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Preliminaries

Here we briefly review some minimal background knowledge that we will assume in the rest of the book. Besides a small amount of that nebulous quality called "mathematical maturity", we only expect some basic concepts from set theory and mathematical indiction. The reader who is familiar with these concepts can safely skip on to the next chapter.

Some notation

We denote the set of natural numbers $\{0, 1, 2, \ldots\}$ by $\mathbb{N}$. There is some inconsistency in the mathematical literature as to whether 0 belongs to the natural numbers or not: some authors choose to include it, other do not. For our purposes it is convenient to include 0 as a natural number. Other number sets which will be of importance to us include the sets of integers $\mathbb{Z}$, positive integers $\mathbb{Z}^+$, rational numbers $\mathbb{Q}$, positive rational numbers $\mathbb{Q}^+$, real numbers $\mathbb{R}$, and positive real numbers $\mathbb{R}^+$.

The product $1 \times 2 \times 3 \times \cdots \times n$ of the first n positive integers turns up in many mathematical situations. It is therefore convenient to have a more compact notation for it. We accordingly define $0! = 1$ and $n! = 1 \times 2 \times 3 \times \cdots \times n$, for $n \geq 1$. We read $n!$ as 'n factorial'. The definition of $0!$ as 1 is not supposed to carry any intuitive meaning: it is simply a useful convention.

Logic and Discrete Mathematics: A Concise Introduction, First Edition. Willem Conradie and Valentin Goranko.

Companion Website: www.wiley.com/go/conradie/logic

1.1. Sets

Sets and elements. By a **set** we intuitively mean a collection of objects of any nature (numbers, people, concepts, sets themselves, etc.) that is considered as a single entity. The objects in that collection are called **elements** of the set. If an object x is an element of a set A, we denote that fact by

$$x \in A;$$

otherwise we write

$$x \notin A.$$

We also say that *x is a member of the set A* or that *x belongs to A*. If a set has finitely many elements (here we rely on your intuition of what *finite* is), we can describe it precisely by listing all of them, for example:

$$A = \{3, 4, 5\}.$$

We often rely on our common intuition and use ellipses, as in

$$A = \{1, 2, \ldots, n\}.$$

We sometimes go further and use the same for *infinite* sets; for example, the set of natural numbers can be specified as

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

Further we will discuss a more universal method of describing sets.

Equality and containment of sets. Two sets are declared **equal** if and only if they have the same elements. In other words, the sets A and B are equal, denoted as usual by $A = B$, if every element of A is an element of B and every element of B is an element of A. For example, the sets $\{a, b, c\}$ and $\{b, c, a\}$ are equal, and so are the sets $\{1, 9, 9, 7\}$, $\{1, 9, 7\}$ and $\{7, 1, 9, 1, 7, 1\}$.

A set A is a **subset** of a set B, denoted $A \subseteq B$, if every element of A is an element of B. If $A \subseteq B$, we also say that A **is included in** B, or that B **contains** A. For example, $\{3, 5\} \subseteq \{5, 4, 3\}$. Note that every set is a subset of itself.

The following facts are very useful. They are direct consequences of the definitions of equality and containment of sets.

- Two sets A and B are equal if, and only if, $A \subseteq B$ and $B \subseteq A$.
- A set A is not a subset of a set B, denoted $A \nsubseteq B$, if, and only if, there is an element of A that is not an element of B.
- A set A is not equal to a set B if A is not a subset of B *or* if B is not a subset of A.

A set A is a **proper subset** of a set B, denoted $A \subset B$ or $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$. In other words, A is a proper subset of B if A is a subset of B and B is *not* a subset of A, i.e. if at least one element of B is not in A. In particular, no set is a proper subset of itself. If A is not a proper subset of B, we denote that by $A \not\subset B$.

The empty set. Amongst all sets there is one that is particularly special. That is the **empty set**, i.e. the set that has no elements. By definition of equality of sets, there is only one empty set. One might think that the empty set is a useless abstraction. On the contrary, it is a very important set, and probably the most commonly used one in mathematics (like the number 0 is the most commonly used number). That is why it has a special notation: $\varnothing$.

Sets and properties. Set-builder notation. We cannot always list the elements of a set, even if it is finite, so we need a more universal method for specifying sets. The commonly used method is to *describe the property that determines membership of the set*, e.g.:

"A is the set of all objects x such that $\dots x \dots$"

where "$\dots x \dots$" is a certain property (predicate) involving x. We use the following convenient notation, called the **set-builder notation** for the set described above:

$$A = \{x \mid \dots x \dots\}.$$

Here are some examples:

- $\{x \mid x \textit{ is a negative real number}\}$ defines the set of negative real numbers;
- $\{x \mid x \textit{ is a student in the MATH3029 class}\}$ defines the set of students in the MATH3029 class.
- $\{x \mid x \in \mathbb{Z} \textit{ and } 3 \geq x > -2\}$ defines the set $\{-1, 0, 1, 2, 3\}$.
- $\{x \mid x = \frac{m}{n}, \textit{ where } m \in \mathbb{Z} n \in \mathbb{Z} \textit{ and } n \neq 0\}$ defines the set of rational numbers.

