1

Statistical Analysis and Spectral Method

Statistical and spectral analyses are commonly used to characterize random phenomena which are seemingly chaotic, irregular, and unpredictable. They can also be used as the effect tools to study and describe turbulent flow, since the turbulent flow is random.

Many different computational approaches have been developed to solve turbulence problem, which is a very difficult problem. The most widely used models include statistical models and pseudospectral methods that are based on statistical and spectral analysis, respectively. In the statistical models, such as turbulent-viscosity models, e.g., $k-\varepsilon$ model, Reynolds-stress models, probability-density function (PDF) methods, and large eddy simulation (LES), the turbulent flow is described in terms of some statistics. In the pseudospectral methods, the turbulent flow is described in terms of the spectral coefficients in the spectral space.

However, in order to extract the essential kinetic and dynamic characteristics of turbulence and interpret it properly from experimental and numerical results, the statistical and spectral analyses are also used to deal with and to analyze these data, amount of which has grown through application of new experimental and computational tools.

Throughout the book, the turbulent control problems are solved numerically by the pseudospectral methods, the experiment and numerical dada involved are dealt with by statistical or spectral analysis, and the discussions on turbulent flow are based on the statistical and spectral descriptions. Therefore, the statistical analysis and spectral method are introduced briefly in Chapter 1.

In Section 1.1, statistical analysis is presented, and in Section 1.2, the statistical representation of turbulent flow is discussed. Sections 1.3 and 1.4 are concerned with spectral series expansions for Fourier series and other orthogonal basis. The fundamental concepts and technical of spectral method, a numerical method for partial differential equations, are introduced in Section 1.5 and its spectacular applications to Navier–Stokes (N-S) equations to turbulent flows are discussed in Section 1.6.

1.1 Statistical Analysis and Spectral Method[1–3]

1.1.1 Average Value

The fluid velocity field in turbulent flow is random, which varies significantly and irregularily in both position and time, and described by random variables. Considering a random

Principles of Turbulence Control, First Edition. Baochun Fan and Gang Dong.

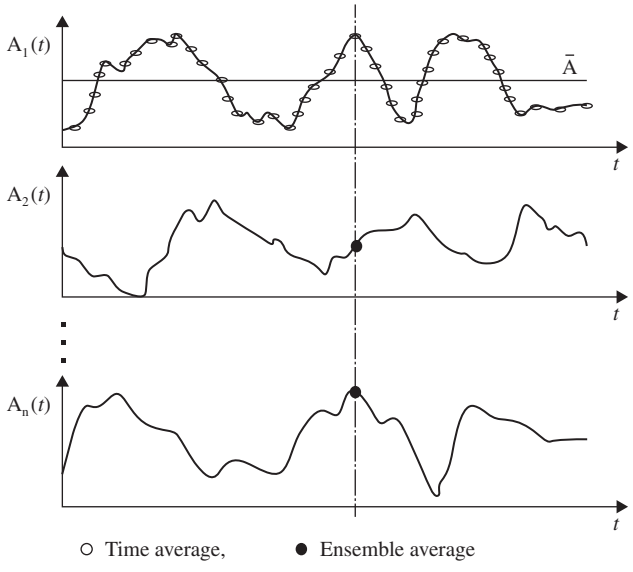

Figure 1.1 Time average and ensemble average of random variables

event expressed by a random variable $A(\mathbf{x}, t)$, which is statistically steady, the experimental measurements are taken on n repetitions at a specified position under the same set of conditions, and measured independent curves, $A_1(t), A_2(t), \ldots, A_n(t)$, are obtained. The curve is called a realization of random events $A(\mathbf{x}, t)$.

As shown in Figure 1.1, the time average for one realization is defined as

$$\overline{A} = \lim_{T\to\infty} \frac{\lim\limits_{T\to\infty} \int_{t}^{t+T} A(t)dt}{T} \tag{1.1}$$

Therefore, a random variable A is decomposed into a mean $\overline{A}$ and a fluctuating part A', representing the deviation from the mean, such that

$$A = \overline{A} + A' \tag{1.2}$$

The mean value of a fluctuating quantity itself is zero.

Similarly, the space average is define as

$$\overline{A} = \lim_{V\to\infty} \frac{\int_{V} A(\mathbf{x})d\mathbf{x}}{V} \tag{1.3}$$

The ensemble average of the realizations under the same set of conditions is defined by

$$\langle A \rangle = \lim_{n\to\infty} \frac{A_1 + A_2 + \cdots + A_n}{n} \tag{1.4}$$

Hence, $A = \langle A \rangle + A'$ (1.5).

1.1.2 Probability Density and Statistical Moments

1.1.2.1 Probability Density

A realization of random events $A(\mathbf{x}, t)$ at a specified position is shown in Figure 1.2.

Define indicator function

$$\varphi(A,t) = \begin{cases} 1 & \text{if} \quad A(t) < A \\ 0 & \text{if} \quad A(t) \geq A \end{cases} \tag{1.6}$$

where A is a given value and $A(t)$ is an arbitrary value. Then we have

$$\lim_{T\to\infty} \int_t^{t+T} \varphi(A,t)dt = \sum \Delta t_i = T_L$$

T_L is a time during which $A(t) < A$.

$$\text{Let } \overline{\varphi} = \frac{T_L}{T} = P(A,t)$$

that is, a ratio between total duration of certain conditions satisfied and total duration of averaging, representing the percentage of time spent by $A(t)$ under the given level. P is called the cumulative distribution function (CDF) ($0 \leq P \leq 1$) and monotonically increases as A increase, as shown in Figure 1.3.

The probability density function (PDF) is defined to be the derivative of CDF:

$$P(A) = \frac{dP}{dA} \tag{1.7}$$

which represents the probability of events with $A < A(t) < A + \Delta A$. That is,

$$\Delta P = P\Delta A = P_{\text{rob}}\{A < A(t) < A + \Delta A\}$$

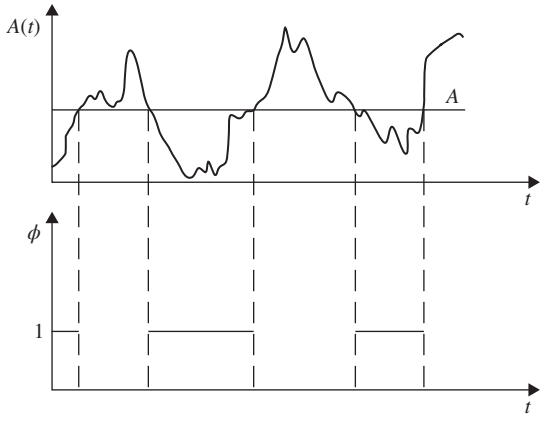

Figure 1.2 Indication function

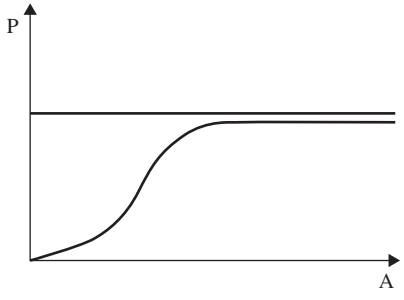

Figure 1.3 Cumulative distribution function

where P_{rob} denotes probability. It satisfies the normalized condition

$$\int_{-\infty}^{+\infty} P(A)dA = 1 \tag{1.8}$$

Obviously, the average (or mean) of the random variable can be obtained by PDF

$$\overline{A} = \int_{-\infty}^{+\infty} AP(A)dA \tag{1.9}$$

1.1.2.2 Statistical Moments

The mean values of the various powers of A are called moments. The nth moment $\overline{A^n}$ is defined by

$$\overline{A^n} = \lim_{T\to\infty} \frac{1}{T} \int_{t}^{t+T} A^n(t)dt = \int_{-\infty}^{+\infty} A^n(t)P(A)dA = \frac{\int_{-\infty}^{+\infty} A^n P(A)dA}{\int_{-\infty}^{+\infty} P(A)dA} \tag{1.10}$$

where the first moment $\overline{A}$, that is, $n = 1$, is the familiar mean value.

The nth central moment $\overline{A'^n}$ is defined by

$$\overline{A'^n(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{t}^{t+T} (A'(t))^n dt = \int_{-\infty}^{\infty} (A - \overline{A})^n P dA \tag{1.11}$$

where A' is fluctuation. The first central moment $\overline{A'}$, of course, is zero. The second central moment $\overline{A'^2(t)}$ is called the variance, characterizing the magnitudes of the fluctuations with respect to its mean, its square root $\sqrt{\overline{A'^2(t)}}$ is the standard deviation, often called root-mean-square (rms), which is the measure of the width of P.

The skewness associated with third central moment is defined by

$$S = \frac{\overline{A'^3}}{\left(\overline{A'^2}\right)^{3/2}} \tag{1.12}$$

which gives an ideal of asymmetry in P about the origin. If the values of all odd moments are zero, P is symmetric about mean. If $S < 0$, the distribution cure of P shifts toward the negative fluctuation direction, the tail of the curve on the left side is longer. The negative fluctuation prevails.

The kurtosis associated with fourth central moment is defined by

$$K = \frac{\overline{A'^4}}{\left(\overline{A'^2}\right)^2} \tag{1.13}$$

which is a measure of whether the curve of P is peaked or flat relative to a normal distribution induced later. The curve of P with high kurtosis has a distinct peak near the mean, declines rather rapidly, and has heavy tails.

1.1.2.3 Characteristic Function

A random variable A can be written in the complex exponential form, that is, $e^{ikA(t)}$, where k is the wave number. The mean called the characteristic function is

$$f(k,t) = \overline{e^{ikA}} = \int_{-\infty}^{+\infty} e^{iAk} P(A,t) dA \tag{1.14}$$

Differentiating Eq. (1.14) with respect to k, we have

$$\frac{\partial^n f}{\partial k^n} = \int_{-\infty}^{+\infty} (iA)^n e^{ikA} P dA$$

$$\left(\frac{\partial^n f}{\partial k^n}\right)_{k=0} = (i)^n \int_{-\infty}^{+\infty} A^n P dA = (i)^n \overline{A^n}$$

$$\overline{A^n} = (i)^n \left(\frac{\partial^n f}{\partial k^n}\right)_{k=0} \tag{1.15}$$

This means moments are related to derivatives of characteristic function $f(k,t)$ at the origin $k = 0$.

The characteristic function can be written as a Taylor series of moments

$$f = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \overline{A^n} \tag{1.16}$$

therefore the characteristic function in principle can be determined from all derivatives.

On substituting Eq. (1.16) into Eq. (1.14) yields

$$P(A,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(k,A) e^{-ikA(t)} dk \tag{1.17}$$

Then, P is given from the determined characteristic function, f.

1.1.2.4 Normal Distribution

In many cases, there is a probability density function called the standard normal distribution (or standard Gaussian distribution) expressed by

$$P = \frac{1}{\sqrt{2\pi}} e^{-\frac{A^2}{2}} \tag{1.18}$$

Obviously, we have $\overline{A} = 0$ and $\overline{A^2} = 1$.

$$\text{If } P = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{A-a}{\sigma}\right)^2} \tag{1.19}$$

then, it is said to be the normal distribution (or Gaussian distribution), in which the first central moment is a, rms is σ, all odd central moments are zero, and even moments B_{2n} are expressed by

$$B_{2n} = (2n-1)!!\sigma^{2n} \tag{1.20}$$

where "!!" denotes the double factorial. Also, skewness, $S = 0$ and kurtosis, $K = 3$.

The characteristic function is

$$f = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikA} e^{-\frac{1}{2}\left(\frac{A-a}{\sigma}\right)^2} dA = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \cos(kA) e^{-\frac{1}{2}\left(\frac{A-a}{\sigma}\right)^2} dA = e^{\left\{ika - \frac{k^2}{2}\sigma^2\right\}} \tag{1.21}$$

The shape of Gaussian distribution is symmetric about the peak at $A = a$, and there exist the grads near the peak, as shown in Figure 1.4. Skewness S and kurtosis K, redefined as $K = \frac{\overline{A'^4}}{\left(\overline{A'^2}\right)^2} - 3$ here, for any arbitrary PDF represent the deviation from the symmetric shape of Gaussian distribution.

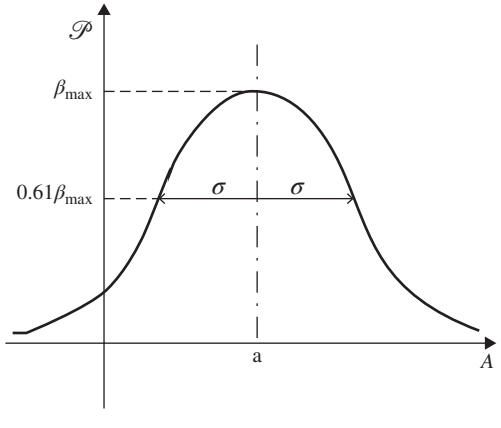

Figure 1.4 Normal distribution function

Let us consider N statistically independent random variables $A_i(t)$, where $A_i(t)$ are said to be independent if $\overline{A_i' A_j'} = 0$. We assume that all $A_i(t)$ have identical probability densities and that their mean values are zero. Then, the sum of all $A_i(t)$ has a Gaussian probability density, that is, A defined by $A = \sum_{i=1}^{N} A_i$ is a normal distribution function, which is called the central limit theorem.

1.1.3 Correlation Function

1.1.3.1 Auto-Correlation Function

Only the distributions of fluctuations at one point in time or space are discussed in previous section. Therefore, the relations between neighboring fluctuations will be discussed further here.

Using superscript i, denoting the measured position, the indicator function is defined by

$$\varphi(A_1, t_1)\varphi(A_2, t_2) = \begin{cases} 1 & A^{(i)}(t_1) < A_1 \text{ and } A^{(i)}(t_2) < A_2 \\ 0 & \text{otherwise} \end{cases} \tag{1.22}$$

where A_1 and A_2 are two given values and $A^{(i)}(t)$ is an arbitrary value at the ith measured position. t_1 and t_2 represent different times.

The cumulative probability distribution function is

$$P_{\text{rob}}\{A(t_1) < A_1; A(t_2) < A_2\} = \overline{\varphi(A_1, t_1)\varphi(A_2, t_2)} = P_2 \tag{1.23}$$

The correlated probability density function is

$$P_2(A_1, A_2; t_1, t_2) = \frac{\partial^2 P_2(A_1, A_2; t_1, t_2)}{\partial A_1 \partial A_2} \tag{1.24}$$

It is the fraction time that the random variable $A(t)$ is between A_1 and $A_1 + dA_1$ at time t_1, as well as is between A_2 and $A_2 + dA_2$ at time t_2. It can also be written as

$$\Delta P_2 = P_2 \Delta A_1 \Delta A_2 = P_{\text{rob}}\{A_1 < A(t_1) < A_1 + \Delta A_1, A_2 < A(t_2) < A_2 + \Delta A_2\} \tag{1.25}$$

which represents a statistical mass of a square area shown in Figure 1.5, if P_2 is regard as a density.

It also satisfies the following equations:

$$\int_{-\infty}^{+\infty} \int P_2(A) dA_1 dA_2 = 1 \tag{1.26}$$

$$\int_{-\infty}^{+\infty} P_2(A) dA_2 = P(A_1, t) \tag{1.27}$$

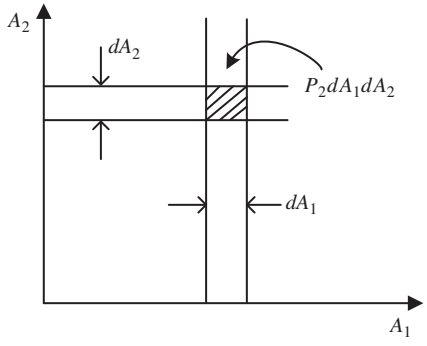

Figure 1.5 Correlated probability density

The central moment is

$$\overline{A^n(t_1)} = \int \int_{-\infty}^{+\infty} A_1^n P_2 dA_1 dA_2 \tag{1.28}$$

and the correlated moment is

$$\overline{A^m(t_1)A^n(t_2)} = \int \int_{-\infty}^{+\infty} A_1^m A_2^n P_2 dA_1 dA_2 \tag{1.29}$$

The most important correlated moment is $\overline{A(t_1)A(t_2)}$, which is defined as

$$\overline{A(t_1)A(t_2)} = \int \int_{-\infty}^{+\infty} A_1 A_2 P_2 dA_1 dA_2 \tag{1.30}$$

This is called the auto-correlation function, which expresses a survival capability of the random event, that is, how many traces remained at the time t_1 can be found at the time t_2 as random event developing. It will vanish as $\Delta t = t_2 - t_1 \to \infty$.

