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Introduction to Lattice Materials

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1.1 Introduction

The word "lattice" implies a certain ordered pattern characterized by spatial periodicity, and hence symmetry. In crystalline solids, for example, atoms are arranged in a spatially periodic pattern or a lattice. Such a crystal lattice is specified by a unit cell and the associated basis vectors defining the directions of tessellation [1, 2]. Spatially repetitive patterns are not unique to atomic length scales. They appear over a wide range of length scales, spanning several disciplines and areas of application; see Figure 1.1 for a representative list. Carbon nanotubes [3] and single-layer graphene sheets [4] are periodic materials with nanoscale features. Microelectromechanical systems (MEMS) for radio frequency applications use microscale periodic architectures to form mechanical filters [5]. Biomedical implants such as cardiovascular stents are periodic cylindrical mesh structures [6, 7]. At macro and mega scales, periodic structural construction is widely used in composites in materials engineering [8, 9], turbomachinery in aerospace engineering [10, 11], and bridge and tower structures in civil engineering [12]. Aircraft surfaces typically use a skin-stinger configuration in the form of a uniform shell, reinforced at regular spatial intervals by identical stiffener/stingers. Similarly, rib-skin aircraft structural components, used in tails and fins, comprise two skins (plates) interconnected by ribs [13]. Interested readers are referred to the book by Gibson and Ashby [14] for further studies on lattice materials and the reviews by Mead and by Hussein et al. [15, 16] on the dynamics of periodic materials in general.

In a closely related research discipline, periodic materials are referred to as *phononic crystals* [17, 18], where strong analogies are drawn with their electromagnetic counterpart, photonic crystals. While there is a significant overlap between lattice materials and phononic crystals [19, 20], the former category is mostly associated with low-density construction and utilization in structural mechanics applications, whereas the latter is mostly connected to applications in applied physics, including filtering [21], waveguiding [22], sensing [23], imaging [24], and, more recently, vibrational energy harvesting [25], thermal transport management at the nanoscale [26], and control of wall-bounded flows [27]. Another class of artificial materials that possess unique

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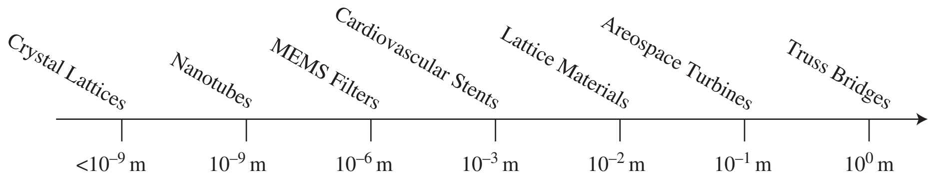


Figure 1.1 Periodic materials and structures across different length scales and disciplines. (MEMS: microelectromechanical systems.)

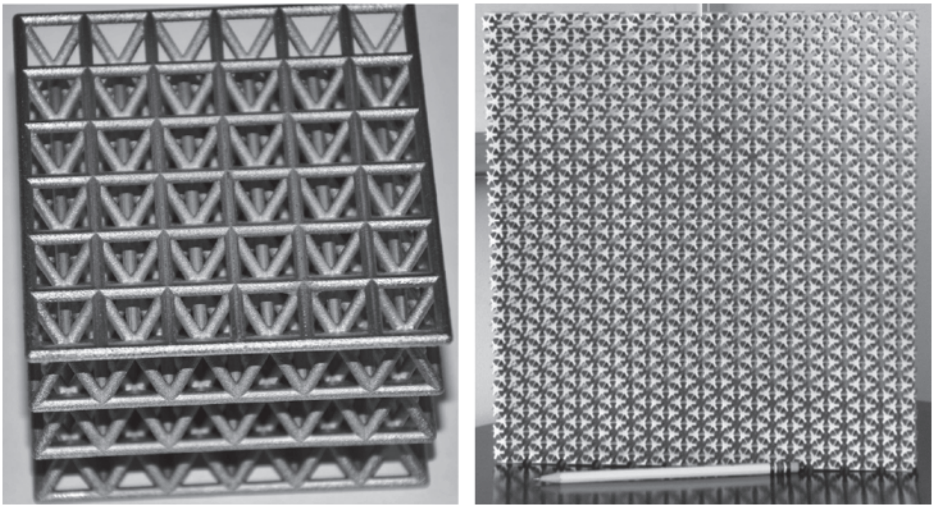
wave-propagation properties is referred to as *acoustic/elastic metamaterials* [28]. These are similar to phononic crystals, with the added feature of local resonators – small oscillating substructures integrally embedded within, or attached to the medium of the host material [29, 30]. However, unlike lattice materials and phononic crystals, periodicity is not a necessity for metamaterials. In addition to controlling sound and vibration, locally resonant "nanophononic metamaterials" have been shown to reduce thermal conductivity [31]. A recent book [32] and review article [16] provide historical background, the state of the art in the analysis and design of phononic crystals and metamaterials, together with their applications. In recent years, a new research community has formed around this discipline, now more broadly termed *phononics*, which incorporates the study and manipulation of "sound" waves in general and across the various spatial and temporal scales [33, 34].

The dynamic response of lattice materials, and structures, and by association phononic crystals and metamaterials, is the overarching theme of the book. We begin with a brief overview of periodic materials and structures, with emphasis on lattice materials, which are considered a new class of periodic materials. A formal classification is presented, followed by a discussion of manufacturing techniques and applications. A link to phononic crystals and acoustic/elastic metamaterials – also a new development in periodic materials – is presented when appropriate. We conclude this introductory chapter with an overview of the book.

1.2 Lattice Materials and Structures

A lattice material is defined as a spatially periodic network of structural elements, such as rods, beams, plates, or shells, whose constituent length scales are generally larger than the load-deformation length scales[1]; see Figure 1.2 for example. It possesses a spatially ordered pattern specified by a unit cell and associated tessellation directions (lattice basis vectors). The unit cell itself is an interconnected network of structural elements. Let us consider a network of flexural beams as an example. The material constituent of each beam can be a single homogeneous isotropic material (such as steel or aluminum) or a hierarchical anisotropic composite. Thus lattice materials, in the form of an interconnected spatially periodic network of composite beams, can be viewed as discrete multiscale materials with hierarchy. The ability to fabricate a spatially periodic network of beams using advanced manufacturing methods has spurred interest in lattice materials; see Fleck et al. [35] for a recent review. When viewed as a porous

1 This condition does not necessarily hold for lattice metamaterials where the size of unit-cell may be smaller than the deformation length scales.