Sometimes, we use the set-builder notation more liberally and, for instance, describe the set of rational numbers as $\left\{\frac{m}{n} \middle| m \text{ and } n \text{ are integers and } n \neq 0\right\}$ or the set of positive real numbers as $\{x \in \mathbb{R} \mid x > 0\}$.

Operations on sets. We describe below the basic operations on sets.

Intersection. The intersection of two sets A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

consisting of all elements that are both in A and in B. If $A \cap B = \varnothing$, then A and B are called **disjoint**.

Union. The union of two sets A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

consisting of all elements that are in at least one of A and B.

Difference. The difference of the sets A and B is the set

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

consisting of all elements that are in A but not in B. An alternative notation for $A - B$ is $A \setminus B$.

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$ then $A \cap B = \{3, 4\}$, $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$, $A - B = \{1, 2\}$ and $B - A = \{5, 6, 7\}$.

Universal sets and complements of sets. Often, all sets that we consider are subsets of one set, called the **domain of discourse**. We also call that set the **universe** or the **universal set**. For example, in arithmetic, the universe is usually the set of natural numbers or the set of integers, while in algebra and calculus, the universe is the set of real numbers; talking about humans, the universe is the set of all humans, etc.

Definition 1.1.1 *Let a universal set* $\mathbf{U}$ *be fixed and* $A \subseteq \mathbf{U}$. *The* complement of A *(with respect to* $\mathbf{U}$*) is the set*

$$A' = \mathbf{U} - A.$$

The complement of a set A *is sometimes also denoted by* $\overline{A}$.

Thus, the complement of A consists of those objects from the universal set that do not belong to A. For example, if the universal set is $\mathbb{R}$, then the complement of the interval $(0, 2]$ is $(-\infty, 0] \cup (2, \infty)$; the complement of $\mathbb{Q}$ is the set of irrational numbers.

Powersets. The *power set of a set* A is the set of all subsets of A:

$$\mathscr{P}(A) = \{X \mid X \subseteq A\}.$$

Here are some examples:

- $\mathscr{P}(\varnothing) = \{\varnothing\}$;
- $\mathscr{P}(\{a\}) = \{\varnothing, \{a\}\}$, in particular, $\mathscr{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$;
- $\mathscr{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$;
- $\mathscr{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Cartesian products of sets. In order to introduce the next operation we need the notion of **ordered pair**. Let a, b be any objects. Intuitively, the ordered pair of a and b, denoted (a, b) (do not confuse this with an open interval of real numbers!), is something like a set consisting of a as a *first element* (or first *component*) and b as a *second element* (or second *component*). Thus, if $a \neq b$, then the ordered pair (a, b) is *different* from the ordered pair (b, a) and each of these is different from the set $\{a, b\}$ because the elements of a set are not ordered. In particular, the ordered pair (a, a) is different from the set $\{a, a\} = \{a\}$. Here is a formal definition of an ordered pair as a set that satisfies the intuition:

Definition 1.1.2 *Given the objects a and b, the* **ordered pair** (a, b) *is the set* $\{\{a\}, \{a, b\}\}$.

Here is the fundamental property of ordered pairs:

Proposition 1.1.3 *The ordered pairs* (a_1, a_2) *and* (b_1, b_2) *are equal if and only if* $a_1 = b_1$ *and* $a_2 = b_2$.

Proof: Exercise.

Definition 1.1.4 *The* **Cartesian product** *of the sets A and B is the set*

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\},$$

consisting of all ordered pairs where the first component comes from A and the second component comes from B. In particular, we denote $A \times A$ *by* A^2 *and call it the* **Cartesian square** *of A.*

For example, if $A = \{a, b\}, B = \{1, 2, 3\}$, then

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\},$$

while

$$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

Note that if A or B is empty, then $A \times B$ is empty too. Moreover, if A has n elements and B has m elements, then $A \times B$ has mn elements (why?). This is one of the reasons for the term "product".

The Cartesian coordinate system in the plane is a representation of the plane as the Cartesian[1] square $\mathbb{R}^2$ of the real line $\mathbb{R}$, where we associate a unique ordered pair of real numbers (its **coordinates**) with every point in the plane.

The notion of an ordered pair can be generalized to **ordered n-tuple**, for any $n \in \mathbb{N}^+$. An n-tuple is an object of the type $(a_1, a_2, \ldots, a_n)$ where the order of the components $a_1, a_2, \ldots, a_n$ matters. We will not give a formal set theoretic definition in the style of Definition 1.2.1, but leave this as an exercise (Exercise 11).

Accordingly, the Cartesian product can be extended to n sets:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\}.$$

As before, we will use the notation A^n for $\underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$.

Relations. Relations, also called **predicates**, are ubiquitous in mathematics. Relations between numbers like "being equal", "being less than" and "being divisible by" come to mind at once. As these examples indicate, many of the relations we commonly encounter are *binary*, i.e., relations relating *two* objects at a time. It will be convenient for us to identify a binary relation with the set of all ordered pairs of elements that stand in that relation. We thus have the following definition:

Definition 1.1.5 *A* **binary relation** *on a set A is any subset of* A^2.

