

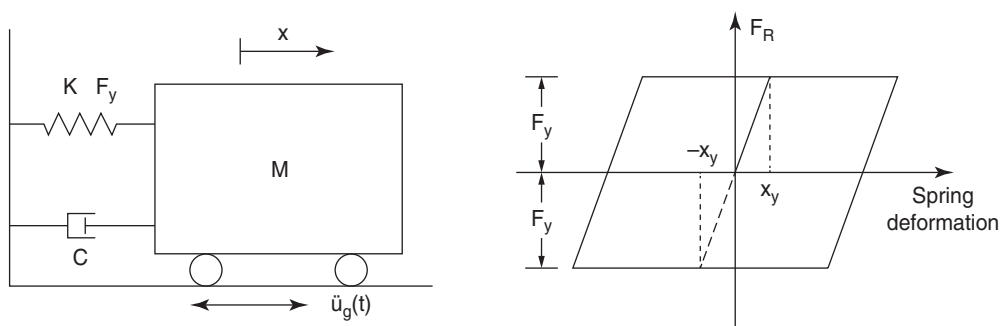
# 1

## Introduction

### 1.1 History of the Force Analogy Method

The force analogy method (FAM) is an analytical tool for solving structural analysis problems with material nonlinearity. It uses the concept of “inelastic displacement”, or more commonly known as the “residual displacement” in the formulation, where the nonlinear stiffness force due to material nonlinearity is represented by a change in displacement instead of a change in stiffness. The original concept of FAM was first introduced by Lin (1968), where the proposed method was actually applied to stress and strain in continuum mechanics with the inelastic behavior defined by plastic strain. Unfortunately, this concept only found limited acceptance because it was developed at approximately the same time as researchers were focusing their attention on studying the deformation of solids using numerical simulation methods, such as the finite element method with the inelastic behavior defined by changing stiffness. Although the finite element method is a powerful tool and widely used, the procedure of the step-by-step numerical integration is unavoidable, time consuming, cumbersome, and costly for practical design in 1980s and even today.

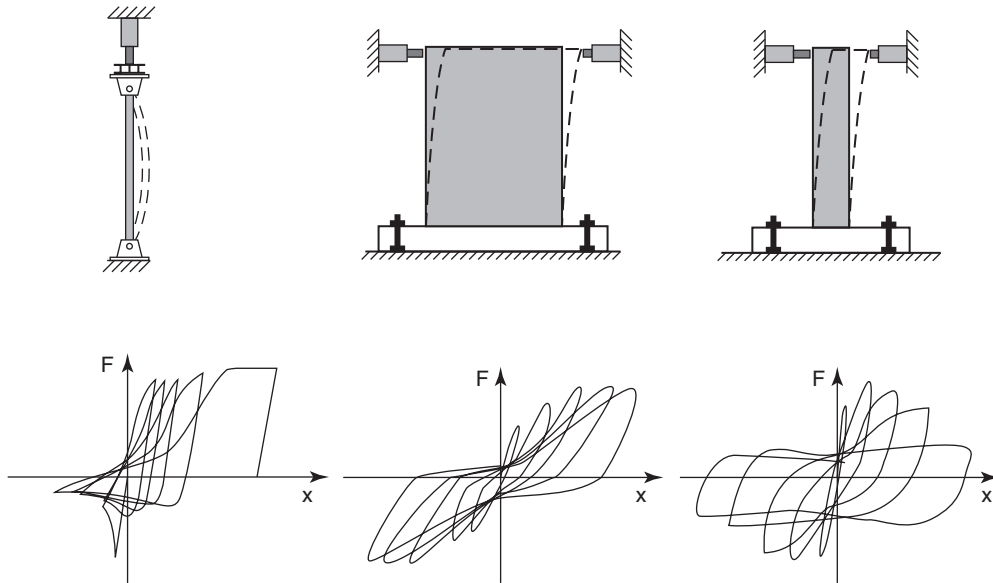
Recognizing that nonlinear finite element method of analysis is a time-consuming process, many structural engineers are constantly seeking a simplified dynamic analysis approach for analyzing nonlinear multi-degree-of-freedom (MDOF) systems to carry out their structural designs. One simplified approach is to represent the nonlinear MDOF system as an elastic system, in which structural response can be estimated by response spectra analysis of using the convenient and efficient modal superposition method. Newmark (1970) proposed a well-known method of extending the elastic response spectra analysis to engineering design of nonlinear systems through the use of inelastic response spectra. However, the method is strictly valid for single-degree-of-freedom (SDOF) systems and thus is inadequate for the analysis