(a) (b)

Figure 1.2 Lattice materials formed from a periodic network of beams: (a) ultralight nanometal truss hybrid lattice; (b) pentamode lattice.

solid, or a hybrid material (of fluid and metal) [36], the high-porosity limit yields a network of beams while the low-porosity limit leads to a continuum with pores. Most of the discussion and examples covered in this book are focused on material configurations at the high-porosity end of this range, although the ideas are usually relevant to low-porosity configurations as well.

1.2.1 Material versus Structure

A spatially periodic network of structural elements, such as beams, can be viewed both as a material and a structure for the following reasons. In engineering applications, employing a truss lattice of beams as a core in a sandwich panel, the length of each lattice beam is of the order of the thickness of the panel, and the thickness of each beam is typically an order of magnitude less. When the deformation processes of interest are at a length scale much larger than the individual beam length, a spatially periodic network of beams is termed a "lattice material" and has its own effective properties. At length scales of the order of the individual beam length, a spatially periodic network of beams behaves as a structure, such as a frame in a building or a truss in a bridge. Thus principles of structural mechanics can be applied to the design of lattice materials [37]. Another avenue for distinguishing between material and structure is in terms of the number unit cells, as well as the internal unit-cell symmetry. It is generally recognized that a for a finite system to exhibit material characteristics, at least a handful of unit cells are needed [38, 39]. In addition, a finite structure based on a repetition of a unit cell with symmetrical internal features is more likely to respond to dynamic loading in a manner consistent with the dispersion band structure of a material theoretically consisting of an infinite number of this unit cell [40, 41].

1.2.2 Motivation

The development of lattice materials is motivated by a desire to design multifunctional materials and structures that are not only light and stiff but also possess a desirable vibroacoustic response and thermal-transport properties, among other features. The need to overcome the limitations of metal foams [42, 43] has propelled the development of lattice materials, a process that has benefited from insights already acquired through studies of cellular solids [37, 44–48]. Similarly, accumulated research on the dynamics of periodic materials and structures (such as aircraft components and conventional composite materials) has provided a valuable knowledge base to build on for the study of wave-propagation characteristics in lattice materials. The following list provides an incomplete but indicative summary of efforts and motivations for current research in lattice materials.

1. Design lightweight and stiff/strong structures with optimal lattice core for multifunctional applications [49–52]. In this line of research, ongoing efforts aim to tailor the effective stiffness and strength of the truss lattice core to achieve high performance with the lowest possible density. The discovery of new unit-cell geometries using topology optimization and other computational methods is a promising avenue for further improvements [53, 54].
2. Advance mathematical modeling and analysis of complex lattice structures. This involves developing homogenization techniques for lattices [55, 56] and in-depth studies on the influence of damping [57–59] and nonlinearities [60–62] on the dispersive behavior of lattices.
3. Develop lattice unit-cell structures with tunable elastodynamic [63–65] and stability [66] properties.
4. Develop lattice-styled metamaterials based on periodic micro-architectures with extraordinary dynamic (acoustic and/or elastic) effective properties, not achievable using conventional materials [67, 68].
5. Create innovative nanostructured lattice materials based on periodic architectures for mechanical [50, 69, 70] and thermal [26, 31, 71] applications.

1.2.3 Classification of Lattices and Maxwell's Rule

Lattices can be classified based on their geometric or their mechanical deformation properties. Geometry-based classification is universally accepted in mathematics and solid-state physics. In 2D, planar lattices are classified into two categories: regular and semi-regular [72]. Regular lattices are obtained by tessellating a single, regular, polygonal unit cell to fill a plane. Here, a regular polygon is defined to be equiangular (all angles are equal) and equilateral (all lengths are equal). Square, triangle, and hexagon are the only plane-filling regular polygons, so there are only three regular planar lattices: square lattice, triangular lattice, and hexagonal lattice. In contrast to regular lattices, semi-regular lattices are obtained by tessellating a unit cell, containing more than one regular polygon, to fill a plane. There are only eight such semi-regular lattices; see Cundy and Rollett [72] for more detail. Kagome or triangular-hexagon lattice is a semi-regular lattice that is widely used in weaving baskets and in architectural construction. A detailed classification of 3D lattices and polyhedra can be found in the literature [72, 73].

Lattices can also be classified into bending- or stretching-dominated categories [37, 73] on the basis of their rigidity. A bending-dominated lattice responds to external loads by cell-wall bending, whereas a stretching-dominated lattice deforms predominantly by stretching. Bending-dominated lattices are less stiff and strong than stretching-dominated lattices, for the same porosity or relative density. Here, relative density, $\overline{\rho}$, is defined as the non-dimensional ratio of the density of the lattice material to the density of the solid. A low value of $\overline{\rho}$ indicates high porosity and $\overline{\rho} = 1$ indicates zero porosity. Thus it is important to identify whether a given lattice is bending- or stretching-dominated.

Maxwell's rule for simply stiff frames [74] provides a rigorous mathematical framework to decide if a given lattice is simply stiff or not. According to Maxwell's rule [74], a *finite* freely-supported pin-jointed lattice with b bars and j frictionless joints is simply stiff provided $b = 2j - 3$ in 2D and $b = 3j - 6$ in 3D. Here, a simply stiff lattice is defined to be both statically and kinematically determinate. Static determinacy implies that all bar tensions due to external forces can be computed from the available number of independent equilibrium equations. States of self-stress are present in statically indeterminate lattices. Similarly, kinematic determinacy implies that joint locations are uniquely determined by the individual lengths of the bars. Mechanisms are present in kinematically indeterminate lattices, such as in a pin-jointed square lattice. Thus it follows from Maxwell's rule that a lattice having j joints requires $3j - 6$ bars in 3D to render it simply stiff. If the bars are fewer, mechanisms exist rendering the lattice kinematically indeterminate. Likewise, if the number of bars exceeds $3j - 6$, states of self-stress exist rendering the lattice statically indeterminate.

Maxwell's rule is a necessary condition. Exceptions to these necessary conditions were recognized by Maxwell in his original work [74, 75] and are put on a firm footing by the generalized Maxwell's rule derived by Pellegrino and Calladine using matrix methods [76]. For example, certain tensegrity structures have fewer bars than are necessary to satisfy Maxwell's rule, and yet are not mechanisms. Their stiffness has been anticipated to be of lower order by Maxwell. Such exceptional cases permit at least one state of self-stress that offers first-order stiffness to one or more infinitesimal mechanisms [75]. This fact has been exploited by Buckminster Fuller in some of his tensegrity structures. Not surprisingly, exceptions to Maxwell's rule occur in biological fibrous structures as well [77, 78].