1.1.3.2 Joint Correlation Function

Considering the probability density for two random variables A and B, the indicator function is defined by

$$\varphi_A(A_1, t, \mathbf{x})\varphi_B(B_1, t, \mathbf{x}) = \begin{cases} 1 & A < A_1, B < B_1 \\ 0 & \text{otherwise} \end{cases} \tag{1.31}$$

where A_1 and B_1 are two given values. The probability for both $A < A_1$ and $B < B_1$ is

$$P_{\text{rob}}\{A < A_1 \text{ and B} < B_1\} = \overline{\varphi_A \varphi_B} = P_2.$$

The joint probability density is

$$P_2 = \frac{\partial^2 P_2}{\partial A \partial B} \tag{1.32}$$

Also, it is written as $\Delta P_2 = P_2 \Delta A \Delta B = P_{\text{rob}}\{A_1 < A < A_1 + \Delta A_1, B_1 < B < B_1 + \Delta B_1\}$.

The most important correlated moment is $\overline{AB}$, which is defined as

$$\overline{AB} = \int_{-\infty}^{+\infty} \int AB P_2 dA dB \tag{1.33}$$

This is called joint correlation function between A and B. When $\overline{AB} = 0$, the two random events A and B are said to be uncorrelated or independent.

1.2 Statistical Analysis of Turbulence

1.2.1 Reynolds Stress and Turbulent Kinetic Energy

Turbulence consists of random velocity fluctuations, so that it must be treated with statistical methods. All fluctuating quantities in turbulence would be decomposed into mean values and fluctuations, representing the mean character of the turbulent flow field and the deviation from the mean at moments, which is referred to as Reynolds decomposition.

In turbulence, a description of the flow at all points in time and space by solving the N-S equations is not feasible. If the Reynolds decomposition is applied to the N-S equations, then we take the average of all terms in the resulting equations, the correlation functions for fluctuating quantities will yield, such as $R_{ij} = \overline{u_i' u_j'}$. In general, the equations for a set statistics at a level (e.g., product of two parameters) contain additional statistics at a high level (e.g., triple products), with more unknowns than equations. Because of these unknown functions, the Reynolds average N-S equations are not closed, unless the additional statistics are modeled.

Reynolds stresses τ_{ij}^t is defined as

$$\tau_{ij}^t = -\rho \overline{u_i' u_j'} \tag{1.34}$$

which plays a crucial role in the equations for the mean velocity field and results in the difference between the instantaneous velocity field and the mean field, by transporting momentums between the turbulence and the mean flow. Therefore, it can be perceived as an agent producing stress in the mean flow.

The second central moment of fluctuating velocities $R_{ii} = \overline{u_i' u_i'} = \overline{u'^2_1 + u'^2_2 + u'^2_3}$, where repeated indices indicate a summation over all three values of the index, is written as

$$K = \frac{1}{2} R_{ii} = \overline{\frac{u'^2_1}{2} + \frac{u'^2_2}{2} + \frac{u'^2_3}{2}} \tag{1.35}$$

It is called the kinetic energy of turbulence, representing the energy of turbulent fluctuation.

$$\text{Let } I = \frac{\sqrt{\frac{1}{3}\left(\overline{u'^2_1} + \overline{u'^2_2} + \overline{u'^2_3}\right)}}{\overline{u}} \tag{1.36}$$

It is called turbulent strength, where $\overline{u} = \sqrt{\overline{u_1^2} + \overline{u_2^2} + \overline{u_3^2}}$. It can also be defined by

$$I_i = \frac{u'_{i\text{rms}}}{\overline{u}}$$

In order to discuss the correlated relations between neighboring velocity fluctuations, we define the spatially auto-correlation coefficient by

$$R(r) = \frac{\overline{u'_A u'_B}}{u'_{A.\text{rms}} u'_{B.\text{rms}}} \tag{1.37}$$

where subscripts A and B represent two neighboring points with the distance of r. Since $\overline{u'_A u'_B} \leq \sqrt{\overline{u'^2_A}} \cdot \sqrt{\overline{u'^2_B}}$, we have $R(r) \leq 1$. As shown in Figure 1.6, the correlativity becomes stronger with the decrease of r, that is, $\lim_{r\to 0} R(r) \to 1$, as $B \to A$, whereas $\lim_{r\to\infty} R(r) \to 0$.

Let

$$l_T = \int_0^\infty R(r)dr \tag{1.38}$$

where l_T is called the characteristic scale of vortex, represented by a square area shown in Figure 1.6. It can be perceived as a scale of an imaginary fluid parcel, which moves in turbulence as a whole, that is, with the same fluctuating frequency and amplitude.

In order to discuss correlated relations between velocity fluctuations at two different times, we define the temporally auto-correlation coefficient by

$$R^*(t) = \frac{\overline{u'(t)u'(t+\Delta t)}}{u'_{t,\text{rms}} u'_{t+\Delta t,\text{rms}}} \tag{1.39}$$

Let $t^* = \int_0^\infty R^*(t)dt$

$$\text{we have } \; l_T^* = u'_{\text{rms}} t^* \tag{1.40}$$

where t^* represents life of an imaginary fluid parcel. l_T^* with the same order of the magnitude with l_T denotes the moving path of the fluid parcel before the disaggregation and the loss of its integral character.

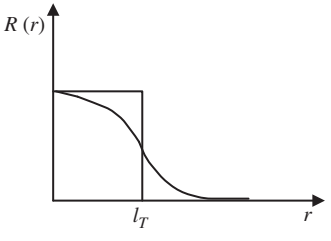

Figure 1.6 Characteristic scale of turbulence

1.2.2 Variable-Interval Time Average Method

Despite the fact that the turbulent flow is seemingly random and chaotic, the near wall turbulence possesses characteristic coherent structures, generally consisting of streaks and streamwise vortices (to be discuss in Chapter 2). The turbulence is formed intermittently and periodically via bursting events related to a violent breakup of a lifted steak. Bursting can be commonly detected by the detection function, then the ensemble average of the flow field around the bursting over all bursting events can be calculated, which may reveal the characteristic flow pattern as the turbulent onset.[4, 5]

1.2.2.1 Variable-Interval Time Average and Variable-Interval Space Average[6, 7]

The variable-interval time average of a fluctuating quantity $A(x_i, t)$, called VITA, is defined as

$$\widehat{A}(x_i, t, T) = \frac{1}{T} \int_{t-\frac{1}{2}T}^{t+\frac{1}{2}T} A(x_i, s)ds \tag{1.41}$$

where T is the averaging time. If one wants to obtain a local average of some particular phenomenon, the averaging time must be of the order of the time scale of the phenomenon under study. Note that

$$\overline{A}(x_i) = \lim_{T\to\infty} \widehat{A}(x_i, t, T)$$

where the bar indicates the conventional time average. The time average value will be fixed, if it is statistically steady.

Similarly, the variable-interval space average of a fluctuating quantity $A(x_i, t)$, called VISA, is defined as

$$\widehat{A}(x_i, t, L) = \frac{1}{L} \int_{x_i-L/2}^{x_i+L/2} A(s, t)ds \tag{1.42}$$

where L is the width of the spatial averaging.

1.2.2.2 Detection Function of Turbulent Bursting[8, 9]

"Bursting" is referred to the production of turbulence in boundary layer, which can be commonly detected by the detection function based on a threshold. For example, the detection criterion is completed using the VITA variance; the detection function $D(t)$ is defined as

$$D(t) = \begin{cases} 1 & \text{if } \overset{\wedge}{\text{Var}} > k \cdot u'^2_{\text{rms}} \text{ and } \partial u'/\partial t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.43}$$

where u' is the fluctuating component of the streamwise velocity. The localized mean of u' is

$$\widehat{u}'(x_i, t, T) = \frac{1}{T} \int_{t-T/2}^{t+T/2} u'(x_i, s)ds$$

and the localized second moment of u' is defined as

$$\widehat{u}'^2(x_i, t, T) = \frac{1}{T} \int_{t-T/2}^{t+T/2} u'^2(x_i, s) ds$$

Finally, the localized variance is defined as

$$\widehat{\mathrm{Var}}(x_i, t, T) = \widehat{u}'^2(x_i, t, T) - [\widehat{u}'(x_i, t, T)]^2 \tag{1.44}$$

Note that, $u'^2_{\mathrm{rms}} = \lim_{T\to\infty} \mathrm{Var}$

k is the threshold level and u'^2_{rms} is the rms of the fluctuating streamwise velocity.

The process of bursting detection from the measured fluctuating streamwise velocity by VITA is shown in Figure 1.7. The measured velocity at a position near the wall is shown in the upper figure, based on which the localized variance is obtained by Eq. (1.44), as shown in the middle. Finally, the distributions of detection function $D(t)$ can be drawn in the bottom one, from which it is clear that only one bursting is detected.

Figure 1.8 shows the variations of the fluctuating velocity with time at some positions with different y^+, where superscript "+" indicates variables normalized by wall-unit scales, to be introduced in Chapter 2. The impulse vertical lines at the horizontal axis shown in Figure 1.8(b) represent the detection functions detected from the streamwise velocity at $y^+ = 15$. It can be seen the bursting events are observed for four times, when the streamwise velocity fluctuate dramatically.

If we define $D(x)$ in the form associated with VISA

$$D(x) = \begin{cases} 1 & \text{if} \quad \mathrm{Var} \ > k \cdot u'^2_{\mathrm{rms}} \ \text{and} \ \partial u'/\partial x < 0 \\ 0 & \text{otherwise} \end{cases}$$

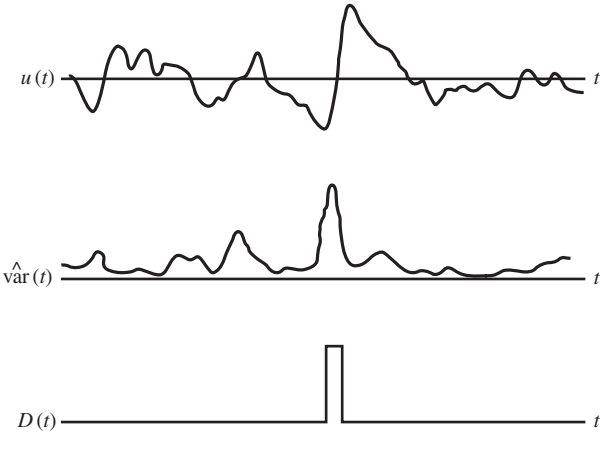

Figure 1.7 Detection process of bursting events.[4] *Source*: Blackwelder 1976. Reproduced with permission of Cambridge University Press

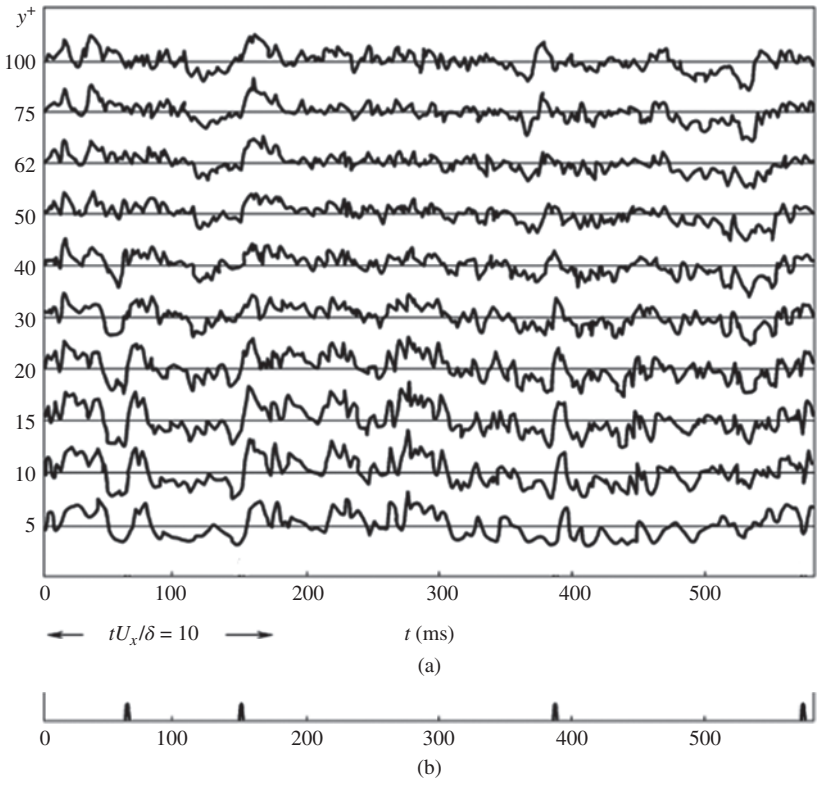

Figure 1.8 Instantaneous streamwise velocities (a) at different y^+ locations and detector function (b) obtained at $y^+ = 15$.[4] *Source*: Blackwelder 1976. Reproduced with permission of Cambridge University Press

where $\widehat{u}'(x_i, t, L) = \frac{1}{L} \int_{x_i - L/2}^{x_i + L/2} u'(s, t) ds$

$$\widehat{u}'^2(x_i, t, L) = \frac{1}{L} \int_{x_i - L/2}^{x_i + L/2} u'^2(s, t) ds$$

$$\widehat{\mathrm{Var}}(x_i, t, L) = \widehat{u}'^2(x_i, t, L) - [\widehat{u}'(x_i, t, L)]^2$$

$D(x) = 1$ denotes the turbulent bursting then the turbulent bursting generated in the instantaneous flow field can be singled out, therefore both the positions, at which the bursting events occurred and the total number of the events detected at a moment are determined.

The detection of bursting phenomenon is very depending on the choice of detection function and threshold values. Since a burst occurs commonly in conjunction with violent outward ejections, the burst can be detected using the quadrant analysis. This technique prescribes the occurrence of an ejection when $u' < 0$ and $v' > 0$ and the $|uv|$ product exceed a set threshold.

Then the detection function $D(x)$ is defined as

$$D(x) = \begin{cases} 1 & \text{if} \quad \left|u'v'\right| > hu'_{\text{rms}}v'_{\text{rms}} \\ 0 & \text{otherwise} \end{cases}$$

Where h is a threshold, and || denotes the absolute value.

1.2.2.3 Conditional Average

Bursting is regard as one of the most important processes in wall-bounded turbulence. By means of the conditional average techniques, it is possible to study the burst phenomenon via the detailed flow pattern associated with the bursting, which will conduce to the understanding of the wall-bounded turbulent phenomenon.

For a reference position x_i where the bursting occurred, the total number of the bursting events detected at x_i, N, and the corresponding moment for each burst, $t_j, j = 1, \ldots, N$, can be obtained using the detection function. The conditional averaging of a quantity A is defined by

$$\langle A(x_i, \tau)\rangle_{y^+} = \frac{1}{N}\sum_{j=1}^{N} A(x_i, t_j + \tau) \tag{1.45}$$

where y^+ indicates the position of detection probe and t_j is the time as the bursting event occurs. The positive or negative τ is used to determine the temporal behavior of A before and after burst.

The variations of the conditional averages of streamwise velocity at different y^+ with the time τ are shown in Figure 1.9. It is evident that, when the bursting event occurs, the velocity varies dramatically for $y^+ \leq 25$, whereas it does not vary considerably for $y > 25$.

In order to obtain a spatial structure rather than the temporal structure, the conditional averaging process is modified in the following way:

$$\langle A(\xi, t_i)\rangle_{y^+} = \frac{1}{N}\sum_{j=1}^{N} A(x_j + \xi, t_i)$$

where $x_j, j = 1, \ldots, N$, denotes the detection point where the bursting event occurs and N is the total number of the bursting events detected at time t_i. The positive or negative ξ is used to determine the spatial behavior of A in the vicinity of the detection points in the $(x - z)$ planes at y^+.

The profiles of conditional averages of streamwise velocity at different y^+ locations are shown in Figure 1.10. The peaks of the profiles occur just upstream of the detected point $\xi = 0$, whereas it occurs after the burst as shown in Figure 1.9.