**Figure 1.1** SDOF system and restoring force model.

of nonlinear MDOF systems due to the changing stiffness matrix. The changing stiffness matrix in the equations of motion for the nonlinear MDOF system is the drawback of this method, since the nonlinearity effect is coupled in each and every mode. Thus, significant effort was spent towards extending the modal superposition method in elastic analysis to inelastic analysis. One effort similar to the FAM, where the restoring force term of nonlinear MDOF systems was expressed by the sum of the elastic restoring force and additional external force, was presented by Villaverde (1988, 1996). After moving the additional external force term to the right-hand side of the equation of motion, the left-hand side of the equation is interpreted as an equivalent linear system. A different approximate modal decomposition method for the equation of motion was subsequently presented by Georgoussis (2008). While the above works emphasized the development of simplified analysis methods, only simple system models and load–deformation relationships, such as those shown in Figure 1.1, were selected to explore the physical significance of the external force term. The relationship between the external force term and inelastic behavior of structural members were ignored at that time.

The same problem was encountered by Wong (1996) during his study on the structural control of nonlinear structures. Since the theory of state space dynamic analysis, as a computing platform for performing structural control calculation, was only applicable to elastic systems, it was an obvious barrier when apply the structural control technique in nonlinear structures. Thus, a method of analyzing the inelastic response of the building by recovering the forces from the states of the building was introduced. Subsequently, Wong and Yang (1999) formally published the first application of the FAM for civil structures where the method was formulated in force–deformation space for inelastic dynamic analysis. The fundamental concept of the FAM is that each inelastic deformation in the structure is formulated as a degree of freedom such that the initial stiffness matrix is computed only once at the beginning and can be used throughout the inelastic analysis. Coupling the FAM with the state space formulation for dynamic analysis provides an accurate, efficient, and stable solution algorithm such that it can be used to analyze structures with various material properties, not only for elastic–plastic property but also for both hardening and softening properties. In addition, the external force term was interpreted as the force analogy, which causes inelastic deformation of structural members at certain locations in the structure. The inelastic deformation includes nonlinear extension of the braces in a braced frame, plastic rotation of the beams and columns in a moment resisting frame, or yielding of the base isolators in a base isolation system. Since then, Zhao and Wong (2006) further



**Figure 1.2** Complex cyclic behaviors of structural members.

developed the FAM by incorporating the geometric nonlinear effect and presented a comprehensive nonlinear approach for inelastic framed structures, including geometric nonlinearity and material nonlinearity. The approach uses finite element formulation to derive the elemental stiffness matrices, particularly to derive the geometric stiffness matrix in a general form. However, the geometric stiffness matrix used in the nonlinear formulation was not exact, and further improvement by Wong (2012, 2013) was recently conducted and will be presented in Chapter 7.

Although Wong and Yang (1999) pointed out that all material properties can be used in the FAM and they have no influence on the algorithm stability, only nonlinear response of a steel moment-resistant frame with simple bilinear moment versus plastic rotation relationship was mentioned in the study. In fact, structural members often exhibit complex cyclic inelastic behavior (i.e. buckling of braces, strength degradation of reinforced concrete members, as shown in Figure 1.2) when they undergo excessive dynamic loadings, and some well-known models have been proposed and developed. It is clear that updating element local stiffness matrices, re-assembling them and performing static condensation to derive system global tangent stiffness matrix is not necessary in the FAM. However, existing material models cannot be applied in the FAM directly because they often reveal a highly nonlinear relation of the external force and total deformation rather than plastic deformation. Since the global behavior of nonlinear structures is closely associated with the relationship between the internal force and plastic deformation, some investigations were carried out to extend the application of the FAM for structural members with different material behaviors.

Chao and Loh (2007) used three different plastic mechanisms to simulate the reinforced concrete beam-column elements in the FAM. The load versus deformation comparison shows that the proposed algorithm gives results very similar to experimental data. Additionally,

the  $P$ -Delta effect also has been considered in this study. Li *et al.* (2013a) implemented an existing brace physical theory model for use in the FAM. In the procedure, the physical theory model developed by Dicleli and Calik (2008) is chosen for implementation in the FAM, because it is a relatively simple and efficient model that has been shown to provide reasonable accuracy. Two sliding plastic mechanisms, which simulate axial displacements produced by transverse brace displacement and the so-called growth effect, are used to represent the inelastic brace behavior. The resulting model is shown to provide good agreement with experimental data. Moreover, this brace model is implemented in a frame where inelastic response occurs in both the frame and braces to demonstrate the value of the brace model and the potential for simulating complex inelastic dynamic behavior of concentric braced frames with the FAM. The model is validated against prior experimental results to be an accurate, efficient, and stable algorithm for conducting dynamic analysis when coupled with the state space formulation.

