}\left(1 + 0.9 \times \frac{3\times10^{-2}}{15\times10^{-2}}\right)} = 8.0295\ \mu\text{H}. \tag{1.314}$$

Note that the inductance calculated in this example is about 15% less than the inductance calculated for a very long inductor because $K = 0.8475$.

1.8.3 Inductance of Inductor with Toroidal Core

A toroidal inductor with a rectangular cross section is shown in Fig. 1.22. The dimensions of the magnetic core are: a is the inner radius, b is the outer radius, and h is the toroid height. The toroid is symmetrical about its axis. The RHR shows that the magnetic flux ϕ is mainly circulating in the φ direction. An idealized toroidal inductor can be thought as a finite length solenoid bent around to close on itself to form a doughnut shape. Assume that the inductor is closely wound coil. Consider a general contour C of radius r, $a \le r \le b$. Applying Ampère's law,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = NI. \tag{1.315}$$

It can be observed that

$$dl = rd\varphi. \tag{1.316}$$

Hence,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_0^{2\pi} Hrd\varphi = Hr \int_0^{2\pi} d\varphi = 2\pi rH. \tag{1.317}$$

Since the path of integration encircles the total current NI, we obtain

$$2\pi rH = NI. \tag{1.318}$$

Hence, the magnetic field intensity inside the toroidal core is

$$H = \frac{NI}{2\pi r} \quad \text{for} \quad a \le r \le b \tag{1.319}$$

and the magnetic flux density inside the toroidal core is given by

$$B = \mu H = \frac{\mu NI}{2\pi r} \quad \text{for} \quad a \le r \le b. \tag{1.320}$$

Since $dS = (dh)(dr)$, the magnetic flux inside the toroidal core is

$$\phi = \int \int_S BdS = \int_a^b \int_0^h \left(\frac{\mu NI}{2\pi r}\right)(dh)(dr) = \frac{\mu NI}{2\pi} \int_0^h dh \int_a^b \frac{dr}{r} = \frac{\mu NIh}{2\pi} \int_a^b \frac{dr}{r}$$
$$= \frac{\mu NIh}{2\pi} \ln\left(\frac{b}{a}\right). \tag{1.321}$$

The flux linkage of the toroidal inductor is

$$\lambda = N\phi = \frac{\mu h N^2 I}{2\pi} \ln\left(\frac{b}{a}\right), \tag{1.322}$$

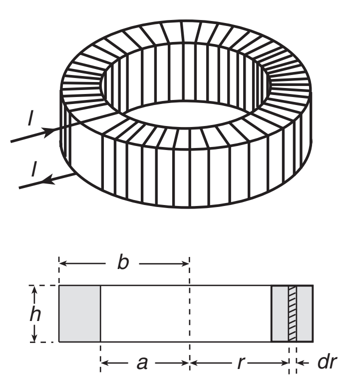

Figure 1.22 Toroidal inductor

resulting in the inductance of a toroidal coil

$$L = \frac{\lambda}{I} = \frac{\mu_{rc}\mu_0 h N^2}{2\pi} \ln\left(\frac{b}{a}\right). \tag{1.323}$$

Alternatively, we may assume that the magnetic flux density within the core is uniform and equal to the average value of the magnetic flux density

$$B = \frac{\phi}{A_c}, \tag{1.324}$$

where $A_c = h(b - a)$ is the cross-sectional area of the core. The magnetic field intensity is

$$H = \frac{B}{\mu} = \frac{\phi}{\mu A_c} = \frac{\phi}{\mu h(b - a)}. \tag{1.325}$$

Using Ampère's law,

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = NI \tag{1.326}$$

yielding

$$\frac{\phi l_c}{\mu A_c} = NI. \tag{1.327}$$

The mean radius of the core is $R = (a + b)/2$ and $l_c = 2\pi R = \pi(a + b)$ is the mean length of the magnetic flux. The magnetic flux inside the core is

$$\phi = \frac{\mu NI A_c}{l_c} = \frac{\mu NIh(b - a)}{\pi(b + a)}, \tag{1.328}$$

which gives the flux linkage

$$\lambda = N\phi = \frac{\mu Ih(b - a)N^2}{\pi(b + a)}, \tag{1.329}$$

Hence, the inductance is

$$L = \frac{\lambda}{I} = \frac{\mu h(b - a)N^2}{\pi(b + a)}. \tag{1.330}$$

An empirical equation for an inductor with a toroidal core is

$$L = 4.6\mu_{rc} N^2 h \log\left(\frac{OD}{ID}\right) \times 10^{-13} \text{ H}, \tag{1.331}$$

where H is the height of the core in meter, and ID and OD are the inner and outer diameters of the core in meters.

Example 1.8

An inductor is wound on a toroidal core, which has $\mu_{rc} = 150$, $h = 1$ cm, $a = 4$ cm, and $b = 5$ cm. The inductor has 20 turns. Find the inductance.

Solution: Using (1.322), the inductance is

$$L = \frac{\mu_{rc}\mu_0 h N^2}{2\pi} \ln\left(\frac{b}{a}\right) = \frac{150 \times 4\pi \times 10^{-7} \times 10^{-2} \times 20^2}{2\pi} \ln\left(\frac{5}{4}\right) = 26.777\ \mu\text{H}. \tag{1.332}$$

From (1.330),

$$L = \frac{\mu_{rc}\mu_0 h(b - a)N^2}{\pi(b + a)} = \frac{150 \times 2\pi \times 10^{-7} \times 10^{-2} \times (5 - 4) \times 10^{-2} 20^2}{\pi(5 + 4) \times 10^{-2}} = 26.667\ \mu\text{H}. \tag{1.333}$$

1.8.4 Inductance of Inductor with Torus Core

The inductance of a torus coil (with a round cross section) can be described by the expression for the inductance of a long solenoid

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = \frac{\mu_{rc}\mu_0 A_c N^2}{2\pi R}, \tag{1.334}$$

where $R = (a + b)/2$ is the mean radius of the core, $l_c = 2\pi R = \pi(a + b)$, $r_o = (b - a)/2$ is the radius of the core cross section, and $A_c = \pi r_o^2 = \pi(b - a)^2/4$ is the cross-sectional area of the core. Hence,

$$L = \frac{\mu_{rc}\mu_0 N^2 (b - a)^2}{4(a + b)}. \tag{1.335}$$

An alternative expression for the inductance of an inductor with a torus core is

$$L = \mu N^2 [R - \sqrt{R^2 - r_o^2}]. \tag{1.336}$$

1.8.5 Inductance of Inductor with Pot Core

The geometry of an inductor with a pot core is very complex and the inductance of these inductors can be determined only approximately. The core cross-sectional area of the pot core is approximately equal to the cross-sectional area of the center post

$$A_c = \frac{\pi d^2}{4}, \tag{1.337}$$

where d is the diameter of the center post. The average diameter of the mean magnetic path is given by

$$D_{av} = \frac{D_i + D_o}{2}, \tag{1.338}$$

where D_i is the inner diameter of the outer core area and D_o is the outer diameter of the outer core area. The mean MPL is given by

$$l_c = D_{av} + 4h = \frac{D_i + D_o}{2} + 4h, \tag{1.339}$$

where h is the height of the core halve. The inductance of an inductor with a pot core can be approximated by

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = \frac{\pi \mu_{rc}\mu_0 d^2 N^2}{2(D_i + D_o) + 16h}. \tag{1.340}$$

1.8.6 Inductance Factor

Equation (1.308) for the inductance can be written as

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c} = A_L N^2. \tag{1.341}$$

The *specific inductance* of a core, also called the *core inductance factor*, is defined as the inductance per single turn

$$A_L = \frac{L}{N^2} = \frac{\mu_{rc}\mu_0 A_c}{l_c} = \frac{1}{\mathcal{R}} = \mathcal{P} \left(\frac{\mathrm{H}}{\mathrm{turn}^2}\right). \tag{1.342}$$

Each core of different materials, shapes, and sizes will have a unique value of A_L, some of which are not easy to predict analytically, especially for complex core shapes. Core manufacturers give the values of A_L in data sheets.

The specific inductance (or the inductance index) A_L is usually specified in Henry per turn, in millihenry per 1000 turns, or in microhenry per 100 turns for cores without and with air gaps. If the specific inductance A_L is expressed in Henry per turn, the number of turns is given by

$$N = \sqrt{\frac{L(\mathrm{H})}{A_L}}. \tag{1.343}$$

If the specific inductance $A_{L(1000)}$ is expressed in millihenry per 1000 turns, the inductance is given by

$$L = \frac{A_{L(1000)}N^2}{(1000)^2}\ (\mathrm{mH}) \tag{1.344}$$

and the number of turns is

$$N = 1000\sqrt{\frac{L(\mathrm{mH})}{A_{L(1000)}}}. \tag{1.345}$$

For most ferrite cores, the specific inductance $A_{L(100)}$ is expressed in μH per 100 turns. In this case, the inductance is given by

$$L = \frac{A_{L(100)}N^2}{(100)^2}\ (\mu\mathrm{H}). \tag{1.346}$$

To compute the required number of turns N for a desired inductance L in microhenries, the following formula can be used for ferrite cores

$$N = 100\sqrt{\frac{L(\mu\mathrm{H})}{A_{L(100)}}}. \tag{1.347}$$

Common values of $A_{L(100)}$ are 16, 25, 40, 63, 100, 250, 400, and so on.

Air-core inductors are linear devices because the relationship $B = \mu_0 H$ is linear. In general, inductors with magnetic cores are nonlinear devices as the B–H relationship is nonlinear. However, for $B < B_s$, inductors can be modeled as linear devices.

Example 1.9

The relative permeability of the Ferroxcube ferrite magnetic core material is $\mu_{rc} = 1800$. The rectangular toroidal core made up of this material has the inner diameter $d = 13.1$ mm, the external diameter $D = 23.7$ mm, and the height $h = 7.5$ mm. Find the specific inductance of this core. What is the inductance of the inductor with this core if the number of turns is $N = 10$?

Solution: The MPL of a toroidal core is

$$l_c = \pi\frac{d + D}{2} = \pi\frac{13.1 + 23.7}{2} = 57.805\ \mathrm{mm} \tag{1.348}$$

and the cross-sectional area of the core is

$$A_c = h\frac{(D - d)}{2} = 7.5 \times 10^{-3} \times \frac{(23.7 - 13.1) \times 10^{-3}}{2} = 39.75 \times 10^{-6}\ \mathrm{m}^2. \tag{1.349}$$

Hence, the specific inductance of the core is

$$A_L = \frac{\mu_{rc}\mu_0 A_c}{l_c} = \frac{1800 \times 4\pi \times 10^{-7} \times 39.75 \times 10^{-6}}{57.805 \times 10^{-3}} = 1.5554\ \mu\mathrm{H/turn}\ . \tag{1.350}$$

The inductance at $N = 10$ is

$$L = N^2 A_L = 10^2 \times 1.5554 \times 10^{-6} = 155.54\ \mu\mathrm{H}. \tag{1.351}$$

1.9 Air Gap in Magnetic Core

1.9.1 Inductance

Gapped core inductors and transformers are useful in a variety of applications, particularly those in which core saturation has to be avoided. The introduction of an air gap into the core of an inductor permits much higher levels of magnetic flux density at the expense of considerably reduced inductance. The overall reluctance of the gapped core $\mathcal{R}$ can be controlled by an air gap length l_g. Therefore, magnetic flux ϕ, magnetic flux density B, and inductance L can also be controlled by the length of the air gap l_g. In addition, gapped cores exhibit enhanced thermal stability and more predictable effective permeability, the overall reluctance, and the inductance.

Air gaps can be bulk or distributed. In a gapped core, a small section of the magnetic flux path is replaced by a nonmagnetic medium, such as air or nylon. It is often filled with a spacer. The air-gap length l_g is usually twice the spacer thickness. Some cores have prefabricated air gaps. Standard values of the air-gap length l_g are 0.5, 0.6, 0.7, …, 5 mm. The same magnetic flux flows in the core and in the gap. Adding an air gap in a core is equivalent to adding a large gap reluctance in series with the core reluctance, that is, a series reluctor. As a result, the magnitude of the magnetic flux ϕ_m at a fixed value of NI_m is reduced. This effect is analogous to adding a series resistor in an electric circuit to reduce the magnitude of the current at a fixed source voltage.