For example the relation $<$ on the set $\mathbb{N}$ of natural numbers is a binary relation, which we identify with the set

$$\{(a, b) \mid a, b \in \mathbb{N} \text{ and } a \text{ is less than } b\}.$$

The relation of "being the mother of" is a binary relation on the set of humans, which we identify with the following set of ordered pairs:

$$\{(x, y) \mid x \text{ and } y \text{ are humans and } x \text{ is the mother of } y\}.$$

Definition 1.1.6 *An* **n-ary relation** *on a set A is any subset of* A^n.

[1] The term "Cartesian" comes from the name of the French mathematician René Descartes (1596–1650), who was the first to introduce coordinate systems and to apply algebraic methods in geometry.

More generally, relations may relate objects from different sets:

Definition 1.1.7 *An* **n-ary relation** *between sets* $A_1, A_2, \ldots, A_n$ *is any subset of* $A_1 \times A_2 \times \cdots \times A_n$.

If $n = 1$ we speak of a **unary relation**, when $n = 2$ of a **binary relation**, when $n = 3$ of a **ternary relation** and so on.

Unary relations correspond to sets. For instance, "being positive" is a unary relation on the set $\mathbb{R}$, which we identify with the set

$$\{a \mid a \in \mathbb{R} \text{ and } 0 < a\}.$$

Conversely, every set A defines the unary relation of "being a member of A".

The set

$$\{(f(x), a) \mid f(x) \text{ is a polynomial, } a \in \mathbb{R} \text{ and } f(a) = 0\}$$

is a binary relation between the set of all polynomials in x and the set $\mathbb{R}$ of real numbers, relating every polynomial to its real roots. The ordered pairs $(x^2 - 4, 2)$ and $(x^2 - 4, -2)$ are in this relation. There is no pair in this relation that has $x^2 + 2$ as first component. Why?

Examples of ternary relations are the relation "A is between B and C" relating triples of points in the plain or the relation "a is greater than $b + c$" between triples of real numbers.

Functions. Informally, a function from a set A to a set B, denoted by $f : A \to B$, can be though a rule that assigns to every element $a \in A$ a unique element $f(a) \in B$. A is called the **domain** of f while B is the **codomain**, or the **target set** of f.

Functions need not take only one argument. The function addition on $\mathbb{R}$, for example, takes two arguments. In general, a function can take any finite number of arguments, but it can always be regarded as a one-argument function if we consider the domain to be the Cartesian product of the domains of the different arguments. For instance, we can think of $+$ as a one-argument function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. Here is the general definition.

Definition 1.1.8 *An* **n-ary function** *f from $A_1 \times A_2 \times \cdots \times A_n$ to B, denoted*

$$f : A_1 \times A_2 \times \cdots \times A_n \to B,$$

is any rule that to every ordered n-tuple $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$ assigns a unique value $f(a_1, a_2, \ldots, a_n) \in B$.

In particular, when $A_1 = A_2 = \cdots = A_n = A$ we call f an n-ary function on the set A, denoted

$$f : A^n \to A.$$

Functions are naturally associated with special relations, called their graphs. For instance, think of some familiar function $f : \mathbb{R} \to \mathbb{R}$, such as $f(x) = x^2$. The graph of this function, as drawn in the plane, consists of the sets of points (x, y) in the Cartesian plane such that $y = x^2$, i.e. the set $\{(x, x^2) \mid x \in \mathbb{R}\}$. This set completely determines the function. Here is the general definition.

Definition 1.1.9 *Given an n-ary function $f : A_1 \times \cdots \times A_n \to B$ the* **graph of f** *is the subset of $A_1 \times A_2 \times \cdots \times A_n \times B$, defined as follows:*

$$G_f = \{(a_1, a_2, \ldots, a_n, f(a_1, a_2, \ldots, a_n)) \mid (a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n\}$$

Sometimes it is technically convenient to identify functions with their graphs. Thus, an n-ary function f can be alternatively defined as a subset of $A_1 \times A_2 \times \cdots \times A_n \times B$, such that for all $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$, there is a unique $b \in B$ such that $(a_1, a_2, \ldots, a_n, b) \in f$. This last condition means that if $(a_1, a_2, \ldots, a_n, b) \in f$ and $(a_1, a_2, \ldots, a_n, b') \in f$ then $b = b'$. Thus an n-ary function can be regarded as a special type of $(n+1)$-ary relation. For instance, the multiplication of two integers can be regarded as the ternary relation:

$$\{(a, b, c) \mid a, b, c \in \mathbb{Z}, \text{and } a \times b = c\}.$$

We will generally use the standard definition of functions as rules but, whenever convenient, we may also adopt the handling functions in terms of their graphs.

1.1.1. *Exercises*

❶ List all elements of the following sets:

(a) $A = \{x \in \mathbb{R} \mid x^2 - 3x = 4\}$

(b) $B = \{y \in \mathbb{Z} \mid (y-1)(y+3)(2y+3)(y+5) = 0\}$

(c) $C = \{x \in \mathbb{Z} \mid -3 \leq x < 3\}$

(d) $D = \{x \in \mathbb{Z} \mid -3 \leq x < 3 \wedge x^2 - 3x = 4\}$

(e) $E = \{x \in \mathbb{Z} \mid -3 \leq x < 3 \wedge x^2 - 3x \neq 4\}$

(f) $F = \{x \in \mathbb{N} \mid x \text{ is an odd single-digit number}\}$

❷ Describe the following sets, using set-builder notation:

(a) $B = \{-\sqrt{3}, \sqrt{3}\}$

(b) $A = \{2, 4, 6, 8\}$

(c) $A = \{2, 4, 6, 8, 10, 12, \ldots\}$

(d) $A = \{\ldots, -7, -4, -1, 2, 5, 8, \ldots\}$

(e) $A = \{0, 1, 8, 27, 64, 125, \ldots\}$

(f) $A = \{-3, -2, -1, 1, 2, 3, 4, 5, 6\}$

(g) $A = \{\ldots, -6, -5, -4, -3, 5, 6, 7, 8, \ldots\}$

❸ Let $A = \{a, b, c\}$. Which of the following are true?