The generalized Maxwell's rule, derived using matrix algebra [75, 76], is $b - 2j + 3 = s - m$ for 2D lattices, and $b - 3j + 6 = s - m$ for 3D lattices, where s and m are the number of states of self-stress and mechanisms. We note that for a special class of lattices with similarly situated nodes (the lattice appears the same when viewed from any node), necessary and sufficient conditions for rigidity have been shown to be $Z = 6$ for planar lattices and $Z = 12$ for 3D lattices, where Z is the nodal connectivity, defined to be the number of bars emanating from a joint [73]. Finally, it has been shown that an *infinite* lattice cannot be simultaneously statically and kinematically determinate [79].

To conclude the discussion on Maxwell's rule, consider the planar Kagome or triangular-hexagonal lattice. It achieves the Hashin–Shtrikman upper bound for effective elastic properties. It has four bars at every joint and hence $Z = 4$. It still exhibits stretching-dominated deformation behaviour under macroscopic loading, and its effective in-plane moduli are linearly proportional to the relative density. This is due to the presence of periodic collapse mechanisms that produce no macroscopic strain [80]. However, in the presence of imperfections, a Kagame lattice exhibits large regions of bending-dominated boundary layers emanating from the edges and interfaces [35, 81]. Such boundary layers, with bending-dominated deformation, have been shown to enhance fracture toughness [82] and hence flaw-tolerance.

1.2.4 Manufacturing Methods

Advances in manufacturing techniques are central to the development of lattice materials, particularly those with complex features of the unit cell. A truss is a natural choice of construction that uses minimum material to fill maximum space, without sacrificing the stiffness and strength requirements. However, manufacturing trusses at mesoscales, with strut lengths of the order of millimetres and diameters of the order of micrometers, is a non-trivial challenge. Several advanced manufacturing techniques have emerged from the microelectronics industry [83] and traditional metal and composite manufacturing industries [84]. Methods for manufacturing metallic lattices include sheet forming, wire assembly, perforated sheet folding/drawing, investment casting, and wire assembly; a comprehensive overview of different techniques can be found in the literature [84–86]. While planar lattices are achievable using microfabrication techniques such as LIGA (LIthographie, Galvano, and Abformung or lithography, electrodeposition, and molding), achieving three-dimensionality is a challenge. 3D lattices can be made either through layer-by-layer addition or by using serial techniques, such as laser micromachining, that carve microstructures from solid objects or “write” 3D microstructures. A small, lightweight, space-filling truss structure is fabricated using soft lithography (rapid prototyping and micro contact printing) in combination with electrodeposition [87]. Three-dimensionality is achieved by folding a 2D silver grid at a specific angle using brass dies, and then assembling to create a 3D silver template. Electrodeposition of nickel on the silver template joins the three layers, two planar lattices, and the truss lattice core grids, and strengthens the overall 3D truss system.

More recently, ultralight ($<10\,mg/cm^3$) hierarchical metallic microlattices with hollow struts have been fabricated, starting with a template formed by self-propagating photopolymer waveguide prototyping, coating the template using electroless nickel plating, and subsequently etching away the template [50]. Three levels of hierarchy and associated length scales are present: unit cell (mm to cm), hollow tube lattice member (mm), and hollow tube wall (nm to mm). Exceptional control over the design and properties of the resulting microlattice is possible due to independent control of each architectural element. Ceramic scaffolds that mimic the length scales and hierarchy of biological materials have been demonstrated [88], leading to the design of biologically inspired hierarchical damage-tolerant engineering materials. Hollow-tube alumina nanolattices have been fabricated using two-photon lithography, atomic layer deposition, and oxygen plasma etching [89]. Nanocrystalline hybrid lattices are shown to have exceptional stiffness and strength properties [69]. Furthermore, advances in 3D printing have been utilized in fabricating hierarchical lattices with fractral microstructures [90] and lattice-based materials with continuously variable mechanical properties [91, 92]. These ongoing developments have prompted studies that consider integration of 3D printing into the design process [93]. This represents a promising research direction for realizing lattice materials and structures with both good manufacturability and performance attributes. In summary, rapid advances in micro and nano manufacturing technologies are enabling the fabrication and exploitation of truss lattice materials at multiple length scales and will continue to open new application areas for lattice materials.

1.2.5 Applications

Applications of lattice materials and structures span several areas of technological interest. This section provides a brief sketch of the rapidly expanding applications base of lattice materials and structures.

Lightweight structures are in high demand in the aerospace, automotive, marine, and other industries. The employment of lattice materials and structures in such multifunctional structural applications [35, 94, 95] is possible due to the open-core architecture that truss lattices provide when used as a core in sandwich structures. Effective thermoelastic stiffness and strength properties of lattices are topology-dependent, stretching-dominated lattices being the stiffest for a given relative density. For example, in Figure 1.3, the in-plane Young's modulus and shear modulus are compared for different planar lattices for a fixed relative density. It can be seen that the symmetry of the lattice topology governs whether a given lattice is isotropic or anisotropic. Further, it is possible to tailor effective properties such as the Poisson ratio with both extreme positive and negative values [46, 96–98]. Lattice morphologies with low coefficients of thermal expansion (CTE) have been studied for aerospace applications [99–101].

Lattices naturally act as filters for mechanical waves in both the frequency (temporal) and wavenumber (spatial) domains due to their spatial periodicity and the induced scattering at unit-cell interfaces [15, 18, 102–104]. Other notable engineering applications employing lattices include reconfigurable lattices [105, 106] and morphing/shape changing structures achieved with inclusion of actuators within a lattice [107–110]. Micro-truss lattice architectures have been used in nano-engineered composites to simultaneously obtain high stiffness and damping [70]. A judicious combination of

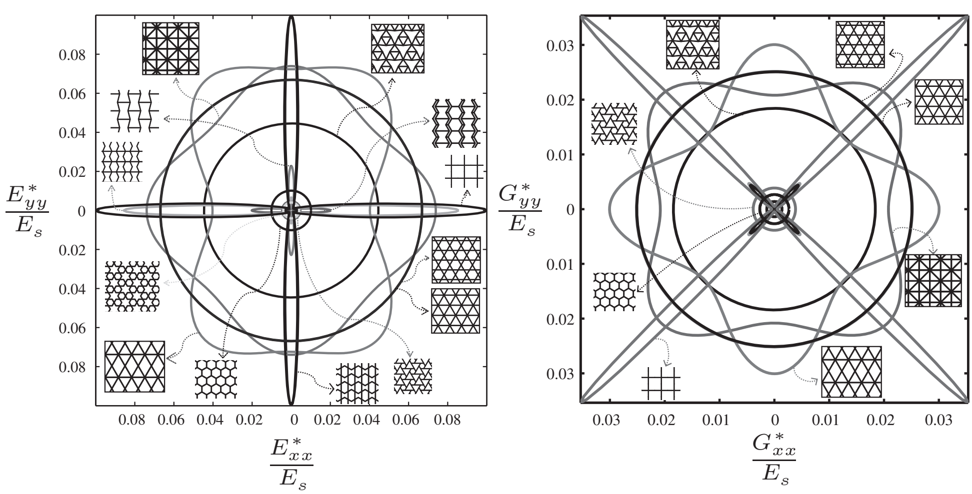


Figure 1.3 Unit-cell geometry-dependent effective in-plane elastic moduli of planar lattices for the same mass: Young's modulus tensor component E^*_{ii} (left) and shear modulus tensor component G^*_{ii} (right) along the Cartesian x (positive to the right) and y (positive upwards) axes. The Young's modulus of the isotropic parent solid is denoted E_s.

carbon nanotube engineered trusses held in a dissipative polymer has been used to design a composite material that simultaneously exhibits both high stiffness and damping, quite high in any single monolithic material. Nano-engineered lattice materials hold significant promise for future structural applications.