Figure 1.23a illustrates an inductor whose core has an air gap. An equivalent magnetic circuit of an inductor with an air gap is shown in Fig. 1.23b. The reluctance of the air gap is

$$\mathcal{R}_g = \frac{l_g}{\mu_0 A_c}, \tag{1.352}$$

the reluctance of the core is

$$\mathcal{R}_c = \frac{l_c - l_g}{\mu_{rc}\mu_0 A_c} \approx \frac{l_c}{\mu_{rc}\mu_0 A_c}, \tag{1.353}$$

and the overall reluctance of the core with the air gap is

$$\mathcal{R} = \mathcal{R}_c + \mathcal{R}_g = \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{l_g}{\mu_0 A_c} = \frac{l_g + l_c/\mu_{rc}}{\mu_0 A_c} = \frac{l_c}{\mu_{rc}\mu_0 A_c}\left(1 + \frac{\mu_{rc} l_g}{l_c}\right) = F_g \mathcal{R}_c, \tag{1.354}$$

where the *air gap factor* is

$$F_g = \frac{\mathcal{R}}{\mathcal{R}_c} = \frac{\mathcal{R}_c + \mathcal{R}_g}{\mathcal{R}_c} = 1 + \frac{\mathcal{R}_g}{\mathcal{R}_c} = 1 + \frac{\mu_{rc} l_g}{l_c}. \tag{1.355}$$

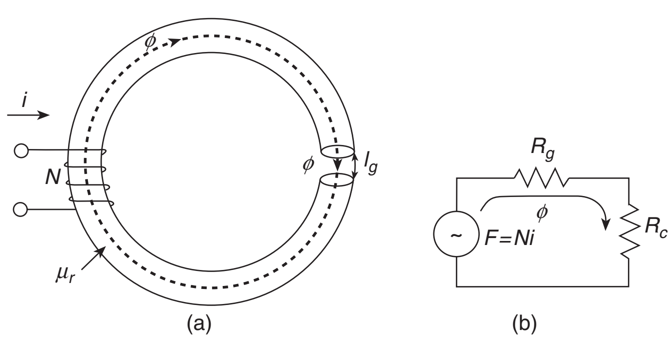

Figure 1.23 Inductor with an air gap. (a) Inductor. (b) Magnetic circuit of an inductor with an air gap

The inductance of a coil with a magnetic core having an air gap at low frequencies is expressed as

$$L = \frac{N^2}{\mathcal{R}} = \frac{N^2}{\mathcal{R}_g + \mathcal{R}_c} = \frac{N^2}{\mathcal{R}_g}\frac{1}{1+\mathcal{R}_c/\mathcal{R}_g} = \frac{N^2}{\frac{l_g}{\mu_0 A_c} + \frac{l_c}{\mu_{rc}\mu_0 A_c}} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c + \mu_{rc} l_g} = \frac{\mu_0 A_c N^2}{l_g + \frac{l_c}{\mu_{rc}}}$$
$$= \frac{\mu_0 A_c N^2}{l_g\left(1+\frac{l_c}{l_g \mu_{rc}}\right)} = \frac{N^2}{\mathcal{R}_g}\frac{1}{1+\frac{l_c}{l_g\mu_{rc}}} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c\left(1+\frac{\mu_{rc} l_g}{l_c}\right)} = \frac{\mu_{re}\mu_0 A_c N^2}{l_c} = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c F_g}, \quad (1.356)$$

where the *effective relative permeability* of a core with an air gap is

$$\mu_{re} = \frac{\mu_{rc}}{1+\frac{\mu_{rc} l_g}{l_c}} = \frac{\mu_{rc}}{F_g}. \quad (1.357)$$

For $\mu_{rc} l_g / l_c \gg 1$,

$$\mu_{re} \approx \frac{l_c}{l_g} \quad (1.358)$$

and

$$L \approx \frac{N^2}{\mathcal{R}_g} = \frac{\mu_0 A_c N^2}{l_g}. \quad (1.359)$$

Thus, the inductance with high-permeability cores is dominated by the air gap.

The air gap causes a considerable decrease in the effective relative permeability. However, it produces a more stable effective permeability and reluctance, resulting in a more predictable and stable inductance. The relative permittivity depends on the temperature. As the temperature increases, the relative permeability increases to reach a maximum value and then decreases to 1. In addition, the relative permeability has a large tolerance, typically $\mu_{rc} = \mu_{rc(nom)} \pm 25\%$. For example, inductors used in resonant circuits should be predictable and stable. Usually, at least 95% of the inductance comes from the air gap for high-permeability cores. The length of the air gap is given by

$$l_g = \frac{\mu_0 A_c N^2}{L} - \frac{l_c}{\mu_{rc}}. \quad (1.360)$$

The number of turns of an inductor whose core has an air gap is given by

$$N = \sqrt{\frac{L\left(l_g + \frac{l_c}{\mu_{rc}}\right)}{\mu_0 A_c}}. \quad (1.361)$$

For high-permeability cores, $l_g \gg l_c/\mu_{rc}$, $\mathcal{R}_g \gg \mathcal{R}_c$,

$$\mathcal{R} \approx \mathcal{R}_g, \quad (1.362)$$

and

$$L \approx \frac{\mu_0 A_c N^2}{l_g} = \frac{N^2}{\mathcal{R}_g}. \quad (1.363)$$

Therefore, the inductance of an inductor with a gapped core is inversely proportional to the air-gap length l_g and is almost independent of the core relative permeability μ_{rc}. The number of turns is

$$N \approx \sqrt{\frac{L l_g}{\mu_0 A_c}} \quad \text{for} \quad l_g \gg \frac{l_c}{\mu_{rc}}. \quad (1.364)$$

The core permeability varies with temperature and flux level. Inductors that carry DC currents and have DC magnetic flux require long air gaps to avoid saturation.

In some applications, the fringing effect of magnetic flux is reduced by multiple air gaps of length l_{g1}, l_{g2}, l_{g3},..., l_{gn} In this case, the total air-gap length is equal to sum of all air gaps

$$l_g = l_{g1} + l_{g2} + l_{g3} + ... + l_{gn}, \quad (1.365)$$

resulting in the effective relative permeability

$$\mu_{re} = \frac{\mu_{rc}}{1 + \frac{\mu_{rc}(l_{g1}+l_{g2}+\ldots+l_{gn})}{l_c}} \tag{1.366}$$

and the inductance of the inductor with the distributed air gap in the core

$$L = \frac{\mu_{rc}\mu_0 A_c N^2}{l_c\left[1 + \frac{\mu_{rc}(l_{g1}+l_{g2}+\ldots+l_{gn})}{l_c}\right]} = \frac{\mu_0 A_c N^2}{l_{g1} + l_{g2} + \ldots + l_{gn} + \frac{l_c}{\mu_{rc}}}. \tag{1.367}$$

1.9.2 Magnetic Field in Air Gap

Using Ampere's law, the MMF of the inductor with an air gap can be written as

$$\mathcal{F} = \int_{lc} \mathbf{H}\cdot d\mathbf{l} + \int_{lg} \mathbf{H}\cdot d\mathbf{l} = Ni = H_c l_c + H_g l_g = \frac{B_c l_c}{\mu_{rc}\mu_0} + \frac{B_g l_g}{\mu_0} = \frac{\phi_c l_c}{A_c \mu_{rc}\mu_0} + \frac{\phi_g l_g}{A_a \mu_0}$$
$$= \mathcal{R}_c\phi_c + \mathcal{R}_g\phi_g \approx (\mathcal{R}_c + \mathcal{R}_g)\phi_c. \tag{1.368}$$

The magnetic flux is

$$\phi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{\mathcal{F}}{\mathcal{R}_g + \mathcal{R}_c} = \frac{Ni}{\mathcal{R}_g + \mathcal{R}_c}. \tag{1.369}$$

The air gap reduces the amount of magnetic flux because it increases the overall reluctance. If $\mathcal{R}_g \gg \mathcal{R}_c$,

$$\phi \approx \frac{\mathcal{F}}{\mathcal{R}_g} = \frac{Ni}{\mathcal{R}_g}. \tag{1.370}$$

Neglecting the fringing flux, $A_g = A_c$, $\lambda_g = \lambda_c$, and $B_g = B_c$. Hence,

$$Ni = B_c\left(\frac{l_c}{\mu_{rc}\mu_0} + \frac{l_g}{\mu_0}\right) = \frac{B_c}{\mu_0}\left(\frac{l_c}{\mu_{rc}} + l_g\right). \tag{1.371}$$

The magnetic flux density in the core with an air gap is given by

$$B_c = B_g = \frac{\mu_0 Ni}{l_g + \frac{l_c}{\mu_{rc}}}. \tag{1.372}$$

The ratio of the magnetic flux density in the ungapped core to the magnetic flux density in the gapped core is

$$\frac{B_{c(ungapped)}}{B_{c(gapped)}} = \frac{l_c}{l_g + \frac{l_c}{\mu_{rc}}} \approx \frac{l_c}{l_g}. \tag{1.373}$$

The maximum flux density in the core with an air gap, which is caused by the DC component of the inductor current I_L and the amplitude of the AC component of the inductor current I_m, is expressed by

$$B_{c(pk)} = B_{DC} + B_m = \frac{\mu_0 N(I_L + I_m)}{l_g + \frac{l_c}{\mu_{rc}}} \le B_s \quad \text{for} \quad T \le T_{max}. \tag{1.374}$$

The magnetic flux density and the magnetic field intensity in the core are

$$B_c = \frac{\phi_c}{A_c} \tag{1.375}$$

and

$$H_c = \frac{B_c}{\mu_{rc}\mu_0}. \tag{1.376}$$

Assuming a uniform magnetic flux density in the air gap and neglecting the fringing flux, magnetic flux, magnetic flux density, and magnetic field intensity in the air gap are

$$\phi_g = \phi_c = A_c B_c = A_g B_g, \tag{1.377}$$

$$B_g = \frac{A_c}{A_g} B_c \approx B_c, \tag{1.378}$$

and

$$H_g = \frac{B_g}{\mu_0} = \frac{B_c}{\mu_0} = \frac{\mu_{rc}\mu_0 H_c}{\mu_0} = \mu_{rc} H_c. \tag{1.379}$$

The maximum MMF is

$$\mathcal{F}_{max} = N_{max} I_{Lmax} = \phi(\mathcal{R}_g + \mathcal{R}_c) = B_{pk} A_c (\mathcal{R}_g + \mathcal{R}_c) \approx B_{pk} A_c \mathcal{R}_g = \frac{B_{pk} l_g}{\mu_0}, \tag{1.380}$$

where $\mathcal{R}_g = l_g/(\mu_0 A_c)$. To avoid core saturation, the maximum number of turns is given by

$$N_{max} = \frac{B_{pk} l_g}{\mu_0 I_{Lmax}}. \tag{1.381}$$

As the air-gap length l_g increases, NI_{Lmmax} can be increased and the core losses can be decreased. However, the number of turns N must be increased to achieve a specified inductance L. The increased number of turns increases the winding loss. In addition, the leakage inductance increases and the air gap radiates a larger amount of electromagnetic interference (EMI).

The behavior of an inductor with an air gap is similar to an amplifier with negative feedback

$$A_f = \frac{A}{1 + \beta A} = \frac{\mu_{rc}}{1 + \frac{l_g}{l_c}\mu_{rc}}. \tag{1.382}$$

Thus, μ_{rc} is analogous to A and l_g/l_c is analogous to β.

Power losses associated with the air gap consist of winding loss, core loss, and hardware loss (e.g., power loss in clamps or bolts). The magnetic field around the core gap can be strong and create localized losses in the winding close to the gap.

Example 1.10

A PQ4220 Magnetics core has $\mu_{rc} = 2500$, $l_c = 4.63$ cm, and $A_c = 1.19$ cm^2. The inductor wound on this core has $N = 10$ turns. The required inductance should be $L = 55.6$ μH. Find the length of the air gap l_g.