(a) $c \in A$

(b) $b \subset A$

(c) $b \subseteq A$

(d) $\{a\} \subset A$

(e) $\{a\} \subseteq A$

(f) $\{a\} \in A$

(g) $\{a, c\} \subset A$

(h) $\{a, c\} \subseteq A$

(i) $\{a, b, c\} \subset A$

(j) $\{a, b, c\} \subseteq A$

❹ Let $A = \{1, 2\},\ B = \{1, \{2\}, A\}$. Which of the following are true?

(a) $1 \in A$
(b) $2 \in B$
(c) $\{2\} \subset B$
(d) $A \subset B$
(e) $2 \subset B$
(f) $\{2\} \subset B$
(g) $\{2\} \subset A$
(h) $\{A\} \subset A$
(i) $\{A\} \subseteq B$
(j) $A \in B$
(k) $\{A\} \in B$
(l) $\{\{2\}\} \subseteq B$
(m) $\{1, 2\} \in B$
(n) $\{\{1, 2\}\} \subseteq B$

❺ Let $\mathbf{U} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set, $A = \{1, 2, 4, 7\}$ and $B = \{0, 2, 5, 6, 7, 9\}$. Determine:

(a) $A \cap B$
(b) $A \cup B$
(c) $A - B$
(d) $B - A$
(e) $A - (A - B)$
(f) $B - (A - B)$
(g) B'
(h) $(A \cap B)'$
(i) $A \cap B'$
(j) $(A \cup B)'$
(k) $(A \cup B)' - (B - A)'$

❻ Let $A = \{0\}$, $B = \{0, A\}$ and $C = \{0, \{0\}, \{A\}\}$. Determine:

(a) $A \cap B$
(b) $A \cup B$
(c) $B - A$
(d) $A \cap (B \cup C)$
(e) $(B \cup C) - A$
(f) $(A \cap B) \cup (A \cap C)$
(g) $(C - B) - A$
(h) $A \cap (C - A)$

❼ If $A = \{1, 2, 3, 4\}$ and $B = \{0, 1\}$, list the elements of $A \times B$ and of $B \times A$.

❽ If $C = \{a, b\}$, $D = \{\alpha, \beta, \gamma, \delta, \theta\}$ and $E = \{a, c, d\}$, how many elements does each of the following sets have?

(a) $C \times D \times E$
(b) $C \times (D \cap E)$
(c) $D \times (C \cap E)$
(d) $D \cap (C \times E)$
(e) $(C \cup D) \times (C \times E)$
(f) $(C \times E) \cap (E \times C)$

❾ Which of the following relations are graphs of functions?

(a) $\{(a, b) \mid a, b \in \mathbb{R} \text{ and } b = a^2\}$
(b) $\{(a, b) \mid a, b \in \mathbb{R} \text{ and } a = b^2\}$
(c) $\{(a, b, c) \mid a, b, c \in \mathbb{Z}$and c divides $a \times b$ without remainder$\}$
(d) $\{(x, y) \mid x$ and y are humans and y is a parent of $x\}$
(e) $\{(x, y) \mid x$ and y are humans and y is the mother of $x\}$

❿ Prove Proposition 1.2.1: two ordered pairs (a_1, a_2) and (b_1, b_2) (seen as special sets as in Definition 1.2.1) are equal if and only if $a_1 = b_1$ and $a_2 = b_2$.

⓫ Generalize Definition 1.2.1 to a formal set-theoretic definition of an ordered n-tuple.

1.2. Basics of logical connectives and expressions

Here we only provide a compendium of basic logical symbols and notation, commonly used in mathematics. We will not discuss here the underlying concepts of logical languages, semantics and deduction; these will be presented in a detailed and systematic way in the chapters on logic, Chapters 3 and 4.

1.2.1. Propositions, logical connectives, truth tables, tautologies

Propositions

The basic concept of propositional logic is the **proposition**. A proposition is a sentence that can be assigned a **truth value**: `true` or `false`.

Some simple examples are:

- The Sun is hot.
- The Earth is made of cheese.
- 2 plus 2 equals 22.
- The 999-th decimal digit of the number π is 9.

Here are some sentences that are not propositions (why?):

- Are you bored?
- Please, don't go away!
- She loves him.
- x is an integer.
- This sentence is false.

Propositional logical connectives

The propositions above are very simple. They have no logical structure, so we call them **primitive** propositions. From primitive propositions one can form **compound** ones by using **logical connectives**. The most commonly used connectives are:

- **not**, called **negation**, denoted by $\neg$;
- **and**, called **conjunction**, denoted by $\wedge$ (or sometimes, by &);
- **or**, called **disjunction**, denoted by $\vee$;
- **if** … **then** …, called **implication**, or **conditional**, denoted by $\rightarrow$;
- … **if and only if** … , called **biconditional**, denoted by $\leftrightarrow$.