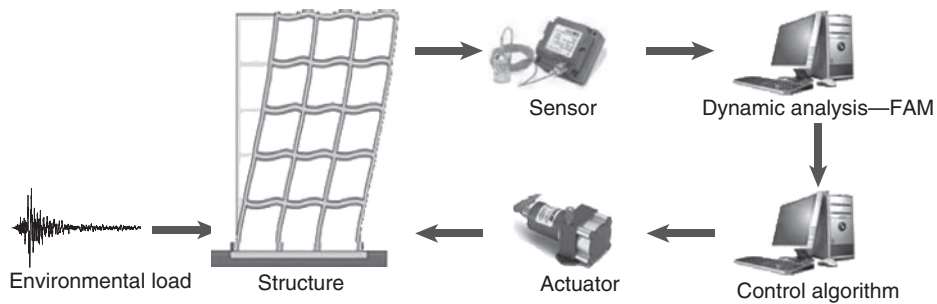
In addition, Li and Zhang (2013b) developed a framework for the seismic damage analysis of reinforced concrete frame structures considering the stiffness degradation based on the FAM. A damage hinge model, which is located at the ends of columns and beams, is proposed for modeling damage behavior due to concrete cracking. As a damage effect is implemented by introducing the damage indices as internal variables, the real-time structural performance and damage level can be evaluated during the computation process. The damage hinge, together with the plastic hinge arising from structural materials, forms a complete inelastic mechanism including stiffness degradation behavior for reinforced concrete frame structures. Since only initial stiffness is used throughout the dynamic computation analysis, the usage of the state space formulation, as an outstanding advantage of the FAM, is retained and makes the real-time damage analysis more accurate, efficient, and stable. As for the reinforced concrete shear wall member, a procedure for modeling the hysteretic response of reinforced concrete shear wall members based on the existing models in the FAM was established and will be discussed in Chapter 6. An reinforced concrete (RC) flexural member model, where the strength deterioration and stiffness degradation effect due to increasing loading cycles, and the pinching behavior that mainly roots in the crack opening and closing during loading reversals are considered, was established and incorporated in the FAM. The methodology will be presented in Chapter 4 together with several examples.

## 1.2 Applications of the Force Analogy Method

Because the FAM has two outstanding benefits in terms of computation efficiency and stability, it has the advantage over other analysis tools for the following applications:

### 1.2.1 Structural Vibration Control

Since the concept of structural vibration control in civil engineering was proposed by J.T.P Yao in 1972, it has made considerable progress in the development of theoretical and experimental researches. A number of structural control techniques and strategies have been developed and applied in practices, specifically in seismic regions. The structural vibration control began in the mechanical engineering in the early 20th century and the majority of control theories,



**Figure 1.3** The framework for the combination of FAM and control algorithm.

includes the linear quadratic regulator, modal control, smart control, H2 control, H $\infty$  control, etc., and algorithms were applied to elastic systems and have been matured. These control algorithms together with dynamic analysis procedure run together for determining controlling force of actuators, as shown in Figure 1.3. However, structural members in civil engineering buildings will always experience inelastic deformation when the buildings are subjected to excessive loadings. This causes significant problems, such as time delay, incompatible program, etc., during the combination of inelastic computation procedures and control algorithms.

The emergence of the FAM provides a way to solve this type of problem because the left-hand side of the equation of motion of nonlinear systems retains the linear properties of corresponding elastic system. Thus, many problems, which are relatively difficult to answer while applying traditional control algorithms to inelastic systems, have been solved to some degrees using the FAM. Wong and Yang (2003) and Wong (2005) proposed inelastic structural control algorithms, which compensates for the time delay that happens in practical control systems, through incorporating the FAM with the predictive instantaneous optimal control algorithm and the predictive instantaneous optimal control algorithm, respectively. Moreover, since the earthquake ground velocity is not at high frequency as compared with the ground acceleration, it can be predicted at certain time steps beforehand in the real-time domain with higher accuracy. Thus, Pang and Wong (2006) proposed a simple control algorithm expressed using the input ground velocity, namely the Predictive Instantaneous Optimal Control algorithm.