Solution: The length of the air gap in the core is

$$l_g = \frac{\mu_0 A_c N^2}{L} - \frac{l_c}{\mu_{rc}} = \frac{4\pi \times 10^{-7} \times 1.19 \times 10^{-4} \times 10^2}{55.6 \times 10^{-6}} - \frac{4.63 \times 10^{-2}}{2500}$$

$$= (0.2689564 - 0.01852) \times 10^{-3} = 0.2504 \text{ mm}. \tag{1.383}$$

1.10 Fringing Flux

1.10.1 Fringing Flux Factor

A fringing flux is present around the air gap whenever the core is excited, as shown in Fig. 1.24. It induces currents in the winding conductors and other conductors, which may cause intense local heating in the vicinity of the gap. Figure 1.25 depicts the fringing flux in an inductor with an EE core and an air gap. The magnetic flux lines bulge outward because the magnetic lines repel each other when passing through a nonmagnetic material. As a result, the cross-sectional area of the magnetic field and the effective length are increased and the flux density is decreased. Typically, 10% is added to the air gap cross-sectional area. This effect is called the *fringing flux effect*. The percentage of the fringing flux in the total magnetic flux increases as the air-gap length l_g increases. As the gap length l_g is increased, the radius of the magnetic flux in the gap also increases. A rule of thumb is that foil

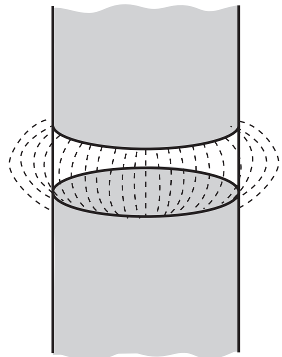

Figure 1.24 Fringing magnetic flux around the periphery of an air gap

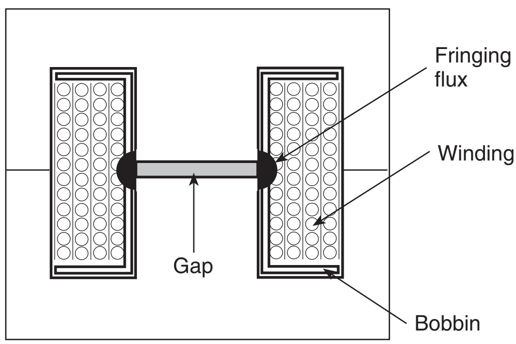

Figure 1.25 Fringing magnetic flux in an inductor with gapped pot core

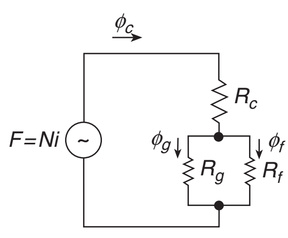

Figure 1.26 Magnetic equivalent circuit of an inductor with an air gap and a fringing magnetic flux

windings should be spaced at least two gap lengths away from the gap to prevent induction of eddy currents in the winding and to prevent overheating. The fringing flux is larger in inductors with a low-permeability core than that in high-permeability core. Figure 1.26 shows a magnetic equivalent circuit for the inductor with an air gap and a fringing flux. The fringing reluctance is shunting the gap reluctance, reducing the equivalent reluctance and slightly increasing the inductance. Thus, the number of turns N required for obtaining a desired inductance should be decreased or the air-gap length l_g should be increased. The effect of the fringing flux on the inductance and the number of turns can be investigated using the concept of the reluctance of the space conducting the fringing magnetic flux. Figure 1.27 shows the magnetic flux distribution in an air-gapped core inductor. The effective cross-sectional area of the gap with fringing flux A_{geff} is obtained by adding the gap length to each of the linear dimensions of the core in the gap area. For a rectangular core, $w_f = l_g$ and $A_{geff} = (a + l_w)(b + l_g)$.

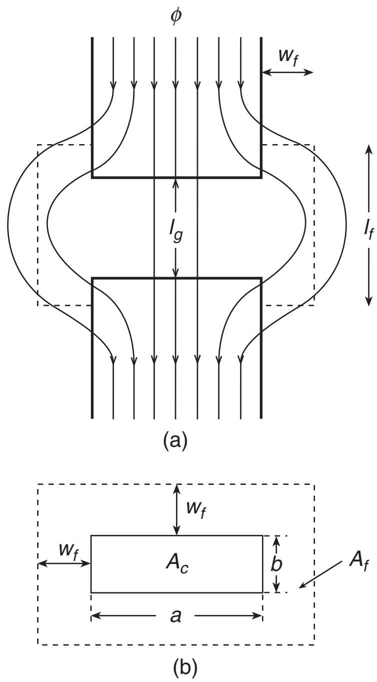

Figure 1.27 Magnetic flux distribution in an inductor with an air gap and a fringing magnetic flux. (a) Flux distribution. (b) Cross-sectional area of the core A_c and effective cross-sectional area of the fringing flux A_f

Due to the continuity of the magnetic flux, the magnetic flux in the core ϕ_c is equal to the sum of the magnetic flux in the air gap ϕ_g and the fringing flux ϕ_f

$$\phi_c = \phi_g + \phi_f. \tag{1.384}$$

The permeance of the core is

$$\mathcal{P}_c = \frac{1}{\mathcal{R}_c} = \frac{\mu_{rc}\mu_0 A_c}{l_c}. \tag{1.385}$$

The permeance of the air gap is

$$\mathcal{P}_g = \frac{1}{\mathcal{R}_g} = \frac{\mu_0 A_c}{l_g}. \tag{1.386}$$

The permeance of the fringing volume is

$$\mathcal{P}_f = \frac{1}{\mathcal{R}_f} = \frac{\mu_0 A_f}{l_f}, \tag{1.387}$$

where A_f is the fringing flux cross-sectional area and l_f is the mean MPL of the fringing flux. Assuming that $A_g = A_c$, the total reluctance is given by

$$\begin{aligned}\mathcal{R} &= \mathcal{R}_c + \mathcal{R}_g \| \mathcal{R}_f = \mathcal{R}_c + \frac{\mathcal{R}_g \mathcal{R}_f}{\mathcal{R}_g + \mathcal{R}_f} = \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{\frac{l_g}{\mu_0 A_g} \times \frac{l_f}{\mu_0 A_f}}{\frac{l_g}{\mu_0 A_g} + \frac{l_f}{\mu_0 A_f}} \\ &= \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{1}{\mu_0 (A_g/l_g + A_f/l_f)} = \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{l_g}{\mu_0 A_c \left(1 + \frac{A_f/l_f}{Ac/l_g}\right)} = \mathcal{R}_c + \frac{\mathcal{R}_g}{F_f} \\ &= \frac{l_c}{\mu_{rc}\mu_0 A_c}\left(1 + \frac{\mu_{rc} A_c}{l_c} \quad \frac{l_g l_f}{l_f A_g + l_g A_f}\right) = \frac{l_c}{\mu_{rc}\mu_0 A_c}\left(1 + \frac{\mu_{rc} l_g}{l_c} \frac{1}{1 + \frac{l_g A_f}{l_f A_g}}\right),\end{aligned} \tag{1.388}$$

where the *fringing factor* is defined as

$$F_f = \frac{\mathcal{R}_g}{\mathcal{R}_g \| \mathcal{R}_f} = 1 + \frac{\mathcal{R}_g}{\mathcal{R}_f} = \frac{\mathcal{P}}{\mathcal{P}_g} = \frac{\mathcal{P}_g + \mathcal{P}_f}{\mathcal{P}_g} = 1 + \frac{\mathcal{P}_f}{\mathcal{P}_g} = 1 + \frac{A_f l_g}{A_c l_f} = 1 + \frac{A_f/l_f}{A_c/l_g} > 1. \quad (1.389)$$

Hence, the inductance of an inductor with an air gap and a fringing flux is given by

$$L_f = \frac{N^2}{\mathcal{R}} = \frac{N^2}{\mathcal{R}_c + \frac{\mathcal{R}_g}{F_f}} > \frac{N^2}{\mathcal{R}_c + \mathcal{R}_g}. \quad (1.390)$$

For high-permeability cores, the core reluctance $\mathcal{R}_c$ can be neglected. The total permeance of the air gap and the fringing area is

$$\mathcal{P} = \mathcal{P}_g + \mathcal{P}_f = \frac{\mu_0 A_c}{l_g} + \frac{\mu_0 A_f}{l_f} = \frac{\mu_0 A_c}{l_g}\left(1 + \frac{A_f l_g}{A_c l_f}\right) = \frac{\mu_0 A_c F_f}{l_g} = F_f \mathcal{P}_g. \quad (1.391)$$

Thus, $\mathcal{R} \approx \mathcal{R}_g / F_f$.

Neglecting the permeance of the core, the inductance of the inductor with an air gap and the fringing flux is

$$L_f = \mathcal{P}N^2 = \frac{\mu_0 A_c N^2}{l_g} + \frac{\mu_0 A_f N^2}{l_f} = \frac{\mu_0 A_c N^2}{l_g}\left(1 + \frac{A_f l_g}{A_c l_f}\right) = \frac{\mu_0 A_c N^2 F_f}{l_g} = F_f L. \quad (1.392)$$

Thus, the fringing effect increases the inductance. The number of turns required for obtaining a desired inductance for the inductor with high-permeability core is

$$N = \sqrt{\frac{L}{\mathcal{P}}} = \sqrt{L\mathcal{R}} = \sqrt{\frac{L l_g}{\mu_0 A_c F_f}}. \quad (1.393)$$

The total reluctance of the inductor is

$$\mathcal{R} = \mathcal{R}_c + \mathcal{R}_g \| \mathcal{R}_f = \mathcal{R}_c + \frac{\mathcal{R}_g \mathcal{R}_f}{\mathcal{R}_g + \mathcal{R}_f} = \frac{l_c}{\mu_{rc}\mu_0 A_c} + \frac{l_g}{\mu_0 A_c F_f}$$
$$= \frac{l_g}{\mu_0 A_c}\left(\frac{l_c}{\mu_{rc} l_g} + \frac{1}{F_f}\right) = \mathcal{R}_g\left(\frac{l_c}{\mu_{rc} l_g} + \frac{1}{F_f}\right) = \frac{\mathcal{R}_g}{F_f} \quad \text{for} \quad \frac{l_c}{l_g} \ll \mu_{rc}. \quad (1.394)$$

The number of turns required for obtaining a desired inductance is

$$N = \sqrt{\frac{L}{\mathcal{P}}} = \sqrt{L\mathcal{R}} = \sqrt{\frac{L l_g}{\mu_0 A_c}\left(\frac{l_c}{\mu_{rc} l_g} + \frac{1}{F_f}\right)}. \quad (1.395)$$

If the air gap is enclosed by the winding, the fringing flux is reduced, lowering the value of F_f. However, the inductor losses increase as much as five times. If a winding is in the vicinity of the air gap, the fringing flux penetrates the winding conductor in the transverse direction, causing air-gap loss. To reduce this loss, the copper winding should be moved away from the air gap vicinity by a distance that is equal to two to three times the air-gap length l_g. This prevents the fringing flux of the gap from affecting the current within the winding conductor. The distance of the winding from the core can be increased by increasing the thickness of the bobbin. A reasonable thickness of the bobbin is 2–4 mm. Short distributed air gaps reduce the fringing flux and the power losses significantly. This is because the radial component of the magnetic flux density is reduced. Cores with a large relative permeability require long air gaps, which increase the fringing flux.

1.10.2 Effect of Fringing Flux on Inductance for Round Air Gap

The effect of the fringing flux on the inductance and the number of turns required to obtain a required inductance can be taken into account using the concept of the mean cross-sectional area of the fringing flux A_f and the mean MPL l_f of the fringing magnetic flux ϕ_f. Consider a round core with a single air gap. The ratio of the mean width of the cross-sectional area of the fringing flux to the air-gap length is defined as

$$\alpha = \frac{w_f}{l_g}. \tag{1.396}$$

The cross-sectional area of a round air gap with diameter D_c is

$$A_g = \frac{\pi}{4}D_c^2. \tag{1.397}$$

The cross-sectional area of the fringing flux for a round air gap is

$$A_f = \frac{\pi}{4}(D_c + 2\alpha l_g)^2 - \frac{\pi}{4}D_c^2 = \pi\alpha l_g(D_c + \alpha l_g). \tag{1.398}$$

The ratio of the cross-sectional areas is

$$\frac{A_f}{A_g} = \frac{4\alpha l_g(D_c + \alpha l_g)}{D_c^2}. \tag{1.399}$$

The ratio of the mean MPL of the fringing flux to the air-gap length is defined as

$$\beta = \frac{l_f}{l_g}. \tag{1.400}$$

The fringing flux factor for a round air gap is

$$F_f = 1 + \frac{A_f}{A_g}\frac{1}{\frac{l_f}{l_g}} = 1 + \frac{A_f}{A_g}\frac{1}{\beta} = 1 + \frac{4\alpha l_g(D_c + \alpha l_g)}{\beta D_c^2}. \tag{1.401}$$

In practice, it is difficult to determine the factors α and β. A good choice is $\alpha = 1$ and $\beta = 2$. The inductance with the fringing effect is

$$L_f = LF_f = L\left[1 + \frac{4\alpha l_g(D_c + \alpha l_g)}{\beta D_c^2}\right]. \tag{1.402}$$

Example 1.11

An inductor with an air gap has a round gap with $D_c = 10$ mm and $l_g = 1$ mm. Find F_f and N_f/N.