Remark 1 *It is usually not grammatically correct to read compound propositions by simply inserting the names of the logical connectives in between the primitive components. A typical problem arises with the negation: one does not say "Not the earth is square". A uniform way to get round that difficulty and to negate a proposition P is to say "It is not the case that P".*

Thus, given the propositions

"Two plus two equals five" and "The Sun is hot"

we can form the propositions

- "It is **not** the case that two plus two equals five."
- "Two plus two equals five **and** the Sun is hot."
- "Two plus two equals five **or** the Sun is hot."
- "If two plus two equals five **then** the Sun is hot."
- "Two plus two equals five **if and only if** the Sun is hot."

Example 1.2.1

Here is a more involved example. Suppose Mary is a particular person, about whom it is known whether she is clever, lazy, and whether she likes logic. Then, we can consider the following sentences as propositions:

"Mary is clever", "Mary is lazy" and "Mary likes logic."

From these, we can compose a proposition (smoothed out a bit), such as

"Mary is not clever or if she likes logic then she is clever and not lazy."

Truth tables

How about the truth value of a compound proposition? It can be ***calculated*** from the truth values of the components (in much the same way as we can calculate the value of the algebraic expression $A \times (B - C) + B/A$ as soon as we know the values of A, B, C) by following the rules of "propositional arithmetic":

- The proposition $\neg A$ is true if and only if the proposition A is false.
- The proposition $A \wedge B$ is true if and only if both A and B are true.
- The proposition $A \vee B$ is true if and only if either of A or B (possibly both) is true.
- The proposition $A \to B$ is true if and only if A is false or B is true, i.e. if the truth of A implies the truth of B.
- The proposition $A \leftrightarrow B$ is true if and only if A and B have the same truth values.

We can systematize these rules in kinds of "multiplication tables". For that purpose, and to make it easier for symbolic (i.e. mathematical) manipulations, we introduce a special notation for the two truth values by denoting the value `true` by `T` and the value `false` by `F`. Another common notation, particularly used in computer science, is to denote `true` by **1** and `false` by **0**.

Now the rules of the "propositional arithmetic" can be tabulated by means of the following *truth tables* (p and q below stand for arbitrary propositions):

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Computing truth values of propositions

One can see from the truth tables that *the truth value of a compound proposition does not depend on the meaning of the component propositions, but only on their truth values*. Therefore, to check the truth of such a proposition we merely need to replace all component propositions by their respective truth values and then 'calculate' the truth of the whole proposition, using the truth tables of the logical connectives. Thus,

- "It is not the case that two plus two equals five" is true;
- "Two plus two equals five and the Sun is hot" is false;
- "Two plus two equals five or the Sun is hot" is true;
- "If two plus two equals five, then the Sun is hot" is true (even though it does not make much sense).

Returning to Example 1.2.1, suppose we know that

"Mary is clever" is true,

"Mary is lazy" is false,

"Mary likes logic" is true.

Then the truth value of the compound proposition

"Mary is not clever or if she likes logic, then she is clever and not lazy"

can be determined just as easily. However, in order to do so, we first have to analyze the *syntactic structure* of the proposition, i.e. to determine how it has been composed or in what order the logical connectives occurring therein have been applied. With algebraic expressions such as $A \times (B - C) + B/A$ that analysis is a little easier thanks to the use of parentheses and the established priority order amongst the arithmetic operations. Let us also make use of parentheses and rewrite the sentence in the way we (presumably) all understand it:

"(Mary is not clever) or (if (she likes logic), then ((she is clever) and (not lazy)))."

The structure of the sentence should be clear now. We can, however, go one step further and make it look exactly like an algebraic expression by using letters to denote the occurring

Table 1.1 Logical connectives paired with set theoretic operations

Logical symbol	Set-theoretic symbol
$\wedge$	$\cap$
$\vee$	$\cup$
$\neg$	$'$

primitive propositions. Let us, for instance, denote

"Mary is clever" by A,

"Mary is lazy" by B,

"Mary likes logic" by C.

Now our compound proposition can be neatly rewritten as

$$(\neg A) \vee (C \to (A \wedge \neg B)).$$

The last step is to calculate the truth value. Here we make use of our observation above and simply replace the primitive propositions A, B and C by their truth values and do the formal calculation following the truth tables step by step:

$$(\neg \mathtt{T}) \vee (\mathtt{T} \to (\mathtt{T} \wedge \neg \mathtt{F})) = \mathtt{F} \vee (\mathtt{T} \to (\mathtt{T} \wedge \mathtt{T})) = \mathtt{F} \vee (\mathtt{T} \to \mathtt{T}) = \mathtt{F} \vee \mathtt{T} = \mathtt{T}.$$

A note on the relation between logic and set theory

The astute reader would have noticed a close connection between basic, "propositional" logical connectives and their properties, studied further in Chapter 3, on the one hand, and the basic set theoretic operations and their properties, on the other. To explore this connection fully would lead us far beyond the scope of this book, but a few comments are nevertheless in order. Firstly, we can compile something like a "translation manual" of the symbols, as depicted in Table 1.1.