To capture the damaging effects during earthquake ground motions, the FAM is used to characterize structures responding in the inelastic domain. Li and Li (2011a) developed an approach based on the FAM to analyze the dynamic response of structure with energy-dissipation devices. The proposed algorithm is applicable to a variety of energy-dissipation devices by turning them to the equivalent force applied at the joints of the frame. Wong (2008) and Wong and Johnson (2009) presented studies on the use of tuned mass dampers as a passive energy-dissipation device to investigate the benefits of using such devices in reducing the inelastic structural responses. In addition, Wong (2011a) presented a simple numerical algorithm based on the combination of the state space method and FAM to calculate the inelastic dynamic analysis of structures with nonlinear fluid viscous dampers. Finally, Li *et al.* (2011b) proposed a control algorithm for inelastic structures through combining the market-based control strategy and force analogy method. The framework of this work will be discussed in Chapter 9.

### 1.2.2 Modal Dynamic Analysis Method

Since each term on the left-hand side of the equation of motion for nonlinear MDOF systems is feasible for modal decomposition like elastic systems, it suggests that the FAM is probably a good baseline for applying the modal dynamic analysis method to solve nonlinear MDOF system problems. Wong (2011b) extended the modal superposition to the nonlinear domain by using the FAM to address material nonlinearity. In addition, because linear modal superposition has found great acceptance in performance-based seismic engineering, geometric nonlinearity is incorporated into the analysis using stability functions. Through the combination of FAM, stability functions, the state space method, and modal superposition, numerical simulations are performed and results are demonstrated to be both accurate and efficient. Moreover, a simple analysis tool for capturing the effect of rigid-end offsets in framed structures under earthquake excitation has been incorporated into the above nonlinear modal analysis methodology by Wong (2012). Author also demonstrated that the equation of motion for nonlinear MDOF systems in the FAM can be uncoupled, but two other governing equations in the FAM relating the internal force, such as the moment and force of structural members are not decomposable. However, uncoupled modal SDOF system responses can be determined by incorporating the FAM with the modal pushover analysis method such that the modal superposition method is suitable for the solution of the nonlinear MDOF system. Although the procedure presented is still an approximation method due to the modal pushover analysis method application, its value and potential for the maximum displacement estimation of the nonlinear MDOF system based on the FAM were validated. The procedure will be discussed in Chapter 8 along with examples.

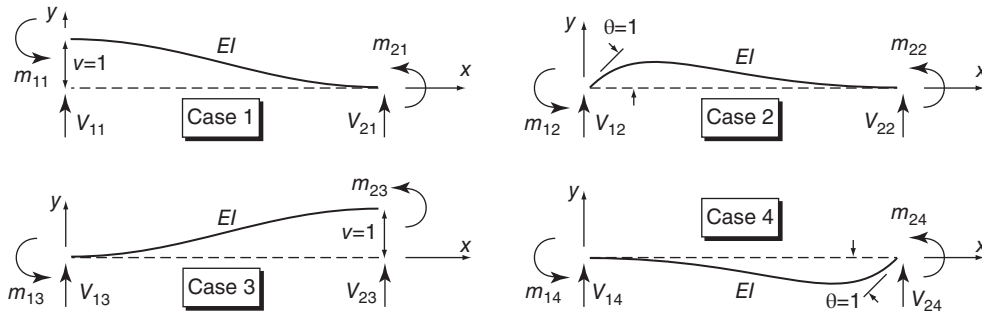
### 1.2.3 Other Design and Analysis Areas

Wong and Yang (2002) derived the plastic energy dissipation of structures based on the FAM and used the energy as the response parameter in evaluating the performance of the structure, and Wong and Wang (2003) extended the energy-balance equation to include control energy as an addition form of energy dissipation to resist earthquake inputs. In these studies, the FAM was modified and extended to analyze real moment-resisting frames with zero rotational mass moment of inertia using the method of static condensation. The static condensation method in the FAM will be discussed in Chapter 2 for static analysis and Chapter 3 for dynamic analysis.

Wang and Wong (2007) introduced the FAM for the first time into the field of stochastic dynamic analysis for inelastic structures. This stochastic FAM maintains the advantage of high efficiency in the numerical computation of the FAM in dynamic analysis. According to the stochastic FAM, the variance covariance functions of inelastic dynamic responses, such as displacement, velocity, inelastic displacement of the entire moment-resisting framed structures, and plastic rotation at individual plastic hinge location, can be produced for structures subjected to random excitation.