Truss lattice materials are also being used in energy-absorption applications, notably the design of impact- and blast-tolerant structures [111–117]. It has been shown that sandwich structures with lattice or foam cores perform better than monolithic plates of the same mass. Fluid–structure interactions in water blast have been shown to significantly enhance the relative performance, due to a reduction in the momentum acquired by the sandwich plate. This enhancement is more pronounced for structures subject to explosive loading in water than in air.

More recently, periodic-architectured biomedical implants have emerged [118]. Various geometrical arrays of cellular, reticulated mesh, and open-cell foams with interconnected porosities have been manufactured in complex, functional, monolithic structures. These complex arrays have the potential for unique bone compatibility as well as the accommodation of more natural bone tissue ingrowth, including vascular system development. Cardiovascular stents also utilize lattice structures. Shape-optimization methods have been developed to design a stress concentration-free lattice for self-expandable Nitinol stent grafts [119]. A systematic study of the influence of cell geometry on expansion characteristics has been reported [120, 121]. Regular and auxetic lattice geometries have been combined to obtain a stent with zero net foreshortening when subjected to large radial plastic deformation during its expansion inside a human artery. These selected examples illustrate the potential for lattice structures in the field of biomedical implant design, including tissue-engineering scaffolds.

Phononic crystals and acoustic/elastic metamaterials [15, 16, 32] add a whole range of applications, as listed in Section 1.1. The close connection between lattice materials/structures and phononic crystals/metamaterials stems mostly from the utilization of periodicity, which is a common theme in both groups and itself draws inspiration (and some analysis tools) from the vast literature on crystalline materials [1, 2].

1.3 Overview of Chapters

The dynamic response, both linear and nonlinear, of lattice materials is the central theme of this book, and where appropriate a structural perspective is also provided. The book is divided into twelve chapters, each dealing with a specific aspect of lattice materials. We begin with the elastostatic response of lattice materials in Chapter 2, with emphasis on homogenization methods. Homogenization methods provide an effective continuum description of lattices, with consequent analytical and computational advantages. Chapter 3 deals with the elastodynamic response of 1D and 2D lattice materials. Of particular interest are elastic wave propagation and related phenomena that are unique to periodic structures. Progressing from 1D lattice configurations, the chapter ends with 2D lattice materials and establishes relevant connections between the solid-state physics and structural dynamics literature along the way. Dissipation-driven phenomena are described in Chapter 4 using state-space transformations, quadratic eigenvalue analysis, and the Bloch–Rayleigh perturbation method. Attention is given to the effects of damping on dispersion relations, considering free waves and frequency-driven waves. Of particular interest is how dissipation alters the wave motion anisotropies stemming from Bragg scattering. Chapter 5 explores weakly nonlinear lattices and their effective exploitation and tailoring for novel functionalities. Analysis tools are described and results are presented from example studies that highlight the potential for exploiting nonlinearity in the design of materials, systems, and devices. It emerges that nonlinear lattices promise to provide a rich platform for developing tunable and controllable phononic devices.

Mechanical instabilities in periodic porous elastic structures may lead to the formation of homogeneous patterns, opening avenues for a wide range of applications that are related to the geometry of the system. Chapter 6 shows how instabilities via buckling can be used to tune the propagation of elastic waves through a square lattice of elastic beams under equibiaxial compression, enhancing the tunability of its dynamic response. Impact and blast responses of body-centered-cubic lattice structures and sandwich structures is assessed in Chapter 7. Tests over a range of strain rates have shown that the lattice structures are rate-sensitive, with the plateau stress increasing by over 20% over roughly six orders of strain rate. A post-test analysis of the samples indicates that the failure mechanisms did not change with strain rate, for a given lattice architecture. Dynamic tests on sandwich panels with carbon-fibre-reinforced polymer skins have shown that the outer composite layers serve to constrain deformation, thereby improving the properties of the lattice structure.

Static and dynamic properties of continuous pentamode materials are reviewed in Chapter 8. Pentamode materials (PM) possess only one stiffness mode. This is analogous to a liquid, which can easily shear but resists compression, but the stiffness of the junctions in pentamode lattice structures ensures small but finite rigidity and hence structural stability. The effective moduli of PMs are determined for periodic lattice structures using simple beam theory for unit cells with coordination number $d + 1$, where $d = 2, 3$ is the spatial dimension. The main result is an explicit relation for the quasi-static stiffness $\boldsymbol{C}$. Example applications to specific lattice microstructures illustrate both isotropic and anisotropic PMs.

Models of lattice unit cells can be rather sizable, especially for 3D lattices. Large computational resources are needed to obtain band structures and other dynamical properties. This problem is even more significant when numerous repetitions of the calculations are needed, such as in parametric sweeps and unit-cell topology optimization. Chapter 9 introduces two reduced-order modeling techniques for band structure calculations and discusses the trade-offs between computational savings and the accuracy of the predictions for each technique. Chapter 10 extends the theme of rapid lattice-material band structure calculations and examines the topic of unit-cell design and optimization. A genetic algorithm uniquely tailored for lattice-material band structures is provided and an optimized 2D lattice topology is reported.

Chapter 11 investigates lattice materials involving antiresonances in the form of local resonances or inertial amplification. A detailed mathematical treatment for obtaining dispersion curves for locally resonant and inertially amplified lattices is provided. The dynamical response of these systems, and how they contrast with each other, is described and discussed using 1D, 2D, and 3D lattice examples considering both infinite and finite systems. Chapter 12 focuses on the nanotruss lattices. These nanostructures are fabricated using polymer 3D printing – convenient for generation of complex geometries – enhanced with electrodeposition of nanocrystalline metals for high strength. Being hybrid polymer–nanometal structures, they have excellent mechanical properties for their mass when optimally designed. Because of the nearly limitless geometric flexibility made available through 3D printing, such lattices can be designed to achieve additional functional goals. This chapter examines the use of polymer–nanometal hybrids in conditions where wave propagation is significant. Concepts for linking Floquet–Bloch analysis to the design of nanolattices with desirable properties are outlined.