Solution: Assume $\alpha = 1$ and $\beta = 2$. The fringing factor is

$$F_f = \frac{L_f}{L} = 1 + \frac{4\alpha l_g(D_c + \alpha l_g)}{\beta D_c^2} = 1 + \frac{4 \times 1 \times 1 \times (10 + 1 \times 1)}{2 \times 10^2} = 1.22. \tag{1.403}$$

The ratio of the turns is

$$\frac{N_f}{N} = \frac{1}{\sqrt{F_f}} = \frac{1}{\sqrt{1.22}} = 0.9054. \tag{1.404}$$

1.10.3 Effect of Fringing Flux on Inductance for Rectangular Air Gap

Consider a core with a single rectangular air gap. The ratio of the mean width of the cross-sectional area of the fringing flux to the air-gap length is defined as

$$\alpha = \frac{w_f}{l_g} \tag{1.405}$$

where w_f is the mean width of the fringing flux. The rectangular air gap has dimensions a and b. The cross-sectional area of the air gap is

$$A_g = ab. \tag{1.406}$$

The cross-sectional area of the fringing flux is

$$A_f = (a + 2\alpha l_g)(b + 2\alpha l_g) - ab = 2\alpha l_g(a + b) + 4\alpha^2 l_g^2 = 2\alpha l_g(a + b + 2\alpha l_g), \tag{1.407}$$

resulting in the ratio of the cross-sectional areas

$$\frac{A_f}{A_g} = \frac{2\alpha l_g(a + b + 2\alpha l_g)}{ab} \tag{1.408}$$

The ratio of the mean MPL of the fringing flux to the air-gap length is defined as

$$\beta = \frac{l_f}{l_g} \tag{1.409}$$

where l_f is the mean length of the fringing flux. The fringing flux factor for a rectangular air gap is

$$F_f = 1 + \frac{A_f}{A_g}\frac{1}{\frac{l_f}{l_g}} = 1 + \frac{A_f}{A_g}\frac{1}{\beta} = 1 + \frac{2\alpha l_g(a + b + 2\alpha l_g)}{\beta ab}. \tag{1.410}$$

In practice, it is difficult to know the factors α and β. Reasonable values are $\alpha = 1$ and $\beta = 2$. The inductance with the fringing flux is

$$L_f = LF_f = L\left[1 + \frac{2\alpha l_g(a + b + 2\alpha l_g)}{\beta ab}\right]. \tag{1.411}$$

Assuming that $\alpha = 1$, the effective cross-sectional area of the fringing flux is

$$A_f = (a + 2w_f)(b + 2w_f) - A_c = 2(a + b)w_f + 4w_f^2. \tag{1.412}$$

Hence, the fringing flux factor is

$$F_f = 1 + \left(\frac{l_g}{A_c}\right)\left(\frac{w_f}{l_f}\right)[2(a + b) + 4w_f] \approx 1 + 2(a + b)\left(\frac{l_g}{A_c}\right)\left(\frac{w_f}{l_f}\right). \tag{1.413}$$

Note that F_f is directly proportional to l_g. The inductance is

$$L_f = F_f L = \frac{\mu_0 A_c N^2}{l_g}\left[1 + 2(a + b)\left(\frac{l_g}{A_c}\right)\right] = \frac{\mu_0 A_c N^2}{l_g} + 2(a + b)\mu_0 N^2 \frac{w_f}{l_f}. \tag{1.414}$$

The first term describes the inductance L without fringing flux. The second term is independent of l_g and constitutes a constant excess inductance. The empirical value is $w_f/l_f = 1.1322$.

Example 1.12

An inductor with a single rectangular air gap has $a = 10$ mm, $b = 20$ mm, and $l_g = 1$ mm. Find F_f and N_f/N.

Solution: Assume that $\alpha = 1$ and $\beta = 2$. The fringing flux factor is

$$F_f = \frac{L_f}{L} = 1 + \frac{2\alpha l_g(a + b + 2\alpha l_g)}{\beta ab} = 1 + \frac{2 \times 1 \times (10 + 20 + 2 \times 1 \times 1)}{2 \times 10 \times 20} = 1.16. \tag{1.415}$$

The ratio of the turns is

$$\frac{N_f}{N} = \frac{1}{\sqrt{F_f}} = \frac{1}{\sqrt{1.16}} = 0.9285. \tag{1.416}$$

1.10.4 Method of Effective Air Gap Cross-Sectional Area

Another method of deriving F_f uses an effective area of the air gap and is as follows. The cross-sectional area of the round core is

$$A_c = \frac{\pi D_c^2}{4} \tag{1.417}$$

and the effective cross-sectional area of the air gap is

$$A_e = \frac{\pi (D_c + l_g)^2}{4}. \tag{1.418}$$

Assuming that $l_g/l_f = 1$, we get F_f for a round air gap

$$F_f = \frac{L_f}{L} = \frac{A_e}{A_c} = \left(1 + \frac{l_g}{D_c}\right)^2. \tag{1.419}$$

For the rectangular cross section of the air gap,

$$F_f = \frac{L_f}{L} = \frac{A_e}{A_c} = \frac{(a + l_g)(b + l_g)}{ab}. \tag{1.420}$$

1.10.5 Method of Effective Length of Air Gap

The permeance of the air gap with the physical gap length l_g and the gap cross-sectional area $A_c + A_f$ is

$$\mathcal{P}_g = \frac{\mu_0 (A_c + A_f)}{l_g}, \tag{1.421}$$

where A_f is the effective cross-sectional area of the fringing flux. The permeance of the air gap with the effective gap length l_{eff} and the air gap flux area equal to the core cross-sectional area A_c is

$$\mathcal{P}_g = \frac{\mu_0 A_c}{l_{eff}}. \tag{1.422}$$

The two permeances are equal, which results in

$$\frac{\mu_0 (A_c + A_f)}{l_g} = \frac{\mu_0 A_c}{l_{eff}}. \tag{1.423}$$

The effective air-gap length for a rectangular gap is given by

$$l_{eff} = l_g \frac{A_c}{A_c + A_f} = A_c \frac{l_g}{(a + l_g)(b + l_g)} = A_c \frac{l_g}{ab\left(1 + \frac{l_g}{a}\right)\left(1 + \frac{l_g}{b}\right)}$$
$$= \frac{l_g}{\left(1 + \frac{l_g}{a}\right)\left(1 + \frac{l_g}{b}\right)} = \frac{l_g}{F_f} \tag{1.424}$$

where

$$F_f = \left(1 + \frac{l_g}{a}\right)\left(1 + \frac{l_g}{b}\right). \tag{1.425}$$

1.10.6 Patridge's Fringing Factor

The fringing flux factor given in Refs [6, 7, 13] is described by

$$F_f = 1 + \frac{a l_g}{N_g \sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right) \approx 1 + \frac{l_g}{N_g \sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right) = 1 + \frac{w}{N_g \sqrt{A_c}} \left(\frac{l_g}{w}\right) \ln \frac{2}{\left(\frac{l_g}{w}\right)}, \tag{1.426}$$

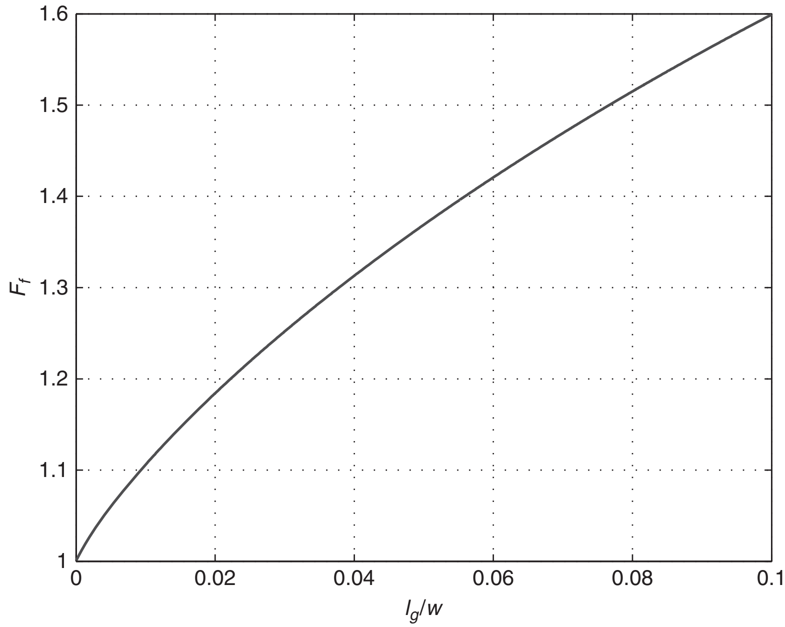

Figure 1.28 Fringing flux factor F_f as a function l_g/w at $A_c = 1\ \text{cm}^2$ and $w = 2$ cm

where l_g is the total gap length, N_g is the number of air gaps in the magnetic path, w is the width of the core window, $a = 0.85$–0.95 for round cores, and $a = 1$–1.1 for rectangular cores. The typical values of the fringing flux factor F_f are in the range 1.1–1.4. The fringing flux reduces the total reluctance of the magnetic path $\mathcal{R}$, and therefore it increases the inductance L. Figure 1.28 shows a plot of the fringing flux factor F_f as a function of l_g/w at $N_g = 1$, $A_c = 1\ \text{cm}^2$, and $w = 2$ cm.

The inductance is increased due to the fringing effect and is given by

$$L_f = F_f L = \left[1 + \frac{a l_g}{\sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right)\right] L = \left[1 + \frac{a l_g}{\sqrt{A_c}} \ln\left(\frac{2w}{l_g}\right)\right] \frac{\mu_{rc}\mu_0 A_c N^2}{l_c(1 + \mu_{rc} l_g)}$$

$$= \frac{\mu_0 A_c N^2 F_f}{l_g + \frac{l_c}{\mu_{rc}}}. \tag{1.427}$$

Therefore, the number of turns N to obtain a required inductance L of an inductor with an air gap and the fringing effect should be reduced to

$$N_f = \sqrt{\frac{L_f\left(l_g + \frac{l_c}{\mu_{rc}}\right)}{\mu_0 A_c F_f}} = \frac{N}{\sqrt{F_f}}. \tag{1.428}$$

Fringing flux generates eddy currents, which cause hot spots in both the core and the winding, resulting in power losses. The winding, banding, and clips should be kept away from the fringing flux to reduce power losses. If a long single air gap is used, a high-fringing magnetic field is induced. Many short air gaps (a distributed air gap) along the magnetic path reduce the fringing flux and the winding loss as compared to the winding loss due to a long single air gap. It is important that there is a sufficient distance between air gaps. The distance between adjacent air gaps should be greater than four times the length of one air gap. Otherwise, the fringing fluxes from the adjacent air gaps will overlap and the air gaps will be shunted by the fringing reluctance.