## 1.3 Background of the Force Analogy Method

The first step in learning the force analogy method for solving nonlinear structure problems is to understand the matrix method of structural analysis. Because understanding each term in the stiffness matrix (i.e.  $12EI/L^3$ ,  $6EI/L^2$ ,  $4EI/L$ , and  $2EI/L$ ) is so important



**Figure 1.4** Displacement patterns and the corresponding fixed-end forces.

to the presentations in the subsequent chapters, it is appropriate and worthwhile to derive the elastic stiffness matrix for bending in this section for the completeness of the book.

Consider a beam of length  $L$  with uniform elastic modulus  $E$  and moment of inertia  $I$  that is subjected to loadings at the two ends. Due to the loadings, the deformation at the two ends (i.e. translations and rotations) of the beam can be related to the amount of shear and moments at the two ends through a stiffness matrix expressed in the following form:

$$\begin{bmatrix} V_1 \\ m_1 \\ V_2 \\ m_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \quad (1.1)$$

where  $V$  is the shear,  $m$  is the moment,  $v$  is the transverse displacement, and  $\theta$  is the rotation. The subscript '1' represents the quantities at the 1-end, and the subscript '2' represents the quantities at the 2-end. Finally,  $k$  represents the entries in the stiffness matrix. To determine the stiffness matrix, four cases of a beam deflection are separated as shown in Figure 1.4 using the unit displacement method. Here,  $V_{1i}$ ,  $m_{1i}$ ,  $V_{2i}$ , and  $m_{2i}$  represent the fixed-end shears and moments of the beam, and  $i = 1, \dots, 4$  represents the four cases of unit displacement patterns of beam deflection.

Using the classical Bernoulli–Euler beam theory with “plane sections remain plane”, where the moment is proportional to the curvature, the governing equilibrium equation describing the deflection of the beam member can be written as

$$(EIv'')'' = 0 \quad (1.2)$$

By assuming  $EI$  is constant along the member, the solution to the fourth-order ordinary differential equation is:

$$v = Ax^3 + Bx^2 + Cx + D \quad (1.3)$$

and the corresponding, rotation, moment, and shear equations become:

$$\theta(x) = v' = 3Ax^2 + 2Bx + C \quad (1.4a)$$

$$m(x) = EIv'' = EI(6Ax + 2B) \quad (1.4b)$$

$$V(x) = EIv''' = EI(6A) \quad (1.4c)$$

In order to solve for the constants in Eq. (1.3), the following four cases of boundary conditions are considered.

**Case 1:** Imposing the boundary conditions  $v(0) = 1$ ,  $v'(0) = 0$ ,  $v(L) = 0$ , and  $v'(L) = 0$  gives

$$v(0) = 1: \quad D = 1 \quad (1.5a)$$

$$v'(0) = 0: \quad C = 0 \quad (1.5b)$$

$$v(L) = 0: \quad AL^3 + BL^2 + CL + D = 0 \quad (1.5c)$$

$$v'(L) = 0: \quad 3AL^2 + 2BL + C = 0 \quad (1.5d)$$

Solving simultaneously for the constants in Eq. (1.5) gives

$$A = 2/L^3, \quad B = -3/L^2, \quad C = 0, \quad D = 1 \quad (1.6)$$

Now substituting the constants in Eq. (1.6) into the shear equation in Eq. (1.4c) and the moment equation in Eq. (1.4b) evaluated at appropriate end points gives the fixed-end forces as labeled in Figure 1.4 (Case 1) as:

$$V_{11} = EIv'''(0) = EI(6A) = 12EI/L^3 \quad (1.7a)$$

$$m_{11} = -EIv''(0) = -EI(2B) = 6EI/L^2 \quad (1.7b)$$

$$V_{21} = -EIv'''(L) = -EI(6A) = -12EI/L^3 \quad (1.7c)$$

$$m_{21} = EIv''(L) = EI(6AL + 2B) = 6EI/L^2 \quad (1.7d)$$

In matrix form, this is given as

$$\begin{bmatrix} V_{11} \\ m_{11} \\ V_{21} \\ m_{21} \end{bmatrix} = \begin{bmatrix} 12EI/L^3 & \times & \times & \times \\ 6EI/L^2 & \times & \times & \times \\ -12EI/L^3 & \times & \times & \times \\ 6EI/L^2 & \times & \times & \times \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.8)$$

Note that the minus signs in front of the calculations for  $m_{1i}$  and  $V_{2i}$  are used because of the differences in sign convention between the classical beam theory and the theory for stiffness method of structural analysis.