1.3 Fourier Transform and Spectrum

1.3.1 Harmonic Wave

Harmonic vibration around an undeflected position is described by

$$x(t) = a\cos\left(\frac{2\pi}{T}t + \alpha\right) \tag{1.46}$$

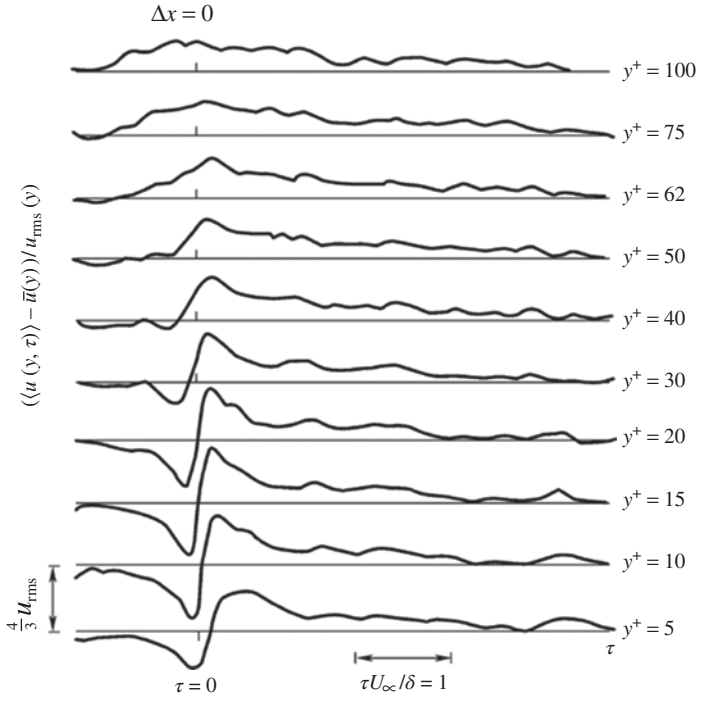

Figure 1.9 Conditionally averaged streamwise velocities at different y^+ locations.[4] *Source*: Blackwelder 1976. Reproduced with permission of Cambridge University Press

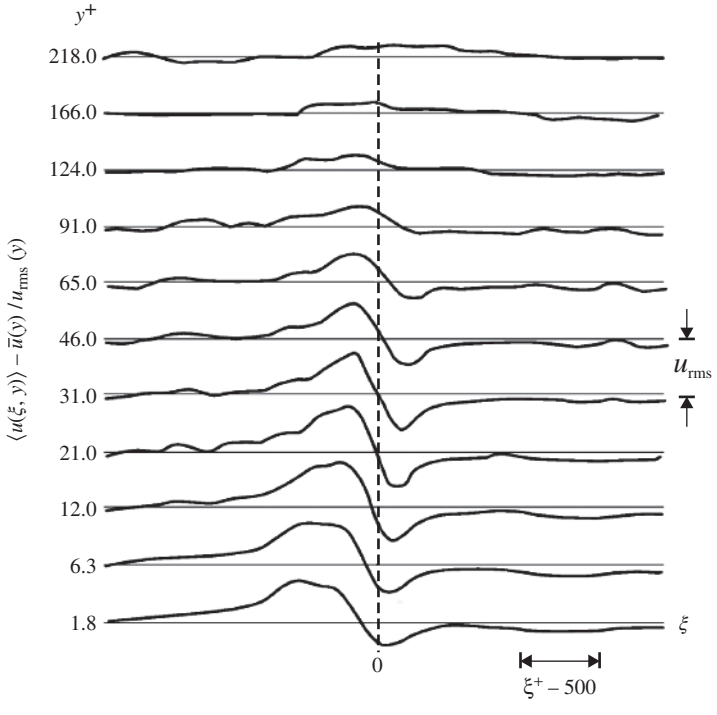

Figure 1.10 Conditionally averaged streamwise velocities at different y^+ locations as a function of streamwise coordinate.[6] *Source*: Kim J 1983. Reproduced with permission of AIP Publishing LLC

where $x(t)$ denotes the displacement, a is the amplitude of vibration, T is the period, $f = \frac{1}{T}$ is the frequency, $\omega = \frac{2\pi}{T}$ circular frequency, α is the initial phase, and $\omega t + \alpha$ is the phase.

Harmonic vibration can be represented by a projection of a circularly moving point on the horizontal axis and is written in the complex form as

$$A = a\cos(\omega t + \alpha) + ia\sin(\omega t + \alpha) = R(A) + iI(A) \tag{1.47}$$

where $a = |A| = \sqrt{[R(A)]^2 + [I(A)]^2}$

$$\omega t + \alpha = \arg A = \tan^{-1}\frac{I(A)}{R(A)}$$

where R denotes the real part of complex A, I is the image part, and a is the radius of the circle, that is, amplitude. At $t = 0$, the moving point is located at A_0 with the initial phase angle α.

Based on the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Equation (1.47) can be written in the complex exponential form as

$$A = ae^{i(\omega t+\alpha)} = ae^{i\alpha}e^{i\omega t} = Fe^{i\omega t} \tag{1.48}$$

where F is a complex number:

$$a = |F| = \sqrt{[R(F)]^2 + [I(F)]^2}$$

$$\alpha = \arg F = \tan^{-1}\frac{I(F)}{R(F)}$$

1.3.2 Fourier Transform[10]

1.3.2.1 Fourier Series of Periodic Function

Fourier series associated with an arbitrary periodic function $f_T(t)$, with the period T, is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\lambda t + b_n\sin n\lambda t) \quad n = 1, 2, 3, \ldots, \tag{1.49}$$

where $\lambda = \frac{2\pi}{T}$ is the base frequency:

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)\cos n\lambda t\, dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t)\sin n\lambda t\, dt$$

Writing in the complex exponential form, it becomes

$$f_T(t) = \sum_{-\infty}^{+\infty} F(n\lambda) e^{in\lambda t} \tag{1.50}$$

where the sum is over the infinite discrete frequencies. The complex Fourier coefficients are

$$F(n\lambda) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{in\lambda t} dt = \frac{\lambda}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{in\lambda t} dt$$

which satisfy conjugate symmetry:

$$F(n\lambda) = F^*(-n\lambda)$$

where an asterisk denotes the complex conjugate. $F(0)$ is the zero frequency component with respect to $n = 0$:

$$F(n\lambda)e^{in\lambda t} + F(-n\lambda)e^{-in\lambda t} = 2|F(n\lambda)| \cos(n\lambda t + \arg F(n\lambda)) \tag{1.51}$$

where

$$|F(n\lambda)| = \sqrt{[R(F(n\lambda))]^2 + [I(F(n\lambda))]^2}$$

$$\alpha = \arg F(n\lambda) = \tan^{-1} \frac{I(F(n\lambda))}{R(F(n\lambda))}$$

Hence,

$$f_T(t) = \sum_{-\infty}^{+\infty} 2|F(n\lambda)| \cos(n\lambda t + \arg F(n\lambda)) \tag{1.52}$$

$$F(n\lambda) = \frac{\lambda}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{in\lambda t} dt \tag{1.53}$$

It can be concluded that an arbitrary periodic function $f_T(t)$ can be regarded as a superimposition of infinite harmonic waves, in which discrete circular frequencies are integer multiple of base frequency, amplitudes are $2|F(n\lambda)|$, and initial phases are $\arg(F(n\lambda))$.

1.3.2.2 Fourier Transform of Nonperiodic Function

Let $\omega = \frac{n2\pi}{T}$, $\Delta\omega = \omega_{n+1} - \omega_n = \lambda$. As $T \to \infty$, that is, nonperiodic function, $d\omega = \frac{2\pi}{T}$, so that the integration is adapted in series expansion defined by Eq. (1.50) instead of the sum. We have

$$f(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{+\infty} f(\tau) e^{-i\omega t} d\tau e^{i\omega t} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega = \mathcal{F}^{-1}[F(\omega)] \tag{1.54}$$

or

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt = \mathsf{F}(f(t)) \tag{1.55}$$

where $F(\omega)$ is a function of ω. $\mathsf{F}(t)$ is defined as Fourier transform and $\mathsf{F}^{-1}(\omega)$ is Fourier inverse transform. $f(t)$ and $F(\omega)$ are a transform pair.

Writing Eq. (1.54) in the cosine transform, we obtain

$$f(t) = \frac{1}{\pi}\int_{0}^{+\infty}\left[\int_{-\infty}^{+\infty} f(\tau)\cos\omega(t-\tau)d\tau\right]d\omega \tag{1.56}$$

then

$$\begin{aligned} f(t) &= \int_{-\infty}^{+\infty}\{R(F(\omega))\cos\omega t + I(F(\omega))\sin\omega t\}d\omega \\ &= \int_{-\infty}^{+\infty}|F(\omega)|e^{i(\omega t-\varphi(\omega))}d\omega \end{aligned} \tag{1.57}$$

where

$$|F(\omega)| = \sqrt{[R(F(\omega))]^2 + [I(F(\omega))]^2}$$

$$\alpha = \tan^{-1}\frac{I(F(\omega))}{R(F(\omega))}$$

$R(F(\omega))$ is the even function and $I(F(\omega))$ is the odd function.

1.3.2.3 Fourier Transform of Periodic Function

In order to discuss the Fourier transform of periodic functions, in which the frequencies of harmonic waves are discrete, impulse function δ is induced, defined by

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \tag{1.58}$$

We have $\int_{-\infty}^{+\infty}\delta(t)dt = 1$

When $t \neq a$, $\delta(t-a) = 0$

$$\int_{-\infty}^{+\infty}\delta(t-a)\varphi(t)dt = \varphi(a) \tag{1.59}$$

Hence, $\mathsf{F}[\delta(t)] = \int_{-\infty}^{+\infty}\delta(t)e^{-i\omega t}dt = e^0 = 1$

$$\mathsf{F}^{-1}[2\pi\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}2\pi\delta(\omega)e^{-i\omega t}d\omega = 1$$

and

$$F^{-1}[2\pi\delta(\omega-a)]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}2\pi\delta(\omega-a)e^{-i\omega t}d\omega=e^{iat} \tag{1.60}$$

Therefore, $\delta(t)$ and 1 are a transform pair, 1 and $2\pi\delta(\omega)$ are a transform pair as well as e^{iat} and $2\pi\delta(\omega-a)$ are also a transform pair.

By substituting Eq. (1.50) into Eq. (1.55), we obtain

$$F((f(t))=\int_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty}F(n\lambda)e^{in\lambda t}e^{-i\omega t}dt=\sum_{-\infty}^{+\infty}F(n\lambda)\int_{-\infty}^{+\infty}e^{in\lambda t}e^{-i\omega t}dt=\sum_{-\infty}^{+\infty}F(n\lambda)F(e^{in\lambda t}) \tag{1.61}$$

By substituting Eq. (1.50) into Eq. (1.61), we obtain

$$F((f(t))=\sum_{-\infty}^{+\infty}F(n\lambda)F(e^{in\lambda t})=\sum_{-\infty}^{+\infty}2\pi F(n\lambda)\delta(\omega-n\lambda)=F(\omega) \tag{1.62}$$

In the same way, we obtain

$$F^{-1}[F(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty}2\pi F(n\lambda)\delta(\omega-n\lambda)e^{i\omega t}d\omega=\sum_{-\infty}^{+\infty}F(n\lambda)e^{in\lambda t}=f(t) \tag{1.63}$$

It is proved that $f(t)$ and $F(\omega)=\sum_{-\infty}^{+\infty}2\pi F(n\lambda)\delta(\omega-n\lambda)$ are a transform pair for periodic functions, that is,

$$F(f(t))=F(\omega)=\int_{-\infty}^{+\infty}f(t)e^{-i\omega t}dt$$

$$F^{-1}(F(\omega))=f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)e^{i\omega t}d\omega$$

with $F(\omega)=\sum_{-\infty}^{+\infty}2\pi F(n\lambda)\delta(\omega-n\lambda)$.

$F(\omega)$ is regarded as a spectrum of the waveform function $f(t)$ as defined in electrical engineering.

If the signal $f(t)$ is applied to a filter, regarded as a physical system having an input and an output, then the output $f_{\text{out}}(t)$ will be produced. This filtering procedure can be described as follows: first, one analyzes $f(t)$ into its spectrum $F(\omega)$, then multiplies each spectral component by the corresponding transfer factor $T(\omega)$ to obtain the spectrum of $f_{\text{out}}(t)$, that is, $F_{\text{out}}(\omega)$, and finally synthesizes $f_{\text{out}}(t)$ from its spectrum. Thus,

$$F_{\text{out}}(\omega)=T(\omega)F(\omega)$$

and then $f_{\text{out}}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}T(\omega)F(\omega)e^{i\omega t}d\omega$

Since multiplication of transform corresponds to convolution of original functions, that is,

$$f_{\text{out}}(t) = G(t) * f(t) = \int G(\tau, t) f(t-\tau) d\tau$$

where $G(t)$ is the Fourier transform of $T(\omega)$, which would be the characteristic of the filter.

1.3.3 Energy Spectrum

The squared modulus of a transform is defined as energy spectrum, which has the character of energy density measured per unit of ω; that is, $|F(\omega)|^2$ is the energy spectrum of $f(t)$.

Based on Parseval's theorem for Fourier series

$$\int_{-\infty}^{+\infty} f^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) d\omega \tag{1.64}$$

where $S(\omega) = |F(\omega)|^2$. The integral, representing the amount of energy in the system, has usually a finite value for nonperiodic function, whereas it would have to become infinite, if the function $f(t)$ is periodic. Now we can consider an average power defined as

$$P = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} f^2(t) dt = \sum_{-\infty}^{+\infty} |F(n\lambda)|^2$$

where $S(n\lambda) = |F(n\lambda)|^2$ is called the discrete power spectrum.

It can be written in the form of Fourier transform as

$$P = \int_{-\infty}^{+\infty} \left| \frac{1}{2\pi} F(\omega) \right|^2 d\omega \tag{1.65}$$

where $F(\omega) = \sum_{-\infty}^{+\infty} 2\pi F(n\lambda) \delta(\omega - n\lambda)$.

1.4 Spectral Series Expansion of Function

1.4.1 Orthogonal Basis

Consider an infinite sequence formed by basis functions

$$\phi_1(x), \phi_2(x), \ldots, \phi_k(x), \ldots \tag{1.66}$$

This series is the so-called linear independence as any one basis function in the series is independent linearly with others and is said to be orthogonal with respect to the weight function $w(x)$ further, if

$$(\phi_i, \phi_j) = \delta_{ij} \nu_i^2$$

where the inner product $(a, b) = \int_{\alpha}^{\beta} abwdx$, and a and b are two functions defined on the interval $[\alpha, \beta]$. δ_{ij} is the delta function defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and v_i is the normalized constant.

Let us consider an unknown function u that can be expressed by a sum of the orthogonal basis:

$$u = \sum_{k=-\infty}^{\infty} \tilde{u}_k \phi_k$$

where $\tilde{u}_k$ is a spectral coefficient. Actually, this expansion gives a mapping between the function u in the physical space and the spectral coefficients $\tilde{u}_k$ in the spectral space. u can be obtained by $\tilde{u}_k$,vice versa.

Fourier series and Chebyshev polynomials are in common use as the basis functions in the spectral method, the former is appropriate for the periodic problems and the latter for nonperiodic.

1.4.2 Fourier Series

As discussed in Section 1.3.2, a periodic function $u(x)$ defined on the interval $(0, 2\pi)$ can be expanded into the Fourier series in the complex form as

$$u(x) = \sum_{k=-\infty}^{\infty} \tilde{u}_k \phi_k(x) \tag{1.67}$$

where $\phi_k(x) = e^{ikx}$ and k is wave number. Then

$$\int_0^{2\pi} u(x) e^{-ilx} dx = \int_0^{2\pi} \sum_{k=-\infty}^{\infty} \tilde{u}_k \phi_k(x) \phi^*{}_l(x) dx = 2\pi \tilde{u}_k$$

Hence, the spectral coefficients are

$$\tilde{u}_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-ikx} dx \qquad k = 0, \pm 1, \pm 2, \ldots . \tag{1.68}$$

Since

$$\int_0^{2\pi} \phi_k(x) \phi_l^*(x) dx = 2\pi \delta_{kl} \tag{1.69}$$

where $\{\phi_k\}$ is orthogonal over the interval $(0, 2\pi)$. If $u(x)$ is a real function, its spectral coefficients are complex.

By differentiating Eq. (1.67), we obtain

$$u'(x) = \sum_{k=-\infty}^{\infty} ik\tilde{u}_k e^{ikx}$$

$$u^{(l)}(x) = \sum_{k=-\infty}^{\infty} (ik)^l \tilde{u}_k e^{ikx} \tag{1.70}$$

where superscript "′" represents first-order derivative, and "l" represents lth derivative. It means that lth derivative in spectral (Fourier transform) space consists of multiplying each spectral coefficient by $(ik)^l$.

1.4.3 Chebyshev Polynomials

The Chebyshev polynomial of the first kind $\{T_k(x)\}$ is the polynomial of degree k defined on $-1 \le x \le 1$ by

$$T_k(x) = \cos(k\cos^{-1}x), \quad k = 0, 1, 2, \ldots, \tag{1.71}$$

Therefore, $-1 \le T_K \le 1$. By setting $x = \cos\theta$, we have

$$T_k = \cos k\theta$$

The differentiation of T_k gives

$$T_k' = k \sin\ k\theta / \sin\theta$$

By the application of trigonometrical formulas, we obtain a recurrence relation on the derivative

$$2T_k(x) = \frac{1}{k+1}T'_{k+1}(x) - \frac{1}{k-1}T'_{k-1}(x) \quad k \ge 1 \tag{1.72}$$

or

$$T'_k(x) = 2k\sum_{n=0}^{K} \frac{1}{c_{k-1-2n}} T_{k-1-2n}(x) \tag{1.73}$$

where $K = (k-1)/2$. A similar formula for the lth derivative is obtained by successive differentiation of Eq. (1.72).