1.10.7 Distribution of Fringing Magnetic Field

Assume that the skin effect in the core is negligible. For a core with a rectangular cross section, the magnitudes of the components of the fringing magnetic field are [54]

$$H_x(x,y) = \frac{H(0,0)}{2\pi} \ln \left[\frac{x^2 + (y - l_g/2)^2}{x^2 + (y + l_g/2)^2} \right] \tag{1.429}$$

and

$$H_y(x,y) = -\frac{H(0,0)}{\pi} \left[\arctan\left(\frac{x l_g}{x^2 + y^2 - l_g^2/4} \right) + m\pi \right], \tag{1.430}$$

where $H(0,0) = 0.9NI/l_g$ is the magnetic field at the center edge of the air gap, $m = 0$ for $x^2 + y^2 \geq l_g^2/4$, and $m = 1$ for $x^2 + y^2 < l_g^2/4$. The magnitude of the total fringing magnetic field is

$$H(x,y) = \sqrt{H_x^2(x,y) + H_y^2(x,y)}. \tag{1.431}$$

1.11 Inductance of Strip Transmission Line

Consider a strip transmission line, where d is the distance between the conductors, l is the length of the strip, and w is the width of the strip. The magnetic field intensity between conducting parallel plates is

$$H = \frac{I}{w} \tag{1.432}$$

resulting in the flux linkage

$$\lambda = \int\int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^l \int_0^d \frac{\mu I}{w} dxdz = \frac{\mu I l d}{w}. \tag{1.433}$$

Hence, the inductance of the strip transmission line is given by

$$L = \frac{\lambda}{I} = \frac{\mu d l}{w}. \tag{1.434}$$

1.12 Inductance of Coaxial Cable

An axial current I flows in the inner conductor of radius a and returns in the outer conductor of radius b, inducing a circumferential magnetic field in the inner conductor, outer conductor, and between the conductors. Assume a uniform current distribution in both the conductors. The magnetic field between the conductors is

$$H_\phi = \frac{I_r}{2\pi r} \quad \text{for} \quad a \leq r \leq b. \tag{1.435}$$

The magnetic flux between radii a and b is

$$\phi_{ext} = \int\int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^{l_w} dz \int_a^b \mu \left(\frac{\mu I}{2\pi r} \right) dr = \frac{\mu l_w I}{2\pi} \ln \left(\frac{b}{a} \right). \tag{1.436}$$

The external inductance of a coaxial cable due to the magnetic energy stored in the magnetic field between the inner and outer conductors is given by

$$L_{ext} = \frac{\phi_{ext}}{I} = \frac{1}{I} \int\int_S \mathbf{B} \cdot d\mathbf{S} = \frac{\mu l_w}{2\pi} \ln \left(\frac{b}{a} \right), \tag{1.437}$$

where a is the radius of the inner conductor, b is the radius of the outer conductor, and l_w is the cable length.

Consider the internal inductance due to the magnetic field inside the inner conductor. The current I_r enclosed by a circle of radius r is

$$I_r = \left(\frac{r}{a}\right)^2 I. \quad (1.438)$$

The magnetic field in the inner conductor is

$$H_\phi(r) = \frac{Ir}{2\pi a^2} \quad \text{for} \quad r < a. \quad (1.439)$$

The instantaneous magnetic energy stored in the magnetic field in the inner conductor is

$$W_m(t) = \frac{1}{2}L_{int(i)}I^2 = \frac{1}{2}\int\int\int_V \mu H^2 dV = \frac{1}{2}\int_0^{l_w} dz \int_0^a \mu\left(\frac{Ir}{2\pi a^2}\right)^2 2\pi r dr = \frac{\mu l_w I^2}{16\pi}. \quad (1.440)$$

Hence, the internal inductance of the inner conductor is

$$L_{int(i)} = \frac{\mu l_w}{8\pi}. \quad (1.441)$$

The magnetic field in the outer conductor is

$$H_\phi(r) = \frac{I}{2\pi(c^2 - b^2)}\left(\frac{c^2}{r} - r\right) \quad \text{for} \quad b \le r \le c \quad (1.442)$$

where c is the outer radius of the outer conductor. The instantaneous magnetic energy stored in the magnetic field in the outer conductor is

$$\frac{1}{2}L_{int(o)}I^2 = \frac{1}{2}\int\int\int_V \mu H^2 dV = \frac{1}{2}\int_0^{l_w} dz \int_0^a \frac{\mu I}{2\pi(c^2 - b^2)}\left(\frac{c^2}{r} - r\right)^2 2\pi r dr \quad (1.443)$$

Hence, the internal inductance of the outer conductor is

$$L_{int(o)} = \frac{\mu l_w}{2\pi}\left[\frac{c^4 \ln(c/b)}{(c^2 - b^2)^2} + \frac{b^2 - 3c^2}{4(c^2 - b^2)}\right]. \quad (1.444)$$

The total inductance of the coaxial transmission line is

$$L = L_{ext} + L_{int} = L_{ext} + L_{int(i)} + L_{int(o)} = \frac{\mu l_w}{2\pi}\ln\left(\frac{b}{a}\right) + \frac{\mu l_w}{8\pi}$$
$$+ \frac{\mu l_w}{2\pi}\left[\frac{c^4 \ln(c/b)}{(c^2 - b^2)^2} + \frac{b^2 - 3c^2}{4(c^2 - b^2)}\right]. \quad (1.445)$$

1.13 Inductance of Two-Wire Transmission Line

The magnetic flux is given by

$$\phi = -\frac{\mu I}{4\pi}\ln\left[\frac{(x - \sqrt{(d/2)^2 - a^2})^2 + y^2}{(x + \sqrt{(d/2)^2 + a^2})^2 + y^2}\right]. \quad (1.446)$$

The magnetic flux difference between its value at $x = d/2 - a$ and $x = -d/2 + a$ is

$$\Delta\phi = -\frac{\mu I}{\pi}\ln\left|\frac{d/2 - a - \sqrt{(d/2)^2 - a^2}}{d/2 - a + \sqrt{(d/2)^2 - a^2}}\right|. \quad (1.447)$$

The internal inductance of a two-wire transmission line for a round conductor of radius a, distance between the conductor centers d, and length l_w is

$$L = \frac{\Delta\phi}{I} = \frac{\mu l_w}{\pi}\cosh^{-1}\left(\frac{d}{2a}\right) = \frac{\mu l_w}{\pi}\ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right]$$
$$\approx \frac{\mu l_w}{\pi}\ln\left(\frac{d}{a}\right) \quad \text{for} \quad \left(\frac{d}{2a}\right)^2 \gg 1, \quad (1.448)$$

where $\cosh^{-1}x \approx \ln(2x)$ for $x \ll 1$.

1.14 Magnetic Energy and Magnetic Energy Density

1.14.1 Magnetic Energy Density

The instantaneous magnetic energy density is

$$w_m(t) = \frac{1}{2}\mathbf{B} \cdot \mathbf{H}. \tag{1.449}$$

For an isotropic medium, $\mathbf{B} = \mu\mathbf{H}$ and the instantaneous magnetic energy density becomes

$$w_m(t) = \frac{1}{2}\mu\mathbf{H} \cdot \mathbf{H} = \frac{1}{2}\mu H^2 = \frac{1}{2}\mathbf{B} \cdot \frac{\mathbf{B}}{\mu} = \frac{1}{2}\frac{B^2}{\mu}. \tag{1.450}$$

For harmonic fields, the instantaneous magnetic energy density is

$$w_m(t) = \frac{1}{2}Re\{\mathbf{B}e^{j\omega t}\} \cdot Re\{\mathbf{H}e^{j\omega t}\}. \tag{1.451}$$

The time-average magnetic energy density for harmonic fields is

$$w_{m(AV)} = \frac{1}{4}Re\{\mathbf{H} \cdot \mathbf{B}^*\}. \tag{1.452}$$

For an isotropic medium,

$$w_{m(AV)} = \frac{1}{4}\mu|H|^2. \tag{1.453}$$

The instantaneous magnetic energy is

$$w_m(t) = \frac{1}{2}\lambda(t)i(t) = \frac{1}{2}N\phi(t)i(t) = \frac{1}{2}NAB(t)i(t) = \frac{1}{2}NA\mu H(t)i(t) = \frac{1}{2}Li^2. \tag{1.454}$$

1.14.2 Magnetic Energy Stored in Inductors with Ungapped Core

Consider first the magnetic energy stored in inductors with ungapped cores. The instantaneous reactive power of an inductor is

$$p(t) = i_L(t)v_L(t) = i_L\left(L\frac{di_L}{dt}\right) = Li_L\frac{di_L}{dt}. \tag{1.455}$$

Power is the time rate of change of energy $P = W/\Delta t$. The instantaneous *magnetic energy* stored in the magnetic field of an inductor without an air gap is given by

$$\begin{aligned} W_m(t) &= \int_0^t p(t)dt = \int_0^t i_L(t)v_L(t)dt = \int_0^t i_L L\frac{di_L}{dt}dt = L\int_0^{i_L} i_L di_L = \frac{1}{2}Li_L^2(t) \\ &= \frac{1}{2}\lambda(t)i_L(t) = \frac{\lambda^2(t)}{2L} = \frac{1}{2}\frac{N^2}{\mathcal{R}}i_L^2(t) = \frac{N^2\phi^2(t)}{2L} = \frac{1}{2}\mathcal{R}\phi^2(t) \\ &= \frac{1}{2}\frac{N^2}{\frac{l_c}{\mu_{rc}\mu_0 A_c}}\left(\frac{l_c H(t)}{N}\right)^2 = \frac{1}{2}\mu_{rc}\mu_0 H^2(t)A_c l_c = \frac{B^2(t)l_c A_c}{2\mu_{rc}\mu_0} = \frac{B^2(t)V_c}{2\mu_{rc}\mu_0} \text{ (J)}, \end{aligned} \tag{1.456}$$

where $V_c = l_c A_c$ is the core volume, $v_L = Ldi_L/dt$, $i_L = \lambda/L$, $L = N^2/\mathcal{R}$, and $H = B/\mu$. The magnetic energy is proportional to the core volume V_c and the magnetic flux density B, and it is inversely proportional to the core relative permeability μ_{rc}.

The maximum energy stored in an inductor with a core without an air gap is given by

$$W_{c(max)} = \frac{B_{pk}^2 l_c A_c}{2\mu_{rc}\mu_0} = \frac{B_{pk}^2 V_c}{2\mu_{rc}\mu_0}. \tag{1.457}$$

The maximum energy that can be stored in an inductor is limited by the core saturation flux density B_s, the temperature rise caused by core losses, the core volume V_c, and the core relative permeability μ_{rc}. At $B_{pk} = B_s$,

$$W_{c(sat)} = \frac{B_s^2 l_c A_c}{2\mu_{rc}\mu_0} = \frac{B_s^2 V_c}{2\mu_{rc}\mu_0}. \tag{1.458}$$

The *instantaneous magnetic energy density* stored in an inductor with ungapped core is

$$w_m(t) = \frac{W_m(t)}{V_c} = \frac{B^2(t)}{2\mu_{rc}\mu_0} = \frac{1}{2}\mu_{rc}\mu_0 H^2(t) = \frac{1}{2}\mu H^2(t) \quad \left(\frac{\mathrm{J}}{\mathrm{m}^3}\right). \tag{1.459}$$

1.14.3 Magnetic Energy Stored in Inductors with Gapped Core

For an inductor with a gapped core, the magnetic energy stored in the gap is

$$W_g(t) = \frac{B^2(t) l_g A_g}{2\mu_0} = \frac{B^2(t) V_g}{2\mu_0} \approx \frac{B^2(t) l_g A_c}{2\mu_0}, \tag{1.460}$$

where $V_g = l_g A_g \approx l_g A_c$ is the air-gap volume and $A_g \approx A_c$. The instantaneous magnetic energy stored in the core is

$$W_c(t) = \frac{B^2(t) l_c A_c}{2\mu_{rc}\mu_0} = \frac{B^2(t) V_c}{2\mu_{rc}\mu_0} \tag{1.461}$$

where the core volume is $V_c = l_c A_c$. The total energy stored in an inductor with an air gap $W_m(t)$ is equal to the sum of the energy stored in the gap $W_g(t)$ and the energy stored in the core $W_c(t)$

$$W_m(t) = W_g(t) + W_c(t) = \frac{B^2(t) A_c}{2\mu_0}\left(l_g + \frac{l_c}{\mu_{rc}}\right). \tag{1.462}$$

The maximum magnetic energy stored in the core is

$$W_{c(max)} = \frac{B_{pk}^2 l_c A_c}{2\mu_{rc}\mu_0} = \frac{B_{pk}^2 V_c}{2\mu_{rc}\mu_0}. \tag{1.463}$$

For high-permeability cores with $l_g \gg l_c/\mu_{rc}$, almost all the inductor energy is stored in the air gap

$$W_m(t) \approx W_g(t) = \frac{B^2(t) l_g A_g}{2\mu_0} \approx \frac{B^2(t) V_g}{2\mu_0}. \tag{1.464}$$

The maximum magnetic energy that is stored in the air gap is

$$W_{g(max)} = \frac{B_{pk}^2 l_g A_c}{2\mu_0} = \frac{B_{pk}^2 V_g}{2\mu_0}. \tag{1.465}$$

At $B_{pk} = B_s$,

$$W_{g(sat)} = \frac{B_s^2 l_g A_c}{2\mu_0} = \frac{B_s^2 V_g}{2\mu_0}. \tag{1.466}$$

The total maximum energy that can be stored in an inductor with a gapped core is

$$W_{m(max)} = W_{g(max)} + W_{c(max)} = \frac{B_s^2 A_c}{2\mu_0}\left(l_g + \frac{l_c}{\mu_{rc}}\right). \tag{1.467}$$

Hence, the length of the air gap required to obtain a specified maximum magnetic energy $W_{m(max)}$ is

$$l_g = \frac{2\mu_0 W_{m(max)}}{A_c B_s^2} - \frac{l_c}{\mu_{rc}}. \tag{1.468}$$

The ratio of the two energies is

$$\frac{W_{g(max)}}{W_{c(max)}} = \frac{l_g}{l_c}\mu_{rc}. \tag{1.469}$$

The instantaneous *magnetic energy density* stored in the air gap is

$$w_g(t) = \frac{W_g(t)}{V_g} = \frac{B^2(t)}{2\mu_0} = \frac{1}{2}\mu_0 H_g^2(t) \ (\text{J/m}^3) \tag{1.470}$$

and in the core is

$$w_c(t) = \frac{W_c(t)}{V_c} = \frac{B^2(t)}{2\mu_{rc}\mu_0} = \frac{1}{2}\mu_{rc}\mu_0 H_c^2(t) \ (\text{J/m}^3). \tag{1.471}$$

Example 1.13

An infinitely long round solid straight conductor of radius r_o conducts sinusoidal current $i(t) = I_m \cos \omega t$ in steady state at low frequencies (with no skin effect). Determine the amplitudes of the magnetic energy density and the total magnetic energy stored inside and outside the conductor.