Using this table, one can translate any proposition into a set-theoretic expression (by regarding the primitive propositions as names of sets) and conversely. For example, the set theoretic translation of the propositional $A \vee \neg A$ would be $A \cup A''$, with A regarded as the name of a set. How are these related? Intuitively, two propositions have the same logical meaning precisely when their "set-theoretic translations" represent equal sets. We will give a more precise answer in Chapter 3.

1.2.2. Individual variables and quantifiers

Using only propositional logical connectives we cannot express very simple mathematical statements such as

"$x + 2$ is greater than 5",

"There exists y such that $y^2 = 2$",

"For every real number x, if x is greater than 0, then there exists a real number y such that y is less than 0 and y^2 equals x", etc.

Indeed, an expression such as "$x + 2$ is greater than 5" is *not* a proposition, for it can be true or false depending on the choice of x. Neither is "There exists y such that $y^2 = 2$" a proposition until the range of possible values of y is specified: if y is an integer, then the statement is false, but if y can be any real number, then it is true. As for the third sentence above, it *is* a proposition, but its truth heavily depends on its internal logical structure and mathematical meaning of all phrases involved, and those are not tractable on a propositional level. All these sentences are from the realm of **first-order logic**, often just called **classical logic**, which we study in Chapter 4. Here we only introduce the basic symbols of first-order logic as they are widely used in the mathematical literature, and will also be used in Section 2.4, before we study logic itself.

Individual variables

Often in mathematics we deal with unknown or unspecified objects (individuals) from the domain of discourse: integers, real numbers, geometric points, elements of abstract sets, etc. In order to be able to reason and make statements about such objects, we use **individual variables** to denote them. For instance, talking about numbers, we use phrases like "Take a positive integer n", "For every real number x greater than $\sqrt{2} \dots$", etc. In natural language instead of variables we usually use pronouns or other syntactic constructions but that can often lead to awkwardness or even ambiguity (e.g. "If a man owes money to another man then that man hates the other man"). Therefore, the use of variables in mathematics is indispensable. We also use variables as *placeholders* for the arguments of the various predicates and functions we deal with. Usually, individual variables will be denoted by $x, y, z, \dots$, possibly with indices.

Quantifiers

In both mathematical discourse and in everyday speech, we often use special phrases used to quantify over objects of our discourse, such as (note the use of individual variables here):

- "(for) every (objects) x",
- "there is (an object) x such that ",
- "for most (objects) x",
- "there are at least 5 (objects) x such that",
- "there are more (objects) x than y such that ", etc.

Such phrases are called **quantifiers**. The first two of those listed above are particularly important and most commonly used in mathematics, and many others can be expressed by means of them, so they are given special names and notation:

– The quantifier "for every " is called **universal quantifier**, denoted by $\forall$.
– The quantifier "there exists " is called **existential quantifier**, denoted by $\exists$.

Besides the phrases above, the universal quantifier is usually represented by *"all"*, *"for all"* and *"every"* while the existential quantifier can appear as *"there is"*, *"some"* and *"for some"*, particularly, in a nonmathematical discourse.

Mathematical terms and formulae

Typically, we build mathematical expressions by using individual variables and constants (names) for specific individuals (e.g. the numbers 0,–5, $\sqrt{2}$, π) and functions and relations that we apply to these. We also use auxiliary symbols, such as parentheses, and **logical symbols**, including the propositional connectives introduced earlier, as well as equality = and quantifiers. Using all these components and following certain syntactic rules, we can compose formal expressions that allow us to symbolically represent mathematical statements, to reason about them and to prove them in precise, well-structured and logically correct ways, which we will present and discuss in detail in Chapter 4.

There are two main types of expressions used in mathematics:

1. **Terms**, used to denote objects, which we build from names and variables for such objects, possibly by applying functions to them. Some examples of mathematical terms are: $1, x, (x+1), (x+1)(x-2), \sqrt{(x+1)(y-z)}, \sin(x+\pi/2)$, etc.
2. **Formulae**, used to make statements about objects, by applying relations between terms, and then composing more complex formulae from simpler ones by applying logical symbols. Here are some examples of formulae used to make statements about numbers:

   - $x > 0$, saying that (the value of) x is greater than 0. This could be true or false, depending on the actual value of$x$.

   - $x \times y = y \times x$, saying that multiplication is commutative. This is true for all real values of x and y but not necessarily true if x and y denote, say, matrices.

   - $x < y \to \neg y < x$, saying that if (the value of) x is less than (the value of) y then it is not the case that y is less than x. Again, this is true for all numerical values of x and y, because of the standard meaning of < and >.

   - $\exists x(x < 0)$, saying that there is a (value of) x that is less than 0. This is true if x is an integer or real, but false if x represents a natural number.

   - $\forall x \forall x(x < y \to \neg y < x)$, saying that for all (values of) x and y, if x is less than y then it is not the case that y is less than x. As noted above, this is true if x and y represent natural numbers, because of the meaning of < and > on real numbers.

   - $\forall x(x > 0 \to \exists y(y > 0 \land y < x))$, saying that for every positive (value of) x there is a lesser, still positive (value of) y. This is true if x and y represent rational or real numbers, but false if they represent integers, when x is taken as 1.