The orthogonality property is

$$\int_{-1}^{1} T_m(x)T_n(x)\frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi} \cos n\theta \cos m\theta d\theta = \begin{cases} 0, & m \ne n \\ c_m \frac{\pi}{2} & m = n \end{cases} \tag{1.74}$$

where $c_m = \begin{cases} 2 & m = 0 \\ 1 & m \ge 1 \end{cases}$.

Therefore, the Chebyshev polynomials are orthogonal over the interval $[-1, 1]$ with the weight $w(x) = (1 - x^2)^{-\frac{1}{2}}$.

From the trigonometrical identity

$$\cos(k+1)\theta + \cos(k-1)\theta = 2\cos\theta\cos k\theta,$$

the recurrence relationship of $\{T_k(x)\}$ is deduced as

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \tag{1.75}$$

That allows us to deduce the expression of the polynomials T_k, $k \geq 2$ from $T_0(x) = 1$ and $T_1(x) = x$.

Let $|u| < 1$, then

$$\sum_{k=0}^{\infty} u^k e^{ik\theta} = \frac{1}{1 - ue^{i\theta}}$$

The real part is

$$\sum_{k=0}^{\infty} u^k \cos k\theta = \frac{1 - u\cos\theta}{1 - 2u\cos\theta + u^2}$$

that is,

$$F(u, x) = \frac{1 - ux}{1 - 2ux + u^2} = \sum_{k=0}^{\infty} T_k(x)u^k \quad x \in [-1,\ 1] \tag{1.76}$$

It is shown that the polynomials $T_k(x)$ are coefficients of the expansion of $F(u, x)$ on the variable u in the exponential series. $F(u, x)$ is called the generation function of $T_k(x)$.

Chebyshev series expansion of the function $u(x)$ defined in the interval $[-1, 1]$ is

$$u(x) = \sum_{k=0}^{\infty} \tilde{u}_k T_k(x) \tag{1.77}$$

Based on the orthogonality property of Eq. (1.74), we have

$$\int_{-1}^{1} uT_k(x)\frac{dx}{\sqrt{1 - x^2}} = \int_{-1}^{1} \sum_{m=0}^{N} \tilde{u}_m T_m(x)T_k(x)\frac{dx}{\sqrt{1 - x^2}} = \tilde{u}_k c_k \frac{\pi}{2}$$

That is,

$$\tilde{u}_k = \frac{2}{c_k \pi} \int_{-1}^{1} u(x)T_k(x)\frac{dx}{\sqrt{1 - x^2}} \tag{1.78}$$

Then the expansion coefficients $\tilde{u}_k$ can be determined from known $u(x)$. Equations (1.77) and (1.78) are a transform pair between the physical space and the spectral space.

From Eqs. (1.77) and (1.72), we obtain

$$u' = \sum_{m=0}^{\infty} \tilde{u}_m T'_m = \sum_{m=0}^{\infty} \tilde{u}_m^{(1)} T_m(x) \tag{1.79}$$

$$u'' = \sum_{m=0}^{\infty} \tilde{u}_m T''_m = \sum_{m=0}^{\infty} \tilde{u}_m^{(2)} T_m(x) \tag{1.80}$$

where superscript "′" represents first-order derivative and "″" represents second-order derivative:

$$\tilde{u}_m^{(1)} = \frac{2}{c_m} \sum_{p=m+1}^{\infty} p\tilde{u}_p \tag{1.81}$$

$$\tilde{u}_m^{(2)} = \frac{1}{c_m} \sum_{p=m+2}^{\infty} p\,(p^2 - m^2)\,\tilde{u}_p \tag{1.82}$$

Obviously, both $\tilde{u}_m^{(1)}$ and $\tilde{u}_m^{(2)}$ are polynomials of $\tilde{u}_k$. The derivative in spectral (Chebyshev transform) space can be represented in a sum of each spectral coefficient multiplying with corresponding weight.

1.5 Fundamentals of Spectral Methods[11–14]

1.5.1 Fundamental Concepts

1.5.1.1 Weighted Residual Method

Considering a differential equation written here as

$$L(u) = f \quad \alpha < x < \beta \tag{1.83}$$

with the following boundary conditions:

$$B_- u = g_- \text{ at } x = \alpha, \quad B_+ u = g_+ \quad \text{at } x = \beta \tag{1.84}$$

where L is an operator, B_- and B_+ correspond to Dirichlet, Neumann, or mixed conditions.

If the solution of the equation, that is, function $u(x)$, is approximated by u^N, the truncated series expansion after N terms is defined as

$$u^N = \sum_{k=0}^{N} \tilde{u}_k \phi_k \quad u^N \in X_N$$

where ϕ_k is trial (or basis) function. The choice of the trial functions is a key issue in the spectral method. The chosen trial functions are orthogonal. If the solution is periodic, use Fourier series, and if the domain is finite, but the solution is not periodic, Chebyshev polynomials are best.

Now we introduce the residual $R_N(x)$ defined by

$$R_N(x) = L_N(u^N) - f(u^N)$$

Spectral methods belong to the general class of weighted residual methods, based on the residual-minimizing conditions, defined through the inner product

$$(u, v) = \int_{\alpha}^{\beta} u\, v\, \omega\, dx = 0$$

where $u(x)$ and $v(x)$ are two functions defined on $[\alpha, \beta]$ and $\omega(x)$ is some given weight function. Therefore, we have

$$(R_N, \psi)_N = \int_{\alpha}^{\beta} R_N \;\; \psi \;\; \omega_* dx = 0 \quad \psi \in Y_N \tag{1.85}$$

where ψ is the test function and ω_* is the weight function. The dimension of the discrete set Y_N depends on the problem under consideration.

1.5.1.2 Collocation Method

The choice of the test functions and of the weights is associated with the spectral method. If the inner product in Eq. (1.85) is evaluated by a type of numerical quadrature known as "Gaussian integration," a set of points $\{x_i\}$, called the collocation points, might be obtained. In turn, the collocation method, one of the formulations in the spectral method, corresponds to the choice

$$\psi_i = \delta(x - x_i) \quad \text{and} \quad \omega_* = 1$$

From Eq. (1.85), we simply obtain

$$R_N(x_i) = 0$$

or

$$(L_N u^N)_{x_i} = f(x_i) \tag{1.86}$$

Therefore, in the collocation method, the residual is exactly zero at the collocation points.

Now, we will present a way to determine the collocation points. As we know, the main idea of the numerical integration is to fit a polynomial of degree N, $P_N(x)$ to the integrand $f(x)$ and then integrate $P_N(x)$, by summation of $P_N(x_i)$ with weight $\{w_i\}$ over the set of points $\{x_i\}$, which is formularized with

$$\int_a^b f(x)dx \approx \sum_{i=0}^{N} w_i P_N(x_i)$$

If we allow $\{x_i\}$ as well as $\{w_i\}$ to be unknowns, we can maximize the accuracy of the numerical integration by the choice of $\{x_i\}$ and $\{w_i\}$.

Based on Gauss–Jacobi integration theorem,[15] for the set of orthogonal polynomials $\{p_j(x)\}$ on $x \in [-1, 1]$ with respect to the weight function $\rho(x)$, the collocation points $\{x_i\}$ are chosen to be the solutions of the following equation:

$$p_{N+1}(x) = 0 \tag{1.87}$$

And the points $\{w_i\}$ are the solutions of the following set of linear equations:

$$\sum_{i=0}^{N} x_i^k w_i = \int_{-1}^{1} x^k \rho(x) dx \quad k = 0, \ldots, N \tag{1.88}$$

Then the Gauss–Jacobi integration

$$\int_a^b p_j(x)\rho(x)dx = \sum_{i=0}^{N} w_i p_j(x_i) \tag{1.89}$$

is exact for all $p_j(x)$ which are polynomials of at most degree $(2N + 1)$. The positive numbers $\{w_i\}$ are called "weights."

However, the roots $\{x_i\}$, which correspond to the collocation points, are all in the interior of $[-1, 1]$, in order to impose boundary conditions at one or both end points; the points $\{x_i\}$ are chosen to be the solutions of the following equation:

$$g(x) = p_{N+1}(x) + ap_N(x) = 0 \tag{1.90}$$

where $a = -p_{N+1}(-1)/p_N(-1)$, thus $g(-1) = 0$, we have $x_0 = -1$, the first point x_0 is fixed on the left boundary. Let the points $\{w_i\}$ be the solutions of Eq. (1.88), the quadrature formula

$$\int_{-1}^{1} p_j(x)\rho(x)dx = \sum_{i=0}^{N} w_i p_j(x_i) \tag{1.91}$$

is called Gauss–Radau integration. It can be proved that Gauss–Radau integration is exact for all $p_j(x)$ which are polynomials of at most degree $2N$.

If the points $\{x_i\}$ are chosen to be the solutions of the following equation:

$$g(x) = p_{N+1}(x) + ap_N(x) + bp_{N-1}(x) = 0 \tag{1.92}$$

where the coefficients a and b are determined by equations $g(\pm 1) = 0$, we have

$$x_0 = -1 \text{ and } x_N = 1$$

the first and final points x_0 and x_N are chosen at the two boundaries.

Let the points $\{w_i\}$ be the solutions of Eq. (1.88), then the quadrature formula

$$\int_{-1}^{1} p_j(x)\rho(x)dx = \sum_{i=0}^{N} w_i p_j(x_i) \tag{1.93}$$

is called Gauss–Lobatto integration, where all $p_j(x)$ are polynomials of at most degree $(2N-1)$. The proof of this result is similar to the previous one.

These Gauss-type quadratures show that the integration with the integrand $p_j(x)\rho(x)$, where $p_j(x)$ and $\rho(x)$ are orthogonal polynomial and its weight function respectively, can be exactly displaced by a summation of $w_i p_j(x_i)$, where the collocation points $\{x_i\}$ and the weight points $\{w_i\}$ are determined on the special equations.

Considering the Chebyshev polynomials, e.g., the collocation points as well as the weight points are determined by Gauss-type quadratures, that is, Gauss–Jacobi integration, in terms of Eqs. (1.87) and (1.88),

$$x_j = \cos\frac{(2j+1)\pi}{2N+2} \quad w_j = \frac{\pi}{N+1} \quad j = 0, \ldots, N \tag{1.94}$$

Gauss–Radau integration, in terms of Eqs. (1.90) and (1.88),

$$x_j = \cos\frac{2\pi j}{2N+1} \qquad w_j = \begin{cases} \dfrac{\pi}{2N+1} & j = 0 \\ \dfrac{2\pi}{2N+2} & 1 \le j \le N-1 \end{cases} \tag{1.95}$$

Gauss–Lobatto integration, in terms of Eqs. (1.92) and (1.88)

$$x_j = \cos\frac{\pi j}{N} \qquad w_j = \begin{cases} \dfrac{\pi}{2N} & j = 0, N \\ \dfrac{\pi}{N} & 1 \le j \le N-1 \end{cases} \tag{1.96}$$

which is used most commonly.

1.5.2 Fourier–Galerkin Method

1.5.2.1 Discrete Fourier Series

The more familiar spectral method is the Fourier–Galerkin method where the trial functions are trigonometric functions $\{e^{ikx}\}$. Therefore, the solution of Eq. (1.83) is sought in the form of the truncated Fourier series

$$u(x) \approx u^N(x) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k e^{ikx}$$

where $u^N(x)$ is a polynomial of degree N.

When the technique of collocation is applied, where the collocation points $\{x_i\}$ associated with the Fourier series are defined by

$$x_i = \frac{2\pi}{N} i \qquad i = 0, 1, \ldots, N-1 \tag{1.97}$$

we have

$$u(x_i) = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k e^{ikx_i} \tag{1.98}$$

Consequently,

$$\tilde{u}_k = \frac{1}{N} \sum_{i=0}^{N-1} u(x_i) e^{-ikx_i} \tag{1.99}$$

If the function $u(x)$ is assumed to be periodic, it satisfies

$$u(x_0) = u(x_n)$$

The solutions at the noncollocation points can be obtained by the interpolation among the collocation points in the form

$$u(x) = \sum_{i=0}^{N-1} u(x_i) g_i(x) \tag{1.100}$$

where the function $g_i(x)$ is reduced to

$$g_i(x) = \frac{1}{N} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}-1} e^{ip(x-x_i)} = \frac{1}{N} \sin\left[\frac{N}{2}\left(x - x_i\right)\right] \cot\frac{x - x_i}{2}$$

is a trigonometric polynomial and satisfies

$$g_i(x_j) = \delta_{ij}.$$

1.5.2.2 Differentiation at Collocation Point

Then from Eq. (1.98), the lth derivative at a given collocation point x_j is expressed as

$$u^{(l)}(x_j) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} (ik)^l \tilde{u}_k e^{ikx_j} \tag{1.101}$$

By differentiating Eq. (1.100), we can construct differentiation formulas expressing the derivative, of any order, in physical space at a given collocation point, in terms of the values of the function itself at all collocation points, that is,

$$(u(x_j))^{(l)} = \sum_{s=0}^{\frac{N}{2}-1} u(x_s) g_s^{(l)}(x_j) = \sum_{s=0}^{\frac{N}{2}-1} (D_l)_{js} u(x_s) \tag{1.102}$$

where

$$(D_1)_{js} = \begin{cases} \frac{1}{2}(-1)^{j+s} \cot \dfrac{x_j - x_s}{2}, & s \neq j \\ 0, & s = j \end{cases}$$

$$(D_2)_{js} = \begin{cases} \frac{1}{2}(-1)^{j+s+1} \dfrac{1}{\sin^2((x_j - x_s)/2)}, & s \neq j \\ -\dfrac{2N^2+1}{6}, & s = j \end{cases}$$

1.5.2.3 Fourier–Galerkin Method

In the Fourier–Galerkin method, the trial functions are trigonometric functions, $\{e^{ikx}\}$, which satisfy the boundary conditions, and the collocation points $\{x_i\}$ associated with the text function $\psi_i = \delta(x - x_i)$ are defined by

$$x_i = \frac{2\pi}{N} i \quad i = 0, 1, \ldots, N-1 \tag{1.103}$$

We have

$$(L_N u^N)_{x_j} = f(x_j) \tag{1.104}$$

Using the discrete orthogonality property of the trial function $\{e^{ikx}\}$ and the derivative character expressed by Eq. (1.101), a closed system furnishing $N+1$ equations can be obtained for determining the $N+1$ spectral coefficients $\tilde{u}_k$ from Eq. (1.86). This is because boundary conditions are satisfied by the trial functions and we have $X_N = Y_N$.

1.5.3 Chebyshev–Tau Method[16]

1.5.3.1 Discrete Chebyshev Series

When the solution is not periodic, taking Chebyshev polynomials $\{T_k(x)\}$ as a trial function is more suitable in the spectral method, called Chebyshev–Tau method. Here, the solution of Eq. (1.83) is sought in the form of the truncated Chebyshev series

$$u(x) \approx u^N(x) = \sum_{k=0}^{N} \tilde{u}_k T_k(x) \tag{1.105}$$

where $u^N(x)$ is a polynomial of degree N. When the technique of collocation is applied, we have

$$u(x_i) = \sum_{k=0}^{N} \tilde{u}_k T_k(x_i)$$

Consequently,

$$\tilde{u}_k = \frac{1}{\gamma_k} \sum_{j=0}^{N} u(x_j) T_k(x_j) w_j \tag{1.106}$$

where $\gamma_k = \frac{\pi}{2} c_k \quad k < N$

$$\gamma_N = \begin{cases} \frac{\pi}{2} & \text{for Gauss and Gauss–Radau} \\ \pi & \text{for Gauss–Lobatto} \end{cases}$$

The solutions at the noncollocation points can be obtained by the interpolation among the collocation points in the form

$$u(x) = \sum_{i=0}^{N-1} u(x_i) \psi_i(x) \tag{1.107}$$

where the function $\psi_i(x)$ is Lagrange interpolated polynomial with $\psi_i(x_j) = \delta_{ij}$.

For Gauss–Lobatto integration,

$$\psi_j(x) = \frac{(-1)^{j+1}(1 - x^2) T'_N(x)}{\bar{c}_j N^2 (x - x_j)} \quad j = 0, 1, \ldots, N \tag{1.108}$$

where $\bar{c}_0 = \bar{c}_N = 2$, $\bar{c}_j = 1$, $1 \le j \le N - 1$.