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References

1 L. Brillouin, *Wave Propagation in Periodic Structures*, 2nd edn. Dover Publications, 1953.
2 C. Kittel, *Introduction to Solid State Physics*, 8th edn. Wiley Publishers, 2004.
3 S. Iijima, "Helical microtubules of graphitic carbon," *Nature*, vol. 354, pp. 56–58, 1991.
4 K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, Y. Zhang, S. V. Dubonos, I. V. Grigorieva, and A. A. Firsov, "Electric field effect in atomically thin carbon films," *Science*, vol. 306, pp. 666–669, 2004.
5 I. El-Kady, R. H. Olsson, and J. G. Fleming, "Phononic band-gap crystals for radio frequency communications," *Applied Physics Letters*, vol. 92, p. 233504, 2008.
6 J. Vincent, *Structural Biomaterials*, 3rd edn. Princeton University Press, 2012.
7 L. Gibson, M. Ashby, and B. Harley, *Cellular Materials in Nature and Medicine*. Cambridge University Press, 2010.
8 C. T. Sun, J. D. Achenbach, and G. Herrmann, "Time-harmonic waves in a stratified medium propagating in the direction of the layering," *Journal of Applied Mechanics-Transactions of the ASME*, vol. 35, pp. 408–411, 1968.
9 R. Esquivel-Sirvent and G. Cocoletzi, "Band-structure for the propagation of elastic-waves in superlattices," *Journal of the Acoustical Society of America*, vol. 95, pp. 86–90, 1994.
10 D. J. Ewins, "Vibration characteristics of bladed disk assemblies," *Journal of Mechanical Engineering Science*, vol. 15, pp. 165–186, 1973.
11 M. P. Castanier, G. Ottarsson, and C. Pierre, "A reduced order modeling technique for mistuned bladed disks," *Journal of Vibration and Acoustics-Transactions of the ASME*, vol. 119, pp. 439–447, 1997.
12 M. Brun, A. B. Movchan, and I. S. Jones, "Phononic band gap systems in structural mechanics: Finite slender elastic structures and infinite periodic waveguides," *Journal of Vibration and Acoustics-Transactions of the ASME*, vol. 135, p. 041013, 2013.

13 A. L. Abrahamson, *The response of periodic structures to aero-acoustic pressures with particular reference to aircraft skin-rib-spar structures*. PhD Thesis, University of Southampton, 1973.

14 L. Gibson and M. Ashby, *Cellular Solids: Structure and Properties*, 2nd edn. Cambridge Solid State Science Series, Cambridge University Press, 1999.

15 D. Mead, "Wave propagation in continuous periodic structures: Research contributions from Southampton 1964–1995," *Journal of Sound and Vibration*, vol. 190, no. 3, pp. 495–524, 1996.

16 M. I. Hussein, M. J. Leamy, and M. Ruzzene, "Dynamics of phononic materials and structures: Historical origins, recent progress, and future outlook," *Applied Mechanics Reviews*, vol. 66, no. 4, pp. 040802–040802, 2014.

17 M. M. Sigalas and E. N. Economou, "Elastic and acoustic wave band structure," *Journal of Sound and Vibration*, vol. 158, no. 2, pp. 377–382, 1992.

18 M. S. Kushwaha, P. Halevi, L. Dobrzynski, and B. Djafari-Rouhani, "Acoustic band structure of periodic elastic composites," *Physical Review Letters*, vol. 71, pp. 2022–2025, 1993.

19 E. Yablonovitch, "Inhibited spontaneous emission in solid-state physics and electronic," *Physical Review Letters*, vol. 58, pp. 2059–2062, 1987.

20 S. John, "Strong localization of photons in certain disordered dielectric superlattices," *Physical Review Letters*, vol. 58, pp. 2486–2489, 1987.

21 M. I. Hussein, G. M. Hamza, Hulbert, R. A. Scott, and K. Saitou, "Multiobjetive evolutionary optimization of periodic layered materials for desired wave dispersion characteristics," *Structural and Multidisciplinary Optimization*, vol. 31, pp. 60–75, 2006.

22 M. M. Sigalas, "Defect states of acoustic waves in a two-dimensional lattice of solid cylinders," *Journal of Applied Physics*, vol. 84, pp. 2026–3030, 1998.

23 R. Lucklum and J. Li, "Phononic crystals for liquid sensor applications," *Measurement Science and Technology*, vol. 20, p. 124014, 2009.

24 L. S. Chen, C. H. Kuo, and Z. Ye, "Acoustic imaging and collimating by slabs of sonic crystals made from arrays of rigid cylinders in air," *Applied Physics Letters*, vol. 85, pp. 1072–1074, 2004.

25 M. Carrara, M. R. Cacan, M. J. Leamy, M. Ruzzene, and A. Erturk, "Dramatic enhancement of structure-borne wave energy harvesting using an elliptical acoustic mirror," *Applied Physics Letters*, vol. 100, no. 20, 2012.

26 J. K. Yu, S. Mitrovic, D. Tham, J. Varghese, and J. R. Heath, "Reduction of thermal conductivity in phononic nanomesh structures," *Nature Nanotechnology*, vol. 5, pp. 718–721, 2010.

27 M. I. Hussein, S. Biringen, O. R. Bilal, and A. Kucala, "Flow stabilization by subsurface phonons," *Proceedings of the Royal Society A*, vol. 471, p. 20140928, 2015.

28 Z. Y. Liu, X. X. Zhang, Y. W. Mao, Y. Y. Zhu, Z. Y. Yang, C. T. Chan, and P. Sheng, "Locally resonant sonic materials," *Science*, vol. 289, no. 5485, pp. 1734–1736, 2000.

29 Y. Pennec, B. Djafari-Rouhani, H. Larabi, J. O. Vasseur, and A.-C. Ladky-Hennion, "Low-frequency gaps in a phononic crystal constituted of cylindrical dots deposited on a thin homogeneous plate," *Physical Review B*, vol. 78, p. 104105, 2008.

30 T. T. Wu, Z. G. Huang, T. C. Tsai, and T. C. Wu, "Evidence of complete band gap and resonances in a plate with periodic stubbed surface," *Applied Physics Letters*, vol. 93, p. 111902, 2008.

31 B. L. Davis and M. I. Hussein, "Nanophononic metamaterial: Thermal conductivity reduction by local resonance," *Physical Review Letters*, vol. 112, p. 055505, 2014.