Solution: From Example 1.1, the amplitude of the magnetic field density inside the conductor is given by

$$H_m(r) = \frac{I_m r}{2\pi r_o^2} \quad \text{for} \quad 0 \le r \le r_o. \tag{1.472}$$

The waveform of the magnetic field intensity is

$$H(r,t) = \frac{I_m r}{2\pi r_o^2} \cos \omega t \quad \text{for} \quad 0 \le r \le r_o. \tag{1.473}$$

Thus, the amplitude of the magnetic energy intensity inside the conductor is

$$w_m(r) = \frac{1}{2}\mu H_m^2(r) = \frac{\mu I_m^2 r^2}{8\pi^2 r_o^4} = \frac{\mu I_m^2}{8\pi^2 r_o^2}\left(\frac{r}{r_o}\right)^2 \quad \text{for} \quad 0 \le r \le r_o. \tag{1.474}$$

The maximum magnetic energy density at a given radius r is

$$w_{m(max)}(r) = \frac{\mu I_m^2 r^2}{4\pi^2 r_o^4}. \tag{1.475}$$

Hence, the waveform of the magnetic energy density inside the conductor is

$$w_m(r,t) = \frac{1}{2}\mu H^2(r,t) = \frac{\mu I_m^2 r^2}{8\pi^2 r_o^4}\cos^2 \omega t \quad \text{for} \quad 0 \le r \le r_o. \tag{1.476}$$

The small volume of a cylindrical shell of radius r, thickness dr, and length l_w is

$$dV = (2\pi r)(dr)(l_w) = 2\pi l_w r dr. \tag{1.477}$$

Assuming that μ is uniform for the entire conductor, the amplitude of the magnetic energy stored in the magnetic field inside the conductor is given by

$$W_{m(max)} = \int\int\int_V w_m(r) dV = \frac{1}{2}\int\int\int_V \mu H_m^2(r) dV = \frac{\mu l_w I_m^2}{4\pi r_o^4}\int_0^{r_o} r^3 dr = \frac{\mu l_w I_m^2}{16\pi} \tag{1.478}$$

or using the internal inductance $L_{int} = \mu l_w/(8\pi)$

$$W_{m(max)} = \frac{1}{2}L_{int} I_m^2 = \frac{1}{2}\left(\frac{\mu l_w}{8\pi}\right) I_m^2 = \frac{\mu l_w I_m^2}{16\pi}. \tag{1.479}$$

From Example 1.1, the amplitude of the magnetic field intensity outside the conductor is

$$H_m(r) = \frac{I_m}{2\pi r} \quad \text{for} \quad r \ge r_o. \tag{1.480}$$

The waveform of the magnetic field intensity outside the conductor is

$$H(r,t) = \frac{I_m}{2\pi r}\cos \omega t \quad \text{for} \quad r \ge r_o. \tag{1.481}$$

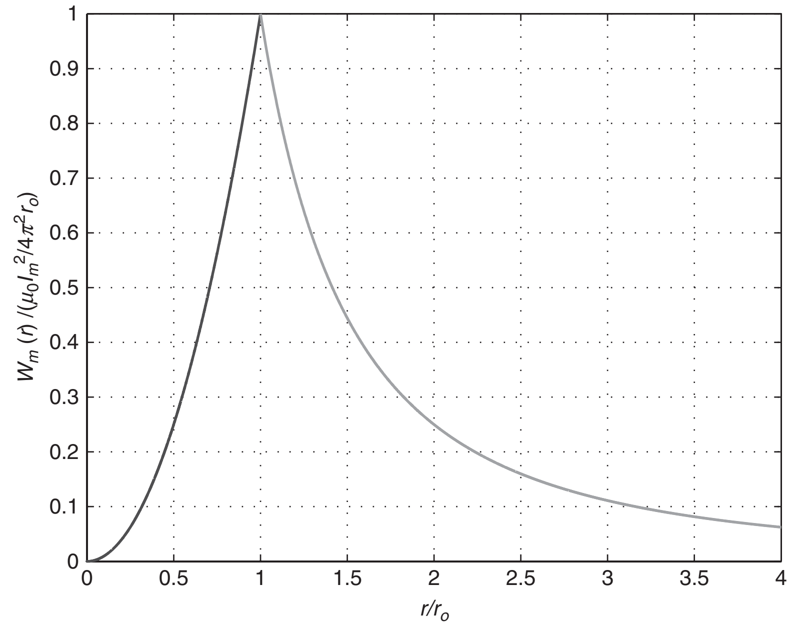

Figure 1.29 Normalized magnetic energy density $w_m(r)/[\mu I_m^2/(4\pi^2 r_o^2)]$ stored inside and outside a long, round, solid conductor conducting a sinusoidal current at low frequencies (with no skin effect)

Therefore, the amplitude of the magnetic energy outside the conductor is

$$w_{m(max)}(r) = \frac{1}{2}\mu H_m^2(r) = \frac{\mu I_m^2}{8\pi^2 r^2} = \frac{\mu I_m^2}{8\pi^2 r_o^2}\frac{1}{\left(\frac{r}{r_o}\right)^2} \quad \text{for} \quad r \geq r_o. \tag{1.482}$$

Hence, the waveform of the magnetic energy density inside the conductor is

$$w_m(r,t) = \frac{1}{2}\mu H_m^2(r,t) = \frac{\mu I_m^2}{8\pi^2 r^2}\cos^2 \omega t \quad \text{for} \quad r \geq r_o. \tag{1.483}$$

Assuming that μ is uniform for the entire area outside the conductor, the magnetic energy stored in the magnetic field outside the conductor is given by

$$W_m = \int\int\int_V w_m(r)dV = \frac{1}{2}\int\int\int_V \mu H_m^2(r)dV = \frac{\mu l_w I_m^2}{8\pi}\int_{r_o}^{\infty}\frac{dr}{r} = \frac{\mu l_w I_m^2}{8\pi}\ln r\Big|_{r_o}^{\infty} = \infty. \tag{1.484}$$

Figure 1.29 shows the normalized magnetic energy density $w_m(r)/[\mu I_m^2/(4\pi^2 r_o^2)]$ inside and outside a long, solid round conductor carrying a sinusoidal current at low frequencies. The reason for the infinite magnetic energy is that the model of the conductor is too ideal, which does not take into account the return path of the current. The presence of this path changes the magnetic field distribution and the stored magnetic energy.

Example 1.14

A Ferroxcube ferrite magnetic core 528T500-4C4 has $A_c = 1.17$ cm^2, $l_c = 8.49$ cm, and $\mu_{rc} = 125$. (a) Determine the maximum magnetic energy that can be stored in the inductor core. (b) Determine the maximum magnetic energy that can be stored in the air gap $l_g = 0.5$ mm. (c) Find the ratio of the maximum magnetic energies.

Solution: The saturation flux density B_s for ferrite cores is $B_s = 0.3\ T$ at room temperature $T = 20\,^{\circ}$C. At $T = 100\,^{\circ}$C, the saturation flux density B_s for ferrite cores decreases by a factor of 2. Thus,

$$B_s = \frac{0.3}{2} = 0.15\ \text{T}. \tag{1.485}$$

The maximum magnetic energy that can be stored in the magnetic core is

$$W_{c(max)} = \frac{B_s^2 l_c A_c}{2\mu_{rc}\mu_0} = \frac{0.15^2 \times 8.49 \times 10^{-2} \times 1.17 \times 10^{-4}}{2 \times 125 \times 4\pi \times 10^{-7}} = 0.711\ \text{mJ}. \tag{1.486}$$

The maximum magnetic energy that can be stored in the air gap is

$$W_{g(max)} = \frac{B_s^2 l_g A_c}{2\mu_0} = \frac{0.15^2 \times 0.5 \times 10^{-3} \times 1.17 \times 10^{-4}}{2 \times 4\pi \times 10^{-7}} = 0.5237\ \text{mJ}. \tag{1.487}$$

Hence,

$$\frac{W_{g(max)}}{W_{c(max)}} = \frac{0.5237}{0.711} = 0.7366. \tag{1.488}$$

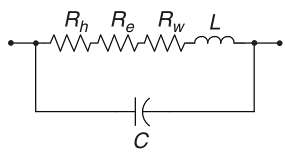

Figure 1.30 Model of an inductor

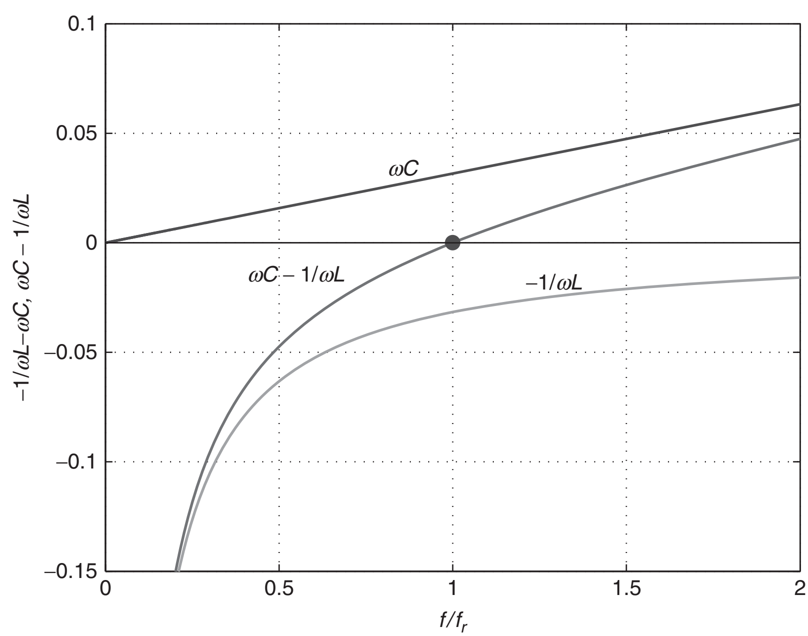

Figure 1.31 Plots of the susceptances $B_C = \omega C$, $B_L = -1/(\omega L)$, and $B = \omega C - 1/(\omega L)$ as functions of frequency for an inductor

1.15 Self-Resonant Frequency

Figure 1.30 shows an equivalent model of an inductor, where L is the inductance, R_w is the winding resistance, $R_{cs} = R_h + R_e$, R_h is the core hysteresis equivalent series resistance (ESR), R_e is the core eddy-current series resistance, and C is the self-capacitance. The distributed capacitance between the winding turns acts like a shunt capacitance, conducting a high-frequency current. This capacitance is called a *stray capacitance* or a *self-capacitance* C [33, 34]. It depends on the winding geometry, the proximity of turns, core, and shield, and the permittivity of the dielectric insulator, in which the winding wire is coated. The core should be insulated to increase the distance between the turns and the core, and therefore reduce the capacitance between the winding and the core. The inductance and the self-capacitance form a parallel resonant circuit, which has the first (fundamental or parallel) SRF

$$f_r = \frac{1}{2\pi\sqrt{LC}}. \tag{1.489}$$

Figure 1.31 shows the plots of the susceptances $B_C = \omega C$, $B_L = -1/(\omega L)$, and $B = B_C + B_L = \omega C - 1/(\omega L)$ as functions of frequency for inductance $L = 1\ \mu\text{H}$ and $C = 1$ nF. At the SRF f_r, the total susceptance of an inductor is zero. Below the SRF f_r, the inductor reactance is inductive. Above the SRF f_r, the inductor reactance is capacitive. Therefore, the operating frequency range of an inductor is usually from DC to $0.9f_r$.