**Case 2:** Imposing the boundary conditions  $v(0) = 0$ ,  $v'(0) = 1$ ,  $v(L) = 0$ , and  $v'(L) = 0$  gives

$$v(0) = 0: \quad D = 0 \quad (1.9a)$$

$$v'(0) = 1: \quad C = 1 \quad (1.9b)$$

$$v(L) = 0: \quad AL^3 + BL^2 + CL = 0 \quad (1.9c)$$

$$v'(L) = 0: \quad 3AL^2 + 2BL + C = 0 \quad (1.9d)$$

Solving simultaneously for the constants in Eq. (1.9) gives

$$A = 1/L^2, \quad B = -2/L, \quad C = 1, \quad D = 0 \quad (1.10)$$

Now substituting the constants in Eq. (1.10) into the shear equation in Eq. (1.4c) and the moment equation in Eq. (1.4b) evaluated at appropriate end points gives the fixed-end forces as labeled in Figure 1.4 (Case 2) as:

$$V_{12} = EIv'''(0) = EI(6A) = 6EI/L^2 \quad (1.11a)$$

$$m_{12} = -EIv''(0) = -EI(2B) = 4EI/L \quad (1.11b)$$

$$V_{22} = -EIv'''(L) = -EI(6A) = -6EI/L^2 \quad (1.11c)$$

$$m_{22} = EIv''(L) = EI(6AL + 2B) = 2EI/L \quad (1.11d)$$

In matrix form, this is given as

$$\begin{bmatrix} V_{12} \\ m_{12} \\ V_{22} \\ m_{22} \end{bmatrix} = \begin{bmatrix} \times & 6EI/L^2 & \times & \times \\ \times & 4EI/L & \times & \times \\ \times & -6EI/L^2 & \times & \times \\ \times & 2EI/L & \times & \times \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (1.12)$$

**Case 3:** Imposing the boundary conditions  $v(0) = 0$ ,  $v'(0) = 0$ ,  $v(L) = 1$ , and  $v'(L) = 0$  gives

$$v(0) = 0: \quad D = 0 \quad (1.13a)$$

$$v'(0) = 0: \quad C = 0 \quad (1.13b)$$

$$v(L) = 1: \quad AL^3 + BL^2 + CL = 1 \quad (1.13c)$$

$$v'(L) = 0: \quad 3AL^2 + 2BL + C = 0 \quad (1.13d)$$

Solving simultaneously for the constants in Eq. (1.13) gives

$$A = -2/L^3, \quad B = 3/L^2, \quad C = 0, \quad D = 1 \quad (1.14)$$

Now substituting the constants in Eq. (1.14) into the shear equation in Eq. (1.4c) and the moment equation in Eq. (1.4b) evaluated at appropriate end points gives the fixed-end forces as labeled in Figure 1.4 (Case 3) as:

$$V_{13} = EIv'''(0) = EI(6A) = -12EI/L^3 \quad (1.15a)$$

$$m_{13} = -EIv''(0) = -EI(2B) = -6EI/L^2 \quad (1.15b)$$

$$V_{23} = -EIv'''(L) = -EI(6A) = 12EI/L^3 \quad (1.15c)$$

$$m_{23} = EIv''(L) = EI(6AL + 2B) = -6EI/L^2 \quad (1.15d)$$

In matrix form, this is given as

$$\begin{bmatrix} V_{13} \\ m_{13} \\ V_{23} \\ m_{23} \end{bmatrix} = \begin{bmatrix} \times & \times & -12EI/L^3 & \times \\ \times & \times & -6EI/L^2 & \times \\ \times & \times & 12EI/L^3 & \times \\ \times & \times & -6EI/L^2 & \times \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (1.16)$$