1.5.3.2 Differentiation at Collocation Point

At a given collocation point, Eqs. (1.79) and (1.80) are written as

$$u'(x_j) = \sum_{m=0}^{N} \tilde{u}_m T'_m(x_j) = \sum_{m=0}^{N} \tilde{u}_m^{(1)} T_m(x_j) \tag{1.109}$$

$$u''(x_j) = \sum_{m=0}^{N} \tilde{u}_m T''_m(x_j) = \sum_{m=0}^{N} \tilde{u}_m^{(2)} T_m(x_j) \tag{1.110}$$

where

$$\tilde{u}_m^{(1)} = \frac{2}{c_m} \sum_{\substack{p=m+1 \\ p+m \text{ odd}}}^{N} p \tilde{u}_p \quad m = 0, \ldots, N - 1 \tag{1.111}$$

$$\tilde{u}_N^{(1)} = 0, \quad \tilde{u}_{N-1}^{(1)} = 2N \tilde{u}_N \tag{1.112}$$

$$\tilde{u}_m^{(2)} = \frac{1}{c_m} \sum_{\substack{p=m+2 \\ p+m \text{ even}}}^{N} p(p^2 - m^2)\tilde{u}_p \quad m = 0, \ldots, N-2 \tag{1.113}$$

$$\tilde{u}_{N-1}^{(2)} = \tilde{u}_N^{(2)} = 0 \tag{1.114}$$

where $c_m = \begin{cases} 2 & m = 0 \\ 1 & m \geq 1 \end{cases}$.

For lth derivative, we have

$$u^l(x_j) = \sum_{m=0}^{N} \tilde{u}_m^{(l)} T_m(x_j)$$

In terms of Eqs. (1.72) and (1.73), we obtain

$$2k\tilde{u}_k = c_{k-1}\tilde{u}_{k-1}^{(1)} - \tilde{u}_{k+1}^{(1)} \quad k \geq 1 \tag{1.115}$$

Since $u_k^{(1)} = 0$ at $k \geq N$, hence

$$c_k\tilde{u}_k^{(1)} = \tilde{u}_{k+2}^{(1)} + 2(k+1)\tilde{u}_{k+1} \quad 0 \leq k \leq N-1 \tag{1.116}$$

By successive differentiations, the general recurrence formula for $\tilde{u}_k^{(l)}$ is obtained

$$c_k\tilde{u}_k^{(l)} = \tilde{u}_{k+2}^{(l)} + 2(k+1)\tilde{u}_{k+1}^{(l-1)} \quad 0 \leq k \leq N-1 \tag{1.117}$$

By differentiating Eq. (1.107), the derivative in physical space at a given collocation point is given by

$$(u(x_j))' = \sum_{s=0}^{N} u(x_s)\psi_s' \ (x_j) = \sum_{s=0}^{N} (D_N)_{js} u(x_s) \tag{1.118}$$

For Gauss–Lobatto integration, in terms of Eq. (1.108), we have

$$(D_N)_{lj} = \begin{cases} \dfrac{\overline{c}_l}{\overline{c}_j} \dfrac{(-1)^{l+j}}{x_l - x_j} & l \neq j \\ \dfrac{-x_j}{2(1-x_j^2)} & 1 \leq l = j \leq N-1 \\ \dfrac{2N^2+1}{6} & l = j = 1 \\ -\dfrac{2N^2+1}{6} & l = j = N \end{cases} \tag{1.119}$$

1.5.3.3 Chebyshev–Tau Method

In the Chebyshev–Tau method, the trial functions are Chebyshev polynomials $\{T_k(x)\}$, which cannot satisfy the boundary conditions so that $Y_N < X_N$, and the collocation points $\{x_i\}$ associated with the text function $\psi_i = \delta(x - x_i)$ might be given by Eqs. (1.94)–(1.96), depended on which type of Gaussian integrations is chosen. Then, we have

$$(L_N u^N)_{x_j} = f(x_j) \tag{1.120}$$

By means of discrete Chebyshev series (Eq. (1.106)) and the derivative character expressed by Eqs. (1.111)–(1.114), a system only furnishing $N-1$ equations related to $\tilde{u}_k$ can be obtained, which is not closed for determining the $N+1$ spectral coefficients $\tilde{u}_k$. The complement equations should be derived from boundary conditions. Considering a channel flow, on upper and lower walls, we have

$$u(-1) = u(1) = 0$$

Since $T_k(1) = 1$ and $T_k(-1) = (-1)^k$, then

$$\sum_{k=0}^{N} \tilde{u}_k = 0 \tag{1.121}$$

$$\sum_{k=0}^{N} (-1)^k \tilde{u}_k = 0 \tag{1.122}$$

Now, we obtain a closed system to determine all spectral coefficients, $\tilde{u}_k$. This so-called Tau method is a modification of the Galerkin method allowing the use of trial functions not satisfying the boundary conditions.

1.5.4 Helmholtz Equation

Helmholtz equation is a one-dimensional ordinary differential equation written as

$$\frac{d^2u}{dy^2} - \lambda u = f \tag{1.123}$$

Where u is unknown, f is a function of y, and λ is a known constant.
Based on the collocation method, the Chebyshev–Tau approximation of equation (1.123) is

$$\tilde{u}_n^{(2)} - \lambda \tilde{u}_n = \tilde{f}_n \; n = 0, 1, \ldots, N-2 \tag{1.124}$$

When combined with Eq. (1.113), this yields

$$\sum_{\substack{p=n+2 \\ p+n \text{ even}}}^{N} p(p^2 - n^2)\tilde{u}_p - \lambda \tilde{u}_n = \tilde{f}_n \quad n = 0, 1, \ldots, N-2 \tag{1.125}$$

Using the recursion equation (1.116), that is, $2n\tilde{u}_n^{(1)} = c_{n-1}\tilde{u}_{n-1}^{(2)} - \tilde{u}_{n+1}^{(2)}$, Eq. (1.124) is also written as

$$2n\tilde{u}_n^{(1)} = c_{n-1}(\tilde{f}_{n-1} + \lambda\tilde{u}_{n-1}) - (\tilde{f}_{n+1} + \lambda\tilde{u}_n) \;\; n = 1, \ldots, N-3$$

Institution of Eq. (1.115), that is, $c_k\tilde{u}_k^{(1)} = \tilde{u}_{k+2}^{(1)} + 2(k+1)\tilde{u}_{k+1}$, we have

$$\begin{aligned} 2n\tilde{u}_n^{(1)} = {} & \frac{c_{n-1}}{2(n-1)}[c_{n-2}(\tilde{f}_{n-2} + \lambda\tilde{u}_{n-2}) - (\tilde{f}_n + \lambda\tilde{u}_n)] \\ & - \frac{1}{2(n+1)}[c_n(\tilde{f}_n + \lambda\tilde{u}_n) - (\tilde{f}_{n+2} + \lambda\tilde{u}_{n+2})] \\ & n = 2, \ldots, N-4 \end{aligned}$$

This simplifies to

$$\begin{aligned} & \frac{c_{n-2}}{4n(n-1)}\lambda\tilde{u}_{n-2} + \left(1 - \frac{\lambda}{2\left(n^2-1\right)}\right)\tilde{u}_n + \frac{\lambda}{4n(n+1)}\tilde{u}_{n+2} \\ & = \frac{c_{n-2}}{4n(n-1)}\tilde{f}_{n-2} - \frac{1}{2(n^2-1)}\tilde{f}_n + \frac{1}{4n(n+1)}\tilde{f}_{n+2} \\ & n = 2, \ldots, N-4 \end{aligned} \tag{1.126}$$

Noting that the four equations are dropped in going from Eqs. (1.124) to (1.126), we can write Eq. (1.124) as

$$\begin{aligned} & \frac{c_{n-2}}{4n(n-1)}\lambda\tilde{u}_{n-2} + \left(1 - \frac{\lambda\beta_n}{2\left(n^2-1\right)}\right)\tilde{u}_n + \frac{\lambda\beta_{n+2}}{4n(n+1)}\tilde{u}_{n+2} \\ & = \frac{c_{n-2}}{4n(n-1)}\tilde{f}_{n-2} - \frac{\beta_n}{2(n^2-1)}\tilde{f}_n + \frac{\beta_{n+2}}{4n(n+1)}\tilde{f}_{n+2} \\ & n = 2, \ldots, N \end{aligned} \tag{1.127}$$

where $\beta_n = \begin{cases} 1 & 0 \le n \le N-2 \\ 0 & n > N-2 \end{cases}$

There are three types of boundary conditions for Eq. (1.123):

1. *Dirichlet boundary conditions:*

$$u(\pm 1) = u_{\pm}$$

which may also be written as

$$\sum_{n=0}^{N} \tilde{u}_n = u_+$$

$$\sum_{n=0}^{N} (-1)^n \tilde{u}_n = u_-$$

or

$$\sum_{\substack{n=0\\ n \text{ even}}}^{N} \tilde{u}_n = \frac{u_+ + u_-}{2} = c_{\text{even}} \tag{1.128}$$

and

$$\sum_{\substack{n=0\\ n \text{ odd}}}^{N} \tilde{u}_n = \frac{u_+ - u_-}{2} = c_{\text{odd}} \tag{1.129}$$

Since the even and odd coefficients are uncoupled in Eqs. (1.127)–(1.129), two complete linear systems are obtained, in which the structure for the coefficients is quasi-tridiagonal. For example, the linear algebraic system for the even coefficients is

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & & & & 1 \\ A & A & A & & & & & \\ & A & A & A & & & & \\ & & A & A & A & & & \\ & & & & \vdots & & & \\ & & & & A & A & A & \\ & & & & & A & A & \\ & & & & & & A & A \end{pmatrix} \begin{pmatrix} \tilde{u}_0 \\ \tilde{u}_2 \\ \tilde{u}_4 \\ \\ \vdots \\ \tilde{u}_{N-4} \\ \tilde{u}_{N-2} \\ \tilde{u}_N \end{pmatrix} = \begin{pmatrix} c_{\text{even}} \\ B_0 \\ B_2 \\ \\ \vdots \\ B_{N-6} \\ B_{N-4} \\ B_{N-2} \end{pmatrix} \tag{1.130}$$

where A denotes the nonzero coefficient from Eq. (1.127) and B is the right-hand side of Eq. (1.127).

Helmholtz equation with the Dirichlet boundary conditions may be solved by solving the quasi-tridiagonal systems for the even and odd coefficients, respectively, for $\tilde{u}_n$, and then reversing the Chebyshev transform on $\tilde{u}_n$ to produce u_n.

2. *Neumann boundary conditions:*

$$\frac{du}{dy}(\pm 1) = a_{\pm} \tag{1.131}$$

The Chebyshev–Tau approximation may be written as

$$\sum_{n=0}^{N} \tilde{u}_n^{(1)} = a_+$$

$$\sum_{n=0}^{N} (-1)^n \tilde{u}_n^{(1)} = a_-$$

or

$$\sum_{\substack{n=0\\ n \text{ even}}}^{N} \tilde{u}_n^{(1)} = \frac{a_+ + a_-}{2} = a_{\text{even}} \tag{1.132}$$

$$\sum_{\substack{n=0\\ n \text{ odd}}}^{N} \tilde{u}_n^{(1)} = \frac{a_+ - a_-}{2} = a_{\text{ odd}} \tag{1.133}$$

Since

$$\tilde{u}_n^{(1)} = \frac{2}{c_n} \sum_{\substack{p=n+1 \\ p+\ n\text{odd}}}^{N} p\tilde{u}_p \quad n = 0, \ldots, N-1 \tag{1.111}$$

We have

$$\sum_{\substack{n=0 \\ n\ \text{odd}}}^{N} \tilde{u}_n^{(1)} = \sum_{\substack{n=0 \\ n\ \text{even}}}^{N} n^2 \tilde{u}_n = a_{\text{odd}} \tag{1.134}$$

$$\sum_{\substack{n=0 \\ n\ \text{even}}}^{N} \tilde{u}_n^{(1)} = \sum_{\substack{n=0 \\ n\ \text{odd}}}^{N} n^2 \tilde{u}_n = a_{\text{even}} \tag{1.135}$$

Since the even and odd coefficients decouple in Eqs. (1.127), (1.134), and (1.135), two complete linear systems are obtained, which is the same as that of the Dirichlet problem.

If $\lambda = 0$, the compatibility condition for Eqs. (1.123) and (1.131) is

$$\int_{-1}^{1} f dy = a_+ - a_-$$

Written in the discrete form

$$\sum_{n=0}^{N-2} \frac{-2}{n^2 - 1} \tilde{f}_n = a_+ - a_- \tag{1.136}$$

3. *Robin boundary conditions:*

$$\frac{du}{dy}(\pm 1) + b_{\pm} u(\pm 1) = B_{\pm} \tag{1.137}$$

which may also be written as

$$\sum_{n=0}^{N} \tilde{u}_n^{(1)} + b_+ \sum_{n=0}^{N} \tilde{u}_n = B_+$$

$$\sum_{n=0}^{N} (-1)^n \tilde{u}_n^{(1)} + b_- \sum_{n=0}^{N} (-1)^n \tilde{u}_n = B_-$$

or equivalently

$$\sum_{n=0}^{N} (n^2 + b_+) \tilde{u}_n = B_+ \tag{1.138}$$

$$\sum_{n=0}^{N} (-1)^{n+1} (n^2 - b_-) \tilde{u}_n = B_- \tag{1.139}$$

Equations (1.127), (1.138), and (1.139) form a complete linear system, which solution process is more costly since the even and odd modes do not decoupled.

1.6 Spectral Method of Navier–Stokes Equations

The incompressible N-S equations are written as

$$\nabla \cdot \mathbf{u} = 0 \tag{1.140}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \tag{1.141}$$

where all variables are normalized with respect to the channel half-width and center line velocity for a channel flow. Equation (1.140) is the mass conservation equation, and Eq. (1.141) is the momentum conservation equation. $\nu = \frac{1}{\mathrm{Re}}$, Re is the Reynolds number, $\mathbf{u}$ is velocity vector, and p is the pressure.

Writing in a general form

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u}, \mathbf{x}, t) \tag{1.142}$$

For numerical calculations, it is in general most efficient to apply spectral methods only to the spatial dependence, since the time dependence can be marched forward from one time level to another. Therefore, the discretization of time derivative in Eq. (1.142) should be discussed first.

1.6.1 Time Integration Method [17, 18]

1.6.1.1 Time-Marching Method

For a time-dependent equation, time marching means that the solution at time level $n + 1$ can calculated from the known values at old time level, where n is the running index in the t direction. After we have known values at time level $n + 1$, then the same procedure is used to calculate values at time level $n + 2$. In this fashion, the solution is progressively obtained by marching in steps of time. There are three type approaches used commonly for time advancement, they are explicit, implicit, and semi-implicit.

Explicit is the simpler numerical scheme, the unknown values at time level $n + 1$ can be obtained from the known values at the time levels directly, that is,

$$\mathbf{u}^{n+1} = \mathbf{G}(\mathbf{u}^n, \mathbf{u}^{n-1}, \ldots)$$

where schemes can be distinguished according to the number of involved time levels, such as the one-step (e.g., Runge–Kutta schemes) and the multistep methods (e.g., Adams–Bushforth schemes).

The second-order and third-order Adams–Bushforth schemes denoted by AB2 and AB3, respectively, are expressed, respectively, by

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left\{ \frac{3}{2}\mathbf{F}\left(\mathbf{u}^n, \mathbf{x}, t^n\right) - \frac{1}{2}\mathbf{F}\left(\mathbf{u}^{n-1}, \mathbf{x}, t^{n-1}\right) \right\} \tag{1.143}$$

and

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left\{ \frac{23}{12}\mathbf{F}\left(\mathbf{u}^n, \mathbf{x}, t^n\right) - \frac{4}{3}\mathbf{F}\left(\mathbf{u}^{n-1}, \mathbf{x}, t^{n-1}\right) + \frac{5}{12}\mathbf{F}\left(\mathbf{u}^{n-2}, \mathbf{x}, t^{n-2}\right) \right\} \tag{1.144}$$

where Δt is the interval of a time step. The stringent time step restrictions should be considered in the explicit approach for stability.