At this stage it is sufficient to understand the meaning of such mathematical expressions and statements informally and to be able to use logical symbols to make such simple statements. In Chapter 4 these skills will be lifted to a higher level and be made much more precise.

1.2.3. *Exercises*

❶ Which of the following are propositions?

(a) $2^3 + 3^2 = 17$

(b) $2^3 + 3^2 = 71$

(c) $2^3 + 3^2 = x$

(d) Will you marry me?

(e) John married on January 1st, 1999.

(f) Mary is not happy.

(g) I told her about John.

(h) If John loves Lisa then Mary is not happy.

❷ If A and B are true propositions and C and D are false ones, determine the truth values of the following compound propositions:

(a) $A \wedge (\neg B \vee C)$

(b) $\neg(\neg A \vee D) \to \neg B$

(c) $\neg(\neg D \wedge (B \to \neg A))$

(d) $(C \to A) \to D$

(e) $C \to (A \to D)$

(f) $\neg(C \to A) \vee (C \to D)$

(g) $(A \to \neg A) \vee (C \leftrightarrow \neg B)$

❸ Write each of the following composite propositions in a symbolic form by identifying its atomic propositions and logical structure. Then determine its truth value.

(a) The Sun is hot and if the Earth is larger than Jupiter then there is life on Jupiter.

(b) If the Sun rotates around the Earth or the Earth rotates around the Moon then the Sun rotates around the Moon.

(c) The Moon does not rotate around the Earth if the Sun does not rotates around the Earth and the Earth does not rotate around the Moon.

(d) The Earth rotate around itself only if the Sun rotates around the Earth or the Moon does not rotate around the Earth.

(e) The Earth rotates around itself if and only if the Sun does not rotate around itself or the Moon does not rotate around itself.

❹ Determine the truth value of the proposition A in each of the following cases:

(a) B and $B \to A$ are true.

(b) $A \to B$ is true and B is false.

(c) $\neg B$ and $A \vee B$ are true.

(d) Each of $B \rightarrow \neg A$, $\neg B \rightarrow \neg C$ and C is true.

(e) Each of $\neg C \wedge B$, $C \rightarrow (A \vee B)$ and $\neg(A \vee C) \rightarrow C$ is true.
(Hint: consider cases for the truth of B and C.)

❺ Translate the following formulae into English and determine which of them are true propositions in the set of natural numbers $\mathcal{N}$ and which are true propositions in the set of real numbers $\mathcal{R}$.

(a) $\neg\forall x(x \neq 0)$

(b) $\forall x(x^3 \geq x)$

(c) $\forall x(x = x^2 \rightarrow x > \mathbf{0})$

(d) $\exists x(x = x^2 \wedge x < \mathbf{0})$

(e) $\forall x(x > \mathbf{0} \rightarrow x^2 > x)$

(f) $\forall x(x = \mathbf{0} \vee \neg x + x = x)$

(g) $\forall x\forall y(x > y \vee y > x)$

(h) $\forall x\exists y(x > y^2)$

(i) $\forall x\exists y(x > y^2 \vee y > \mathbf{0})$

(j) $\forall x(x \geq \mathbf{0} \rightarrow \exists y(y > \mathbf{0} \wedge x = y^2))$

(k) $\forall x\exists y(x > y \rightarrow x > y^2)$

(l) $\forall x\exists y(\neg x = y \rightarrow x > y^2)$

(m) $\exists x\forall y(x > y)$

(n) $\exists x\forall y(x + y = x)$

(o) $\exists x\forall y(x + y = y)$

(p) $\exists x\forall y(x > y \vee -x > y)$

(q) $\exists x\forall y(x > y \vee \neg x > y)$

(r) $\exists x\forall y(y > x \rightarrow y^2 > x)$

(s) $\exists x\forall y(x > y \rightarrow x > y^2)$

(t) $\exists x\exists y(xy = x + y)$

(u) $\forall x\forall y(x > y \rightarrow \exists z(x > z \wedge z > y))$

1.3. Mathematical induction

Mathematical induction is a method for proving statements about all natural numbers. Recall that we assume the set of natural numbers to be $\mathbb{N} = \{0, 1, 2, \ldots\}$. Here is the most common version of the Principle of Mathematical Induction (PMI):

Theorem 1.3.1 (Principle of Mathematical Induction) *Suppose that some property P of natural numbers holds for 0 and whenever P holds for some natural number k, then it also holds for $k + 1$. Then P holds for every natural number.*

We write $P(n)$ to say that the property P holds for the number n. Thus, using the PMI to prove a statement of the type "$P(n)$ holds for every natural number n" requires proving the following two steps:

1. *Base step*: $P(0)$ holds.
2. *Induction step*: for any $k \in \mathbb{N}$, if $P(k)$ holds, then $P(k + 1)$ also holds.

We illustrate the use of the PMI with the following example.

Example 1.3.2

Prove that for every natural number n,

$$0 + 1 + \cdots + n = \frac{n(n+1)}{2}. \tag{1.1}$$

Proof: Let us denote the statement in Equation (1.1) by $P(n)$. First, we have to verify the Base step, i.e. to show that $P(0)$ is true. Here:

$$0 = \frac{0(0+1)}{2}.$$

Then, to verify the Induction step, suppose that $P(k)$ is true for some $k \geq 0$. This assumption is called the *Inductive Hypothesis (IH)*. We now have to show that $P(k + 1)$ is also true. Here we go:

$$\begin{aligned} 0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

where we use the Inductive Hypothesis to justify the second equality.