**Case 4:** Finally, imposing the boundary conditions  $v(0)=0$ ,  $v'(0)=0$ ,  $v(L)=0$ , and  $v'(L)=1$  gives

$$v(0)=0: D=0 \quad (1.17a)$$

$$v'(0)=0: C=0 \quad (1.17b)$$

$$v(L)=0: AL^3 + BL^2 + CL=0 \quad (1.17c)$$

$$v'(L)=1: 3AL^2 + 2BL + C=1 \quad (1.17d)$$

Solving simultaneously for the constants in Eq. (1.17) gives

$$A = 1/L^2, \quad B = -1/L, \quad C = 1, \quad D = 0 \quad (1.18)$$

Now substituting the constants in Eq. (1.18) into the shear equation in Eq. (1.4c) and the moment equation in Eq. (1.4b) evaluated at appropriate end points gives the fixed-end forces as labeled in Figure 1.4 (Case 4) as:

$$V_{14} = EIv'''(0) = EI(6A) = 6EI/L^2 \quad (1.19a)$$

$$m_{14} = -EIv''(0) = -EI(2B) = 2EI/L \quad (1.19b)$$

$$V_{24} = -EIv'''(L) = -EI(6A) = -6EI/L^2 \quad (1.19c)$$

$$m_{24} = EIv''(L) = EI(6AL + 2B) = 4EI/L \quad (1.19d)$$

In matrix form, this is given as

$$\begin{bmatrix} V_{14} \\ m_{14} \\ V_{24} \\ m_{24} \end{bmatrix} = \begin{bmatrix} \times & \times & \times & 6EI/L^2 \\ \times & \times & \times & 2EI/L \\ \times & \times & \times & -6EI/L^2 \\ \times & \times & \times & 4EI/L \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (1.20)$$

In summary, based on the construction of the stiffness matrix using above four cases as shown in Eqs. (1.8), (1.12), (1.16), and (1.20), the stiffness equation of the  $i$ th beam member becomes:

$$\begin{bmatrix} V_1 \\ m_1 \\ V_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \quad (1.21)$$

From Eq. (1.21), it can be seen that:

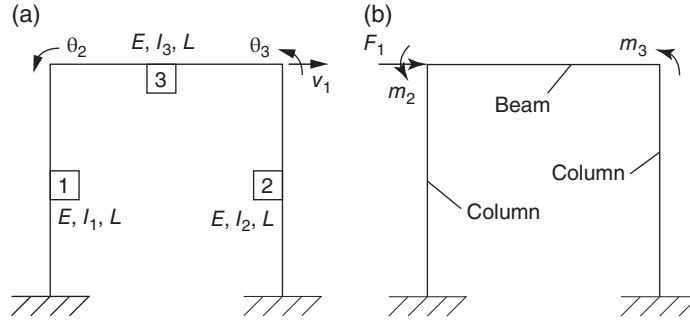
- The stiffness relating the transverse displacement at one end of the beam with end shears is  $12EI/L^3$ .
- The stiffness relating the transverse displacement at one end of the beam with end moments is  $6EI/L^2$ .
- The stiffness relating the rotation at one end of the beam with end shears is  $6EI/L^2$ .
- The stiffness relating the rotation at one end of the beam with end moment at the same end is  $4EI/L$ .
- The stiffness relating the rotation at one end of the beam with end moment at the opposite end is  $2EI/L$ .

### Example 1.1 One-Story One-Bay Frame

Consider a one-story one-bay frame as shown in Figure 1.5(a). Assume the members are axially rigid, this results in a three degrees of freedom system, one floor translation and two joint rotations, as labeled in Figure 1.5(a) as  $v_1$ ,  $\theta_2$ , and  $\theta_3$ . Also assume that the beam and the two columns are of the same length  $L$  and elastic modulus  $E$ , but the moment of inertias of each member are as labeled in the figure. The global stiffness matrix relates  $v_1$ ,  $\theta_2$ , and  $\theta_3$  at the degrees of freedom with the corresponding applied forces, i.e.

$$\begin{bmatrix} F_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \quad (1.22)$$

where  $F_1$ ,  $m_2$ , and  $m_3$  are the global applied force and moments at the degrees of freedom as shown in Figure 1.5(b).



**Figure 1.5** One-story one-bay moment-resisting frame: (a) Three degrees of freedom system; (b) Applied forces at the degrees of freedom.

The objective here is to construct the global stiffness matrix in Eq. (1.22). Three displacement patterns are used to construct this stiffness matrix as follows:

**Case 1:** Imposing the boundary conditions  $v_1 = 1$ ,  $\theta_2 = 0$ , and  $\theta_3 = 0$

Figure 1.6 shows the displacement pattern and the corresponding applied forces required to induce such displacement pattern. These applied forces give the first column of the stiffness matrix as

$$\begin{bmatrix} F_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} 12EI_1/L^3 + 12EI_2/L^3 & \times & \times \\ 6EI_1/L^2 & \times & \times \\ 6EI_2/L^2 & \times & \times \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (1.23)$$