32 P. A. Deymier, *Acoustic Metamaterials and Phononic Crystals*. Springer, 2013.

33 M. I. Hussein and I. El-Kady, "Preface to special topic: Selected articles from Phononics 2011: The First International Conference on Phononic Crystals, Metamaterials and Optomechanics, 29 May–2 June, 2011, Santa Fe, NM," *AIP Advances*, vol. 1, p. 041301, 2011.

34 M. I. Hussein, I. El-Kady, B. Li, and J. Sánchez-Dehesa, "Preface to special topic: Selected articles from Phononics 2013: The Second International Conference on Phononic Crystals/Metamaterials, Phonon Transport and Optomechanics, 2–7 June, 2013, Sharm El-Sheikh, Egypt," *AIP Advances*, vol. 4, p. 124101, 2014.

35 N. A. Fleck, V. S. Deshpande, and M. F. Ashby, "Micro-architectured materials: past, present and future," *Proceedings of the Royal Society A*, vol. 466, no. 2121, pp. 2495–2516, 2010.

36 M. F. Ashby, "Hybrids to fill holes in material property space," *Philosophical Magazine*, vol. 85, no. 26-27, pp. 3235–3257, 2005.

37 M. F. Ashby, "The properties of foams and lattices," *Philosophical Transactions of the Royal Society: Mathematical, Physical and Engineering Sciences*, vol. 364, no. 1838, pp. 15–30, 2006.

38 D. Mead, "Wave propagation and natural modes in periodic systems: I. Mono-coupled systems," *Journal of Sound and Vibration*, vol. 40, pp. 1–18, 1975.

39 M. I. Hussein, G. M. Hulbert, and R. A. Scott, "Dispersive elastodynamics of 1D banded materials and structures: Analysis," *Journal of Sound and Vibration*, vol. 289, pp. 779–806, 2006.

40 D. Mead, "Wave propagation and natural modes in periodic systems: II. Multi-coupled systems, with and without damping," *Journal of Sound and Vibration*, vol. 40, pp. 19–39, 1975.

41 L. Raghavan and A. S. Phani, "Local resonance bandgaps in periodic media: Theory and experiment," *Journal of the Acoustical Society of America*, vol. 134, pp. 1950–1959, 2013.

42 J. Banhart, "Light-metal foams – history of innovation and technological challenges," *Advanced Engineering Materials*, vol. 15, no. 3, pp. 82–111, 2013.

43 M. F. Ashby, T. Evans, N. A. Fleck, L. J. Gibson, J. W. Hutchinson, and H. N. G. Wadley, *Metal Foams: A Design Guide*. Butterworth-Heinemann, 2000.

44 R. Lakes, "Materials with structural hierarchy," *Nature*, vol. 361, no. 6412, pp. 511–515, 1993.

45 L. J. Gibson, M. F. Ashby, G. S. Schajer, and C. I. Robertson, "The mechanics of two-dimensional cellular materials," *Proceedings of the Royal Society of London A*, vol. 382, no. 1782, pp. 25–42, 1982.

46 L. J. Gibson and M. F. Ashby, *Cellular Solids: Structure and Properties*, 2nd edn. Cambridge Solid State Science Series, Cambridge University Press, 1997.

47 S. Torquato, L. Gibiansky, M. Silva, and L. Gibson, "Effective mechanical and transport properties of cellular solids," *International Journal of Mechanical Sciences*, vol. 40, no. 1, pp. 71–82, 1998.

48 R. Christensen, "Mechanics of cellular and other low-density materials," *International Journal of Solids and Structures*, vol. 37, no. 1–2, pp. 93–104, 2000.

49 V. S. Deshpande, N. A. Fleck, and M. F. Ashby, "Effective properties of the octet-truss lattice material," *Journal of the Mechanics and Physics of Solids*, vol. 49, pp. 1747–1769, 2001.

50 T. A. Schaedler, A. J. Jacobsen, A. Torrents, A. E. Sorensen, J. Lian, J. R. Greer, L. Valdevit, and W. B. Carter, "Ultralight metallic microlattices," *Science*, vol. 334, pp. 962–965, 2011.

51 X. Zheng, H. Lee, T. H. Weisgraber, M. Shusteff, J. DeOtte, E. B. Duoss, J. D. Kuntz, M. M. Biener, Q. Ge, J. A. Jackson, S. O. Kucheyev, N. X. Fang, and C. M. Spadaccini1, "Ultralight, ultrastiff mechanical metamaterials," *Science*, vol. 344, pp. 1373–1377, 2014.

52 L. R. Meza, A. J. Zelhofer, N. Clarke, A. J. Mateos, D. M. Kochmann, and J. R. Greer, "Resilient 3D hierarchical architected metamaterials," *Proceedings of the National Academy of Sciences*, vol. 112, pp. 11502–11507, 2015.

53 O. Sigmund, "Tailoring materials with prescribed elastic properties," *Mechanics of Materials*, vol. 20, no. 4, pp. 351–368, 1995.

54 A. Evans, J. Hutchinson, N. Fleck, M. Ashby, and H. Wadley, "The topological design of multifunctional cellular metals," *Progress in Materials Science*, vol. 46, no. 3–4, pp. 309–327, 2001.

55 S. Gonella and M. Ruzzene, "Homogenization of vibrating periodic lattice structures," *Applied Mathematical Modelling*, vol. 32, pp. 459–482, 2008.

56 S. Nemat-Nasser, J. R. Willis, A. Srivastava, and A. V. Amirkhizi, "Homogenization of periodic elastic composites and locally resonant sonic materials," *Physical Review B*, vol. 83, p. 104103, 2011.

57 R. P. Moiseyenko and V. Laude, "Material loss influence on the complex band structure and group velocity in phononic crystals," *Physical Review B*, vol. 83, p. 064301, 2011.

58 A. S. Phani and M. I. Hussein, "Analysis of damped Bloch waves by the Rayleigh perturbation method," *Journal of Vibration and Acoustics-Transactions of the ASME*, vol. 135, p. 041014, 2013.

59 M. J. Frazier and M. I. Hussein, "Metadamping: An emergent phenomenon in dissipative metamaterials," *Journal of Sound and Vibration*, vol. 332, pp. 4767–4774, 2013.

60 R. K. Narisetti, M. J. Leamy, and M. Ruzzene, "A perturbation approach for predicting wave propagation in one-dimensional nonlinear periodic structures," *Journal of Vibration and Acoustics-Transactions of the ASME*, vol. 132, p. 031001, 2010.

61 R. Khajehtourian and M. I. Hussein, "Dispersion characteristics of a nonlinear elastic metamaterial," *AIP Advances*, vol. 4, p. 124308, 2014.