Thus, by the Principle of Mathematical Induction, we conclude that $P(n)$ holds for every natural number n.

Sometimes, we only need to prove that a property holds for all natural numbers greater than some $m \geq 0$. Then the PMI works just as well, by starting with a Base step $P(m)$ and applying the Induction step only to natural numbers $k \geq m$.

Example 1.3.3

Prove that $n^2 \leq 2^n$ for every natural number $n \geq 4$.

Proof: Let $P(n)$ be the statement $n^2 \leq 2^n$.

Base step. Prove that $P(4)$ is true:

$$4^2 = 16 \leq 16 = 2^4$$

Induction step. Assume that $P(k)$ is true for some $k \geq 4$ (the Inductive Hypothesis (IH)), i.e. $k^2 \leq 2^k$.

Now we have to prove that $(k+1)^2 \leq 2^{(k+1)}$. We begin with:

$$(k+1)^2 = k^2 + 2k + 1 = k\left(k + 2 + \frac{1}{k}\right)$$

Our goal is to rewrite this equation so that we can use the IH. Note that $1/k < 1$ since $k \geq 4$. Therefore

$$k\left(k + 2 + \frac{1}{k}\right) \leq k(k + 2 + 1) = k(k+3) \leq k(k+k) = 2k^2$$

We can now use the IH to get

$$2k^2 \leq 2.2^k = 2^{(k+1)}$$

Therefore we have that $(k+1)^2 \leq 2^{(k+1)}$, as required.

By the PMI, we conclude that $n^2 \leq 2^n$ for all natural numbers $n \geq 4$.

Later, in the beginning of the chapter on number theory (Chapter 5), we will extend and generalize the Principle of Mathematical Induction.

1.3.1. Exercises

❶ Prove, by using the PMI, that $n^2 + n$ is even for all natural numbers n.
❷ Prove that $2^n < n!$ for all natural numbers $n \geq 4$.
❸ Show that for every natural number $n \geq 1$, the power set of the set $\{1, 2, 3, \ldots, n\}$ has 2^n elements.

Paradoxes

A *paradox* is a statement or an argument that is an apparent contradiction, either with well-known and accepted truths or simply with itself. Unlike a *fallacy*, a paradox is not due to an incorrect reasoning, but it could be based on a play of words, or on using self-referential statements, or on a very subtle ambiguity in the assumptions or concepts involved. Some set-theoretic paradoxes, however, such as *Russell's paradox* (see the box at the end of Section 2.1 in Chapter 2), lead to deeper foundational issues of mathematics.

Probably the best-known logical **Paradox of the Liar** is the person saying "I am lying now", i.e. "This sentence is false". If the sentence is true, then it truly claims that it is false and, if it is false, then it falsely states so; hence it must be true.

There are may variations of this paradox, one of them being **Jourdain's card paradox**. This is a card, on both sides of which there is a written statement. The statement on one side of the card says "The sentence on the other side of this card is *true*" and the one on the other side says "The sentence on the other side of this card is *false*". Think of each of these as to whether it is true or false.

We will see more on logical fallacies and set theory paradoxes further. Here we offer an assortment of some simple mathematical fallacies and genuine paradoxes.

The heap paradox. How many grains of wheat, put together, form a heap? A hundred, a million … ? Well, neither! Indeed, one grain is certainly not a heap and, if k grains do not form a heap, adding just one more will not make a heap either. Therefore, by induction, for every $n \in \mathbb{N}$, a collection of n grains is *not a heap*. What is wrong with this argument?

All horses have the same colour. Here is another proof by induction, of the claim that *all horses have the same colour*. Indeed, there are only finitely many horses, so it suffices to prove that for every $n \in \mathbb{N}$, in any set of n horses they all have the same colour. When $n = 1$ this is certainly true. Now, suppose this claim is true for n and let us take any set of $n + 1$ horses, say $H_1, H_2, \ldots, H_{n+1}$. Then, by the inductive hypothesis, the first n of them, $H_1, H_2, \ldots, H_n$, have the same colour, in particular H_1, H_2 have the same colour and the last n of them, $H_2, \ldots, H_{n+1}$, have the same colour. Therefore, they all have the same colour. Do you agree? Why?

All natural numbers are interesting. Surely, there are many interesting natural numbers, for instance 0 is the first natural number, 1 is interesting for being the first positive natural number, 2 is the first prime number, etc. However, are there any natural numbers that are *not* interesting? Suppose there are. Then, there is a least number amongst them, say k. However, then k is *the least non interesting natural number*, which is certainly quite an interesting property, isn't it? Therefore, k is in fact interesting, which is a contradiction. Therefore, all natural numbers are interesting!

Berry paradox. Every natural number can be defined in English with sufficiently many words. However, if we bound the number of words that can be used, then only finitely many natural numbers can be defined, so there will be ones that cannot be defined with that many words. Therefore, there must be a *smallest* so undefinable natural number. Now, consider the following sentence: "The least natural number that is not definable in English with less than twenty words." There is a uniquely determined natural number that satisfies this description, so it *is a definition* in English, correct? Well, count how many words it uses.