**Case 2:** Imposing the boundary conditions  $v_1 = 0$ ,  $\theta_2 = 1$ , and  $\theta_3 = 0$

Figure 1.7 shows the displacement pattern and the corresponding applied forces required to induce such displacement pattern. These applied forces give the second column of the stiffness matrix as

$$\begin{bmatrix} F_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \times & 6EI_1/L^2 & \times \\ \times & 4EI_1/L + 4EI_3/L & \times \\ \times & 2EI_3/L & \times \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (1.24)$$

**Case 3:** Imposing the boundary conditions  $v_1 = 0$ ,  $\theta_2 = 0$ , and  $\theta_3 = 1$

Figure 1.8 shows the displacement pattern and the corresponding applied forces required to induce such displacement pattern. These applied forces give the third column of the stiffness matrix as

$$\begin{bmatrix} F_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} \times & \times & 6EI_2/L^2 \\ \times & \times & 2EI_3/L \\ \times & \times & 4EI_2/L + 4EI_3/L \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (1.25)$$

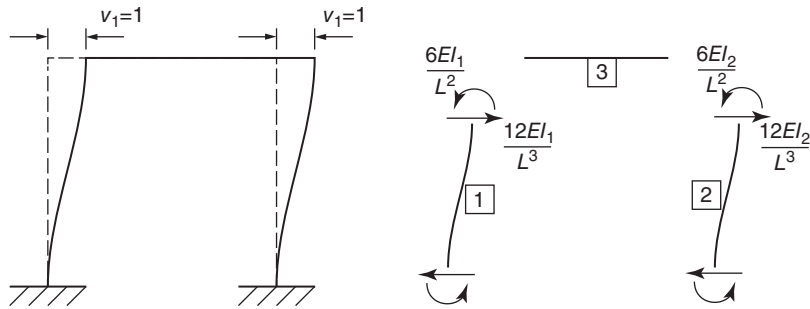


Figure 1.6 Displacement pattern using unit displacement at  $v_1$ .

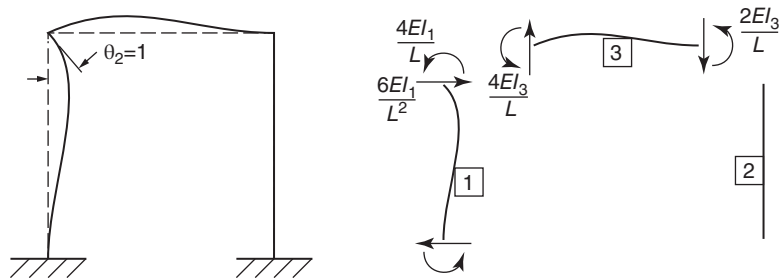


Figure 1.7 Displacement pattern using unit displacement at  $\theta_2$ .

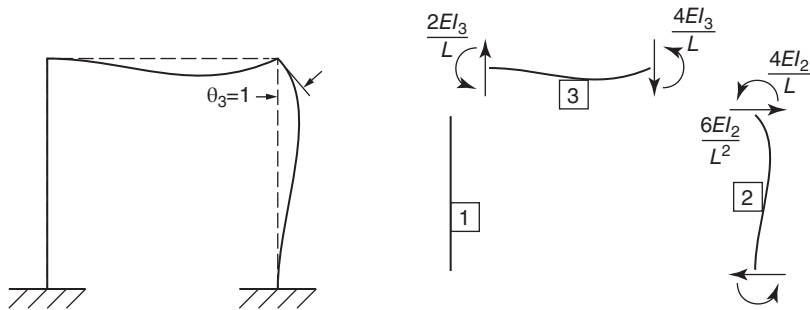


Figure 1.8 Displacement pattern using unit displacement at  $\theta_3$ .

Based on Eqs. (1.23), (1.24), and (1.25), the global stiffness matrix is therefore constructed as:

$$\mathbf{K} = \begin{bmatrix} 12EI_1/L^3 + 12EI_2/L^3 & 6EI_1/L^2 & 6EI_2/L^2 \\ 6EI_1/L^2 & 4EI_1/L + 4EI_3/L & 2EI_3/L \\ 6EI_2/L^2 & 2EI_3/L & 4EI_2/L + 4EI_3/L \end{bmatrix} \begin{matrix} \leftarrow v_1 \\ \leftarrow \theta_2 \\ \leftarrow \theta_3 \end{matrix} \quad (1.26)$$

Note that the global stiffness matrix  $\mathbf{K}$  in Eq. (1.26) is symmetric – an important property that is observed throughout this book.

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