1.16 Quality Factor of Inductors

A winding represents a series combination of an inductance and a frequency-dependent resistance. The *quality factor* of an inductor with a magnetic core at a given frequency f is defined as

$$\begin{aligned} Q_{Lo} &= \frac{\text{Reactance at } f}{\text{Total resistance at } f} = \frac{X_L}{r_L} = \frac{\omega L}{r_L} = \frac{\omega L}{R_w + R_{cs}} \\ &= \frac{1}{\frac{R_w}{\omega L} + \frac{R_{cs}}{\omega L}} = \frac{1}{\frac{1}{\omega L/R_w} + \frac{1}{\omega L/R_{cs}}} = \frac{1}{\frac{1}{Q_{LRw}} + \frac{1}{Q_{LRcs}}} = \frac{Q_{LRw}Q_{LRcs}}{Q_{LRw} + Q_{LRcs}}, \end{aligned} \tag{1.490}$$

where $r_L = R_w + R_{cs}$ is the ESR of an inductor at frequency f, R_w is the winding resistance, and R_{cs} is the core series resistance, the quality factor of an inductor due to the winding resistance is

$$Q_{LRw} = \frac{\omega L}{R_w}, \tag{1.491}$$

and the quality factor of an inductor due to the core series resistance is

$$Q_{LRcs} = \frac{\omega L}{R_{cs}}. \tag{1.492}$$

The quality factor of an air-core inductor is defined as

$$Q_{Lo} = Q_{LRw} = \frac{\omega L}{R_w}. \tag{1.493}$$

1.17 Classification of Power Losses in Magnetic Components

Figure 1.32 shows a classification of power losses in magnetic components. These losses can be categorized into winding (or copper) losses P_{Rw} and core losses P_C. The winding losses can be divided into the DC loss and the AC loss.

$$P_{Rw} = P_{wDC} + P_{wAC}. \tag{1.494}$$

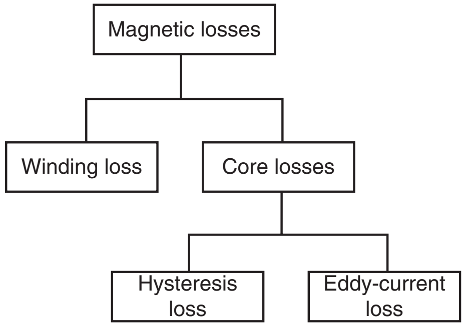

Figure 1.32 Classification of power losses in magnetic components

In turn, core losses can be divided into hysteresis loss P_H and eddy-current loss P_E

$$P_C = P_H + P_E. \tag{1.495}$$

Hence, the total inductor power loss P_L is given by

$$P_L = P_{Rw} + P_C = P_{Rw} + P_H + P_E = P_{wDC} + P_{wAC} + P_H + P_E. \tag{1.496}$$

There are two kinds of eddy-current winding losses: the skin-effect loss and the proximity-effect loss. Both these effects cause current crowding. Eddy-current losses are magnetically induced losses. The winding power losses are increased at high frequencies by eddy currents induced in conductors by magnetic fields. Consider a case where a sinusoidal current is applied to an inductor

$$i_L(t) = I_{Lm} \sin \omega t. \tag{1.497}$$

At a high frequency, the inductor winding carries the applied current $i_L(t)$ and the eddy current $i_{ec}(t) = I_{ecm} \sin \omega t$. The applied current density is uniform and therefore the winding resistance presented to the applied current is equal to the DC winding resistance R_{wDC}. Hence, the power loss due to the applied current is

$$P_{wDC} = \frac{1}{2} R_{wDC} I_{Lm}^2. \tag{1.498}$$

The eddy current density is not uniform and therefore the winding resistance presented to the eddy current R_{wec} is higher than the DC winding resistance presented to the applied sinusoidal current R_{wDC}. The total AC winding resistance R_w is given by

$$R_w = R_{wDC} + R_{wec} = R_{wDC}\left(1 + \frac{R_{wec}}{R_{wDC}}\right) = R_{wDC} + \alpha R_{wDC} = R_{wDC}(1+\alpha) = F_R R_{wDC}, \tag{1.499}$$

where the ratio of the eddy current winding resistance to the DC winding resistance is

$$\alpha = \frac{R_{wec}}{R_{wDC}} \tag{1.500}$$

and the ratio of the AC-to-DC winding resistance is

$$F_R = \frac{R_w}{R_{wDC}} = \frac{R_{wDC} + R_{wec}}{R_{wDC}} = 1 + \frac{R_{wec}}{R_{wDC}} = 1 + \alpha. \tag{1.501}$$

The total winding power loss is equal to the power loss due to the conduction of the applied current P_{wDC} of uniform density and the power losses due to the conduction of the eddy current P_{wec} of nonuniform density

$$\begin{aligned} P_{wAC} &= P_w = P_{wDC} + P_{wec} = \frac{1}{2} R_w I_{Lm}^2 = \frac{I_{Lm}^2}{2}(R_{wDC} + R_{wec}) = \frac{I_{Lm}^2}{2} R_{wDC} + \frac{I_{Lm}^2}{2} \alpha R_{wDC} \\ &= \frac{I_{Lm}^2}{2} R_{wDC} + \frac{I_{ecm}^2}{2} R_{wDC}. \end{aligned} \tag{1.502}$$

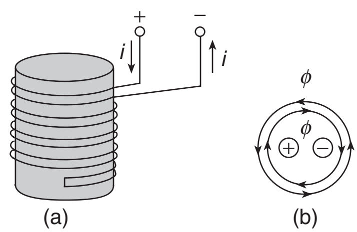

Figure 1.33 Noninductive coil. (a) Bifilar winding. (b) Magnetic flux cancellation

Hence, the amplitude of the eddy current is

$$I_{ecm} = \sqrt{\alpha} I_{Lm} = \sqrt{F_R - 1} \quad I_{Lm} = \sqrt{\alpha} \quad I_{Lm}. \tag{1.503}$$

1.18 Noninductive Coils

In some applications, it is desired to have a noninductive coil. Precision resistors are usually noninductive. For example, current probes require noninductive resistors. A noninductive coil is usually made using closely spaced, parallel windings, called the *bifilar winding*, as illustrated in Fig. 1.33a. Therefore, every coil turn has an adjacent turn, which carries current in the opposite direction. The magnetic fields generated by the adjacent turns cancel each other, as shown in Fig. 1.33b. As a result, the coil does not store magnetic flux and presents no self-inductance.

1.19 Summary

Magnetic Laws

- A field is a spatial distribution of a scalar or vector quantity.
- Field lines may be used for visualization of the behavior of a field.
- A field for which the line integral around the closed path is zero is conservative. A field is conservative or irrotational if $\nabla \times \mathbf{B} = 0$.
- The sources of magnetic fields **H** and **B** are moving charges, that is, the electric current i.
- The source of electric fields **E** and **D** is the electric charge Q.
- The divergence represents the rate of change of a flow.
- The curl represents the rotation of a flow.
- The MMF $\mathcal{F} = Ni$ is a source magnetic flux in a magnetic circuit.
- Magnetic fields can be categorized as self, proximity, mutual, and fringing magnetic fields.
- The instantaneous magnetic field vector is a function of position and time.
- The phasor of magnetic field vector is a function of position only.
- A time-varying current in an inductor produces a changing magnetic flux, which induces a voltage between the terminals of an inductor.
- The RHR states that if the fingers of the right hand are placed around the coil in the direction of the current, the magnetic flux produced by the current is in the direction of the thumb.

- Ampère's law describes the relationship between the (conduction, convection, or displacement) current and the magnetic field intensity produced by this current.
- Both conductive and displacement currents induce the magnetic field.
- Ampère's law states that the line integral of magnetic field intensity **H** around a closed contour C is equal to the current enclosed by the contour.
- Faraday's law states that an AC voltage is induced in a coil, which contains a time-varying magnetic flux, regardless of the source of the magnetic flux.
- According to Faraday's law, the voltage (EMF) induced in closed circuit is equal to the time rate of change of the magnetic flux linkage $v = -d\lambda/dt = -Nd\phi/dt$.
- In the inductor, the induced voltage is proportional to the number of turns N and the time rate of change of the magnetic flux $d\phi/dt$.
- A magnetostatic magnetic field produces no current flow; however, a time-varying magnetic field produces an induced voltage (EMF) in a closed circuit.
- Ohm's law describes the relationship between the conduction (or drift) current density J_{cond} and the electric field intensity E, that is, $\mathbf{J}_{cond} = \sigma \mathbf{E}$.
- Convection current and displacement current do not obey Ohm's law.
- A curl-free vector field is called a irrotational or a conservative field.
- The Biot–Savart's law allows us to calculate the magnetic field intensity produced by a small current element at some point in space. This law states that the differential field intensity $d\mathbf{H}$ produced by the differential current element $Id\mathbf{l}$ at a point P is proportional to the product Idl and sin of the angle θ between the element and the line connecting P and the element and inversely proportional to the square of the distance R between P and the element.
- According to Lenz's law, the direction of the EMF is such that the current forced by the EMF induces a magnetic field that opposes the change in the applied magnetic field. The induced currents never support and always oppose the changes by which they are induced.
- Power is defined as the time rate of change of energy.
- The Poynting vector represents the direction and the magnitude of the surface power flow density of electromagnetic fields at any point in space, that is, the rate of energy transfer per unit area. This vector is equal to the cross product of the electric and magnetic fields $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.
- The Poynting theorem states that the rate of decrease in the energy stored in electric and magnetic fields in volume V, less the energy dissipated as heat, is equal to the power leaving the closed surface S bounding the volume V.
- The magnetic flux density **B** outside a very long current-carrying wire is inversely proportional to the distance from the axis of the wire.
- Joule's law states that the total power loss in a volume V is

$$P_D = \int\int\int_V \mathbf{J} \cdot \mathbf{E} dV. \tag{1.504}$$

Reluctance

- The reluctance is directly proportional to the length of the core mean magnetic path l_c and inversely proportional to the relative permeability μ_{rc} and the core cross-sectional area A_c through which the magnetic flux ϕ flows.

- The magnetic flux always takes the path with the highest permeability μ because the lowest reluctance occurs at the highest permeability.
- The magnetic flux always flows through the lowest reluctance.

Inductance

- The inductor sinusoidal current legs the inductor sinusoidal voltage by 90°.
- The inductance (or self-inductance) of a wire-wound inductor depends on its geometry and is proportional to the square of the number of turns N.
- The inductance is proportional to the ratio of the core cross-sectional area to the MPL A_c/l_c.
- The inductance of an inductor with a ferromagnetic core is μ_{rc} times higher than that of an air-core inductor.
- An inductor has a maximum value of the ampere-turn product $(\mathcal{F} = NI_m)_{max}$ limited by the core saturation flux density B_s.
- The self-inductance can be categorized as an internal inductance and the external inductance.
- The internal inductance is related to the magnetic field inside a conductor itself.
- The external inductance is related to the magnetic field outside a conductor.
- The winding turns should be evenly spaced to achieve consistent inductance and reduce leakage inductance.

Core Saturation

- At the core saturation, the magnetic flux density B approaches its maximum value known as the saturation flux density B_s. For $B > B_s$, $\mu_r = 1$.
- Core saturation can be avoided by reducing the peak value of the magnetic flux ϕ in the core or by increasing core cross-sectional area A_c so that $\phi/A_c < B_s$.
- It is difficult to avoid core saturation during transient circuit operation when the transient inductor current is large.