62 B. Yousefzade and A. S. Phani, "Energy transmission in finite dissipative nonlinear periodic structures from excitation within a stop band," *Journal of Sound and Vibration*, vol. 354, pp. 180–195, 2015.

63 M. Ruzzene and F. Scarpa, “Directional and band-gap behavior of periodic auxetic lattices,” *Physica Status Solidi (B)*, vol. 242, pp. 665–680, 2005.

64 O. Sigmund and J. Søndergaard Jensen, “Systematic design of phononic band gap materials and structures by topology optimization,” *Philosophical Transactions of the Royal Society of London A*, vol. 361, no. 1806, pp. 1001–1019, 2003.

65 O. R. Bilal and M. I. Hussein, “Ultrawide phononic band gap for combined in-plane and out-of-plane waves,” *Physical Review E*, vol. 84, p. 065701, 2011.

66 K. Bertoldi and M. C. Boyce, “Mechanically triggered transformations of phononic band gaps in periodic elastomeric structures,” *Physical Review B*, vol. 77, p. 052105, 2008.

67 D. Torrent and J. Sánchez-Dehesa, “Acoustic cloaking in two dimensions: a feasible approach,” *New Journal of Physics*, vol. 10, p. 063015, 2008.

68 A. N. Norris, “Acoustic cloaking theory,” *Proceedings of the Royal Society of London A*, vol. 464, pp. 2411–2434, 2008.

69 B. Bouwhuis, J. McCrea, G. Palumbo, and G. Hibbard, “Mechanical properties of hybrid nanocrystalline metal foams,” *Acta Materialia*, vol. 57, no. 14, pp. 4046–4053, 2009.

70 J. Meaud, T. Sain, B. Yeom, S. J. Park, A. B. Shoultz, G. Hulbert, Z.-D. Ma, N. A. Kotov, A. J. Hart, E. M. Arruda, and A. M. Waas, “Simultaneously high stiffness and damping in nanoengineered microtruss composites,” *ACS Nano*, vol. 8, no. 4, pp. 3468–3475, 2014. PMID: 24620996.

71 C. A. Steeves and A. G. Evans, “Optimization of thermal protection systems utilizing sandwich structures with low coefficient of thermal expansion lattice hot faces,” *Journal of the American Ceramic Society*, vol. 94, pp. s55–s61, 2011.

72 H. Cundy and A. Rollett, *Mathematical Models*. Tarquin Publications, 1981.

73 V. Deshpande, M. Ashby, and N. Fleck, “Foam topology: bending versus stretching dominated architectures,” *Acta Materialia*, vol. 49, no. 6, pp. 1035–1040, 2001.

74 J. C. Maxwell, “On the calculation of the equilibrium and stiffness of frames,” *Philosophical Magazine Series 6*, vol. 27, pp. 294–299, 1864.

75 C. Calladine, “Buckminster Fuller’s ‘tensegrity’ structures and Clerk Maxwell’s rules for the construction of stiff frames,” *International Journal of Solids and Structures*, vol. 14, no. 2, pp. 161–172, 1978.

76 S. Pellegrino and C. Calladine, “Matrix analysis of statically and kinematically indeterminate frameworks,” *International Journal of Solids and Structures*, vol. 22, no. 4, pp. 409–428, 1986.

77 C. P. Broedersz, X. Mao, T. C. Lubensky, and F. C. MacKintosh, “Criticality and isostaticity in fibre networks,” *Nature Physics*, vol. 7, no. 12, pp. 983–988, 2011.

78 E. van der Giessen, “Materials physics: Bending Maxwell’s rule,” *Nature Physics*, vol. 7, no. 12, pp. 923–924, 2011.

79 S. Guest and J. Hutchinson, “On the determinacy of repetitive structures,” *Journal of the Mechanics and Physics of Solids*, vol. 51, no. 3, pp. 383–391, 2003.

80 R. Hutchinson and N. Fleck, “The structural performance of the periodic truss,” *Journal of the Mechanics and Physics of Solids*, vol. 54, no. 4, pp. 756–782, 2006.

81 A. S. Phani and N. A. Fleck, “Elastic boundary layers in two-dimensional isotropic lattices,” *Journal of Applied Mechanics*, vol. 75, p. 021020, 2008.

82 N. A. Fleck and X. Qiu, “The damage tolerance of elastic-brittle, two-dimensional isotropic lattices,” *Journal of the Mechanics and Physics of Solids*, vol. 55, no. 3, pp. 562–588, 2007.

83 M. Madou, *Fundamentals of Microfabrication: The Science of Miniaturization*, 2nd edn. Taylor & Francis, 2002.

84 H. Wadley, “Cellular metals manufacturing,” *Advanced Engineering Materials*, vol. 4, no. 10, pp. 726–733, 2002.

85 H. N. Wadley, “Multifunctional periodic cellular metals,” *Philosophical Transactions of the Royal Society of London A*, vol. 364, no. 1838, pp. 31–68, 2006.

86 K.-J. Kang, “Wire-woven cellular metals: The present and future,” *Progress in Materials Science*, vol. 69, no. 0, pp. 213–307, 2015.

87 S. T. Brittain, Y. Sugimura, O. J. Schueller, A. Evans, and G. M. Whitesides, “Fabrication and mechanical performance of a mesoscale space-filling truss system,” *Journal of Microelectromechanical Systems*, vol. 10, no. 1, pp. 113–120, 2001.

88 D. Jang, L. R. Meza, F. Greer, and J. R. Greer, “Fabrication and deformation of three-dimensional hollow ceramic nanostructures,” *Nature Materials*, vol. 12, no. 10, pp. 893–898, 2013.

89 L. R. Meza, S. Das, and J. R. Greer, “Strong, lightweight, and recoverable three-dimensional ceramic nanolattices,” *Science*, vol. 345, no. 6202, pp. 1322–1326, 2014.

90 R. Oftadeh, B. Haghpanah, D. Vella, A. Boudaoud, and A. Vaziri, “Optimal fractal-like hierarchical honeycombs,” *Physical Review Letters*, vol. 113, p. 104301, 2014.

91 P. S. Chang and D. W. Rosen, “The size matching and scaling method: A synthesis method for the design of mesoscale cellular structures,” *International Journal of Computer Integrated Manufacturing*, vol. 26, pp. 907–927, 2013.

92 T. Stankovic, J. Mueller, P. Egan, and K. Shea, “A generalized optimality criteria method for optimization of additively manufactured multimaterial lattice structures,” *Journal of Mechanical Design-Transactions of the ASME*, vol. 137, p. 111405, 2015.

93 J. Mueller, K. Shea, and C. Daraio, “Mechanical properties of parts fabricated with inkjet 3D printing through efficient experimental design,” *Materials and Design*, vol. 86, pp. 902–912, 2015.