The implicit scheme is somewhat more complicated comparing with the explicit scheme, where the unknowns are not only expressed on the left-hand side but also occur in a function on the right-hand side. Such as

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \mathbf{F}\left(\mathbf{u}^{n+1}, t^{n+1}\right) \quad \text{Backward Euler scheme (BE)} \tag{1.145}$$

$$\frac{(3/2)\mathbf{u}^{n+1} - 2\mathbf{u}^n + (1/2)\mathbf{u}^{n-1}}{\Delta t} = \mathbf{F}\left(\mathbf{u}^{n+1}, t^{n+1}\right)$$

Second-order backward finite difference (BFD2) (1.146)

$$\frac{(11/6)\mathbf{u}^{n+1} - 3\mathbf{u}^n + (3/2)\mathbf{u}^{n-1} - (1/3)\mathbf{u}^{n-2}}{\Delta t} = \mathbf{F}\left(\mathbf{u}^{n+1}, t^{n+1}\right)$$

Third-order backward finite difference (BFD3) (1.147)

$$\frac{(25/12)\mathbf{u}^{n+1} - 4\mathbf{u}^n + 3\mathbf{u}^{n-1} - (4/3)\mathbf{u}^{n-2} + (1/4)\mathbf{u}^{n-3}}{\Delta t} = \mathbf{F}\left(\mathbf{u}^{n+1}, t^{n+1}\right)$$

Fourth-order backward finite difference (BFD4) (1.148) and

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \frac{\mathbf{F}(\mathbf{u}^{n+1}, t^{n+1}) + \mathbf{F}(\mathbf{u}^n, t^n)}{2} \quad \text{Crank–Nichoson scheme (CN)} \tag{1.149}$$

In the semi-implicit scheme, the explicit scheme is applied for some terms, as well as the implicit scheme is applied for others. A nonlinear equation system is expressed by

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{F}(\mathbf{u}, \mathbf{x}, t) + L(\mathbf{u}, \mathbf{x}, t) \tag{1.150}$$

where $\mathbf{F}$ and $\mathbf{L}$ represent nonlinear and linear operators, respectively. If the explicit scheme AB3 is applied to the nonlinear part and the implicit scheme CN to the linear part, then we

obtain a semi-implicit scheme, called AB3CN, expressed as

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \left\{ \frac{23}{12}\mathbf{F}\left(\mathbf{u}^n, \mathbf{x}, t^n\right) - \frac{4}{3}\mathbf{F}\left(\mathbf{u}^{n-1}, \mathbf{x}, t^{n-1}\right) + \frac{5}{12}\mathbf{F}\left(\mathbf{u}^{n-2}, \mathbf{x}, t^{n-2}\right) \right\}$$
$$+\frac{\Delta t}{2}\left\{\mathbf{L}\left(\mathbf{u}^{n+1}, t^{n+1}\right) + \mathbf{L}\left(\mathbf{u}^n, t^n\right)\right\} \tag{1.151}$$

Semi-implicit backward finite differences with different accuracies for N-S equation (1.141), denoted as SBFD, can be expressed by

$$\frac{1}{\Delta t}\sum_{j=0}^{k} a_j \mathbf{u}^{n+1-j} = \sum_{j=0}^{k-1} b_j \mathbf{F}\left(\mathbf{u}^{n-j}\right) \tag{1.152}$$

where superscript n denotes the time level, and a_j and b_j are constant given in Table 1.1.

Thus, the scheme SBFD3 can expressed as

$$\frac{1}{\Delta t}\left(\frac{11}{6}\mathbf{u}^{n+1} - 3\mathbf{u}^n + \frac{3}{2}\mathbf{u}^{n-1} - \frac{1}{3}\mathbf{u}^{n-2}\right) - 3\mathbf{F}\left(\mathbf{u}^n\right) + 3\mathbf{F}\left(\mathbf{u}^{n-1}\right) - \mathbf{F}\left(\mathbf{u}^{n-2}\right) = 0 \tag{1.153}$$

or

$$\frac{11}{6}\mathbf{u}^{n+1} = -\left(-3\mathbf{u}^n + \frac{3}{2}\mathbf{u}^{n-1} - \frac{1}{3}\mathbf{u}^{n-2}\right) + \Delta t[3\mathbf{F}\left(\mathbf{u}^n\right) - 3\mathbf{F}\left(\mathbf{u}^{n-1}\right) + \mathbf{F}\left(\mathbf{u}^{n-2}\right)] \tag{1.154}$$

1.6.1.2 Time Splitting Method[19, 20]

The terms contained in N-S equations represent the different physical process, such as, $\frac{\partial \mathbf{u}}{\partial t}$ is a local derivative describing the velocity changing with time at an instantaneous position, flow is steady if $\frac{\partial \mathbf{u}}{\partial t} = 0$; $\mathbf{u} \cdot \nabla \mathbf{u}$, called inertia or a convective term, denotes the effect of flow on the velocity field; ∇p, called pressure adjustment, represents the effect of pressure difference; $\nu \nabla^2 \mathrm{u}$, called diffusion or a dissipation term, represents the effect of viscosity.

The key idea of time-splitting method is the replacement of simultaneous processes by sequential steps. For example, the split can be by physics: advection on one fractional step, pressure adjustment on another, and diffusion/viscosity on a third, described as follows and

Table 1.1 Coefficients of SBDF

Scheme	Order	a_0	a_1	a_2	a_3	a_4	b_0	b_1	b_2	b_3
SBDF1	1	1	−1				1			
SBDF2	2	3/2	−2	1/2			2	−1		
SBDF3	3	11/6	−3	3/2	−1/3		3	−3	1	
SBDF4	4	25/12	−4	3	−4/3	1/4	4	−6	4	−1

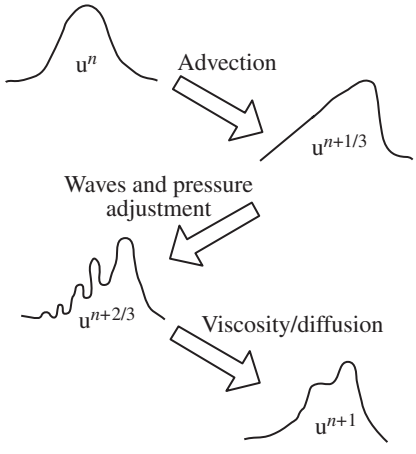

Figure 1.11 Splitting by a physical process.[14] *Source*: Boyd J P 2001. Reproduced with permission of Dover Publication

shown in Figure 1.11:

$$\frac{D\mathbf{u}^{n+1/3}}{Dt} = 0 \tag{1.155}$$

$$\frac{\partial \mathbf{u}^{n+2/3}}{\partial t} = -\nabla p^{n+2/3} \tag{1.156}$$

$$\frac{\partial \mathbf{u}^{n+1}}{\partial t} = -\nu \nabla^2 \mathbf{u}^{n+1} \tag{1.157}$$

where Du/Dt denotes the total derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

Each fractional equation can be solved individually using a different reasonable numerical method.

1.6.2 Spectral Method based on Time Marching Algorithms (1)

1.6.2.1 Semi-Implicit Time Marching Algorithm

Consider an incompressible fluid flow in a rectangular channel as shown in Figure 1.12. For a turbulent channel flow, the velocity and pressure fields are decomposed into base and fluctuating parts, called base-fluctuation decomposition:

$$\mathbf{u}(\mathrm{x}, t) = U(y)\mathbf{e}_x + \mathbf{u}'(\mathrm{x}, t) \tag{1.158}$$

$$p(\mathbf{x}, t) = \Pi_x(t)x + p'(\mathbf{x}, t) \tag{1.159}$$

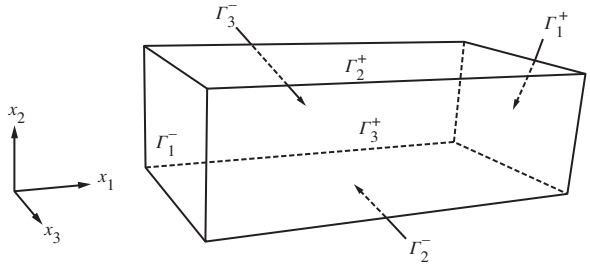

Figure 1.12 Schematic of channel flow

where $\mathbf{x} = (x, y, z)$ or $\mathbf{x} = (x_1, x_2, x_3)$ represents the streamwise, normal, and spanwise directions, respectively. $U(y)$ is the base flow varying only in the y direction, and $\mathbf{e}_x$ is a unit vector in the x direction. The base pressure $\Pi_x(t)x$ is linear in the x direction, where $\Pi_x(t)$ varies linearly with an average drag coefficient on the wall. Then the pressure gradient is

$$\nabla p(\mathbf{x}, t) = \Pi_x(t)\mathbf{e}_x + \nabla p'(\mathbf{x}, t) \tag{1.160}$$

Substituting Eqs. (1.158) and (1.160) into Eqs. (1.140) and (1.141), respectively, gives

$$\nabla \cdot \mathbf{u}' = 0 \tag{1.161}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + \nabla p' - \nu \nabla^2 \mathbf{u}' = -\mathrm{N}(\mathbf{u}') + \mathbf{C} \tag{1.162}$$

where nonlinear term $\mathbf{N}(\mathbf{u}') = \mathbf{u}' \cdot \nabla \mathrm{u}'$ and spatially constant term $\mathbf{C} = \left[\nu \frac{\partial^2 U}{\partial y^2} - \Pi_x\right] \mathrm{e}^x$.

Substituting Eq. (1.161) into the divergence of Eq. (1.171) leads to

$$\nabla^2 p' = \nabla \cdot \mathbf{C} \tag{1.163}$$

Equations (1.161)–(1.163) constitute the equations for fluctuating quantities.

Taking the SBDF3 scheme, that is, Eq. (1.154) in temporal discretization, Eq. (1.162) can then be written as

$$\nu \nabla^2 \mathbf{u}' - \frac{11}{6\Delta t}\mathbf{u}' - \nabla p' = -\mathbf{R} \tag{1.164}$$

where superscript "$n + 1$" is suppressed to convenience writing:

$$\mathbf{R} = -\frac{1}{\Delta t}\left[-3\mathbf{u}'^n + \frac{3}{2}\mathbf{u}'^{n-1} - \frac{1}{3}\mathbf{u}'^{n-2}\right] - [3\mathbf{N}(\mathbf{u}'^n) - 3\mathbf{N}(\mathbf{u}'^{n-1}) + \mathbf{N}(\mathbf{u}'^{n-2})] + \mathbf{C}$$

Obviously, $\nabla \cdot \mathbf{R} = \nabla \cdot \mathbf{C}$.

Suppressing the time superscripts gives

$$\nu \nabla^2 \mathbf{u} - \varepsilon \mathbf{u} - \nabla p = -\mathbf{R} \tag{1.165}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{1.166}$$

$$\nabla^2 p = \nabla \cdot \mathbf{R} \tag{1.167}$$

Considering the coherent structures of wall turbulence, to be presented in the next chapter, the boundary conditions in the x and z directions are periodic, while the upper and lower walls give rise to no-slip boundary conditions. Thus, we have

$$\begin{cases} \mathbf{u}(x, y, z, t) = W(x, \pm 1, z, t) \\ \mathbf{u}(x + L_x, y, z, t) = \mathbf{u}(x, y, z, t) \\ \mathbf{u}(x, y, z + L_z, t) = \mathbf{u}(x, y, z, t) \end{cases} \tag{1.168}$$

where L_x and L_z are the periodic intervals in the x and z directions, respectively, and W is the wall velocity.

1.6.2.2 Fourier–Galerkin Method

One-dimensional Fourier transform as described by Eq. (1.58) can easily be extended into a two-dimensional system $(x,\ z)$, expressed as

$$\tilde{u}_{\mathbf{k}} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} u(x, z) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$$

where $\mathbf{x} = 2\pi \left(\frac{x}{L_x} \frac{z}{L_z} \right)^T$, $\mathbf{k} = \begin{pmatrix} k_x \\ k_z \end{pmatrix}$

and its inverse transform

$$u(x, z) = \sum_{k_x=-\infty}^{\infty} \sum_{k_z=-\infty}^{\infty} \tilde{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{1.169}$$

By differentiating Eq. (1.169), we have

$$\frac{\partial u}{\partial x_j} = \frac{2\pi i k_j}{L_j} \sum_{k_x=-\infty}^{\infty} \sum_{k_z=-\infty}^{\infty} \tilde{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\frac{\partial^2 u}{\partial x_j^2} = -4\pi^2 \left(\frac{k_j}{L_j} \right)^2 \sum_{k_x=-\infty}^{\infty} \sum_{k_z=-\infty}^{\infty} \tilde{u}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

where subscript j represents the component in the j direction.

Hence, from Eqs. (1.165) and (1.167), we have

$$\nu \frac{d^2 \tilde{u}_{\mathbf{k}}}{dy^2} - \lambda \tilde{u}_{\mathbf{k}} - i2\pi \frac{k_x}{L_x} \tilde{p}_{\mathbf{k}} = -\tilde{R}_{x\mathbf{k}} \tag{1.170}$$

$$\nu \frac{d^2 \tilde{v}_{\mathbf{k}}}{dy^2} - \lambda \tilde{v}_{\mathbf{k}} - \frac{d\tilde{p}_{\mathbf{k}}}{dy} = -\tilde{R}_{y\mathbf{k}} \tag{1.171}$$

$$\nu\frac{d^2\tilde{w}_{\mathbf{k}}}{dy^2} - \lambda\tilde{w}_{\mathbf{k}} - i2\pi\frac{k_z}{L_z}\tilde{p}_{\mathbf{k}} = -\tilde{R}_{z\mathbf{k}} \tag{1.172}$$

$$\frac{d^2\tilde{p}_{\mathbf{k}}}{dy^2} - 4\pi^2\left(\frac{k_x^2}{L_x^2} + \frac{k_z^2}{L_z^2}\right)\tilde{p}_{\mathbf{k}} = i2\pi\frac{k_x}{L_x}\tilde{R}_{x\mathbf{k}} + i2\pi\frac{k_z}{L_z}\tilde{R}_{z\mathbf{k}} + \frac{d\tilde{R}_{y\mathbf{k}}}{dy} \tag{1.173}$$

where $\lambda = 4\pi^2\nu\left(\frac{k_x^2}{L_x^2} + \frac{k_z^2}{L_z^2}\right) + \varepsilon$. Equations (1.170)–(1.173) constitute a complete system of equations, which can be written in a brief form as

$$\tilde{p}'' - \boldsymbol{\kappa}^2\tilde{p} = \tilde{F} \tag{1.174}$$

$$\nu\tilde{v}'' - \lambda\tilde{v} - \tilde{p}' = -\tilde{R}_y \tag{1.175}$$

$$\nu\tilde{u}'' - \lambda\tilde{u} - i\kappa_x\tilde{p} = -\tilde{R}_x \tag{1.176}$$

$$\nu\tilde{w}'' - \lambda\tilde{w} - i\kappa_z\tilde{p} = -\tilde{R}_z \tag{1.177}$$

where superscript "′" and "″" represent first-order and second-order derivatives individually and $\boldsymbol{\kappa} = 2\pi\left(\frac{k_x}{L_x}\frac{k_z}{L_z}\right)^T$.

1.6.2.3 Aliasing Removal (3/2 Rule)

Since $\mathbf{N}(\mathbf{u}) = \mathbf{u}\cdot\nabla\mathbf{u}$ in Eq. (1.162) is a nonlinear quadratic term, the Fourier–Galerkin treatment of this term should be considered when the equation is solved by the spectral method. Consider the quadratic term

$$w(x) = u(x)v(x)$$

In case of an infinite series expansion, the Fourier–Galerkin approximation takes the form of convolution sum:

$$\tilde{w}_k = \sum_{p+q=k}\tilde{u}_p\tilde{v}_q \tag{1.178}$$

Where $u(x) = \sum_{p=-\infty}^{\infty}\tilde{u}_p e^{ipx}$

$$v(x) = \sum_{q=-\infty}^{\infty}\tilde{v}_q e^{iqx}$$

$$\tilde{w}_k = \frac{1}{2\pi}\int_0^{2\pi} w(x)e^{-ikx}dx$$

In the collocation method, where collocation points $x_j = 2\pi j/N$, the Fourier–Galerkin approximation takes the form

$$u_j^N = \sum_{k=-N/2}^{N/2-1} \tilde{u}_k \exp(ikx_j) \quad j = 0, \ldots, N-1$$

$$v_j^N = \sum_{k=-N/2}^{N/2-1} \tilde{v}_k \exp(ikx_j) \quad j = 0, \ldots, N-1$$

where the series expansion is truncated after N terms, and the maximal wave number K in the spectra is $K = N/2 - 1$.

Since $w_j^N = u_j^N v_j^N \quad j = 0, 1, \ldots, N-1$

$$\widehat{w}_k = \frac{1}{N}\sum_{j=0}^{N-1} w_j^N e^{-ikx_j} = \sum_{\substack{p+q=k \\ |p|,|q| \le N/2}} \tilde{u}_p \tilde{v}_q \tag{1.179}$$

thus, Eq. (1.179) is replaced with

$$\begin{aligned} \widehat{w}_k &= \frac{1}{N}\sum_{j=0}^{N-1} \left(\sum_{p=-N/2}^{N/2-1} \tilde{u}_p \exp\left(ipx_j\right) \right) \left(\sum_{q=-N/2}^{N/2-1} \tilde{v}_q \exp\left(iqx_j\right) \right) \\ &= \sum_{p=-N/2}^{N/2-1} \sum_{q=-N/2}^{N/2-1} \tilde{u}_p \tilde{v}_q \left(\frac{1}{N}\sum_{j=0}^{N-1} \exp\left[i\left(p+q-k\right)x_j\right] \right) \end{aligned} \tag{1.180}$$

Using the orthogonality relation

$$\frac{1}{N}\sum_{j=0}^{N-1} \exp[i(p+q-k)x_j] = \begin{cases} 1 & p+q-k = 0, \pm N, \pm 2N, \ldots \\ 0 & p+q-k = \text{other} \end{cases} \tag{1.181}$$

Since $-N/2 \le p+q-k \le N/2-1$, $\widehat{w}_k$ is nonzero only if

$$p+q-k = 0 \quad \text{or} \quad p+q-k = \pm N$$

We have

$$\widehat{w}_k = \sum_{p+q=k} \tilde{u}_p \tilde{v}_q + \sum_{p+q=k\pm N} \tilde{u}_p \tilde{v}_q = \tilde{w}_k + \sum_{p+q=k\pm N} \tilde{u}_p \tilde{v}_q \tag{1.182}$$

The first term on the right-hand side is the convolution sum defined by Eq. (1.178), and the second term is the aliasing error produced by truncation of an infinite series expansion.