Air Gap

- An air gap is used to prevent core saturation and to make the inductance almost independent of μ_{rc}, yielding good inductance repeatability.
- Most of the MMF $\mathcal{F} = Ni$ is dropped across the air gap.
- The air gap contains nearly all of the magnetic field energy for high-permeability cores.
- Typically, 95% of inductance comes from the gap for inductors with high-permeability cores.
- An air gap in the core increases the energy storage capability of an inductor or a transformer.
- The core relative permeability μ_{rc} varies considerably with temperature and current. In contrast, the effective relative permeability is less dependent on the temperature and current. Therefore, it is desirable to maintain $\mathcal{R}_c \ll \mathcal{R}_g$ to achieve a predictable and stable inductance.
- The effective relative permeability of the core is proportional to the ratio l_c/l_g.
- The inductance of an inductor with an air gap is lower than the inductance of an inductor without an air gap.

Fringing Effect

- Whenever the core is excited, the fringing flux is present around the air gap, reducing the reluctance, increasing the inductance, and causing power losses.
- Fringing flux represents a larger percentage of the total flux for larger gaps.
- Fringing flux and inductor losses can be reduced by dividing a large air gap into several shorter air gaps.
- The fringing flux reduces the total reluctance $\mathcal{R}$ and increases the inductance L. Therefore, the number of turns should be reduced if the exact value of the inductance is required.
- The fringing field decreases substantially within one gap length distance l_g from the edge of the core.

Power Losses

- Power losses in inductors and transformers consist of winding and core losses.
- Eddy currents are induced in conductors by time-varying magnetic fields.
- Core losses consist of hysteresis loss and eddy-current loss.
- A distributed air gap along the magnetic path reduces the winding loss as compared to the winding loss due to a single gap. This is because the radial component of the magnetic flux is reduced.
- The impact of the radial component of the magnetic flux can be reduced by increasing the distance between the winding and the core. This distance can be increased by increasing the thickness of the bobbin.
- The winding should be moved away from the air gap by a distance, which is equal to twice the air-gap length $2l_g$.

Shielding

- A shield can be used to reduce EM emission by inductors and transformers.
- The thickness of the shield foils should be low compared to the skin depth.
- As the distance between the shield and the inductor decreases, the inductance also decreases.

Self-Resonant Frequency

- The SRF of an inductor is the resonant frequency of the resonant circuit formed by the inductance and the stray capacitance.
- The inductor impedance is capacitive above the SRF $f > f_r$.

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1.21 Review Questions

1.1. What is the MMF?

1.2. What is the magnetic flux?

1.3. What is the magnetic field intensity?

1.4. What is the magnetic flux density?

1.5. What is the magnetic linkage?

1.6. Define relative permeability.

1.7. What is the reluctance of a core?

1.8. What is the magnetic circuit? Give an example.

1.9. Can magnetic field exist in a good conductor?

1.10. State Ampère's circuital law.

1.11. State Faraday's law.

1.12. State Lenz's law.

1.13. What is Joule's law?

1.14. What is the point (microscopic) form of Ohm's law?

1.15. Write Maxwell's equations in differential and integral forms.

1.16. Write Maxwell's equations for good conductors.

1.17. State Poynting's law.

1.18. Write Biot–Savart's law.

1.19. Derive Joule's law.

1.20. Define power.

1.21. What is core saturation?

1.22. Define an inductance of a linear inductor.

1.23. Define an inductance of a nonlinear inductor.

1.24. What is the core inductance factor?

1.25. How is the inductance of a coil related to its number of turns?

1.26. What is the effect of an air gap on the inductance?

1.27. What is the fringing factor?

1.28. What is the effect of an air gap on core saturation?

1.29. Where is the magnetic energy stored in an inductor with an air gap?

1.30. Is the magnetic field intensity in the air gap higher or lower than that in the core?

1.31. Is the magnetic flux density in the air gap higher or lower than that in the core?

1.32. What is the volt–second balance?

1.33. Give expressions for magnetic energy in terms of H and B.

1.34. What are the mechanisms of power losses in magnetic components?

1.35. What are winding losses?

1.36. What is hysteresis loss?

1.37. What is eddy-current loss?

1.38. What are the effects of eddy currents on winding conductors and magnetic cores?

1.39. What is the SRF?

1.40. What is the difference between fringing flux and leakage flux?

1.41. The line integral of the magnetic field intensity **H** over a closed contour is zero. What is the net current flowing through the surface enclosed by the contour?

1.22 Problems

1.1. A current flows in the inner conductor of a long coaxial cable and returns through the outer conductor. What is the magnetic field intensity in the region outside the coaxial cable? Explain why.

1.2. Sketch the shape of the magnetic field around a current-carrying conductor and show how the direction of the field is related to the direction of the current in the conductor.

1.3. A toroidal inductor has the number of turns $N = 20$, the inner radius $a = 1$ cm, the outer radius $b = 2$ cm, and the height $h = 1$ cm. The core relative permeability is $\mu_{rc} = 100$. Find the inductance.

1.4. An inductor has $N = 300$ turns, $B = 0.5\ T$, and carries a current I of 0.1 A. The length $l_c = 15$ cm and cross-sectional area $A_c = 4\ \text{cm}^2$. Find the magnetic flux intensity, magnetic flux, and flux linkage.

1.5. An inductor has $\mu_{rc} = 800$, $N = 700$, $\phi = 4 \times 10^{-4}$ Wb, $l_w = 22$ cm, and $A_c = 4 \times 10^{-4}\ \text{m}^2$. Find the current I.

1.6. An inductor has $L = 100\ \mu\text{H}$, $l_c = 2.5$ cm, and $A_c = 2\ \text{cm}^2$. Find the number of turns N.

(a) For $\mu_{rc} = 1$.

(b) For $\mu_{rc} = 25$.

(c) For $\mu_{rc} = 25$ and $l_g = 3$ mm.

(d) For $\mu_{rc} = 2500$ and $l_g = 3$ mm.

1.7. A core has $A_L = 30\ \mu\text{H}/100$ turns. Find N to make an inductor of $1\ \mu\text{H}$.

1.8. A toroidal core has $N = 500$, $\mu_{rc} = 200$, $A_c = 4\ \text{cm}^2$, $r = 2$ cm, $I_m = 1c$A, $f = 10$ MHz, $\rho_w = \rho_{Cu} = 1.724 \times 10^{-6}\ \Omega\cdot\text{cm}$, and $\rho_c = 10^5\ \Omega\cdot\text{m}$. Find L, A_L, $\mathcal{R}$, H_m, B_m, ϕ_m, and λ_m.

1.9. A toroidal core of $\mu_{rc} = 3000$ has a mean radius $R = 80$ mm and a circular cross section with radius $b = 25$ mm. The core has an air gap $l_g = 3$ mm and a current I flows in a 500-turn winding to produce a magnetic flux of 10^{-4} Wb. Neglect the leakage flux.

(a) Determine the reluctance of the air gap, the reluctance of the core, and the total reluctance of the core with air gap.

(b) Find B_g and H_g in the air gap and B_c and H_c in the core.

(c) Find the required current I.

1.10. An inductor has $N = 100$, $A_c = 1\ \text{cm}^2$, $B_s = 0.3\ T$, $v_L = 10 \cos \omega t$ (V). Find $\lambda(t)$ and f_{min}.

1.11. Derive an expression for the internal and external inductances of a two-wire transmission line consisting of two parallel conducting wires of radius a that carry currents I in opposite directions. The axis-to-axis distance between the two wires is $d \gg a$.

1.12. The number of turns of a 100-μH inductor is doubled, while maintaining its cross-sectional area, length, and core material. What is the new inductance?

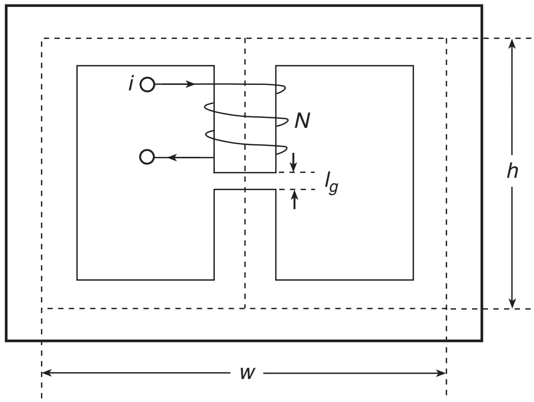

Figure 1.34 An inductor with an EE core and an air gap

1.13. An inductor with an EE magnetic core has an air gap in the center leg of length $l_g = 1$ mm and $\mu_{rc} = 3000$, as shown in Fig. 1.34. The height of the mean magnetic path is $h = 10$ cm. The width of the mean magnetic path at the core base is $w = 16$ cm. The cross-sectional area of all the legs is $A_c = 4$ cm^2. The winding is placed on the center leg. The number of turns is $N = 100$. The inductor current is $i_L = I_m \sin \omega t = 2 \sin 2\pi 60t$ (A).

(a) Draw the magnetic circuit and its single-loop equivalent circuit.

(b) Find the reluctance of each leg, the reluctance of the air gap, and the total reluctance of the core with the air gap.

(c) Find the amplitude of the magnetic flux in each leg and the air gap.

(d) Find the amplitude of the magnetic flux density in each leg and the air gap.

(e) Find the amplitude of the magnetic field intensity in each leg and the air gap.

(f) Determine the inductance.

1.14. An inductor with an air-gapped CC cut core shown in Fig. 1.35 has $\mu_{rc} = 10^5$, $N = 66$, $l_c = 17$ cm, and the length of the air gap on each side of the CC core is 0.5 mm. The cross section of all legs is a rectangular with dimensions $a = 1.28$ cm and $b = 0.98$ cm. Neglect the fringing effect of the magnetic flux.

(a) Determine the reluctance of the core, the reluctance of the gap, and the total reluctance.

(b) Find the ratio of the gap reluctance to the core reluctance.

(c) Determine the inductance.

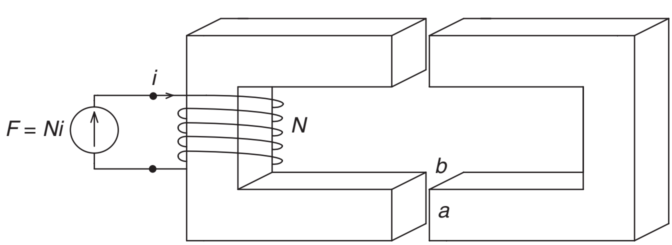

Figure 1.35 Inductor with a CC cut core

1.15. Consider the inductor described in Problem 1.14, neglecting the fringing flux. The inductor current is $i = I_m \sin \omega t = 0.5 \sin \omega t$ (A).

(a) Find the magnetic flux densities in the core and the gaps.

(b) Sketch the magnetic flux densities in the core and the gaps.

(c) Find the magnetic field intensities in the core and the gaps.

(d) Sketch the magnetic field intensities in the core and the gaps.

(e) Find the maximum magnetic energies stored in the core and both gaps.

(f) Find the maximum magnetic energy densities stored in the core and both gaps.

1.16. An air-gapped inductor with a CC supermalloy cut core shown in Fig. 1.35 has $a = 12.8$ mm, $b = 9.8$ mm, $\mu_{rc} = 10^5$, and $N = 66$. The air-gap length on each side is 0.5 mm. Find F_f, L, and L_f.

1.17. An inductor with an air-gapped CC cut core shown in Fig. 1.35 has $\mu_{rc} = 10^5$, $B_s = 1.5$ T, $N = 66$, $l_c = 17$ cm, and the length of the air gap on each side of the CC core is 0.5 mm. The cross section of all legs is rectangular with dimensions $a = 1.28$ cm and $b = 0.98$ cm. The inductor current is sinusoidal. Neglect the fringing effect of the magnetic flux.

(a) Determine the maximum amplitude of magnetic flux density in the core and in the gaps for operation just below core saturation.

(b) Determine the maximum amplitude of magnetic field intensity in the core and in the gaps for operation just below the core saturation.

(c) Determine the amplitudes of magnetic energy densities stored in the core and the air gaps.

(d) Determine the amplitudes of magnetic energies stored in the core and the air gaps.

(e) Determine the maximum amplitude of the inductor current just below the core saturation.

1.18. An inductor has the inductance $L = 100$ μH, ESR $r_L = 1.5$ Ω, and self-capacitance $C = 10$ pF.

(a) Find the quality factor of the inductor at $f = 1$ MHz.

(b) Find the SRF of the inductor.