94 N. Wicks and J. W. Hutchinson, “Optimal truss plates,” *International Journal of Solids and Structures*, vol. 38, pp. 5165–5183, 2001.

95 A. Evans, J. Hutchinson, and M. Ashby, “Multifunctionality of cellular metal systems,” *Progress in Materials Science*, vol. 43, no. 3, pp. 171–221, 1998.

96 R. Lakes, “Foam structures with a negative Poisson’s ratio,” *Science*, vol. 235, no. 4792, pp. 1038–1040, 1987.

97 K. E. Evans and A. Alderson, “Auxetic materials: Functional materials and structures from lateral thinking!,” *Advanced Materials*, vol. 12, no. 9, pp. 617–628, 2000.

98 G. W. Milton, “Composite materials with Poisson’s ratios close to −1,” *Journal of the Mechanics and Physics of Solids*, vol. 40, no. 5, pp. 1105–1137, 1992.

99 R. Lakes, “Cellular solid structures with unbounded thermal expansion,” *Journal of Materials Science Letters*, vol. 15, no. 6, pp. 475–477, 1996.

100 C. A. Steeves, S. L. dos Santos e Lucato, M. He, E. Antinucci, J. W. Hutchinson, and A. G. Evans, “Concepts for structurally robust materials that combine low thermal expansion with high stiffness,” *Journal of the Mechanics and Physics of Solids*, vol. 55, no. 9, pp. 1803–1822, 2007.

101 C. Steeves, C. Mercer, E. Antinucci, M. He, and A. Evans, “Experimental investigation of the thermal properties of tailored expansion lattices,” *International Journal of Mechanics and Materials in Design*, vol. 5, no. 2, pp. 195–202, 2009.

102 R. Langley, N. Bardell, and H. M. Ruivo, “The response of two-dimensional periodic structures to harmonic point loading: A theoretical and experimental study of a beam grillage,” *Journal of Sound and Vibration*, vol. 207, no. 4, pp. 521–535, 1997.

103 A. S. Phani, J. Woodhouse, and N. A. Fleck, “Wave propagation in two-dimensional periodic lattices,” *The Journal of the Acoustical Society of America*, vol. 119, no. 4, pp. 1995–2005, 2006.

104 M. Ruzzene, F. Scarpa, and F. Soranna, “Wave beaming effects in two-dimensional cellular structures,” *Smart Materials and Structures*, vol. 12, no. 3, p. 363, 2003.

105 J. Shim, S. Shan, A. Kosmrlj, S. Kang, E. Chen, J. Weaver, and K. Bertoldi, “Harnessing instabilities for design of soft reconfigurable auxetic/chiral materials,” *Soft Matter*, vol. 9, pp. 8198–8202, 2013.

106 A. Q. Liu, W. M. Zhu, D. P. Tsai, and N. I. Zheludev, “Micromachined tunable metamaterials: a review,” *Journal of Optics*, vol. 14, no. 11, p. 114009, 2012.

107 S. dos Santos e Lucato, J. Wang, P. Maxwell, R. McMeeking, and A. Evans, “Design and demonstration of a high authority shape morphing structure,” *International Journal of Solids and Structures*, vol. 41, no. 13, pp. 3521–3543, 2004.

108 S. Mai and N. Fleck, “Reticulated tubes: effective elastic properties and actuation response,” *Proceedings of the Royal Society of London A*, vol. 465, no. 2103, pp. 685–708, 2009.

109 N. Wicks and S. Guest, “Single member actuation in large repetitive truss structures,” *International Journal of Solids and Structures*, vol. 41, no. 3–4, pp. 965–978, 2004.

110 R. Hutchinson, N. Wicks, A. Evans, N. Fleck, and J. Hutchinson, “Kagome plate structures for actuation,” *International Journal of Solids and Structures*, vol. 40, no. 25, pp. 6969–6980, 2003.

111 N. A. Fleck and V. S. Deshpande, “The resistance of clamped sandwich beams to shock loading,” *Journal of Applied Mechanics*, vol. 71, no. 3, pp. 386–401, 2004.

112 J. W. Hutchinson and Z. Xue, “Metal sandwich plates optimized for pressure impulses,” *International Journal of Mechanical Sciences*, vol. 47, no. 4–5, pp. 545–569, 2005.

113 K. P. Dharmasena, H. N. Wadley, Z. Xue, and J. W. Hutchinson, “Mechanical response of metallic honeycomb sandwich panel structures to high-intensity dynamic loading,” *International Journal of Impact Engineering*, vol. 35, no. 9, pp. 1063–1074, 2008.

114 Z. Xue and J. W. Hutchinson, “A comparative study of impulse-resistant metal sandwich plates,” *International Journal of Impact Engineering*, vol. 30, no. 10, pp. 1283–1305, 2004.

115 N. Kambouchev, R. Radovitzky, and L. Noels, "Fluid–structure interaction effects in the dynamic response of free-standing plates to uniform shock loading," *Journal of Applied Mechanics*, vol. 74, no. 5, pp. 1042–1045, 2006.
116 J. Harrigan, S. Reid, and A. S. Yaghoubi, "The correct analysis of shocks in a cellular material," *International Journal of Impact Engineering*, vol. 37, no. 8, pp. 918–927, 2010.
117 A. Evans, M. He, V. Deshpande, J. Hutchinson, A. Jacobsen, and W. Carter, "Concepts for enhanced energy absorption using hollow micro-lattices," *International Journal of Impact Engineering*, vol. 37, no. 9, pp. 947–959, 2010.
118 L. E. Murr, S. M. Gaytan, F. Medina, H. Lopez, E. Martinez, B. I. Machado, D. H. Hernandez, L. Martinez, M. I. Lopez, R. B. Wicker, and J. Bracke, "Next-generation biomedical implants using additive manufacturing of complex, cellular and functional mesh arrays," *Philosophical Transactions of the Royal Society of London A*, vol. 368, no. 1917, pp. 1999–2032, 2010.
119 E. M. K. Abad, D. Pasini, and R. Cecere, "Shape optimization of stress concentration-free lattice for self-expandable nitinol stent-grafts," *Journal of Biomechanics*, vol. 45, no. 6, pp. 1028–1035, 2012.
120 G. R. Douglas, A. S. Phani, and J. Gagnon, "Analyses and design of expansion mechanisms of balloon expandable vascular stents," *Journal of Biomechanics*, vol. 47, no. 6, pp. 1438–1446, 2014.
121 T. W. Tan, G. R. Douglas, T. Bond, and A. S. Phani, "Compliance and longitudinal strain of cardiovascular stents: Influence of cell geometry," *Journal of Medical Devices*, vol. 5, no. 4, pp. 041002–041002, 2011.