The aliasing error can be removed by extension of spectra from K to K', where $K' = M/2 - 1$. M will be found later to be equal to $(3/2)N$. The extended spectra is defined as

$$\tilde{U}_k = \begin{cases} \tilde{u}_k & |k| \le N/2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{V}_k = \begin{cases} \tilde{v}_k & |k| \le N/2 \\ 0 & \text{otherwise} \end{cases} \tag{1.183}$$

Thus, the $\tilde{U}_k$ (or $\tilde{V}_k$) coefficients are the $\tilde{u}_k$ (or $\tilde{v}_k$) coefficients padded with zeros for the additional wave numbers.

Let

$$y_j = 2\pi j/M$$

$$W_j = U_j V_j$$

$$U_j = \sum_{k=-M/2}^{M/2-1} \tilde{U}_k e^{iky_j} \quad j = 0, 1, \ldots, M-1$$

$$V_j = \sum_{k=-M/2}^{M/2-1} \tilde{V}_k e^{iky_j}$$

Likewise, let

$$\widehat{W}_k = \frac{1}{M} \sum_{j=0}^{M-1} W_j e^{-iky_j} \quad k = -M/2, \ldots, M/2-1$$

Then

$$\widehat{W}_k = \sum_{p+q=k} \tilde{u}_p \tilde{v}_q + \sum_{p+q=k\pm M} \tilde{U}_p \tilde{V}_q \tag{1.184}$$

If $M = 3N/2$, when $-N/2 \le k \le N/2 - 1$, the second sum of Eq. (1.184) is taken on

$$-2N \le p + q \le -N - 1 \text{ or } N \le p + q \le 2N - 1$$

Since $-M/2 \le p \le M/2 - 1$ and $-M/2 \le q \le M/2 - 1$, the above-mentioned constraint conditions are simplified further as

$$p < -N/2 \text{ or } p > N/2 - 1$$

$$q < -N/2 \text{ or } q > N/2 - 1$$

which leads to $\tilde{U}_p$ and $\tilde{V}_q$ being zero, such that the second sum vanishes. This de-aliasing technique is called the 3/2 rule.

1.6.2.4 Influence Matrix Technique

The pressure equation (1.174) and the normal velocity equation (1.175) form a complete set for $\tilde{p}$ and $\tilde{v}$:

$$\begin{cases} \tilde{p}'' - \boldsymbol{\kappa}^2 \tilde{p} = \tilde{F} & \tilde{v}'(\pm 1) = h_\pm \\ \nu \tilde{v}'' - \lambda \tilde{v} - \tilde{p}' = -\tilde{R}_y & \tilde{v}(\pm 1) = g_\pm \end{cases} \tag{1.185}$$

It is difficult to solve, because the pressure $\tilde{p}_{\pm}$ at the wall is unknown a priori while $\tilde{p}$ appears in the $\tilde{v}$ differential equation. A method called the influence matrix technique is employed usually to obtain the solutions of these equations.

$$\text{Let} \quad \begin{pmatrix} \tilde{p} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} \tilde{p}_1 \\ \tilde{v}_1 \end{pmatrix} + \begin{pmatrix} \tilde{p}_2 \\ \tilde{v}_2 \end{pmatrix} \tag{1.186}$$

where $(\tilde{p}_1, \tilde{v}_1)$ and $(\tilde{p}_2, \tilde{v}_2)$ satisfy the following equations, respectively:

$$\begin{cases} \tilde{p}_1'' - \boldsymbol{\kappa}^2 \tilde{p}_1' = \tilde{F} & \tilde{p}_1(\pm 1) = 0 \\ \nu \tilde{v}_1'' - \lambda \tilde{v}_1 - \tilde{p}_1' = -\tilde{R}_y & \tilde{v}_1(\pm 1) = g_{\pm} \end{cases} \tag{1.187}$$

and

$$\begin{cases} \tilde{p}_2'' - \boldsymbol{\kappa}^2 \tilde{p}_2' = 0 & \tilde{p}_2(\pm 1) = \delta_{\pm} \\ \nu \tilde{v}_2'' - \lambda \tilde{v}_2 - \tilde{p}_2' = 0 & \tilde{v}_2(\pm 1) = 0 \end{cases} \tag{1.188}$$

The solution of Eq. (1.187) can be obtained by a method solving Helmholtz equation presented in Section 1.6.4. However, δ_+ and δ_- in Eq. (1.188) are unknown, therefore, $(\tilde{p}_2, \tilde{v}_2)$ should be decomposed further:

$$\begin{pmatrix} \tilde{p}_2 \\ \tilde{v}_2 \end{pmatrix} = \delta_+ \begin{pmatrix} \tilde{p}_+ \\ \tilde{v}_+ \end{pmatrix} + \delta_- \begin{pmatrix} \tilde{p}_- \\ \tilde{v}_- \end{pmatrix} \tag{1.189}$$

where $(\tilde{p}_+, \tilde{v}_+)$ and $(\tilde{p}_-, \tilde{v}_-)$ satisfy the following equations, respectively:

$$\begin{cases} \tilde{p}_+'' - \boldsymbol{\kappa}^2 \tilde{p}_+ = 0 & \tilde{p}_+(+1) = 1, \quad \tilde{p}_+(-1) = 0 \\ \nu \tilde{v}_+'' - \lambda \tilde{v}_+ - \tilde{p}_+' = 0 & \tilde{v}_+(\pm 1) = 0 \end{cases} \tag{1.190}$$

and

$$\begin{cases} \tilde{p}_-'' - \boldsymbol{\kappa}^2 \tilde{p}_- = 0 & \tilde{p}_-(+1) = 0, \quad \tilde{p}_-(-1) = 1 \\ \nu \tilde{v}_-'' - \lambda \tilde{v}_- - \tilde{p}_-' = 0 & \tilde{v}_-(\pm 1) = 0 \end{cases} \tag{1.191}$$

$(\tilde{p}_+, \tilde{v}_+)$ and $(\tilde{p}_-, \tilde{v}_-)$ as well as $\tilde{v}_{\pm}'(\pm 1)$ and $\tilde{v}_1'(\pm 1)$ are obtained by solving the Helmholtz equations, that is, Eqs. (1.191) and (1.199), meanwhile h_+ and h_- can be determined by iteration. Then δ_+ and δ_- are determined further by solving the following linear algebra equations:

$$\mathbf{A} \begin{bmatrix} \delta_+ \\ \delta_- \end{bmatrix} = \begin{bmatrix} \tilde{v}_+'(+1) & \tilde{v}_-'(+1) \\ \tilde{v}_+'(-1) & \tilde{v}_-'(-1) \end{bmatrix} \begin{bmatrix} \delta_+ \\ \delta_- \end{bmatrix} = \begin{bmatrix} h_+ - \tilde{v}_1'(+1) \\ h_- - \tilde{v}_1'(-1) \end{bmatrix} \tag{1.192}$$

which is produced from the requirement of the boundary conditions of pressure equation. The matrix $\mathbf{A}$ is called the influence matrix.

1.6.2.5 Tau Correction

The pressure equation (1.167) is derived from the continuity equation (1.166) and the momentum equation (1.165) based on the properties of the continuous differentiation operators. Errors will be introduced into the pressure equation when discretized, such that the solutions do not satisfy the continuity equation (1.166). In order to correct the errors, additional terms, called tau term, are introduced into the right-hand side of the equations, which yields

$$\nu\tilde{u}_m^{(2)} - \lambda\tilde{u}_m - i\kappa_x\tilde{p}_m = -\tilde{R}_{x,m} - \tilde{\tau}_{x,m} \tag{1.193}$$

$$\nu\tilde{v}_m^{(2)} - \lambda\tilde{v}_m - \tilde{p}_m^{(1)} = -\tilde{R}_{y,m} - \tilde{\tau}_{y,m} \tag{1.194}$$

$$\nu\tilde{w}_m^{(2)} - \lambda\tilde{w}_m - i\kappa_z\tilde{p}_m = -\tilde{R}_{z,m} - \tilde{\tau}_{z,m} \tag{1.195}$$

$$\tilde{d}_m = i\kappa_x\tilde{u}_m + i\kappa_z\tilde{w}_m + \tilde{v}_m^{(1)} = 0 \tag{1.196}$$

$$m = 0, 1, \ldots, N$$

The application of the discrete divergence to the above-mentioned complete system yields

$$\tilde{p}_m^{(2)} - \boldsymbol{\kappa}^2\tilde{p}_m = \tilde{F}_m + \left(i\kappa_x\tilde{\tau}_{x,m} + i\kappa_z\tilde{\tau}_{z,m} + \tilde{\tau}_{y,m}^{(1)}\right)$$

Since $\tilde{\tau}_{x,m}$, $\tilde{\tau}_{y,m}$, and $\tilde{\tau}_{z,m}$ vanish for $0 \le m \le N-2$, we have

$$\tilde{p}_m^{(2)} - \boldsymbol{\kappa}^2\tilde{p}_m = \tilde{F}_m + \tilde{\tau}_{y,m}^{(1)} \quad m = 0, 1, \ldots, N-2 \tag{1.197}$$

Hence, the complete set for $\tilde{p}$ and $\tilde{v}$ is

$$\begin{cases} \tilde{p}_m^{(2)} - \boldsymbol{\kappa}^2\tilde{p}_m = \tilde{F}_m + \tilde{\tau}_{y,m}^{(1)} & \tilde{v}^{(1)}(\pm 1) = \tilde{h}_\pm \quad m = 0, \ldots, N-2 \\ \nu\tilde{v}_m^{(2)} - \lambda\tilde{v}_m - \tilde{p}_m^{(1)} = -\tilde{R}_{y,m} - \tilde{\tau}_{y,m} & \tilde{v}(\pm 1) = \tilde{g}_\pm \quad m = 0, \ldots, N \end{cases} \tag{1.198}$$

Where $\tilde{\tau}_y = (0, \ldots, 0, \tilde{\tau}_{y,N-1}, \tilde{\tau}_{y,N})^T$.
According to the recurrence relation (1.116),

$$\tilde{\tau}_y^{(1)} = ((N-1)\tilde{\tau}_{y,N-1}, 2N\tilde{\tau}_{y,N}, 2(N-1)\tilde{\tau}_{y,N-1}, 2N\tilde{\tau}_{y,N}, \ldots, 2(N-1)\tilde{\tau}_{y,N-1}, 2N\tilde{\tau}_{y,N}, 0)^T,$$

all spectral coefficients, even if $0 \le m \le N-2$, are disturbed by the truncation errors. Hence, Eq. (1.198) is modified further, using $\tilde{\sigma}_m$ to displace $\tilde{\tau}_{y,m}$

$$\begin{cases} \tilde{p}_m^{(2)} - \boldsymbol{\kappa}^2\tilde{p}_m = \tilde{F}_m + \tilde{\sigma}_m^{(1)} & \tilde{v}^{(1)}(\pm 1) = \tilde{h}_\pm \quad m = 0, \ldots, N-2 \\ \nu\tilde{v}_m^{(2)} - \lambda\tilde{v}_m - \tilde{p}_m^{(1)} = -\tilde{R}_{y,m} - \tilde{\sigma}_m & \tilde{v}(\pm 1) = \tilde{g}_\pm \quad m = 0, \ldots, N \end{cases} \tag{1.199}$$

where $\tilde{\sigma}^{(1)} = ((N-1)\tilde{\sigma}_{N-1}, 2N\tilde{\sigma}_N, 2(N-1)\tilde{\sigma}_{N-1}, 2N\sigma_N, \ldots, 2(N-1)\tilde{\sigma}_{N-1}, 2N\tilde{\sigma}_N, 0)^T$.
Obviously, for $m \le N-1$,

$$\tilde{\sigma}_m^{(1)} = \frac{2}{c_m} m'\tilde{\sigma}_{m'} \tag{1.200}$$

where

$$m' = \begin{cases} N-1 & m \text{ even} \\ N & m \text{ odd} \end{cases} \tag{1.201}$$

To solve Eq. (1.199), consider the B_1 problem

$$\begin{cases} \tilde{p}_{1,m}^{(2)} - \boldsymbol{\kappa}^2 \tilde{p}_{1,m} = \tilde{F}_m & \tilde{v}^{(1)}(\pm 1) = \tilde{h}_\pm \quad m = 0, \ldots, N-2 \\ \nu \tilde{v}_{1,m}^{(2)} - \lambda \tilde{v}_{1,m} - \tilde{p}_{1,m}^{(1)} = -\tilde{R}_{y,m} & \tilde{v}(\pm 1) = \tilde{g}_\pm \quad m = 0, \ldots, N-2 \end{cases} \tag{1.202}$$

and the B_0 problem

$$\begin{cases} \tilde{p}_{0,m}^{(2)} - \boldsymbol{\kappa}^2 \tilde{p}_{0,m} = \frac{2}{c_m} m' & \tilde{v}^{(1)}(\pm 1) = \tilde{h}_\pm \quad m = 0, \ldots, N-2 \\ \nu \tilde{v}_{0,m}^{(2)} - \lambda \tilde{v}_{0,m} - \tilde{p}_{0,m}^{(1)} = 0 & \tilde{v}(\pm 1) = \tilde{g}_\pm \quad m = 0, \ldots, N-2 \end{cases} \tag{1.203}$$

m' is determined by Eq. (1.201).

Equation (1.202) identifying with Eq. (1.185) is a complete set for unmodified $\tilde{p}_1$ and $\tilde{v}_1$. Equation (1.203) embodied by the tau terms is a supplementary equation for the tau correction. Both problems for $\tilde{p}_{0,m}$,$\tilde{v}_{0,m}$ and $\tilde{p}_{1,m}$,$\tilde{v}_{1,m}$ can be solved by the influence matrix technique.

Substituting Eqs. (1.113) and (1.114) into the v equations in Eqs. (1.192) and (1.193), respectively, yields

$$\begin{cases} \tilde{\sigma}_{1,N-1} = \lambda \tilde{v}_{1,N-1} + 2N p_{1,N} - \tilde{R}_{y,N-1} \\ \tilde{\sigma}_{1,N} = \lambda \tilde{v}_{1,N} - \tilde{R}_{y,N} \end{cases} \tag{1.204}$$

and

$$\begin{cases} \tilde{\sigma}_{0,N-1} = \lambda \tilde{v}_{0,N-1} + 2N p_{0,N} \\ \tilde{\sigma}_{0,N} = \lambda \tilde{v}_{0,N} \end{cases} \tag{1.205}$$

The unknown variables in the above-mentioned equations, that is, $\tilde{\sigma}_{1,N}$, $\tilde{\sigma}_{1,N-1}$, $\tilde{\sigma}_{0,N}$, and $\tilde{\sigma}_{0,N-1}$ would be obtained straightforwardly based on the solutions for Eqs. (1.202) and (1.203).

Let $\tilde{p}_m = \tilde{p}_{1,m} + \beta \tilde{p}_{0,m}$

Substituting into the p equations in Eqs. (1.199), (1.202), and (1.203) gives

$$\tilde{\sigma}_m^{(1)} = \frac{2}{c_m} m' \beta \tag{1.206}$$

Compared with Eq. (1.200),

$$\beta = \tilde{\sigma}_{m'} \tag{1.207}$$

Let $\tilde{v}_m = \tilde{v}_{1,m} + \beta \tilde{v}_{0,m}$

Substituting into the v equations in Eqs. (1.199), (1.202), and (1.203)

$$\tilde{\sigma}_m = \tilde{\sigma}_{1,m} + \beta \tilde{\sigma}_{0,m} \tag{1.208}$$

From Eqs. (1.207) and (1.208),

$$\tilde{\sigma}_{m'} = \tilde{\sigma}_{1,m'} / (1 - \tilde{\sigma}_{0,m'}) \quad (m' = N-1, N) \tag{1.209}$$

Then we can show that

$$\begin{cases} \tilde{p}_m = \tilde{p}_{1,m} + \tilde{\sigma}_{m'} \tilde{p}_{0,m} & m = 0, \ldots, N \\ \tilde{v}_m = \tilde{v}_{1,m} + \sigma_{(m+1)'} \tilde{v}_{0,m} \end{cases} \tag{1.210}$$

Now, the spectral coefficients resulting from a combined Fourier transform in $x - z$ and a Chebyshev transform in y, such as $\tilde{\mathbf{u}}_{k_x,n_y,k_z}$ and $\tilde{p}_{k_x,n_y,k_z}$, can be computed numerically.

Finally, the primary functions in N-S equations, that is, Eqs. (1.140) and (1.141), will be achieved by the inverse transform, from the mathematical form, such as,

$$\mathbf{u}(\mathbf{x}) = \sum_{k_x=-N_x/2+1}^{N_x/2} \sum_{n_y=0}^{N_y-1} \sum_{k_z=-N_z/2+1}^{N_z/2} \tilde{\mathbf{u}}_{k_x,n_y,k_z} T_{n_y} e^{2\pi i(k_x x/L_x + k_z z/L_z)}$$

1.6.3 Spectral Method based on Time Marching Algorithms (2)

For incompressible flow, there is no connection between pressure and density, and pressure satisfies the Poisson equation that is usually solved numerically by the finite difference method. In order to solve N-S equations by the spectral method, Eqs. (1.161) and (1.162) can be written as

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1.211}$$

$$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + H_i + \nu \frac{\partial^2 u_i}{\partial x_j^2} \tag{1.212}$$

where H_i denotes a sum of convective term and mean pressure gradient.

Introducing two quantities ϕ and g as dependent variables, defined by

$$\phi = \frac{\partial^2 v}{\partial x_j^2} \tag{1.213}$$

$$g = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{1.214}$$

thus

$$\frac{\partial \phi}{\partial t} = h_v + \nu \frac{\partial^2 \phi}{\partial x_j^2} \tag{1.215}$$

$$\frac{\partial g}{\partial t} = h_g + \nu \frac{\partial^2 g}{\partial x_j^2} \tag{1.216}$$

where

$$h_v = -\frac{\partial}{\partial y}\left(\frac{\partial H_1}{\partial x} + \frac{\partial H_3}{\partial z}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) H_2$$

$$h_g = \frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}$$

Utilizing AB2CN to Eqs. (1.215) and (1.216), we have

$$\begin{cases} \left(1-\frac{\nu\Delta t}{2}\nabla^2\right)g^{n+1}=\frac{\Delta t}{2}\left(3h_g^n-h_g^{n-1}\right)+\left(1+\frac{\nu\Delta t}{2}\nabla^2\right)g^n \\ g(\pm 1)=0 \end{cases} \tag{1.217}$$

and

$$\begin{cases} \left(1-\frac{\nu\Delta t}{2}\nabla^2\right)\phi^{n+1}=\frac{\Delta t}{2}\left(3h_v^n-h_v^{n-1}\right)+\left(1+\frac{\nu\Delta t}{2}\nabla^2\right)\phi^n \\ \nabla^2 v^{n+1}=\phi^{n+1} \\ v^{n+1}(\pm 1)=a_\pm \\ \frac{\partial v^{n+1}}{\partial y}(\pm 1)=b_\pm \end{cases} \tag{1.218}$$

Note that Eq. (1.218) is analogous to Eq. (1.185) formally, then let

$$v^{n+1}=v_1^{n+1}+c_+v_+^{n+1}+c_-v_-^{n+1} \tag{1.219}$$

where v_1,v_+ and v_- satisfy the following equations, respectively:

$$\begin{cases} \left(1-\frac{\nu\Delta t}{2}\nabla^2\right)\phi_1^{n+1}=\frac{\nabla t}{2}\left(3h_v^n-h_v^{n-1}\right)+\left(1+\frac{\nu\Delta t}{2}\nabla^2\right)\phi_1^n & \phi_1^{n+1}(\pm 1)=0 \\ \nabla^2 v_1^{n+1}=\phi_1^{n+1} & v_1^{n+1}(\pm 1)=a_\pm \end{cases} \tag{1.220}$$

$$\begin{cases} \left(1-\frac{\nu\Delta t}{2}\nabla^2\right)\phi_+^{n+1}=0 & \phi_+^{n+1}(1)=1,\ \ \phi_+^{n+1}(-1)=0 \\ \nabla^2 v_+^{n+1}=\phi_+^{n+1} & v_+^{n+1}(\pm 1)=0 \end{cases} \tag{1.221}$$

$$\begin{cases} \left(1-\frac{\nu\Delta t}{2}\nabla^2\right)\phi_-^{n+1}=0 & \phi_-^{n+1}(1)=0,\ \ \phi_-^{n+1}(-1)=1 \\ \nabla^2 v_-^{n+1}=\phi_-^{n+1} & v_-^{n+1}(\pm 1)=0 \end{cases} \tag{1.222}$$

Those equations can be solved numerically by spectral method, that is, the Fourier–Galerkin method in the streamwise and spanwise directions, and the Chebyshev–Tau in the wall normal direction, and finally we obtain v_1^{n+1}, v_+^{n+1}, and v_-^{n+1}∘

Based on the influence matrix method, the algebraic system is written as

$$\begin{pmatrix} \tilde{v}_+'(+1) & \tilde{v}_-'(+1) \\ \tilde{v}_+'(-1) & \tilde{v}_-'(-1) \end{pmatrix}\begin{pmatrix} \tilde{c}_+ \\ \tilde{c}_- \end{pmatrix}=\begin{pmatrix} \tilde{b}_+-\tilde{v}_1'(+1) \\ \tilde{b}_--\tilde{v}_1'(-1) \end{pmatrix} \tag{1.223}$$

by which the constants $\tilde{c}_+$ and $\tilde{c}_-$ can be determined, then $\tilde{v}^{n+1}$ will be obtained from Eq. (1.219). Likewise, the normal vorticity g is computed from Eq. (1.215) by the spectral method. Finally, the streamwise velocity u and the spanwise velocity w are obtained from Eqs. (1.211) and (1.214).

1.6.4 Spectral Method based on Time-Split Method[21, 22]

Time-split method is in general a method of approximation of the N-S equations, replacing simultaneous processes by sequential steps. Here it is split into two fractional steps:

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_j}u_i u_j + \nu\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i \quad \text{(convective and diffusive step)} \tag{1.224}$$

and

$$\frac{\partial u_i}{\partial t} = -\frac{\partial \phi}{\partial x_i} \quad \text{(pressure adjustment step)} \tag{1.225}$$

To ensure the velocity field to be divergence-free, a scale ϕ to be determined is induced in Eq. (1.225) instead of original pressure p. If the semi-implicit scheme AB2CN is applied, then

$$\frac{\widehat{u}_i - u_i^n}{\Delta t} = \frac{1}{2}\left(3H_i^n - H_i^{n-1}\right) + \frac{1}{2}\nu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\left(\widehat{u}_i + u_i^n\right) \tag{1.226}$$

where $H_i = -(\partial/\partial x_j)u_i u_j$ is the convective term.

And Eq. (1.225) is written as

$$\frac{u_i^{n+1} - \widehat{u}_i}{\Delta t} = -\frac{\partial \phi^{n+1}}{\partial x_i} \tag{1.227}$$

In order to ensure

$$\frac{\partial u_1^{n+1}}{\partial x_1} + \frac{\partial u_2^{n+1}}{\partial x_2} + \frac{\partial u_3^{n+1}}{\partial x_3} = 0$$

we have

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\phi^{n+1} = \frac{1}{\Delta t}\left(\frac{\partial \widehat{u}_1}{\partial x_1} + \frac{\partial \widehat{u}_2}{\partial x_2} + \frac{\partial \widehat{u}_3}{\partial x_3}\right) \tag{1.228}$$

and

$$p = \phi + (\Delta t/2\nu)\nabla^2\phi \tag{1.229}$$

So, Eq. (1.226) can be written as

$$\nu\nabla^2\widehat{\mathbf{u}} - \frac{2}{\Delta t}\widehat{\mathbf{u}} = -\mathbf{R} \tag{1.230}$$

which is the governing equation about the intermediate velocity fields in time splitting method. Only the boundary conditions for the velocity field are given, and those of the intermediate velocity field are unknown. Here, the periodic conditions are still imposed for both streamwise and spanwise directions in intermediate velocity field, but the boundary conditions at the upper and lower walls would be obtained by the following derivatives.

Let $\mathbf{u}_i^*(\mathbf{x}, t_n + \Delta t)$ satisfy the differential equation

$$\begin{cases} \frac{\partial u_i^*}{\partial t} = H_i^* + \nu \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_j} u_i^* \\ u_i^* \left(\mathbf{x}, t_n\right) = u_i(\mathbf{x}, t_n) \end{cases} \tag{1.231}$$

where $\mathbf{u}_i(\mathbf{x}, t_n)$ is the solution of the N-S equation (1.141). Hence,

$$\begin{aligned} \widehat{u}_i &\approx u_i^*(\mathbf{x}, \ t_n + \Delta t) \\ &= u_i^*(\mathbf{x}, \ t_n) + \Delta t \frac{\partial u_i^*}{\partial t} + O(\Delta t^2) \\ &= u_i^*(\mathbf{x}, \ t_n) + \Delta t (H_i^* + \nu \nabla^2 u_i^*) + O(\Delta t^2) \end{aligned}$$

Since $u_i^*(\mathbf{x}, t_n) = u_i(\mathbf{x}, t_n)$,

$$\begin{aligned} \widehat{u}_i &= u_i^*(\mathbf{x}, \ t_n) + \Delta t (H_i^* + \nu \nabla^2 u_i^*) + O(\Delta t^2) \\ &= u_i(\mathbf{x}, \ t_n) + \Delta t (H_i + \nu \nabla^2 u_i) + O(\Delta t^2) \\ &= u_i(\mathbf{x}, \ t_{n+1}) + \Delta t \frac{\partial p}{\partial x_i} + O(\Delta t^2) \end{aligned}$$

From Eq. (1.229), $\widehat{u}_i = u_i^{n+1} + \Delta t \frac{\partial \phi^n}{\partial x_i}$ Also, it is written as

$$\widehat{\mathbf{u}}|_w = \mathbf{u}^{n+1}|_w + \left. \Delta t \frac{\partial \phi^n}{\partial \mathbf{x}} \right|_w \tag{1.232}$$

It is the boundary condition at the upper and lower walls for Eq. (1.230).
Equation (1.228) can be written in the form as

$$\nabla^2 \phi = f \tag{1.233}$$

The boundary conditions at streamwise and spanwise directions are periodic, and the conditions at the upper and lower walls are written as

$$\left. \frac{\partial \phi^{n+1}}{\partial \mathbf{x}} \right|_w = \left. \frac{\partial \phi^n}{\partial \mathbf{x}} \right|_w \tag{1.234}$$

Equations (1.230) and (1.233) would appear in the same form with that of Eqs. (1.165) and (1.167), hence $\widehat{\mathbf{u}}$ and ϕ^{n+1} can be calculated by the spectral method, in succession, the solutions $\mathbf{u}^{n+1}$ are obtained by Eq. (1.227).

1.7 Closed Remarks

The realization of a random process appears unpredictable, but some properties, that is, its statistical properties, are quite reproducible. The probability density function (PDF) representing the probability of the event involved in the process is commonly used to describe the statistical properties, which can be constructed by the infinite statistical moments possessed in random variables. Several statistical moments, such as mean value, rms, skewness, kurtosis, Reynolds stress, turbulent kinetic energy, etc., are traditionally used in discussing the turbulent problem. The turbulence is formed intermittently via bursting events that can be detected by the detection function. By means of the conditional average the flow field around the bursting, the characteristic flow pattern associated with the bursting will be revealed.

By decomposition of state variables of flow fields, such as Reynolds decomposition, LES decomposition (low-pass filtering) etc, a closed equation describing the statistical properties of turbulence can be induced based on the statistical models, this widely used approach equations can be solved numerically.

A periodic function $f(\mathbf{x})$ can be regarded as a superimposition of infinite harmonic waves, or Fourier series in the complex form, by Fourier transform, where the weight function $F(\mathbf{k})$ is called the spectrum of $f(\mathbf{x})$. Actually, this transform gives a mapping between the function $f(\mathbf{x})$ in the physical space $\mathbf{x}$ and the spectrum $F(\mathbf{k})$ in the spectral space $\mathbf{k}$. Likewise, a function $f(\mathbf{x})$ can also be expressed by a sum of orthogonal polynomials, such as Legendre polynomials, Chebyshev polynomials, etc. The mapping between the physical space and the spectral space can also be performed by the transform.

Since the derivative in spectral space can be represented in terms of all spectral coefficients, the partial differential equations in physical space are then transformed into a set of algebraic equations in spectral space, which conduces to the numerical computation. Subsequently, the resulting spectral coefficients are transformed back to physical space. The method solving the equations in spectral space via the discrete technique of collocation is called the pseudospectral method.

The choice of the basis (or trial) function and the test function is essentially associated with the spectral method. In the collocation method, the test function is taken as a delta function at the collocation points given by special formulas, such as Gaussian integrations. Based on this method, the choice of the basis function is then considered. For example, the basis functions are trigonometric functions in the Fourier–Galerkin method for the periodic problems, which satisfy the boundary conditions, so that a closed system can be obtained. Furthermore, the basis functions are Chebyshev polynomials in the Chebyshev–Tau method for the nonperiodic problems, which cannot satisfy the boundary conditions, so that the complement equations should be derived from boundary conditions.

The near-wall turbulence governed by the N-S equations can be simulated numerically by the spectral methods. In calculations, it is in general most efficient to apply spectral methods only to the spatial dependence, since the time-dependence can be marched forward from one time level to another. Both a Chebyshev–Tau method in the wall normal direction and a dealiazed Fourier method in the homogeneous directions, that is, streamwise and spanwise directions, are used for the spatial derivatives. In order to ensure that the computed solutions satisfy both the incompressibility constraint and the momentum equation when the time advancement is carried out using a semi-implicit time marching schemes, a Chebyshev–Tau influence-matrix method, including a Tau-correction step, is employed for the linear term and

the pressure term. Aliasing errors in the streamwise and spanwise directions are removed by the spectral truncation method referred to as 3/2 rule.

References

[1] Tennekes H and Lumley D L. A First Course in Turbulence, Academic Press, Inc., New York, 1972.
[2] Panchev S. Random Functions and Turbulence, Pergamon Press, Oxford, 1971.
[3] Pope S B. Turbulent Flows, Cambridge University Press, Cambridge, 2000.
[4] Blackwelder R F and Kaplan R E. On the wall structure of the turbulent boundary layer. J. Fluid Mech., 1976, 76:89–112.
[5] Blackwelder R F and Eckelmann H. Streamwise vortices associated with the bursting phenomenon. J. Fluid Mech., 1979, 94:577–594.
[6] Kim J. On the structure of wall-bounded turbulent flows. Phys. Fluids, 1983, 26(8):2088–2097.
[7] Kim J. Turbulence structures associated with the bursting event. Phys. Fluids, 1985, 26(1):52–58.
[8] Bogard D G and Tiederman W G. Burst detection with single point velocity measurements. J. Fluid Mech., 1986, 162:389–413.
[9] Lu S S and Willmarth W W. Measurements of the structure of the Reynolds stress in a turbulent boundary layer. J. Fluid Mech., 1973, 60:481–511.
[10] Bracewell R N. The Fourier Transform and its Applications, McGraw-Hill Companies, Inc., New York, 2000.
[11] Adrian R J, Christensen K T and Liu Z C. Analysis and interpretation of instantaneous turbulent velocity fields. Exp. Fluid, 2000, 29:275–290.
[12] Canuto C M, Hussaini Y, Quarteroni A and Zang T A. Spectral Methods in Fluid Dynamics, Springer-Verlag, Berlin, 1988.
[13] Peyret R. Spectral Methods for Incompressible Viscous Flow, Springer-Verlag, Inc., 2002.
[14] Boyd J P. Chebyshev and Fourier Spectral Methods, Dover Publication, Inc., 2001.
[15] Davis P J and Rabinowitz P. Methods of Numerical Integration, Academic Press, 1984.
[16] Kleiser L and Schumann U. Treatment of incompressibility and boundary conditions in 3-D numerical spectral simulations of plane channel flows. In: Hirschel EH (ed.), Third GAMM Conference Numerical Methods in Fluid Mechanics. Vieweg, Braunschweigh, 1980, pp. 165–173.
[17] Chorin A J. On the convergence of discrete approximation to the Navier–Stokes equations. Math. Comp., 1969, 23:341–353.
[18] Teman R. Navier–Stokes Equations Theory and Numerical Analysis, 3rd ed. North-Holland, Amsterdam, 1987, pp. 395–426.
[19] Kim J, Moin P and Moser R. Turbulence statistics in fully developed channel flow at low Reynolds number. J. Fluid Mech., 1987, 177:133–166.
[20] Kim J and Moin P. Application of a fractional step method to incompressible Navier–Stokes equations. J. Comput. Phys, 1985, 59:308–323.
[21] Choi H, Moin P and Kim J. Active turbulence control for drag reduction in wall-bounded flows. J. Fluid Mech., 1994, 262:75–110.
[22] LeVeque R L and Oliger J. Numerical Analysis Project. Manuscript NA-81-16, Computer Science Department, Stanford University, Stanford, CA, 1981.